ON SOME NON-LINEAR ELLIPTIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

BY

PHILIP HARTMAN and GUIDO STAMPACCHIA

The John Hopkins University, Baltimore, Md., U.S,A. and Università di Pisa, Pisa, Italy(1)

The object of this paper is to obtain existence and uniqueness theorems for (weak) uniformly Lipschitz continuous solutions $u(x)$ of Dirichlet boundary value problems associated with non-linear elliptic differential-functional equations of the form

$$[a_j(\operatorname{grad} u)]_{x_j} + F[u](x) = 0, \tag{0.1}$$

where, for a fixed x, $F[u](x)$ is a non-linear functional of u. The results to be obtained can be considered as generalizations of some theorems of Gilbarg [5] and Stampacchia [14] in the case $F[u] \equiv 0$ and of some theorems of Stampacchia [14] in certain cases $F[u] \not\equiv 0$.

Part I deals with the functional analysis basis for the proofs. It gives existence theorems for the solutions of certain non-linear, functional inequalities. By a weak solution of (0.1) on a domain Ω is usually understood a function $u(x)$ having a gradient u_x in some sense and satisfying

$$\int_\Omega \left\{ a_j(u_x)\eta_{x_j} - F[u]\eta \right\} dx = 0 \tag{0.2}$$

for all continuously differentiable $\eta(x)$ with compact support in Ω, i.e., $\eta \in C_0^1(\Omega)$. Part I will imply existence and uniqueness theorems for functions $u(x)$, to be called quasi solutions, satisfying

$$\int_\Omega \left\{ a_j(u_x)\eta_{x_j} - F[u]\eta \right\} dx \geqslant 0 \tag{0.3}$$

for η in certain subsets of $C_0^1(\Omega)$ depending on u. A particular case of this situation arises, for example, if one seeks the solution of a variational problem

$$\min \int_\Omega \{ f(u_x) + \dots \} dx$$

(1) This research was partially supported by the Air Force Office of Scientific Research under Contract AF 49 (638)–1382 and Grant AF EOAR 65–42.

in a convex set of functions u. If the minimum is attained at an interior point u of the convex set, one expects $u(x)$ to be a weak solution of the corresponding Euler equation, say (0.2), for all $\eta \in C_0^1(\Omega)$. But if the minimum is attained at a boundary point of the convex set, one can only expect to obtain inequalities of the type (0.3) for a more restricted class of test functions.

Professor H. Lewy called our attention to the technique of his paper [9]. In the variational case, this involves the consideration of the desired solution as a limit, as $K \to \infty$, of a minimizing function for the case when the competing functions are restrained to be uniformly Lipschitz continuous with a Lipschitz constant not exceeding K. This idea is the motivation for our introduction of quasi solutions; cf. also [15].

Part II will deal with a priori estimates for quasi solutions. The methods will be similar to, but simpler than, those of [14]. One of the main simplifications (which permits the avoidance of results of De Giorgi [4] and their extension to the boundary) arises from an adaptation of an idea of Rado [12], p. 63; cf. the proof of Lemma 10.0 below. A similar use of Rado's device occurs in Miranda [11].

The first two sections of Part III give existence and uniqueness theorems for Dirichlet boundary value problems associated with (0.3). One of the novel features of the results below is the fact that the equations considered involve non-linear functionals, rather than functions, of the unknown u. The last section is concerned with the regularity (beyond that of Lipschitz continuity) for solutions. The results of De Giorgi and their extensions are used only in the last section.

Part I. Functional analysis

1. An existence theorem. Let X be a reflexive Banach space over the reals and X' its strong dual (=conjugate space). The pairing of X' and X will be denoted by (u', u).

Let Y be a closed linear manifold in X. Suppose that Y is also a Banach space with a norm $\|\cdot\|_Y$ which may be different from that of $\|\cdot\|_X$. By the closed graph theorem, there exist constants $0 < \theta_1, \theta_2 \leq 1$ such that

$$\theta_1 \|y\|_X \leq \|y\|_Y \leq \|y\|_X / \theta_2 \quad \text{for} \quad y \in Y. \tag{1.1}$$

The pairing of Y' and Y will be denoted by $\langle y', y \rangle$.

If S is a subset of X and $\varphi \in X$, then $S + \varphi$ will denote the translation of S by φ; i.e., $S + \varphi = \{u\colon u = s + \varphi,\ s \in R\}$.

In the theorems of Part I, $\mathfrak{K}$ will denote a closed convex subset of X with the property that

$$u_1, u_2 \in \mathfrak{K} \quad \Rightarrow \quad u_1 - u_2 \in Y. \tag{1.2}$$

This is the case if and only if there exists a closed convex set $\mathfrak{K}_0$ in Y,

$$0 \in \mathfrak{K}_0 \subset Y, \tag{1.2'}$$

and an element $\varphi \in X$ such that $\mathfrak{K} = \mathfrak{K}_0 + \varphi$. It is clear that $\mathfrak{K} = \mathfrak{K}_0 + \varphi$ has the property (1.2); conversely, if $\varphi \in \mathfrak{K}$, then $\mathfrak{K}_0 = \mathfrak{K} - \varphi$ has property (1.2′) and $\mathfrak{K} = \mathfrak{K}_0 + \varphi$.

THEOREM 1.1.(1) *Let X, Y be as above, $\mathfrak{K}$ a closed convex set in X satisfying (1.2). For every $u \in \mathfrak{K}$, let $A(u)$ be a bounded linear functional on Y, with the metric induced by X, and let $A(u)$ have the following properties;*

(i) *if M is any linear manifold in Y with $\dim M < \infty$ and $\varphi \in \mathfrak{K}$, then $(A(u), v)$ is a continuous function of u, v for $u \in \mathfrak{K} \cap (M + \varphi)$, $v \in M$;*

(ii) *$A(u)$ is monotone*, i.e.,

$$(A(u_2) - A(u_1), u_2 - u_1) \geq 0 \quad for \quad u_1, u_2 \in \mathfrak{K}; \tag{1.3}$$

(iii) *when $\mathfrak{K}$ is not bounded, $A(u)$ is coercive in the sense that there exists some $\varphi_0 \in \mathfrak{K}$ satisfying*

$$(A(u) - A(\varphi_0), u - \varphi_0)/\|u - \varphi_0\|_X \to \infty \quad as \quad \|u\|_X \to \infty, \quad u \in \mathfrak{K}. \tag{1.4}$$

Let $u \to C(u)$ be a mapping from $\mathfrak{K}$ to Y' which is completely continuous (i.e., is continuous from the weak topology of $\mathfrak{K} \subset X$ to the strong topology of Y') and which is bounded,

$$\|C(u)\|_{Y'} \leq L \quad for \quad u \in \mathfrak{K}, \tag{1.5}$$

L constant.

Then there exists at least one $u_0 \in \mathfrak{K}$ satisfying

$$(A(u_0), v - u_0) \geq \langle C(u_0), v - u_0 \rangle \quad for \quad v \in \mathfrak{K}. \tag{1.6}$$

Remark. Since $v - u_0$ occurs linearly, it follows that (1.6) holds for all v in the cone $\{v : v = u_0 + tw,\ w \in \mathfrak{K} - u_0 \text{ and } t \geq 0\}$ with vertex u_0. This cone contains $\mathfrak{K}$ and becomes $Y + u_0$ when 0 is an interior point of the subet $\mathfrak{K} - u_0$ of Y. In the latter case, equality holds in (1.6).

Theorem 1.1 contains, as a special case, the main result of [15]. We had originally formulated this theorem with a monotony condition stronger than (1.3). The question of the validity of the theorem, as stated above, was suggested to us by J. L. Lions.

(1) Added in proof (Jan. 18, 1966). After this paper was submitted for publication, the authors received a preprint of the article F. BROWDER, *Nonlinear monotone operators and convex sets in Banach spaces*, which has now appeared in *Bull. Amer. Math. Soc.* 71 (1965), 780–785. This article contains Theorem 1.1 with $C(u) \equiv 0$. Our proof is similar to Browder's in that it involves first the case $\dim X < \infty$ and then a passage to a limit. In contrast to our Lemma 3.1, Browder's proof in the finite dimensional case uses the monotony of $A(u)$ (and hence requires $C(u) \equiv 0$).

Actually, in this theorem, there is no loss of generality in assuming that $\|\cdot\|_X = \|\cdot\|_Y$ and $(\cdot,\cdot) = \langle\cdot,\cdot\rangle$; cf. the part (a) of Section 4. The formulation of the theorem involving two norms for Y is suggested by applications.

In order to illustrate the significance of the different assumptions and the way that this theorem will be applied, let $X = H^{1,2}(\Omega)$ for some bounded open $\Omega \subset E^n$ and $Y = H_0^{1,2}(\Omega)$; cf. Section 7 for definitions. Let $\varphi(x)$ be a function which is uniformly Lipschitz continuous on $\bar{\Omega}$ and $\mathfrak{K} = \mathcal{K}_\varphi^K$ the subset of $H^{1,2}(\Omega)$ consisting of uniformly Lipschitz continuous functions $u(x)$ with a Lipschitz constant not exceeding K and satisfying $u(x) = \varphi(x)$ for $x \in \partial\Omega$. If K is as large as the Lipschitz constant of $\varphi(x)$, then $\varphi(x) \in \mathcal{K}_\varphi^K$ and $\mathfrak{K}_0 = \mathcal{K}_\varphi^K - \varphi$ satisfies (1.2'). In this case, $\mathfrak{K} = \mathcal{K}_\varphi^K$ is bounded and so, no coercivity condition (1.5) is needed. Let $A(u)$ be defined by

$$(A(u), v) = \int_\Omega a_i(u_x) v_{x_i}\, dx$$

for $u \in \mathcal{K}_\varphi^K$, $v \in H_0^{1,2}(\Omega)$, where $u_x = \operatorname{grad} u = (u_{x_1}, \dots, u_{x_n})$. In this case, the continuity condition (i) holds if $a_i(p) = a_i(p_1, \dots, p_n)$, for $i = 1, \dots, n$, are real-valued, continuous functions of p in the Euclidean sphere $|p| \leqslant K$. The weak ellipticity condition

$$[a_i(p) - a_i(q)](p_i - q_i) \geqslant 0$$

implies the monotony (1.3).

It will be clear from the proof that if $\mathfrak{K}$ is unbounded (so that (1.4) holds), then (1.5) can be relaxed to

$$\limsup \|C(u)\|_{Y'} / \|u\|_X < \infty, \tag{1.7}$$

or even to

$$\|C(u)\|_{Y'} \|u - \varphi_0\|_Y / (A(u) - A(\varphi), u - \varphi_0) \to 0, \tag{1.8}$$

as $u \in \mathfrak{K}$, $\|u\|_X \to \infty$.

COROLLARY 1.1. *Assume the conditions of Theorem 1.1. Let $u_1, u_2, \dots$ be elements of $\mathfrak{K}$ such that $u_0 = u_m$ satisfies (1.6) for $m = 1, 2, \dots$. Let $u_m \to u_\infty$ weakly in X as $m \to \infty$. Then $u_\infty \in \mathfrak{K}$ and $u_0 = u_\infty$ satisfies (1.6).*

This assertion is a consequence of Lemma 2.3 below. For, by this lemma, (1.6) is equivalent to

$$(A(v), v - u_0) \geqslant \langle C(u_0), v - u_0 \rangle \quad \text{for} \quad v \in \mathfrak{K}. \tag{1.9}$$

Since $\mathfrak{K}$ is convex and closed, it is weakly closed, so that $u_\infty \in \mathfrak{K}$. It is clear that u_0 can be replaced by u_m in (1.9), and letting $m \to \infty$ gives the corresponding relation with $u_0 = u_\infty$ (since $C(u)$ is completely continuous).

COROLLARY 1.2. *If, in Theorem 1.1, $A(u)$ and $C(u)$ satisfy*

$$(A(u_2)-A(u_1),\, u_2-u_1) > \langle C(u_2)-C(u_1),\, u_2-u_1\rangle \quad \textit{for} \quad u_1, u_2(\neq u_1) \in \mathfrak{K}, \tag{1.10}$$

then u_0 is unique.

Condition (1.10) holds, for instance, if $C(u)\equiv y'$ is independent of u and $A(u)$ satisfies

$$(A(u_2)-A(u_1),\, u_2-u_1) > 0 \quad \text{for} \quad u_1, u_2(\neq u_1) \in \mathfrak{K}.$$

In order to prove the last corollary, let u_0, u_1 be two solutions of (1.6), so that

$$(A(u_0),\, u_1-u_0) \geqslant \langle C(u_0),\, u_1-u_0\rangle, \qquad (A(u_1),\, u_0-u_1) \geqslant \langle C(u_1),\, u_0-u_1\rangle.$$

Adding these inequalities gives

$$(A(u_0)-A(u_1),\, u_0-u_1) \leqslant \langle C(u_0)-C(u_1),\, u_0-u_1\rangle.$$

Hence $u_0=u_1$ by (1.10).

The proof of Theorem 1.1 will be given in two parts: first, the case where Y is a finite dimensional manifold (Section 4) and, second, a limit process (Section 5). The second part depends on an application of arguments of Minty generalized by Browder (cf., in particular, the proof of Theorem 4 in [3]).

2. A priori bounds. In what follows, φ_0 denotes a fixed element of $\mathfrak{K}$, chosen so as to satisfy (1.4) if $\mathfrak{K}$ is unbounded.

LEMMA 2.1. *Let X, Y, $\mathfrak{K}$, $A(u)$ be as in Theorem 1.1 and let $y' \in Y'$, $\|y'\|_{Y'} \leqslant L$. Then there exists a constant $R=R(L)$ such that any solution $u_0 \in \mathfrak{K}$ of*

$$(A(u_0),\, v-u_0) \geqslant \langle y',\, v-u_0\rangle \quad \textit{for} \quad v \in \mathfrak{K} \tag{2.1}$$

satisfies

$$\|u_0\|_X \leqslant R. \tag{2.2}$$

Proof. Let $v=\varphi_0$ in (2.1) and rewrite the resulting inequality as

$$(A(u_0),\, u_0-\varphi_0) \leqslant \langle y',\, u_0-\varphi_0\rangle.$$

Hence

$$(A(u_0)-A(\varphi_0),\, u_0-\varphi_0) \leqslant \langle y',\, u_0-\varphi_0\rangle - (A(\varphi_0),\, u_0-\varphi_0).$$

The right side is majorized by $L\|u_0-\varphi_0\|_X/\theta_2 + \|A(\varphi_0)\|\cdot\|u_0-\varphi_0\|_X$ by (1.1). Thus

$$(A(u_0)-A(\varphi_0),\, u_0-\varphi_0) \leqslant (L/\theta_2 + \|A(\varphi_0)\|)\|u_0-\varphi_0\|_X.$$

If $\mathfrak{K}$ is bounded, the lemma is trivial. If $\mathfrak{K}$ is unbounded, the assertion follows from (1.4).

LEMMA 2.2. *In the proof of Theorem 1.1, there is no loss of generality in assuming that $\mathfrak{K}$ is bounded; e.g., that $\mathfrak{K}$ is replaced by $\mathfrak{K} \cap \{\|u\|_X \leqslant r\}$, where $r > R(L)$ and $R(L)$ is given in Lemma 2.1.*

This is a consequence of Lemma 2.1 and the Remark following Theorem 1.1.

LEMMA 2.3. *Let $X, Y, \mathfrak{K}$, $A(u)$ be as in Theorem 1.1 and $y' \in Y'$. Then $u_0 \in \mathfrak{K}$ satisfies (2.1) if and only if*

$$(A(v), v-u_0) \geq \langle y', v-u_0 \rangle \quad \textit{for} \quad v \in \mathfrak{K}.$$

The proof depends on a device introduced by Minty [10].

Proof. The inequality (2.1) implies (2.3) by the monotone condition (1.3). In order to deduce the converse, assume (2.3). Let $w \in \mathfrak{K}$ be arbitrary. Then

$$v = u_0 + t(w-u_0) = tw + (1-t)\,u_0$$

is in the convex set $\mathfrak{K}$ for $0 \leq t \leq 1$. Thus, (2.3) gives

$$t(A(u_0+t(w-u_0)), w-u_0) \geq t\langle y', w-u_0 \rangle \quad \text{for} \quad w \in \mathfrak{K}.$$

Dividing by $t>0$ and letting $t \to 0$ gives (2.1) by virtue of the continuity condition (i). This proves the lemma.

COROLLARY 2.1. *In Lemma 2.3, the set of solutions u_0 of (2.1) is convex.*

3. Finite dimensional case. In this section, we shall prove the finite dimensional analogue of Theorem 1.1. Actually, no monotony assumption is involved.

LEMMA 3.1. *Let $\mathfrak{C}$ be a compact convex set in E^n and $B(u)$ a continuous map of $\mathfrak{C}$ into E^n. Then there exists $u_0 \in \mathfrak{C}$ such that*

$$(B(u_0), v-u_0) \geq 0 \quad \textit{for} \quad v \in \mathfrak{C}, \tag{3.1}$$

where $(\cdot\,,\,\cdot)$ denotes the scalar product in E^n.

Proof. If $\mathfrak{C}$ is a point, the lemma is trivial. If $\mathfrak{C}$ is not a point, then it can be supposed that $\mathfrak{C}$ has interior points for otherwise, without loss of generality, E^n is replaced by a suitable subspace of E^n containing $\mathfrak{C}$. Since a translation of the space E^n does not affect the assumption or assertion, it can be supposed that $u=0$ is an interior point of $\mathfrak{C}$.

Let $u_0 \in \partial\mathfrak{C}$. Then (3.1) holds if and only if there is a hyperplane π through u_0, supporting $\mathfrak{C}$ such that if $N \neq 0$ is a vector orthogonal to π and pointing into the half-space not containing $\mathfrak{C}$, then $B(u_0) = -tN$ for some $t \geq 0$.

Case 1. *$\partial\mathfrak{C}$ is of class C^1.* Assume that (3.1) fails to hold for all $u_0 \in \partial\mathfrak{C}$. We shall show that

$$B(u) = 0 \tag{3.2}$$

has a solution $u_0 \in \mathfrak{C}$ (which satisfies (3.1) trivially).

Let $N(u_0)$ be the outward, unit normal vector at $u_0 \in \partial\mathfrak{C}$. Then

$$B(u_0, t) = (1-t)\,B(u_0) + tN(u_0),\ 0 \leq t \leq 1,$$

is a deformation of the vector field $B(u_0)$, $u_0 \in \partial\mathfrak{C}$, into the vector field $N(u_0)$. The assumption that (3.1) does not hold for $u_0 \in \partial\mathfrak{C}$ implies that $B(u_0, t) \neq 0$ for $u_0 \in \partial\mathfrak{C}$, $0 \leqslant t \leqslant 1$. Hence the indices of the vector fields $B(u_0)$, $N(u_0)$ with respect to $u=0$ are identical.

There is a deformation $D(u_0, s) = (1-s)N(u_0) + su_0$, $0 \leqslant s \leqslant 1$, of $N(u_0)$ into u_0 and $D(u_0, s) \neq 0$ since $u=0$ is an interior point of $\mathfrak{C}$. Since the vector field u_0, $u_0 \in \partial\mathfrak{C}$, has index 1 with respect to $u=0$, the index of $N(u_0)$ and, hence, of $B(u_0)$ is 1. This proves that (3.2) has solutions in $\mathfrak{C}$.

Case 2. *$\partial\mathfrak{C}$ is not of class C^1.* By a theorem of Minkowski (cf. [1], pp. 36–37), there exists a sequence of compact convex sets $\mathfrak{C}_1 \subset \mathfrak{C}_2 \subset \dots$ such that $\mathfrak{C}$ is the closure of the union $\mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \dots$ and $\partial\mathfrak{C}_m$ is of class C^1. By Case 1, there exists $u_m \in \mathfrak{C}_m$ satisfying

$$(B(u_m), v-u_m) \geqslant 0 \quad \text{for} \quad v \in \mathfrak{C}_m.$$

After a selection of a subsequence, it can be supposed that $u_0 = \lim u_m$ exists. Then, by continuity, it follows that

$$(B(u_0), v-u_0) \geqslant 0 \quad \text{for} \quad v \in \mathfrak{C}_m,$$

$m = 1, 2, \dots$. This implies (3.1) and completes the proof.

4. Proof of Theorem 1.1. According to Lemma 2.2, it can be supposed that $\mathfrak{K}$ is bounded.

(a) Without loss of generality, it can be supposed that $A(u) \in Y'$ and we can write $\langle A(u), v\rangle$ in place of $(A(u), v)$ for $v \in Y$. (This only affects the norm assigned to $A(u)$).

Let $\varphi \in \mathfrak{K}$ be fixed and $\mathfrak{K}_0 = \mathfrak{K} - \varphi$, so that $\mathfrak{K}_0$ is a closed, bounded, convex set in Y containing 0 and $\mathfrak{K} = \mathfrak{K}_0 + \varphi$. Let M be a linear subspace of Y with $m = \dim M < \infty$, $j: M \to Y$ the injection of M into Y, $j^*: Y' \to M'$ the dual map, and

$$\mathfrak{K}_M = \mathfrak{K}_0 \cap M \subset \mathfrak{K} - \varphi.$$

It will be shown that there exists an element $y_M \in \mathfrak{K}_M$ satisfying

$$\langle j^*A(y_M+\varphi), z-y_M\rangle \geqslant \langle j^*C(y_M+\varphi), z-y_M\rangle \quad \text{for} \quad z \in \mathfrak{K}_M, \tag{4.1}$$

or, equivalently,
$$\langle A(y_M+\varphi) - C(y_M+\varphi), z-y_M\rangle \geqslant 0 \quad \text{for} \quad z \in \mathfrak{K}_M. \tag{4.2}$$

Introduce bases $e_1, \dots, e_m$ on M and $f_1, \dots, f_m$ on M' such that $\langle f_i, e_j\rangle = \delta_{ij}$. For $y_M, y, z \in \mathfrak{K}_M$, write

$$y_M = \sum_{i=1}^{m} (y_M)_i e_i, \quad y = \sum_{i=1}^{m} y_i e_i, \quad z = \sum_{i=1}^{m} z_i e_i,$$

$$j^*[A(y+\varphi) - C(y+\varphi)] = \sum_{i=1}^{m} B_i(y) f_i.$$

Thus (4.1) is equivalent to

$$\sum_{i=1}^{m} B_i(y_M)(z_i-(y_M)_i)\geqslant 0 \quad \text{for } z=\sum_{i=1}^{m} z_i e_i \in \mathfrak{K}_M. \tag{4.3}$$

This shows that the desired result (4.3) does not depend on the norm on M. Thus we can suppose that M carries a Euclidean norm and write (4.3) as (3.1), where

$$B(y)=(B_1(y),\ldots,B_m(y))$$

is a continuous function from $\mathfrak{K}_M\subset M$ to M. Hence, the existence of a $y_M\in\mathfrak{K}_M$ satisfying (4.3) follows from Lemma 3.1.

(b) Put $\mathfrak{K}(M)=\mathfrak{K}_M+\varphi\subset\mathfrak{K}$ and $u_M=y_M+\varphi\in\mathfrak{K}(M)$. Then (4.2) becomes

$$(A(u_M),\, v-u_M)\geqslant\langle C(u_M),\, v-u_M\rangle \quad \text{for} \quad v\in\mathfrak{K}(M). \tag{4.4}$$

By the monotony condition (1.3),

$$(A(v),\, v-u_M)\geqslant\langle C(u_M),\, v-u_M\rangle \quad \text{for} \quad v\in\mathfrak{K}(M). \tag{4.5}$$

(c) For $v\in\mathfrak{K}$, let

$$S(v)=\{u\colon u\in\mathfrak{K},\ (A(v),\, v-u)\geqslant\langle C(u),\, v-u\rangle\}.$$

The sets $S(v)$ are closed with respect to the weak topology on X. For $\mathfrak{K}$ is closed and convex, hence weakly closed, while the complete continuity of $C(u)$ shows that

$$(A(v),\, v-u)-\langle C(u),\, v-u\rangle$$

is a continuous function of $u\in\mathfrak{K}$ from the weak topology on $\mathfrak{K}\subset X$ to the reals.

The collection of sets $\{S(v)\}$, $v\in\mathfrak{K}$, has the finite intersection property. For if $v_1,\ldots,v_m\in\mathfrak{K}$ and M is a finite dimensional manifold of Y such that $v_1,\ldots,v_m\in\mathfrak{K}(M)$, then $u_M\in S(v_1)\cap\ldots\cap S(v_m)$. Since X is reflexive, the set $\mathfrak{K}$ is weakly compact. Thus $S(v)\subset\mathfrak{K}$ implies the existence of an element u_0 such that

$$u_0\in\bigcap_{v\in\mathfrak{K}} S(v)\subset\mathfrak{K}.$$

This element satisfies

$$(A(v),v-u_0)\geqslant\langle C(u),v-u_0\rangle \quad \text{for} \quad v\in\mathfrak{K}.$$

By virtue of Lemma 2.3, u_0 is a solution of (1.6). This proves Theorem 1.1.

5. Another existence theorem. The result of this section is a theorem related to Theorem 1.1 and is a generalization of results of Browder [3] and of Leray and Lions [8] concerning the equation $Au=0$.

THEOREM 5.1. *Let $X, Y, \mathfrak{K}$ be as in Theorem 1.1 and, in addition, assume that X is separable. For $u\in\mathfrak{K}$, let $A(u)$ be a bounded linear functional on Y (considered as a subspace of X) and satisfy the continuity condition* (i) *and the coercivity condition* (iii) *of Theorem 1.1.*

For $u,v \in \mathfrak{K}$, let $A(u,v)$ be a bounded linear functional on Y, considered as a subset of X, satisfying

(i_0) *$A(u, v)$ is bounded on bounded subsets of $\mathfrak{K}\times\mathfrak{K}$;*

(ii_0) *for fixed $u\in\mathfrak{K}$, $A(u,\cdot)$ is a continuous function on every line segment in $\mathfrak{K}$;*

(iii_0) *$A(u, v)$ satisfies the monotony condition*

$$(A(u, u)-A(u, v), u-v)\geqslant 0 \quad \textit{for} \quad u, v\in\mathfrak{K}; \tag{5.1}$$

(iv_0) *if $u_1, u_2, \ldots \in\mathfrak{K}$ satisfy, as $m\to\infty$,*

$$u_m\to u_0 \quad \textit{weakly in} \quad X, \tag{5.2}$$

$$(A(u_m, u_m)-A(u_m, u_0), u_m-u_0)\to 0, \tag{5.3}$$

then

$$(A(u_m, v), w)\to(A(u_0, v), w) \quad \textit{for} \quad v\in\mathfrak{K}, w\in Y; \tag{5.4}$$

(v_0) *If $u_1, u_2, \ldots\in\mathfrak{K}$ satisfy* (5.2) *and*

$$(A(u_m, v), w)\to\langle y', w\rangle \quad \textit{for} \quad w\in Y \tag{5.5}$$

and some fixed $v\in\mathfrak{K}$ and $y'\in Y'$, then

$$(A(u_m, v), v-u_m)\to\langle y', v-u_0\rangle; \tag{5.6}$$

(vi_0) *$A(u)=A(u, u)$ for $u\in\mathfrak{K}$.*

Then there exists $u_0\in\mathfrak{K}$ such that

$$(A(u_0), v-u_0)\geqslant 0 \quad \textit{for} \quad v\in\mathfrak{K}. \tag{5.7}$$

Illustrations of the conditions of this theorem in the theory of non-linear elliptic partial differential equations are given in [3] and [8]; see Section 12 below. The formulation of conditions (i_0)–(vi_0) follows [8].

If $C(u)$ satisfies the condition of Theorem 1.1, the assertion (5.7) can be replaced by (1.6). But this fact is contained in Theorem 5.1 if one replaces $A(u, v)$ by the linear functional on Y defined by $(A(u,v),y)-\langle C(u),y\rangle$ for $y\in Y$.

COROLLARY 5.1. *Assume the conditions of Theorem 5.1. Let $u_m\in\mathfrak{K}$, $m=1,2,\ldots$, and let $u_0=u_m$ satisfy (5.7) and $u_m\to u_\infty$ weakly in X as $m\to\infty$. Then $u_\infty\in\mathfrak{K}$ and $u_0=u_\infty$ satisfies (5.7).*

This will be clear from the proof of Theorem 5.1; cf. the arguments leading from (6.2) to (5.7) below.

6. Proof of Theorem 5.1. By the coercivity condition (iii) of Theorem 1.1 and Lemma 2.1, there is no loss of generality in supposing that $\mathfrak{K}$ is bounded and, hence by (i_0), that $A(u,v)$ is bounded, say

$$|(A(u, v), y)|\leqslant c\|y\|_X \quad \text{for} \quad y\in Y. \tag{6.1}$$

Let $M_1 \subset M_2 \subset \ldots$ be a sequence of finite dimensional subspaces of Y such that $\bigcup M_m$ is dense in Y. Let $\varphi \in \mathfrak{K}$ be fixed. Lemma 3.1 implies that, if $\mathfrak{K}_0 = \mathfrak{K} - \varphi$, then there exist $u_m \in (\mathfrak{K}_0 \cap M_m) + \varphi$ such that

$$(A(u_m, u_m), w - u_m) \geq 0 \quad \text{for} \quad w \in (\mathfrak{K}_0 \cap M_m) + \varphi; \tag{6.2}$$

cf. parts (a), (b) of Section 4. Thus (6.1) and (6.2) show that

$$(A(u_m, u_m), v - u_m) \geq -c \inf \|v - w\|_X \quad \text{for} \quad v \in \mathfrak{K}, \tag{6.3}$$

where the infimum refers to $w \in (\mathfrak{K}_0 \cap M_m) + \varphi$. By the monotony condition (5.1),

$$(A(u_m, v), v - u_m) \geq -c \inf \|v - w\|_X \quad \text{for} \quad v \in \mathfrak{K}. \tag{6.4}$$

After a selection of a subsequence, it can be supposed that there exist $u_0 \in \mathfrak{K}$ and $y' \in X'$ such that, as $m \to \infty$, (5.2) holds and

$$(A(u_m, u_0), y) \to \langle y', y \rangle \quad \text{for} \quad y \in Y. \tag{6.5}$$

By (v_0), it follows that

$$(A(u_m, u_0), u_0 - u_m) \to \langle y', 0 \rangle = 0. \tag{6.6}$$

From (6.3), with $v = u_0$, and the monotony (5.1),

$$(A(u_m, u_0), u_0 - u_m) \geq (A(u_m, u_m), u_0 - u_m) \geq -c \inf \|u_0 - w\|_X.$$

The extreme members of this inequality tend to 0, the first because of (6.6) and the last because $\bigcup M_m$ is dense in Y. Consequently

$$(A(u_m, u_m), u_0 - u_m) \to 0 \quad \text{as} \quad m \to \infty.$$

Thus, by (6.6), the limit relation (5.3) holds and so, (5.4) holds by (iv_0). This fact, together with (v_0), gives

$$(A(u_m, v), v - u_m) \to (A(u_0, v), v - u_0) \quad \text{for} \quad v \in \mathfrak{K};$$

cf. (5.6). Thus, by (6.4),

$$(A(u_0, v), v - u_0) \geq 0 \quad \text{for} \quad v \in \mathfrak{K}.$$

An analogue of the argument of Lemma 2.3 completes the proof of Theorem 5.1.

Part II. A priori bounds

7. Uniformly elliptic linear equations. Let $n \geq 2$, E^n Euclidean n-space, and $|x|$, $|p|$, $|\xi|$ the Euclidean norms of points $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ in E^n. In what follows, Ω is a bounded open subset of E^n, $\partial\Omega$ its boundary, $\bar{\Omega} = \Omega \cup \partial\Omega$ its closure, and $|\Omega|$ its Euclidean measure.

If $m \geq 0$, $C^m(\Omega)$ [or $C^m(\bar{\Omega})$] denotes the set of functions having all continuous partial derivatives of order $\leqslant m$ on Ω [or $\bar{\Omega}$]. $C_0^m(\Omega)$ is is the subset of functions in $C^m(\Omega)$ vanishing near $\partial\Omega$. For $\alpha \geqslant 1$, the $L^\alpha(\Omega)$ norm of $u(x) \in L^\alpha(\Omega)$ will be denoted by $\|u\|_\alpha$ or $\|u\|_{\alpha, \Omega}$.

The completion of $C^m(\bar{\Omega})$ [or $C_0^m(\Omega)$] with respect to the norm

$$\|u\|_{H^{m,\alpha}(\Omega)} = \sum_{0 \leqslant |j| \leqslant m} \|D^j u\|_\alpha$$

will be called $H^{m,\alpha}(\Omega)$ [or $H_0^{m,\alpha}(\Omega)$]. In the last display,

$$D^j u = \partial^{|j|} u / \partial x_1^{j_1} \ldots \partial x_n^{j_n}, \quad |j| = j_1 + \ldots + j_n.$$

For $\alpha = 2$, we write $H^m(\Omega)$ or $H_0^m(\Omega)$ in place of $H^{m,2}(\Omega)$ or $H_0^{m,2}(\Omega)$. If $u(x) \in H^{1,\alpha}(\Omega)$, then we write u_x for $u_x = \operatorname{grad} u(x) = (u_{x_1}, \ldots, u_{x_n})$ and $\|u_x\|_\alpha$ or $\|u_x\|_{\alpha, \Omega}$ for the $L^\alpha(\Omega)$ norm of the Euclidean length $|u_x|$ of $u_x \in E^n$. Similarly, for any vector valued function

$$f(x) = (f_1(x), \ldots, f_n(x)),$$

the $L^\alpha(\Omega)$ norm of $|f(x)|$ is denoted simply by $\|f\|_\alpha$. The norm on $H_0^1(\Omega)$ will be taken to be $\|u_x\|_{2, \Omega}$.

If $1 \leqslant \alpha < \infty$, then α' denotes the Hölder conjugate exponent, $1/\alpha + 1/\alpha' = 1$. If $1 \leqslant \alpha < n$, then α^* denotes the Sobolev exponent

$$1/\alpha^* = 1/\alpha - 1/n.$$

LEMMA 7.1 (*Sobolev*). *Let $1 \leqslant \alpha < n$ and $u \in H_0^{1,\alpha}(\Omega)$. Then there exists a constant S_α depending only on α, n but not on Ω, such that*

$$\|u\|_{\alpha^*} \leqslant S_\alpha \|u_x\|_\alpha. \tag{7.1}$$

We shall make occasional use of the following simple lemma which is an analogue of the Case 1 of Lemma 2.1 of [14].

LEMMA 7.2. *Let $\varrho(t)$ be a non-negative, non-increasing function on $t \geqslant 0$ such that $\varrho(t) \to 0$ as $t \to \infty$ and*

$$-\int_k^\infty (t-k)\, d\varrho(t) \leqslant c[\varrho(k)]^\gamma \tag{7.2}$$

for $0 \leqslant k < \infty$, where $c > 0$, $\gamma > 1$ are constants. Then

$$\varrho(t) = 0 \quad \text{for} \quad t \geqslant c[\varrho(0)]^{\gamma-1}\gamma/(\gamma-1). \tag{7.3}$$

Proof. Define the function $H(k)$, $0 \leqslant k < \infty$, by

$$H(k) = -\int_k^\infty (t-k)\, d\varrho(t) = \int_k^\infty \varrho(t)\, dt, \tag{7.4}$$

since the existence of the integral in (7.2) implies that $t\varrho(t)\to 0$ as $t\to\infty$. Thus, by (7.2),

$$H'(k) = -\varrho(k) \leqslant -[H(k)/c]^{1/\gamma}.$$

Hence a quadrature gives

$$0 \leqslant \gamma H^{1-1/\gamma}(k)/(\gamma-1) \leqslant \gamma H^{1-1/\gamma}(0)/(\gamma-1) - k/c^{1/\gamma}.$$

Consequently, $H(k)=0$ for some $k \leqslant k_0$,

$$k_0 = \gamma c^{1/\gamma} H^{1-1/\gamma}(0)/(\gamma-1);$$

in which case, $\varrho(t)=0$ for $t \geqslant k_0$ by (7.4). Since $H(0) \leqslant c\varrho^{\gamma}(0)$ by (7.2) and (7.4), (7.3) follows.

A function $u \in H^{1,\alpha}(\Omega)$ is said to be bounded from above on $\partial\Omega$ by a constant Φ if there exists a sequence $\{u_m\}$ of functions in $C^1(\bar{\Omega})$ such that $u_m \to u$ in $H^{1,\alpha}(\Omega)$ and $u_m \leqslant \Phi$ on $\partial\Omega$. The least such Φ will be called $\max u(x)$ on $\partial\Omega$. The $\min u(x)$ on $\partial\Omega$ is similarly defined.

Let $a_{ij}(x)$, where $i,j=1,\ldots,n$, be bounded and measurable functions on Ω such that there exists a number $\nu > 0$ satisfying

$$a_{ij}(x)\xi_i\xi_j \geqslant \nu|\xi|^2. \tag{7.5}$$

By a weak subsolution [or supersolution] of

$$\sum_{j=1}^{n} [(a_{ij}(x)u_{x_i})_{x_j} - f_{jx_j}] = 0 \tag{7.6}$$

is meant a function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega (a_{ij}(x)u_{x_i} - f_j)\eta_{x_j}\,dx \leqslant 0 \text{ [or} \geqslant 0] \quad \text{for } 0 \leqslant v \in H_0^1(\Omega). \tag{7.7}$$

Lemma 7.3 (*Maximum principle*). *Let* $f(x)=(f_1,\ldots,f_n)\in L^\alpha(\Omega)$, $\alpha>n$, *and* $u(x)$ *satisfy* (7.7). *Then*

$$u(x) \leqslant \max_{\partial\Omega} u + (S/\nu)\,\|f\|_\alpha\,|\Omega|^{1/n-1/\alpha}, \tag{7.8}$$

$$[u(x) \geqslant \min_{\partial\Omega} u - (S/\nu)\,\|f\|_\alpha\,|\Omega|^{1/n-1/\alpha}, \tag{7.9)]$$

where $S=[(1+1/n-1/\alpha)/(1/n-1/\alpha)]\inf S_\tau$ *and the infimum is taken over the range*

$$1 \leqslant \tau \leqslant 2,\ \tau < n.$$

This result is due to Stampacchia; cf. [14], pp. 387–388. For the sake of completeness, a variant of the proof of [14] will be given here.

Proof. We shall only prove (7.8), as (7.9) is a consequence of (7.8). It is sufficient to suppose that $\Phi=0$, where $\Phi=\max u$ on $\partial\Omega$. For otherwise, we replace $u(x)$ by the function

$v(x) = \max(u(x) - \Phi, 0)$ and $f_j(x)$ by the function which is $f_j(x)$ or 0 according as $v(x) > 0$ or $v(x) = 0$ (i.e., in (7.7), we consider only $\eta(x)$ with support on the support of $v(x)$). Then $v(x)$ is a subsolution of the resulting equation and $\max v(x) = 0$ on $\partial\Omega$.

Let $k \geqslant 0$ and $A(k) = \{x: u(x) \geqslant k\}$. Then the choice $\eta(x) = \max(u(x) - k, 0)$ in (7.7) gives

$$\int_{A(k)} a_{ij}(x) u_{x_i} u_{x_j} dx \leqslant \int_{A(k)} f_j u_{x_j} dx.$$

Then, by (7.5) and Schwarz's inequality, $\nu(\|u_x\|_{2, A(k)})^2 \leqslant \|f\|_{2, A(k)} \|u_x\|_{2, A(k)}$, so that

$$\nu \| u_x \|_{2, A(k)} \leqslant \| f \|_{2, A(k)}. \tag{7.10}$$

Let $1 \leqslant \tau \leqslant 2$ if $n > 2$ or $1 \leqslant \tau < 2$ if $n = 2$. Then Hölder's inequality applied to both sides of (7.10) gives

$$\nu \| u_x \|_{\tau, A(k)} \leqslant \| f \|_{\alpha, A(k)} | A(k) |^{1/\tau - 1/\alpha}.$$

Applying Sobolev's lemma to the function $\max(u(x) - k, 0)$, we see that the left side is not less than $(\nu / S_\tau) \| u - k \|_{\tau^*, A(k)}$. The exponent τ^* can be reduced to 1 by Hölder's inequality.

Thus
$$\| u - k \|_{1, A(k)} \leqslant c \, | A(k) |^\gamma, \tag{7.11}$$
where
$$c = S_\tau \| f \|_{\alpha, \Omega} / \nu, \quad \gamma = 1 + 1/n - 1/\alpha > 1. \tag{7.12}$$

Write (7.11) in the form

$$\int_{A(k)} (u - k) dx \leqslant c \, | A(k) |^\gamma \tag{7.13}$$

or, equivalently, as
$$-\int_k^\infty (t - k) d | A(t) | \leqslant c \, | A(k) |^\gamma. \tag{7.14}$$

Thus, by Lemma 7.2 with $\varrho(t) = | A(t) |$, $| A(t) | = 0$ if $t \geqslant c \gamma \varrho^{\gamma - 1}(0) / (\gamma - 1)$; i.e.,

$$| u(x) | \leqslant c \gamma \varrho^{\gamma - 1}(0) / (\gamma - 1),$$

where $\varrho(0) = | \Omega |$. Hence (7.8) follows from (7.12).

Remark. For applications below, it is important to note that, for the validity of (7.8) [or (7.9)], it is sufficient to know only that (7.7) holds for the functions

$$\eta(x) = \max(u(x) - k, 0 \quad [\text{or } \eta(x) = -\min(k + u(x), 0] \tag{7.15}$$

for
$$k \geqslant \max u \quad [\text{or } k \leqslant -\min u] \quad \text{on } \partial\Omega. \tag{7.16}$$

In fact, it is sufficient to have the inequality (7.10) for $A(k) = \{x: u(x) \geqslant k\}$ [or $A(k) = \{x: u(x) \leqslant -k\}$] for k in (7.16).

8. Quasi solutions and a maximum principle. Let $\mathcal{K}=\mathcal{K}(\Omega)$ denote the set of uniformly Lipschitz continuous functions $u(x)$ on Ω (or, equivalently, on $\bar{\Omega}$). If $u(x)\in\mathcal{K}$, let $\lambda(u)$ denote its best Lipschitz constant

$$\lambda(u)=\sup|u(x^1)-u(x^0)|/|x^1-x^0| \quad \text{for} \quad x^0, x^1\in\Omega. \tag{8.1}$$

Let $\varphi(x)$ be a function defined only on $\partial\Omega$ and uniformly Lipschitz continuous there. We shall also use the notation

$$\lambda(\varphi)=\sup|\varphi(x^1)-\varphi(x^0)|/|x^1-x^0| \quad \text{for} \quad x^0, x^1\in\partial\Omega. \tag{8.2}$$

Let $\mathcal{K}_\varphi$ be the set of functions $u(x)\in\mathcal{K}$ for which $u(x)=\varphi(x)$ on $\partial\Omega$. For a given $K\geqslant 0$, let $\mathcal{K}_\varphi^K$ be the set of functions $u(x)\in\mathcal{K}_\varphi$ satisfying $\lambda(u)\leqslant K$. The sets $\mathcal{K}_0$, $\mathcal{K}_0^K$ correspond, of course, to $\varphi(x)\equiv 0$.

By a quasi or K-quasi solution of

$$[a_j(u_x)]_{x_j}+F[u](x)=0 \tag{8.3}$$

will be meant a function $u(x)\in\mathcal{K}_\varphi^K$ satisfying

$$\int_\Omega [a_j(u_x)(v-u)_{x_j}-F[u](v-u)]dx\geqslant 0 \quad \text{for } v\in\mathcal{K}\varphi^K. \tag{8.4}$$

The object of this section is to obtain a maximum principle (i.e., an a priori bound) for K-quasi solutions under suitable conditions on $a_i(p)$ and $F[u](x)$.

(**A 1**) Let $a(p)=(a_1(p),\dots,a_n(p))\in C^0(E^n)$ and satisfy

$$a_j(p)p_j\geqslant\mu|p|^\alpha-N, \tag{8.5}$$

where $\mu>0$, $1\leqslant\alpha<n$, N are constants.

(**A 2**) Let $u\to F[u]$ be a mapping of $\mathcal{K}$ into $L^1(\Omega)$ such that

$$F[u](x)\operatorname{sgn}u(x)\leqslant\sum_{i=1}^m c_i\|u\|_{\alpha(i)}^{\beta(i)}|u(x)|^{\gamma(i)-1}|u_x(u)|^{\delta(i)}, \tag{8.6}$$

where $c_i\geqslant 0$, $\alpha(i)\geqslant 1$, $\beta(i)\geqslant 0$, $\gamma(i)\geqslant 1$, $\delta(i)\geqslant 0$, and

$$\alpha(i)\leqslant\alpha^*, \quad \beta(i)+\gamma(i)+\delta(i)\leqslant\alpha. \tag{8.7}$$

A simple example of an admissible $F[u]$ is one which has the form

$$F[u](x)=G[u]g(x,u,u_x), \tag{8.8}$$

where $G[u]$ is a non-linear, real-valued functional of u satisfying $0\leqslant G[u]\leqslant\|u\|_{\alpha(1)}^{\beta(1)}$ and $g(x,u,p)\in C^0(\bar{\Omega}\times E^1\times E^n)$ satisfies $ug(x,u,p)\leqslant c_1|u|^{\gamma(1)}|p|^{\delta(1)}$. For example, if one considers a variational problem of the form

$$\min\left\{\int_\Omega f(u_x)\,dx-\left[\int_\Omega h(x,u)\,dx\right]^\beta\right\},$$

then the corresponding Euler equation is of the form (8.3) with $a_j(p)=f_{p_j}(p)$ and $F[u]$ of the type (8.8) with

$$G[u]=\left[\int_\Omega h(x,u(x))\,dx\right]^{\beta-1}\quad\text{and}\quad g(x,u)=\beta h_u(x,u),$$

independent of u_x.

The reason that the right side of (8.6) has been chosen as a sum rather than as one term is, not to obtain greater generality but, to illustrate the fact that two different situations occur according as $\beta(i)+\gamma(i)+\delta(i)<\alpha$ or $\beta(i)+\gamma(i)+\delta(i)=\alpha$. In the first case, there will be no restriction on the constant c_i [and, in fact, c_i can be replaced by a function $c_i(x)\in L^{\varepsilon(i)}(\Omega)$ for a suitable $\varepsilon(i)$]; in the second case, smallness conditions will have to be imposed on c_i.

In order to make this specific, let σ denote a number satisfying

$$\max[\alpha(i),\gamma(i)]\leqslant\sigma\leqslant\alpha^*\quad\text{for}\quad i=1,\dots,m;\tag{8.9}$$

cf. (8.7). Put

$$\Lambda=\Lambda(\sigma,\alpha,\Omega)=\inf\frac{\|u_x\|_\alpha}{\|u\|_\sigma}\quad\text{for}\quad u\in H_0^{1,\alpha}(\Omega).\tag{8.10}$$

For example, if $\sigma=\alpha^*$, then $\Lambda(\alpha^*,\alpha,\Omega)\geqslant 1/S_\alpha$; if $\alpha=\sigma=2$, then $\Lambda^2(2,2,\Omega)$ becomes the first eigenvalue for $\Delta u+\lambda u=0$ on Ω, $u=0$ on $\partial\Omega$. In any case, Sobolev's inequality implies that

$$\Lambda\geqslant 1/S_\alpha|\Omega|^{1/\sigma-1/\alpha^*}.$$

The smallness condition on certain c_i mentioned above will be the following:

(**A 3**) Let the coefficients c_i in (8.6) satisfy

$$\mu-\sum{}'c_i\Lambda^{\delta(i)-\alpha}|\Omega|^{(1-\alpha/\sigma)(1-\delta(i)/\alpha)+\beta(i)/\alpha(i)}>0,\tag{8.11}$$

where Σ' is the sum over the indices i for which $\beta(i)+\gamma(i)+\delta(i)=\alpha$.

The following conditions will not be used in this section but will be stated here for reference later.

(**A 4**) For every number $M>0$, there exists a number $\chi(M)$ such that

$$|u(x)|\leqslant M\quad\text{on}\quad\Omega\quad\Rightarrow\quad|F[u](x)|\leqslant\chi(M)(1+|u_x(x)|^{\delta(0)}),$$

where $0\leqslant\delta(0)<\alpha-1$.

(**A 5**) If $u_h(x)\in\mathcal{K}_\varphi^K$ for $h=0,1,\dots$ and $u_h(x)\to u_0(x)$ in $H^1(\Omega)$ as $h\to\infty$, then $F[u_h](x)\to F[u_0](x)$ in measure on Ω as $h\to\infty$.

This holds, for example, if $F[u]$ has the form (8.8), where $g(x,u,p) \in C^0(\bar{\Omega} \times E^1 \times E^n)$ and $G[u_h] \to G[u_0]$ as $h \to \infty$; cf., e.g., [3], [7].

If $F[u](x)$ depends essentially only on $u(x)$ and not on its gradient $u_x(x)$ (as in the case in (8.8) when $g(x,u,p) = g(x,u)$ does not depend on p), then (A5) can be replaced by the simpler condition:

(**A 5′**) If $u_h(x) \in \mathcal{K}_\varphi^K$ for $h = 0, 1, \ldots$ and $u_h(x) \to u_0(x)$ uniformly on $\bar{\Omega}$ as $h \to \infty$, then $F[u_h](x) \to F[u_0](x)$ in measure in Ω as $h \to \infty$.

Remark. Assumptions (A4) and (A5) [or (A5′)] imply that the map $u \to F[u]$ is continuous [or completely continuous] from $\mathcal{K}_\varphi^K \subset H^1(\Omega)$ to the conjugate space of $H_0^1(\Omega)$, where $F[u]$ is interpreted as the linear functional

$$\int_\Omega F[u](x)\eta(x)\,dx \quad \text{for } \eta \in H_0^1(\Omega).$$

THEOREM 8.1 (*A priori bound*). *Assume* (A1), (A2) *and the inequality* (*8.11*) *in* (A3). *Let* $\Psi = \max|\varphi(x)|$ *on* $\partial\Omega$. *Then there exists a constant* T, *depending on the parameters* $n, \alpha, \mu, N, |\Omega|, \Psi$, *and* $c_i, \alpha(i), \beta(i), \gamma(i), \delta(i)$ *for* $i = 1, \ldots, m$ (*but not on* K), *such that if* $u(x)$ *is a* K-*quasi solution of* (*8.3*), *then*

$$|u(x)| \leqslant T \quad on \quad \Omega. \tag{8.12}$$

The proof of this theorem will depend on modifications and simplifications of the proofs of Theorems 6.1 and 6.2 in [14]. For related results, see [7], [13].

Proof. In the proof, T will denote a constant (not always the same) depending on the parameters mentioned in the theorem. By $T(\varepsilon)$ will be meant a constant depending on an additional parameter ε.

(a) The first step in the proof will be to obtain an inequality of the form $\|u\|_{\alpha^*} \leqslant T$. To this end, we shall first majorize $\|u_x\|_{\alpha, A}$, where A is the set $A = \{x : |u(x)| \geqslant \Psi\}$.

Define the function $U(x)$ to be $u(x) - \Psi$, 0, or $u(x) + \Psi$ according as $u(x) \geqslant \Psi$, $|u(x)| \leqslant \Psi$, or $u(x) \leqslant -\Psi$. Then $U(x) = 0$ on $\Omega - A$ and $U(x) \in \mathcal{K}_0$. Define $v(x)$ by the relation

$$v(x) - u(x) = -U(x);$$

in other words, $v(x) = \Psi$, $u(x)$, $-\Psi$ according as $u(x) \geqslant \Psi$, $|u(x)| \leqslant \Psi$, or $u(x) \leqslant -\Psi$. This makes it clear that $v(x) \in \mathcal{K}_\varphi^K$. Thus (8.4) gives

$$\int_A a_j(u_x)\,u_{x_j}\,dx \leqslant \int_A F[u]\,U\,dx.$$

Since $\Psi \geqslant 0$, $\operatorname{sgn} U = \operatorname{sgn} u$. Consequently, by (8.5) and (8.6),

$$\mu \int_A |u_x|^\alpha dx - N|A| \leqslant \sum_{i=1}^m c_i \|u\|_{\alpha(i)}^{\beta(i)} \int_A |u|^{\gamma(i)-1} |u_x|^{\delta(i)} |U| dx.$$

Since $|U| \leqslant |u|$ on A, the last integrand can be replaced by $|u|^{\gamma(i)} |u_x|^{\delta(i)}$ and so, Hölder's inequality gives, as a majorant for this integral,

$$\int_A |u|^{\gamma(i)} |u_x|^{\delta(i)} dx \leqslant \|u\|_{\sigma(i), A}^{\gamma(i)} \|u_x\|_{\alpha, A}^{\delta(i)}, \tag{8.13}$$

$$\sigma(i) = \gamma(i)\,\alpha/(\alpha - \delta(i)) \leqslant \alpha. \tag{8.14}$$

Using Hölder's inequality again, it is seen that the right side of (8.13) does not exceed

$$\|u\|_\sigma^{\gamma(i)} \|u_x\|_{\alpha, A}^{\delta(i)} |A|^{\gamma(i)(1/\sigma(i) - 1/\sigma)},$$

By a similar use of Hölder's inequality for $\|u\|_{\alpha)i)}$ and $|A| \leqslant |\Omega|$,

$$\mu \|u_x\|_{\alpha, A}^\alpha \leqslant N|\Omega| + \sum_{i=1}^m c_i \|u\|_\sigma^{\beta(i)+\gamma(i)} \|u_x\|_{\alpha, A}^{\delta(i)} |\Omega|^{\tau(i)}, \tag{8.15}$$

$$\tau(i) = \beta(i)[1/\alpha(i) - 1/\sigma] + 1 - \delta(i)/\alpha - \gamma(i)/\sigma. \tag{8.16}$$

Note that, for any number $a > 0$,

$$\int_\Omega |u|^a dx \leqslant \int_A (|U| + \Psi)^a dx + \Psi^a |\Omega|. \tag{8.17}$$

Also, if $\varepsilon > 0$ is arbitrary, there is a number $c = c(\varepsilon, a)$ such that

$$\int_A (|U| + \Psi)^a dx \leqslant (1+\varepsilon) \int_A |U|^a dx + c(\varepsilon, a) \Psi^a |\Omega|,$$

since $(|U| + \Psi)^a \leqslant (1+\varepsilon)^a |U|^a + (1 + 1/\varepsilon)^a \Psi^a$ as can be seen by considering the two possibilities $\Psi \leqslant \varepsilon |U|$ or $\Psi \geqslant \varepsilon |U|$. Consequently, if $\varepsilon > 0$, there is a number $T(\varepsilon)$ such that

$$\mu \|U_x\|_\alpha^\alpha \leqslant (1+\varepsilon) \sum_{i=1}^m c_i \|U\|_\sigma^{\beta(i)+\gamma(i)} \|U_x\|_\alpha^{\delta(i)} |\Omega|^{\tau(i)} + T(\varepsilon)(1 + \Sigma'' \|U_x\|_\alpha^{\delta(i)}),$$

where Σ'' is a sum over indices i for which $\beta(i) + \gamma(i) > 0$, so that $\delta(i) < \alpha$.

Since $U(x) \in \mathcal{K}_0 \subset H_0^1(\Omega)$, (8.10) gives

$$\mu \|U_x\|_\alpha^\alpha \leqslant (1+\varepsilon) \sum_{i=1}^m c_i \Lambda^{-\beta(i)-\gamma(i)} \|U_x\|_\alpha^{\beta(i)+\gamma(i)+\delta(i)} |\Omega|^{\tau(i)} + T(\varepsilon)(1 + \Sigma'' \|U_x\|_\alpha^{\delta(i)}).$$

From (8.16), $\tau(i) = (1 - \alpha/\sigma)(1 - \delta(i)/\alpha) + \beta(i)/\alpha(i)$ when $\beta(i) + \gamma(i) + \delta(i) = \alpha$. Hence (8.11) shows that there exists a constant T such that

$$\|u_x\|_{\alpha, A} = \|U_x\|_\alpha \leqslant T. \tag{8.18}$$

By Sobolev's inequality, $\|U\|_{\alpha^*} \leq TS_\alpha$. Finally, the analogue of (8.17), with $a=\alpha^*$, implies the existence of a constant T such that

$$\|u\|_{\alpha^*} \leq T. \tag{8.19}$$

(b) This result and (8.6) show that

$$F[u](x)\operatorname{sgn} u(x) \leq T\sum_{i=1}^{m} |u(x)|^{\gamma(i)-1}|u_x(x)|^{\delta(i)}. \tag{8.20}$$

(c) We now show that if $a \geq \alpha^*$, then there exists a constant $T(a)$ such that

$$\|u\|_a \leq T(a). \tag{8.21}$$

To this end, let $k \geq \Psi'$ and $A(k)=\{x: u(x)\geq k\}$. Determine v by the relation

$$v-u=-\max(u(x)-k,0),$$

so that $v(x)=k$ or $v(x)=u(x)$ according as $u(x)\geq k$ or $u(x)\leq k$. Thus $v\in\mathcal{K}_\varphi^K$ and (8.4) gives

$$\int_{A(k)} a_j(u_x)u_{x_j}\,dx \leq \int_{A(k)} F[u](u-k)\,dx.$$

By (8.5) and (8.20),

$$\mu\int_{A(k)} |u_x|^\alpha dx \leq N|A(k)| + T\sum_{i=1}^{m}\int_{A(k)} u^{\gamma(i)}|u_x|^{\delta(i)}dx. \tag{8.22}$$

From the inequality

$$|ab| \leq |\varepsilon a|^r/r + |b/\varepsilon|^{r'}/r', \quad 1/r+1/r'=1, \tag{8.23}$$

applied to $r=\alpha/\delta(i)$, $a=|u_x|^{\delta(i)}$, $\varepsilon^r/r=\mu/2T$, $b=u^{\gamma(i)}$, we get

$$\int_{A(k)} |u_x|^\alpha dx \leq T|A(k)| + T\sum_{i=1}^{m}\int_{A(k)} u^{\sigma(i)}dx; \tag{8.24}$$

cf. (8.14), (8.18) and $A(k)\subset A$.

From the relation

$$-\int_{\Psi'}^{\infty}(t-\Psi')^\beta d_t\left(\int_{A(t)}|u_x|^\alpha dx\right) = \int_{A(\Psi')}(u-\Psi')^\beta|u_x|^\alpha dx$$

and obvious integration by parts, (8.24) gives

$$\int_{A(\Psi')}(u-\Psi')^\beta|u_x|^\alpha dx \leq T\int_{A(\Psi')}(u-\Psi')^\beta dx + T\sum_{i=1}^{m}\int_{A(\Psi')}(u-\Psi')^\beta u^{\sigma(i)}dx, \tag{8.25}$$

whenever $\beta\geq 0$ and the right side is finite. In this case, Sobolev's lemma applied to $\max[(u-\Psi')^{1+\beta/\alpha},0]$ gives

$$\int_{A(\Psi)} (u-\Psi)^{\alpha^*(1+\beta/\alpha)}dx \leqslant [(1+\beta/\alpha)S_\alpha]^{\alpha^*}\{\ldots\}^{\alpha^*/\alpha}, \tag{8.26}$$

where $\{\ldots\}$ denotes the right side of (8.25).

From this last inequality, it follows that, for $a \geqslant \alpha^*$, there is a $T(a)$ satisfying

$$\|\max(u-\Psi,0)\|_a \leqslant T(a). \tag{8.27}$$

In fact, the choice $\beta = \alpha^* - \alpha$ in (8.26), where $\beta + \sigma(i) \leqslant \alpha^*$, gives such an inequality for $a = (\alpha^*)^2/\alpha > \alpha^*$. If (8.27) holds for some $a > \alpha^*$, the choice $\beta = a - \alpha$ in (8.26) gives (8.27) when a is replaced by $\alpha^* a/\alpha$. This proves (8.27). Similarly, one obtains $\|\min(u+\Psi,0)\|_a \leqslant T(a)$. Hence (8.21) follows.

(d) *Completion of the proof.* Choose $a = 2n$, so that $\sigma(i)/a \leqslant \alpha/a \leqslant \alpha/2n$. Then, by (8.24) and Hölder's inequality

$$\int_{A(k)} |u_x|^\alpha dx \leqslant T\,|A(k)|^{1-\alpha/2n}.$$

An application of Sobolev's inequality on the left, followed by Hölder's inequality to reduce the exponent from α^* to 1, gives

$$\int_{A(k)} (u-k)\,dx \leqslant T\,|A(k)|^\gamma, \quad \gamma = 1 + 1/2n > 1,$$

for $k \geqslant \Psi$; cf. the deduction of (7.11). An application of Lemma 7.2 shows that $u(x) \leqslant \Psi + T$ on Ω. Similarly, one obtains $-u(x) \leqslant \Psi + T$. This proves Theorem 8.1.

9. Lipschitz constants of quasi solutions. The object of this section is to obtain an a priori bound for the best Lipschitz constant $\lambda(u)$ of a quasi solution $u(x)$. For this purpose, we shall have to impose additional conditions on Ω, $a_i(p)$ and $\varphi(x)$.

(B1) Let $a(p) = (a_1(p), \ldots, a_n(p)) \in C^0(E^n)$ satisfy

$$a(0) = 0, \tag{9.1}$$

$$[a_j(p) - a_j(q)](p_j - q_j) \geqslant \nu_0[1 + \max(|p|^2, |q|^2)]^\tau |p-q|^2, \tag{9.2}$$

where $\nu_0 > 0$ and $-1/2 < \tau \leqslant 0$ are constants.

Remark 1. Condition (9.1) is no loss of generality, for the replacement of $a(p)$ by $a(p) - a(0)$ does not affect (8.4).

Remark 2. Conditions (9.1), (9.2) imply (8.5) with $\mu = \nu_0$ and $\alpha = 2(1+\tau)$, for $q = 0$ in (9.2) gives

$$a_j(p)p_j \geqslant \nu_0[1 + |p|^2]^\tau |p|^2 \geqslant \nu_0 |p|^{2+2\tau} - N.$$

Remark 3. If the functions $a_i(p) \in C^1(E^n)$, then (9.2) is equivalent to

$$\frac{\partial a_j(p)}{\partial p_i}\xi_i\xi_j \geq \nu_0(1+|p|^2)^\tau|\xi|^2. \tag{9.3}$$

In order to see that (9.2) implies (9.3), let $q=p-t\xi$ with $t>0$, use the mean value theorem for the difference $a_j(p)-a_j(q)$, divide the resulting inequality by t^2, and let $t\to 0$ to obtain (9.3). In order to deduce (9.2) from (9.3), note that

$$a_j(p)-a_j(q)=\int_0^1\frac{\partial a_j(tp+(1-t)q)}{\partial p_i}(p_i-q_i)\,dt.$$

By (9.3),

$$[a_j(p)-a_j(q)](p_j-q_j)\geq \nu_0[1+\max_{0\leq t\leq 1}|tp+(1-t)q|^2]^\tau|p-q|^2,$$

so that (9.2) follows from $|tp+(1-t)q|\leq\max(|p|,|q|)$.

(B1′) Let $a_i(p)$ be as in (B1) with (9.2) replaced by

$$[a_j(p)-a_j(q)](p_j-q_j)\geq\nu|p-q|^2,\quad \nu>0. \tag{9.4}$$

(B2) Let $a_j(p)\in C^1(E^n)$ satisfy

$$\lambda_0(p)|\xi|^2\leq\frac{\partial a_j(p)}{\partial p_i}\xi_i\xi_j\leq\lambda_1(p)|\xi|^2, \tag{9.5}$$

$$\lambda_0(p)\geq\nu_0(1+|p|^2)^\tau,\quad -1/2<\tau\leq 0, \tag{9.6}$$

$$\lambda_1(p)/\lambda_0(p)\leq\nu_1(1+|p|^2)^\vartheta,\quad \vartheta\geq 0, \tag{9.7}$$

where ν_0, ν_1 are positive constants.

If $\lambda_0(p)=\nu_0(1+|p|^2)^\tau$, this condition takes the form:

(B2′) Let $a_j(p)\in C^1(E^n)$ satisfy

$$\nu_0(1+|p|^2)^\tau|\xi|^2\leq\frac{\partial a_i}{\partial p_j}\xi_i\xi_j\leq\nu_0\nu_1(1+|p|^2)^{\tau+\vartheta}|\xi|^2, \tag{9.5′}$$

where ν_0, ν_1 are positive constants and $-\frac{1}{2}<\tau\leq 0$, $\vartheta\geq 0$.

(B3) The function $\varphi(x)$, $x\in\partial\Omega$, satisfies a bounded slope condition with constant K_0: for every $x_0\in\partial\Omega$, there is a pair of linear functions $\pi^\pm(x)=\alpha_j^\pm(x^j-x_0^j)+\varphi(x_0)$ of x satisfying, for $x\in\partial\Omega$,

$$\begin{gathered}\alpha_j^-(x^j-x_0^j)+\varphi(x_0)\leq\varphi(x)\leq\alpha_j^+(x^j-x_0^j)+\varphi(x_0),\\ |\pi_x^\pm(x)|^2=\sum_{j=1}^n|\alpha_j^\pm|^2\leq K_0^2.\end{gathered} \tag{9.8}$$

When $n=2$, this is equivalent to the classical 3-point condition. If $\varphi(x)$ is the restriction to $\partial\Omega$ of a linear function of x, then $\varphi(x)$ satisfies (B3). When $\varphi(x)$ satisfies (B3) and is not the restriction of a linear function to $\partial\Omega$, then Ω is necessarily convex.

(B4) The function $\varphi(x)$, $x \in \Omega$, satisfies a generalized bounded slope condition with constants K_0 and Q: for every $x_0 \in \partial\Omega$, there is a pair of functions $\pi^{\pm}(x) \in C^{1,1}(\bar{\Omega})$ (i.e., $\pi^{\pm}(x) \in C^1(\Omega)$ and its partial derivatives are uniformly Lipschitz continuous) such that

$$\pi^{\pm}(x_0) = \varphi(x_0); \tag{9.9}$$

the best Lipschitz constants of $\pi^{\pm}$ and its partial derivatives satisfy

$$\lambda(\pi^{\pm}) \leqslant K_0 \quad \text{and} \quad \lambda(\pi^{\pm}_{x_i}) \leqslant Q \tag{9.10}$$

for $i = 1, \dots, n$; finally,

$$\pi^{-}(x) \leqslant \varphi(x) \leqslant \pi^{+}(x) \quad \text{for} \quad x \in \partial\Omega. \tag{9.11}$$

When $Q = 0$, this condition reduces to the bounded slope condition (B3). A sufficient (but not necessary) condition that $\varphi(x)$ satisfy (B4) is the following.

(B4′) $\varphi(x)$ is the trace of a function of class $C^{1,1}(\bar{\Omega})$, i.e., there exists a function $\Psi(x) \in C^{1,1}(\bar{\Omega})$ such that $\Psi(x) = \varphi(x)$ for $x \in \partial\Omega$. (In this case, $\pi^{\pm}(x) = \Psi(x)$ for each $x_0 \in \partial\Omega$, $K_0 = \lambda(\Psi)$, and $Q = \max(\lambda(\Psi_{x_1}), \dots, \lambda(\Psi_{x_n}))$ are admissible.)

If $\Omega \in C^{1,1}$ is uniformly convex, then (B4′) implies (B3). Conversely, if $\Omega \in C^{1,1}$ is convex (but not necessarily uniformly convex), then (B3) implies (B4′); see [6].

THEOREM 9.1. *Let Ω be convex, $a_i(p)$ satisfy* (B2) *with*

$$0 \leqslant \vartheta < 1/2, \tag{9.12}$$

and $\varphi(x)$ satisfy (B4) [*e.g.,* (B4′)]. *Let $F[u]$ satisfy conditions* (A2), (A3), (A4) *of Section 8 with $\alpha = 2 + 2\tau$, $\mu = \nu_0$. Then there exists a constant T_1 (depending on the parameters specified in Theorem 8,1, on $\delta(0)$ in* (A4), *and on ν_1, ϑ) with the property that if $u(x)$ is a K-quasi solution of* (8.3), *then*

$$\lambda(u) \leqslant T, \quad \textit{where} \quad T = 3K_0 + T_1[1 + Q + Q^{1/(1-2\vartheta)}] \tag{9.13}$$

As to the choice $\alpha = 2 + 2\tau$ and $\mu = \nu_0$, see Remark 2 following (B1) above. This type of result, in which $\vartheta > 0$, is permitted, seems novel. If $Q = 0$ (so that $\varphi(x)$ satisfies (B3)), then the condition on $a_i(p)$ can be reduced from (B2) to (B1).

THEOREM 9.2. *Let Ω be convex, $a_i(p)$ satisfy* (B1), *and $\varphi(x)$ satisfy* (B3). *Let $F[u]$ be as in Theorem 9.1. Then there exists a constant T_2 (depending on the parameters specified in Theorem 8.1 and on $\delta(0)$ in* (A4)) *such that if $u(x)$ is a K-quasi solution of (8.3), then $\lambda(u) \leqslant T$, where $T = 3K_0 + T_2$.*

For the variational case, Stampacchia [14] derived a similar a priori bound for $\lambda(u)$ under the additional condition that Ω is uniformly convex. Theorem 9.2 can be used to reduce the assumption that Ω is uniformly convex to the assumption that Ω is convex in theorems of [14].

The convexity assumption on Ω in Theorem 9.1 can be relaxed if condition (9.12) is strengthened to $\vartheta = 0$.

THEOREM 9.3. *Let* $a_i(p)$ *satisfy* (B2) *with* $\vartheta = 0$; $\varphi(x)$ *satisfy* (B4) [*e.g.*, (B4′)]; *and* Ω *have the property that for every* $x_0 \in \partial\Omega$, *there is closed sphere* $\Sigma(x_0, R)$ *of radius* R (*independent of* x_0) *outside of* Ω *such that the intersection* $\bar{\Omega} \cap \Sigma(x_0, R)$ *is the point* x_0. *Let* $F[u]$ *be as in Theorem 9.1. Then there exists a constant* T_3 (*depending on the parameters specified in Theorem 8.1, on* $\delta(0)$ *in* (A4), *and on* R, ν_1) *with the property that if* $u(x)$ *is a* K *quasi solution of* (8.3), *then* $\lambda(u) \leqslant T$, *where* $T = 3K + T_3(1 + Q)$.

A part of the condition (B4) can be stated as follows: $\pi^{\pm}(x) \in H^{2,\infty}(\Omega)$ and the norms of $\pi^{\pm}(x)$ in $H^{2,\infty}(\Omega)$ are uniformly bounded with respect to the parameter $x_0 \in \partial\Omega$. By a different technique, it will be shown that if (B2) is strengthened to (B2′), then Theorem 9.3 remains correct if the space $H^{2,\infty}(\Omega)$ is replaced by $H^{2,\varkappa}(\Omega)$, $\varkappa > n(n+1)/2$, and that the same is true of Theorem 9.1 if the "convexity" of Ω is replaced by "uniform convexity". Instead of formulating an analogue of (B4), we shall merely use the following relaxation of (B4′):

(B5) $\varphi(x)$, $x \in \partial\Omega$, is the trace of a function $\Psi(x)$, $x \in \Omega$, which is uniformly Lipschitz continuous and of class $H^{2,\varkappa}(\Omega)$ for some $\varkappa > n(n+1)/2$. Let $K_0 = \lambda(\Psi)$ and $Q = \|(\sum\sum |\Psi_{x_i x_j}|)\|_{\varkappa,\Omega}$.

If Ω satisfies a cone condition, then $\Psi \in H^{2,\varkappa}(\Omega)$ implies that $\Psi \in C^1(\bar{\Omega})$. In this case, the assumption that Ψ is uniformly Lipschitz continuous is redundant.

(B6) Ω is a bounded, uniformly convex domain; i.e., there exists a number $m_0 > 0$ such that through every point $x_0 \in \partial\Omega$, there passes a supporting hyperplane π of Ω satisfying $\operatorname{dist}(x, \pi) \geqslant m_0 |x - x_0|^2$ for $x \in \partial\Omega$.

If $\partial\Omega$ is smooth, this means that the curvatures of the 2-dimensional sections of $\partial\Omega$ are bounded away from zero.

THEOREM 9.4. *Let* $a_j(p)$, φ *and* Ω *satisfy* (B2′) *with* $\vartheta < 1/2$, (B5) *and* (B6), *respectively. Let* $F[u]$ *be as in Theorem 9.1. Then the conclusion of Theorem 9.1 remains valid, but* T_1 *also depends on* $\varkappa$ *in* (B5) *and* m_0 *in* (B6).

Remark. If (B6) is replaced by the condition that Ω is convex Theorem 9.4 remains correct provided that the assumption $\vartheta < 1/2$ is strengthened to

$$\vartheta < 1/2[1 + (n-1)(1 + 2/n - 1/\varkappa)/(n+3)];$$

cf. the Remark following Lemma 10.4 a below.

THEOREM 9.5. *Theorem 9.3 is valid if condition* (B4) *on* φ *is relaxed to* (B5), *but* T_3 *depends also on* $\varkappa$.

10. Proofs. The proofs of Theorems 9.1–9.5 will be given in this section and will be based on several lemmas. The first of these (Lemma 10.0) depends on a device of Rado and shows that it is sufficient to derive a priori bounds for the Lipschitz continuity at points of $\partial\Omega$. We then state and prove Lemmas 10.1–10.3 and derive Theorems 9.1–9.3, respectively, from these. Theorem 9.4 will be proved with the use of Lemmas 10.4 and 10.4a. Finally, we indicate the proof of Theorem 9.5.

LEMMA 10.0. *Let $a_j(p)\in C^0(E^n)$ satisfy* (9.4) *in* (B1′), *$\varphi(x)$ be a uniformly Lipschitz continuous function on $\partial\Omega$, and $g(x)\in L^\infty(\Omega)$. Let $u(x)\in\mathcal{K}_\varphi^K$ satisfy*

$$\int_\Omega [a_j(u_x)(v-u)_{x_j}-g(x)(v-u)]dx\geqslant 0 \quad \textit{for } v\in\mathcal{K}_\varphi^k, \tag{10.1}$$

$$|u(x_0)-u(x^0)|\leqslant K_1|x_0-x^0| \quad \textit{for} \quad x_0\in\partial\Omega,\ x^0\in\Omega. \tag{10.2}$$

Then the best Lipschitz constant $\lambda(u)$ of u satisfies

$$\lambda(u)\leqslant K_1+2S|\Omega|^{1/n}\|g\|_\infty/\nu, \tag{10.3}$$

where S is the constant in Lemma 7.3 *with $\alpha=\infty$.*

This lemma involves no convexity assumption on Ω.

Proof. It has to be shown that (10.2) remains valued if "$x_0\in\partial\Omega$" is replaced by "$x_0\in\Omega$" and K_1 by the constant on the right of (10.3). For the sake of simplicity, it will be supposed that x^0-x_0 is in the direction of the x_1-axis. Let $x^0-x_0=(1,0,\ldots,0)\Delta=\Delta e_1$. Introduce the notation

$$u_\Delta(x)=u(x+\Delta e_1)-u(x). \tag{10.4}$$

Let $\Omega_{-\Delta}$ denote the translation $\Omega-\Delta e_1$ of Ω, so that (10.4) is defined on $\Omega\cap\Omega_{-\Delta}$.

Note that if x in (10.4) is on $\partial(\Omega\cap\Omega_{-\Delta})$, then one of the two points x or $x+\Delta e_1$ is on $\partial\Omega$. Hence, by (10.2),

$$|u_\Delta(x)|\leqslant K_1|\Delta| \quad \text{for} \quad x\in\partial(\Omega\cap\Omega_{-\Delta}). \tag{10.5}$$

Let $U(x)=\max(u_\Delta(x)-k,0)$ for $x\in\Omega\cap\Omega_{-\Delta}$ and $U(x)=0$ for $x\in\Omega-\Omega_{-\Delta}$. Thus $U(x)\in\mathcal{K}_0$ if

$$k\geqslant\max u_\Delta(x) \quad \text{on} \quad \partial(\Omega\cap\Omega_{-\Delta}). \tag{10.6}$$

Determine $v(x)$ by the relation

$$v(x)-u(x)=U(x);$$

i.e., $v(x)=u(x+\Delta e_1)-k$ if $u_\Delta(x)\geqslant k$, $x\in\Omega\cap\Omega_{-\Delta}$ and $v(x)=u(x)$ otherwise. Thus, in the case (10.6), $v(x)\in\mathcal{K}_\varphi^K$ and (10.1) gives

$$\int_{A(k,\Delta)} [a_j(u_x)u_{\Delta x_j}(x)-g(x)(u_\Delta(x)-k)]dx\geqslant 0, \tag{10.7}$$

where
$$A(k,\Delta)=\{x\colon x\in\Omega\cap\Omega_{-\Delta},\ u_\Delta(x)\geqslant k\}. \tag{10.8}$$

Similarly, let $V(x)=-\max(u_\Delta(x-\Delta e_1)-k,0)$ for $x\in\Omega\cap\Omega_\Delta$ and $V(x)=0$ otherwise, where
$$k\geqslant\max u_\Delta(x-\Delta e_1)\quad\text{for}\quad x\in\partial(\Omega\cap\Omega_\Delta \tag{10.9}$$

Defining v by $v-u=V(x)$, we get
$$-\int_{B(k,\Delta)}\{a_j(u_x(x))u_{\Delta x_j}(x-\Delta e_1)-g(x)[u_\Delta(x-\Delta e_1)-k]\}\,dx\geqslant 0, \tag{10.10}$$

where
$$B(k,\Delta)=\{x\colon x\in\Omega\cap\Omega_\Delta,\ u_\Delta(x-\Delta e_1)\geqslant k\}.$$

Note that $A(k,\Delta)=B(k,\Delta)-\Delta e_1$. If $x-\Delta e_1$ is introduced as a new integration variable in (10.10),
$$\int_{A(k,\Delta)}[a_j(u_x(x+\Delta e_1))u_{\Delta x_j}(x)-g(x+\Delta e_1)(u_\Delta(x)-k)]dx\leqslant 0. \tag{10.11}$$

Subtracting (10.11) from (10.7),
$$\int_{A(k,\Delta)}\{[a_j(u_x(x+\Delta e_1))-a_j(u_x(x))]u_{\Delta x_j}(x)-g_\Delta(x)(u_\Delta(x)-k)\}dx\leqslant 0, \tag{10.12}$$

and using (9.4),
$$\int_{A(k,\Delta)}[\nu|u_{\Delta x}|^2-g_\Delta(x)(u_\Delta-k)]dx\leqslant 0. \tag{10.13}$$

Let $g_\Delta(x)=0$ for $x\notin\Omega\cap\Omega_{-\Delta}$ and let $\Omega\cap\Omega_{-\Delta}$ be in the half-space $x_1\geqslant c$. Put
$$f_1(x)=\int_c^{x_1}g_\Delta(t,x_2,\ldots,x_n)dt,\quad f_2=\ldots=f_n=0.$$

Then, for $x\in\Omega\cap\Omega_{-\Delta}$,
$$f_1(x)=\int_{c+\Delta}^{x_1+\Delta}g(t,x_2\ldots)dt-\int_c^{x_1}g(t,x_2,\ldots)\,dt.$$

Hence $|f_1(x)|\leqslant 2\chi|\Delta|$. Integrating the last term of (10.13) by parts gives
$$\int_{A(k,\Delta)}[\nu|u_{\Delta x}|^2+f_j(x)u_{\Delta x_j}]dx\leqslant 0.$$

Thus the proof of Lemma 7.3 and the Remark following it show that, for $x\in\Omega\cap\Omega_{-\Delta}$,
$$u_\Delta(x)\leqslant\max_{\partial(\Omega\cap\Omega-\Delta)}u_\Delta(x)+2S|\Omega|^{1/n}\chi|\Delta|/\nu. \tag{10.14}$$

Similarly, we obtain a lower bound for $u_\Delta(x)$. Thus, by (10.5),
$$|u_\Delta(x)|\leqslant(K_1+2S|\Omega|^{1/n}\chi/\nu)|\Delta|$$

for $x\in\Omega\cap\Omega_{-\Omega}$. This proves (10.3).

LEMMA 10.1. *Let $a_j(p), \Omega, \varphi$ satisfy the conditions of Theorem 9.1 and let $g(x) \in L^\infty(\Omega)$. Then there exists a constant $S = S(\Omega, \nu_0, \nu_1, \tau, \vartheta)$ with the property that if $u(x) \in \mathcal{K}_\varphi^K$ satisfies (10.1), then*

$$\lambda(u) < 2K_0 + S[Q + Q^{1/(1-2\vartheta)} + \|g\|_\infty + \|g\|_\infty^{1/(1+2\tau)}]. \tag{10.15}$$

Proof. Let ν, ν^* denote the constants

$$\nu = \nu_0[1 + \lambda^2(u)]^\tau, \quad \nu^* = \nu_1[1 + \lambda^2(u)]^\vartheta; \tag{10.16}$$

so that for $|p|, |q| \leqslant \lambda(u)$,

$$\nu|p - q|^2 \leqslant [a_j(p) - a_j(q)](p_j - q_j); \quad \lambda_0(p) \leqslant \nu; \quad \lambda_1(p)/\lambda_0(p) \leqslant \nu^*. \tag{10.17}$$

By (9.5),
$$0 \leqslant \frac{\partial a_j}{\partial p_j} \leqslant \lambda_1(p), \quad \left|\frac{\partial a_j}{\partial p_i} + \frac{\partial a_i}{\partial p_j}\right| \leqslant 2\lambda_1(p). \tag{10.18}$$

Let $x_0 \in \partial\Omega$. Consider a closed sphere $\Sigma(x_0, R)$ of radius R, outside of Ω, and intersecting $\bar{\Omega}$ only in the point x_0. Let $r = r(x)$ be the distance of a point x from the center of $\Sigma(x_0, R)$. Define $\delta(x)$ by

$$\delta(x) = 1 - e^{a(R-r)}, \tag{10.19}$$

where k, a, R are positive constants to be determined so that

$$\int_\Omega [a_j(\pi^+ + k\delta_x)\eta_{x_j} - \chi\eta]\,dx \geqslant 0 \quad \text{for} \quad 0 \leqslant \eta \in \mathcal{K}_0, \tag{10.20}$$

$\chi = \|g\|_\infty$ and $\pi^+ = \pi^+(x)$ is a function occurring in (B4). (Actually, in this proof, we shall let $a = 1$ and choose k, R suitably. The number a is inserted in (10.19) to facilitate the proof of Lemma 10.3.)

Integration by parts of the first term of (10.20) shows that a sufficient condition for (10.20) is that $-\partial a_j(\pi_x^+ + k\delta_x)/\partial x_j \geqslant \chi$, i.e.,

$$-(\pi^+_{x_j x_i} + k\delta_{x_j x_i})(\partial a_j/\partial p_i) \geqslant \chi.$$

Since $|\pi^+_{x_j x_i}| \leqslant Q$ and (10.19) show that

$$\delta_x = ae^{a(R-r)}x/r, \quad \delta_{x_i x_j} = -[(a + 1/r)x_i x_j/r^2 - \delta_{ij}/r]ae^{a(R-r)}, \tag{10.21}$$

it is easily seen that (10.20) holds if

$$ka[\lambda_0 a - n\lambda_1/r]e^{a(R-r)} \geqslant \chi + n^2 Q\lambda_1 \tag{10.22}$$

for $x \in \Omega$, where λ_0 and λ_1 are evaluated at $p = \pi_x^+ + k\delta_x$. Let $d = d(\Omega)$ satisfy $0 < r - R \leqslant d$ for $x \in \Omega$ and let $a = 1$. Then (10.22) holds if $R \geqslant 2n\lambda_1/\lambda_0$ (so that $\lambda_0 - n\lambda_1/r \geqslant \frac{1}{2}\lambda_0$) and

$$k \geqslant 2[\chi/\nu_0 + n^2 Q\lambda_1/\lambda_0]e^d. \tag{10.23}$$

Thus if $R=2n\nu^*$ and
$$k=2[\chi/\nu+n^2Q\nu^*]e^d, \tag{10.24}$$
then (10.22) holds provided that $|\pi_x^+ + k\delta_x| \leqslant K_0+k \leqslant \lambda(u)$. On the other hand, if $\lambda(u)< K_0+k$, then the end of the proof will show that the lemma is correct.

Determine $v(x)$ by the relation
$$v-u=-\max(u-\pi^+-k\delta, 0).$$
Then $u\leqslant \pi^+$ on $\partial\Omega$ implies that $v-u=\varphi$ on $\partial\Omega$ and $\lambda(v)\leqslant \min(K, \lambda(\pi^+)+k\lambda(\delta))$. Thus if $\lambda(\pi^+)+k\lambda(\delta)\leqslant K$, (10.1) and $\|g\|_\infty=\chi$ give
$$\int_\Lambda a_j(u_x)(u-\pi^+-k\delta)_{x_j}dx \leqslant \chi\int_\Lambda (u-\pi^+-k\delta)dx,$$
where $\Lambda=\{x: u-\pi^+-k\delta\geqslant 0\}$. By the choice $\eta=\max(u-\pi^+-k\delta, 0)$ in (10.20),
$$-\int_\Lambda a_j(\pi_x^+ + k\delta_x)(u-\pi^+-k\delta)_{x_j}dx \leqslant -\chi\int_\Lambda (u-\pi^+-k\delta)dx.$$
Adding these two inequalities and using (10.6),
$$\nu\int_\Lambda |(u-\pi^+-k\delta)_x|^2 dx\leqslant 0. \tag{10.25}$$
Thus $u\leqslant \pi^+ + k\delta$ on Ω. Similarly, we obtain $u\geqslant \pi^- - k\delta$ on Ω.

Since $\lambda(\pi^\pm)\leqslant K_0$ and $\lambda(\delta)\leqslant 1$, by (10.21) and $a=1$, we have for $x_0\in\partial\Omega$, $x^0\in\Omega$,
$$|u(x_0)-u(x^0)|\leqslant |x_0-x^0|\min(K, K_0+k).$$
Consequently, by Lemma 10.0,
$$\lambda(u)\leqslant K_0+k+2S|\Omega|^{1/n}\chi/\nu. \tag{10.26}$$
From (10.16), (10.24) and (10.26), it follows that $\lambda=\lambda(u)$ satisfies an inequality of the form
$$\lambda\leqslant K_0+S_0Q(1+\lambda^{2\vartheta})+S_1\chi(1+\lambda^{-2\tau}), \tag{10.27}$$
where $S_0=S_0(\Omega, \nu_1, \vartheta)$, $S_1=(\Omega, \nu_0, \tau)$. By $2\vartheta<1$, $-2\tau<1$, and the inequality (8.23) for the arithmetic-geometric means, there exist constants S_0', S_1' (depending on the same parameters, respectively) such that
$$S_0Q\lambda^{2\vartheta}\leqslant \lambda/4+S_0'Q^{1/(1-2\vartheta)}, \quad S_1\chi\lambda^{-2\tau}\leqslant \lambda/4+S_1'\chi^{1/(1+2\tau)}.$$
Thus Lemma 10.1 follows.

LEMMA 10.2. *Let $a_j(p)$, Ω, φ satisfy the conditions of Theorem 9.2 and let $g(x)\in L^\infty(\Omega)$. Then there exists a constant $S=S(\Omega, \nu_0, \tau)$ with the property that if $u(x)\in\mathcal{K}_\varphi^K$ satisfies (10.1), then*
$$\lambda(u)\leqslant 2K_0+S[\|g\|_\infty+\|g\|_\infty^{1/(1+2\tau)}]. \tag{10.28}$$

Proof. Suppose first that $a_j(p) \in C^1(E^n)$, so that (9.3) holds. In this case, Lemma 10.2 follows from the proof of Lemma 10.1 if it is noted that $Q=0$ implies that the second term in (10.23) vanishes, so that no estimate for λ_1/λ_0 is needed.

In order to show that the extra assumption $a_j(p) \in C^1(E^n)$ is unnecessary, note that there exist sequences $\{a_{jm}(p)\}$, $m=1,2,\ldots$, a functions of class $C^1(E^n)$ such that $a_{jm}(p) \to a_j(p)$, $m \to \infty$, uniformly on bounded p-sets and $(a_{1m}(p),\ldots,a_{nm}(p))$ satisfies condition (B1), say, with ν_0 replaced by $\nu_0/2$. Theorem 1.1 and the remarks following it show that there exist functions $u_m(x) \in \mathcal{K}_\varphi^K$ satisfying

$$\int_\Omega [a_{jm}(u_{mx})(v-u_m)_{x_j} - g(v-u_m)]dx \geqslant 0 \quad \text{for } v \in \mathcal{K}_\varphi^K.$$

Since $a_{jm} \in C^1$, the function $u=u_m$ satisfies (10.28), where $S=S(\Omega,\nu_0/2,\tau)$. Hence, after a selection of a subsequence, it can be supposed that $\lim u_m = u_0$ exists uniformly on Ω and weakly in $H^1(\Omega)$. Consequently, $u=u_0(x)$ satisfies (10.28). Furthermore, a variant of the proof of Corollary 1.1 shows that $u=u_0(x)$ satisfies (10.1). Since the function $u(x)$ satisfying (10.1) is unique by Corollary 1.2, $u(x)=u_0(x)$. This completes the proof of Lemma 10.2.

LEMMA 10.3. *Let $a_j(p),\Omega,\varphi$ satisfy the assumptions of Theorem 9.3 and let $g(x) \in L^\infty(\Omega)$. Then there exists a constant $S=S(\Omega,\nu_0,\nu_1,\tau,R)$ such that if $u(x) \in K_\varphi^K$ satisfies (10.1), then*

$$\lambda(u) \leqslant 2K_0 + S[Q + \|g\|_\infty + \|g\|_\infty^{1/(1+2\tau)}]. \tag{10.29}$$

Proof. This proof is identical with that of Lemma 10.1 except that, in order to satisfy (10.22), choose $a=1+n\nu^*/R$ and determine k so that

$$ka = (\chi/\nu + n^2 Q\nu^*)e^{ad},$$

where R is given.

Proof of Theorems 9.1–9.3. The function $u(x) \in \mathcal{K}_\varphi^K$ satisfies (10.1) with $g(x)=F[u](x)$. By Theorem 8.1, there is a constant T such that $|u(x)| \leqslant T$ on χ. Hence, by (A4), $\|g\|_\infty \leqslant \chi(T)(1+\lambda^{\delta(0)})$, $\lambda=\lambda(u)$. If this is substituted in (10.15), (10.28), or (10.29), the respective Theorems 9.1, 9.2, or 9.3 follow from $\delta(0)<\alpha-1=1+2\tau<1$.

LEMMA 10.4a. *Let Ω be a bounded, uniformly convex domain and x_0, π, m_0 as in* (B6). *Let $\Sigma(x_0,R)$ be the closed sphere of radius R outside of Ω and tangent to π at x_0. Let $r=r(x)$ be the distance from x to the center of $\Sigma(x_0,R)$. Then there is a number $L=L(s,m_0,\Omega)$, independent of $x_0 \in \partial\Omega$ and $R>0$, such that $\delta(x)=1-e^{R-r}$ satisfies*

$$\int_\Omega \delta^{-s}(x)dx \leqslant L < \infty \quad \text{if } s<(n+1)/2. \tag{10.30}$$

Remark. If Ω is convex (not necessarily uniformly convex), then (10.30) can be replaced by

$$\int_\Omega \delta^{-s}(x)dx \leqslant LR^{s-1} < \infty \quad \text{if } 1 < s < (n+1)/2$$

and $R \geqslant 1$, where $L = L(s, \Omega)$; cf. the proof of Lemma 10.5a below. It will be clear from the proof of Theorem 9.4 that this inequality can be used to prove the Remark following Theorem 9.4.

Proof. If $d \geqslant r - R$, then $\delta(x) \geqslant e^{-d}(r-R)$ and $r - R \geqslant \operatorname{dist}(x, \pi)$. Hence Lemma 10.4a follows from [14], p. 404; cf. also the proof of Lemma 10.5a below.

LEMMA 10.4. *Let $a_j(p)$, φ, Ω satisfy the conditions of Theorem 9.4 and let $g(x) \in L^\infty(\Omega)$. Then there exists a constant $S = S(\Omega, \nu_0, \nu_1, \tau, \vartheta, \varkappa)$ with the property that if $u(x) \in \mathcal{K}^K_\varphi$ satisfies (10.1), then (10.15) holds.*

Proof. Let $x_0 \in \partial\Omega$ and $\Sigma(x_0, R)$ be as in Lemma 10.4a. Let $\nu, \nu^*, d, \chi = \|g\|_\infty, r, \delta(x) = 1 - e^{R-r}$, and $R = 2n\nu^*$ be as in the proof of Lemma 10.1. Let Ψ be the function given in (B5). Choose k_0 to be

$$k_0 = 2e^d\chi/\nu, \tag{10.31}$$

so that

$$-k\delta_{x_j x_i}(\partial a_j(\Psi_x + k\delta_x)/\partial p_i) \geqslant \chi \quad \text{if } k \geqslant k_0.$$

The beginning of the proof of Lemma 10.1 shows that

$$\int_\Omega a_j(\Psi_x + k\delta_x)\eta_{x_j}dx \geqslant \int_\Omega (\chi - \nu\nu^* J(x))\eta\, dx \quad \text{for } 0 \leqslant \eta \in \mathcal{K}_0,$$

where

$$J(x) = \sum_{j=1}^n \sum_{k=1}^n |\Psi_{x_i x_j}(x)|.$$

Letting $\eta = \max(u - \Psi - k\delta, 0)$ gives

$$-\int_{\Lambda(k)} a_j(\Psi_x + k\delta_x)(u - \Psi - k\delta)_{x_j}dx \leqslant \int_{\Lambda(k)} (-\chi + \nu\nu^* J)(u - \Psi - k\delta)dx, \tag{10.32}$$

where $\Lambda(k) = \{x : u - \Psi \geqslant k\delta\}$.

Let k be on the range

$$k_0 \leqslant k \leqslant \lambda(u) - K_0, \tag{10.33}$$

where, without loss of generality, it can be supposed that $\lambda(u) \geqslant K_0 + k_0$. Since $\lambda(\Psi + k\delta) \leqslant K_0 + k$, the last inequality shows that $v = u - \max(u - \Psi - k\delta, 0)$ is in $\mathcal{K}^K_\varphi$. Thus, by (10.1),

$$\int_{\Lambda(k)} a_j(u_x)(u-\Psi-k\delta)_{x_j}dx \leqslant \chi \int_{\Lambda(k)} (u-\Psi-k\delta)dx.$$

This relation, (10.32), and (10.17) show that

$$\nu \int_{\Lambda(k)} |(u-\Psi-k\delta)_x|^2 dx \leqslant \nu\nu^* \int_{\Lambda(k)} J(u-\Psi-k\delta)dx.$$

From Sobolev's and Hölder's inequalities

$$\nu \| u-\Psi-k\delta \|_{2^*, \Lambda(k)} \leqslant S_2^2 \nu\nu^* \| J \|_{2^{*\prime}, \Lambda(k)},$$

where $1/2^{*\prime} = 1/2 + 1/n$. Hölder's inequality applied to both sides gives

$$\nu \| u-\Psi-k\delta \|_{1, \Lambda(k)} \leqslant S_2^2 \nu\nu^* Q |\Lambda(k)|^{1+2/n-1/\varkappa}, \tag{10.34}$$

since $\| J \|_{\varkappa, \Lambda(k)} \leqslant \| J \|_{\varkappa, \Omega} = Q$.

From $\varkappa > n(n+1)/2$, the last exponent satisfies

$$1+2/n-1/\varkappa > 1+2/(n+1) = (n+3)/(n+1).$$

Thus there exists a number θ less than, but near to, $(n+1)/(n+3)$ such that

$$\gamma = \theta(1+2/n-1/\varkappa) > 1 \quad \text{and} \quad s = \theta/(1-\theta) < (n+1)/2. \tag{10.35}$$

From (10.34) and

$$|\Lambda(k)| = \int_{\Lambda(k)} \delta^\theta(x)\delta^{-\theta}(x)dx < \left(\int_{\Lambda(k)} \delta(x)dx\right)^\theta \left(\int_\Omega \delta^{-s}(x)dx\right)^{1-\theta},$$

we get

$$\int_{\Lambda(k)} [(u-\Psi)/\delta - k]\delta dx \leqslant c\left(\int_{\Lambda(k)} \delta(x)dx\right)^\gamma, \tag{10.36}$$

$$c = S_2^2 Q \nu^* L^{(1-\theta)(1+2/n-1/\varkappa)}, \tag{10.37}$$

and L is given by (10.30).

The inequality (10.36) can be written in the form (7.2), where k is on the range (10.33) and

$$\varrho(t) = \int_{\Lambda(t)} \delta(x)dx,$$

Thus Lemma 7.2 implies that

$$u - \Psi \leqslant k\delta \text{ on } \Omega \text{ if } k = k_0 + c[\varrho(k_0)]^{\gamma-1}\gamma/(\gamma-1),$$

provided that this value of k satisfies (10.33). Since $\delta(x) \leqslant 1$ shows that $\varrho(k_0) \leqslant |\Omega|$ and since $\lambda(\delta) \leqslant 1$ and $\lambda(\Psi) = K_0$, there is an $S_0 = S_0(\Omega, \varkappa)$ such that $k \leqslant k_0 + S_0 Q \nu^*$; hence,

$$u(x_0) - u(x^0) \leqslant \{K_0 + k_0 + S_0 Q \nu^*\} |x_0 - x^0|$$

for $x_0 \in \partial\Omega$, $x^0 \in \Omega$, provided that the coefficient of $|x_0 - x^0|$ satisfies $\{...\} \leqslant \lambda(u)$. Obviously this last proviso is unnecessary.

We can obtain a similar lower bound for $u(x_0) - u(x^0)$. Hence, by (10.31),

$$|u(x_0) - u(x^0)| \leqslant \{K_0 + 2e^d\chi/\nu + S_0 Q\nu^*\} |x_0 - x^0|$$

for $x_0 \in \partial\Omega$, $x^0 \in \Omega$. By Lemma 10.0, the same inequality holds for $x_0, x^0 \in \Omega$ if the coefficient of $|x_0 - x^0|$ is increased by $2S|\Omega|^{1/n}\chi/\nu$. The arguments used at the end of the proof of Lemma 10.1 can be used to complete the proof of Lemma 10.4.

Proof of Theorem 9.4. It is clear that this theorem follows from Lemma 10.4 in the same way that Theorem 9.1 follows from Lemma 10.1.

LEMMA 10.5a. *Let Ω and $\Sigma(x_0, R)$ be as in Theorem 9.3, $r = r(x)$ the distance from x to the center of $\Sigma(x_0, R)$, and $\delta(x) = 1 - e^{a(R-r)}$, $a > 0$. Then there exists a constant $L = L(a, R, s, \Omega)$ such that*

$$\int_\Omega \delta^{-s}(x)\,dx \leqslant L < \infty \quad \text{if } s < (n+1)/2. \tag{10.38}$$

Proof. By replacing R by $R/2$, if necessary, it can be supposed that there is a closed sphere $\Sigma(x_0, 2R)$ of radius $2R$, containing $\Sigma(x_0, R)$, tangent to it at x_0, and lying outside of Ω.

Choose a coordinate system such that $x_0 = 0$ and the center of $\Sigma(x_0, R)$ is $(0, 0, ..., 0, -R)$. We can suppose that Ω lies in the half-space $x_n \geqslant -R/2$, for the contribution of $\Omega \cap \{x_n < -R/2\}$ to the integral in (10.38) can be estimated trivially.

First, make the change of integration variables $x_n \to r$ and introduce polar coordinates on the hyperplane $x_n = 0$. Then, at $x \in \Omega$,

$$dx = r\varrho^{n-2}(r^2 - \varrho^2)^{-1/2}\,dr\,d\varrho\,d\omega, \tag{10.39}$$

where $\varrho = |(x_1, ..., x_{n-1}, 0)|$ and ω is the surface of the unit sphere in $x_n = 0$. In fact, (10.39) follows from $r^2 = (x_n + R)^2 + \varrho^2$, so that $\partial r/\partial x_n = (x_n + R)/r = (r^2 - \varrho^2)^{1/2}/r$.

Let $y = y(x)$ be the point where the line joining the center of $\Sigma(x_0, R)$ and $x \in \Omega$ meets the hyperplane $x_n = 0$. Let $\sigma = |y|$. Thus $R/\sigma = (x_n + R)/\varrho$, hence

$$\varrho = \sigma r/(R^2 + \sigma^2)^{1/2}, \quad \partial\varrho/\partial\sigma = rR^2/(R^2 + \sigma^2)^{3/2}$$

and $\sigma = R\varrho(r^2 - \varrho^2)^{-1/2}$. Thus (10.39) implies that

$$dx = Rr^{n-1}\sigma^{n-2}(R^2 + \sigma^2)^{-n/2}\,dr\,d\sigma\,d\omega.$$

For $x \in \Omega$ and $d \geqslant r - R$,

$$Rr^{n-1}(R^2 + \sigma^2)^{-n/2} \leqslant (1 + d/R)^{n-1}.$$

Since $\delta(x) \geqslant e^{-ad}a(r-R)$ and $x \notin \Sigma(x_0, 2R)$,

$$\int_\Omega \delta^{-s}(x)\,dx \leqslant L_0 \int_0^D \sigma^{n-2}\,d\sigma \int_{R+\varepsilon\vartheta^2}^{R+d} (r-R)^{-s}\,dr,$$

where $L_0 = L_0(s, d, a, R)$, $D = D(d, R)$ and $\varepsilon = \varepsilon(R) > 0$. This implies Lemma 10.15a.

It is clear from the proof of Lemma 10.4 that one can derive an analogous lemma leading to the proof of Theorem 9.5.

11. The case $F[u](x) \equiv 0$. A priori bounds for $|u|$ and $\lambda(u)$ are particularly easy in this case.

(**C1**) Let $a(p) = (a_1(p), \ldots, a_n(p)) \in C^0(E^n)$ satisfy

$$a(0) = 0, \tag{11.1}$$

$$[a_j(p) - a_j(q)](p_j - q_j) > 0 \quad \text{if} \quad p \neq q. \tag{11.2}$$

Lemma 11.1. *Let* $a(p)$ *satisfy* (C1) *and* $u(x) \in \mathcal{K}_\varphi^K$ *satisfy*

$$\int_\Omega a_j(u_x)(v-u)_{x_j}\,dx \geqslant 0 \quad \textit{for} \quad v \in \mathcal{K}_\varphi^K. \tag{11.3}$$

Then, for $x \in \Omega$,

$$\min_{\partial\Omega} \varphi \leqslant u(x) \leqslant \max_{\partial\Omega} \varphi. \tag{11.4}$$

Proof. Let $\Phi = \max \varphi(x)$ on $\partial\Omega$ and $v - u = \max(u - \Phi, 0)$. Then (11.3) gives

$$\int_{\{u \geqslant \Phi\}} a_j(u_x)\,u_{x_j}\,dx \leqslant 0.$$

By (11.1) and (11.2), it follows that the function $\max(u(x) - \Phi, 0)$ is a constant. This gives the last inequality in (11.4) and the first is obtained similarly.

Lemma 11.2. *Let* $a(p)$ *satisfy* (C1), Ω *be convex*, $\varphi(x)$ *satisfy a bounded slope condition with constant* K_0 (*cf.* (B3) *in Section 9*). *Let* $u \in \mathcal{K}_\varphi^K$ *satisfy* (*11.3*). *Then*

$$\lambda(u) \leqslant K_0. \tag{11.5}$$

Proof. Let $x_0 \in \partial\Omega$ and $\pi^{\pm}(x)$ the linear functions of x in (9.8). Then

$$\pi^-(x) \leqslant u(x) \leqslant \pi^+(x) \quad \text{for} \quad x \in \Omega;$$

cf. the derivation of (10.25) with $\delta = 0$, $\chi = 0$. Thus

$$|u(x_0) - u(x)| \leqslant K_0 |x_0 - x| \tag{11.6}$$

if $x_0 \in \partial\Omega$, $x \in \Omega$.

Repeating the arguments in the proof of Lemma 10.1, one obtains

$$\int_{A(k,\Delta)} [a_j(u_x(x+\Delta e_1)) - a_j(u_x(x)](u(x+\Delta e_1) - u(x))_{x_j} dx \leqslant 0$$

in place of (10.12). Hence (11.2) implies that $\max(u_\Delta(x) - k, 0) \equiv 0$ on $\Omega \cap \Omega_{-\Delta}$ if (10.6) holds. This proves (11.6) with $x_0, x \in \Omega$, hence (11.5).

Part III. Existence theorems [1]

12. The general case. Let $\Omega \subset E^n$ be a bounded open set, $\varphi(x)$ a function on $\partial\Omega$ which is uniformly Lipschitz continuous, and $\lambda(\varphi)$ the Lipschitz constant defined at the beginning of Section 8.

LEMMA 12.1. *Let $a(p) = (a_1(p), \ldots, a_n(p))$ be continuous for $|p| \leqslant K$ and satisfy*

$$[a_j(p) - a_j(q)](p_j - q_j) \geqslant 0. \tag{12.1}$$

Let $K \geqslant \lambda(\varphi)$. For every $u(x) \in \mathcal{K}_\varphi^K$, let $F[u](x)$ be a measurable function satisfying (A4), (A5′) *in Section 8. Then there exists at least one $u \in \mathcal{K}_\varphi^K$ such that*

$$\int_\Omega [a_j(u_x)(v-u)_{x_j} - F[u](v-u)]dx \geqslant 0 \quad \textit{for } v \in \mathcal{K}_\varphi^K. \tag{12.2}$$

Proof. Let $X = H^1(\Omega)$, $Y = H_0^1(\Omega)$, $\mathfrak{K} = \mathcal{K}_\varphi^K$, *and*

$$(A(u), w) = \int_\Omega a_j(u_x) w_{x_j} dx \quad \text{for } w \in Y,$$

$$\langle C(u), w \rangle = \int_\Omega F[u](x) w(x) dx \quad \text{for } w \in Y.$$

The remarks following Theorem 1.1 show that $X, Y, \mathfrak{K}$, $A(u)$ satisfy the conditions of this theorem. Also, the Remark following (A5′) in Section 8 shows that $u \to C(u)$ from $\mathcal{K}_\varphi^K \subset X$ to Y' is completely continuous. Hence Lemma 12.1 follows from Theorem 1.1.

COROLLARY 12.1. *Let $A(u)$, $C(u)$ in the last display satisfy (1.10), e.g., let $a(p)$ satisfy (11.2) and let*

$$\int_\Omega \{F[u_2] - F[u_1]\}(u_2 - u_1) dx \leqslant 0 \quad \textit{for } u_1, u_2 \in \mathcal{K}_\varphi^K,$$

then the solution $u(x) \in \mathcal{K}_\varphi^K$ of (12.4) in Lemma 12.1 is unique.

This is a consequence of Corollary 1.2.

---

[1] Added in proof (Jan. 18, 1966): More general results can be obtained using the same methods; see P. HARTMAN, On quasi linear elliptic functional-differential equations, *Proceedings of the International Symposium on Differential Equations and Dynamical Systems*, Puerto Rico, 1965.

LEMMA 12.2. *Let $a(p) = (a_1(p), \dots, a_n(p))$ be continuous for $|p| \leq K$ and satisfy*

$$[a_j(p) - a_j(q)](p_j - q_j) \geq \nu |p-q|^2 \quad \textit{for} \quad |p|, |q| \leq K, \tag{12.3}$$

$\nu > 0$ constant. Let $K \geq \lambda(\varphi)$. For $u \in \mathcal{K}_\varphi^K$, let $F[u](x)$ be a measurable function satisfying (A4) *and* (A5) *in Section 8. Then there exists at least one $u \in \mathcal{K}_\varphi^K$ satisfying* (12.2).

Proof. This will be deduced from Theorem 5.1 with the choice $X = H^1(\Omega)$, $Y = H_0^1(\Omega)$, $\mathfrak{K} = \mathcal{K}_\varphi^K$, and

$$(A(u,v), w) = \int_\Omega [a_j(v_x) w_{x_j} - F[u] w] dx$$

for $u, v \in \mathcal{K}_\varphi^K$ and $w \in Y$.

The remarks following Theorem 1.1 and (A5′) in Section 8 show that $A(u) = A(u,u)$ is continuous from $\mathcal{K}_\varphi^K \subset X$ to Y'. Since $\mathcal{K}_\varphi^K$ is bounded, no coercivity is required and (i_0), (ii_0) in Theorem 5.1 follows from (A4), (A5). The monotony condition (5.1) holds; in fact,

$$(A(u,u) - A(u,v), u-v) \geq \nu \|u-v\|_Y^2 \quad \text{for} \quad u, v \in \mathcal{K}_\varphi^K.$$

In order to verify (iv_0), let the conditions (5.2), (5.3) hold. Then the last inequality implies that $u_m \to u_0$ in X as $m \to \infty$. Hence, (5.4) is a consequence of (A4), (A5); i.e., of the continuity of the map $u \to F[u]$ from $\mathcal{K}_\varphi^K \subset X$ to Y'. This gives (iv_0).

In order to verify (v_0), it has to be shown that if (5.2) holds and there is a $y' \in Y'$ such that

$$\int_\Omega F[u_m](x) w(x) dx \to \langle y', w \rangle \quad \text{for } w \in Y, \tag{12.4}$$

then

$$\int_\Omega F[u_m](v - u_m) dx \to \langle y', v - u_0 \rangle \quad \text{for } v \in \mathcal{K}_\varphi^K. \tag{12.5}$$

Since $\{F[u_m](x)\}$ is uniformly bounded on Ω by (A4), there exists an increasing sequence of positive integers $\{m'\}$ and a function $y'(x) \in L^2(\Omega)$ such that

$$F[u_{m'}] \to y'(x) \quad \text{weakly in} \quad L^2(\Omega),$$

$$u_{m'} \to u_0 \quad \text{uniformly in} \quad \Omega.$$

Thus if $\{m\}$ is replaced by $\{m'\}$, (12.4) takes the form

$$\int_\Omega F[u_{m'}](x) w(x) dx \to \int_\Omega y'(x) w(x) dx.$$

It also follows that

$$\int_\Omega F[u_{m'}](x) u_{m'}(x) dx \to \int_\Omega y'(x) u_0(x) dx.$$

This gives (12.5) if $\{m\}$ is replaced by $\{m'\}$. Clearly, (12.5) follows from this. Thus Lemma 12.2 is a consequence of Theorem 5.1.

LEMMA 12.3. *For every $K>0$, assume the conditions of Lemma 12.1 [or Lemma 12.2 with $\nu=\nu(K)$]. Let $K(1)<K(2)<\dots$ and $K(m)\to\infty$ as $m\to\infty$. Let $u=u_m(x)\in\mathcal{K}_\varphi^{K(m)}$ satisfy (12.2) with $K=K(m)$, $\lambda(u_m)\leqslant T$ independent of m, and $u_m(x)\to u_0(x)$ uniformly on $\bar{\Omega}$ as $m\to\infty$. Then $u=u_0(x)$ satisfies*

$$\int_\Omega [a_j(u_x)\eta_{x_j}-F[u]\eta]dx=0 \quad \text{for } \eta\in H_0^1(\Omega). \tag{12.6}$$

Proof. After a selection of a subsequence (if necessary), it can be supposed that $u_m\to u_0$ weakly in $X=H_0^1(\Omega)$ as $m\to\infty$. Since $u=u_m(x)$ satisfies

$$\int_\Omega [a_j(u_x)(v-u)_{x_j}-F[u](v-u)]dx\geqslant 0 \quad \text{for } v\in\mathcal{K}_\varphi^K \tag{12.7}$$

when $K(m)\geqslant K$, it follows from Corollary 1.1 [or Corollary 5.1] and the proof of Lemma 12.1 [or Lemma 12.2] that $u=u_0$ satisfies (12.7) for all $K>0$. Consequently, $u=u_0(x)$ satisfies

$$\int_\Omega [a_j(u_x)\eta_{x_j}-F[u]\eta]dx\geqslant 0 \quad \text{for } \eta\in\mathcal{K}_0. \tag{12.8}$$

This proves the assertion.

THEOREM 12.1. *Let $a(p),\Omega,\varphi(x)$ satisfy the conditions of Theorem 9.1, 9.2, 9.3, 9.4 or 9.5. Let $F[u](x)$ satisfy conditions (A2)–(A5) of Section 8 with $\alpha=2+2\tau$, $\mu=\nu_0$. Then there exists at least one function $u(x)$, uniformly Lipschitz continuous on $\bar{\Omega}$, satisfying (12.6) and the boundary condition $u(x)=\varphi(x)$ on $\partial\Omega$.*

Proof. Let T be the constant supplied by Theorem 9.1–9.5. Let $\max(\lambda(\varphi),T)\leqslant K(1)<K(2)<,\dots,K(m)\to\infty$ as $m\to\infty$. Then, by Lemma 12.2, there exists $u_m(x)\in\mathcal{K}_\varphi^{K(m)}$ such that $u=u_m(x)$ satisfies (12.2) with $K=K(m)$. By Theorem 9.1–9.5, $\lambda(u_m)\leqslant T$, independent of m. It can be supposed that $\{u_m\}$ has a uniform limit $u_0(x)$ on $\bar{\Omega}$. Then, by Lemma 12.3, $u=u_0(x)$ satisfies the assertion of the theorem.

13. The homogeneous case. The last theorem involved the ellipticity conditions given in either (9.2) in (B1) or (9.5)–(9.7) in (B2) or (9.5′) in (B2′). In the homogeneous case ($F[u](x)\equiv 0$ for all u), we obtain existence theorems under the mild ellipticity condition

$$[a_j(p)-a_j(q)](p_j-q_j)\geqslant 0 \tag{13.1}$$

and uniqueness when

$$[a_j(p)-a_j(q)](p_j-q_j)>0 \quad \text{if} \quad p\neq q. \tag{13.2}$$

THEOREM 13.1. *Let* $a(p)=(a_1(p),...,a_n(p))\in C^0(E^n)$ *satisfy (13.1). Let* Ω *be a bounded convex set in* E^n *and let* $\varphi(x)$ *satisfy a bounded slope condition with constant* K_0 *on* Ω *(cf.* (B3) *in Section 9). Then there exists at least one function* $u(x)$*, uniformly Lipschitz continuous on* $\bar{\Omega}$*, satisfying*

$$\int_\Omega a_j(u_x)\eta_{x_j}dx=0 \quad for\ \eta\in H_0^1(\Omega) \tag{13.3}$$

and the boundary condition $u(x)=\varphi(x)$ *on* $\partial\Omega$*; in fact,* $u\in\mathcal{K}_\varphi^K$, $K=K_0$*. The set of solutions* $u\in\mathcal{K}_\varphi$ *of 13.3 is convex. If (13.2) holds, then there is exactly one* $u\in\mathcal{K}_\varphi$ *satisfying (13.3).*

Proof. Under the assumption (13.2), the existence statement and $\lambda(u)\leqslant K_0$ can be obtained by the same proof used for Theorem 12.1 except that one uses the a priori estimate $\lambda(u)\leqslant K_0$ supplied by Lemma 11.2 in place of Theorem 9.1–9.5. Uniqueness in this case follows from Corollary 12.1.

In order to obtain existence under the condition (13.1), let $m=1,2,...$ and put

$$a_{jm}(p)=a_j(p)+p_j/m \quad \text{for} \quad j=1,...,n. \tag{13.4}$$

Then

$$[a_{jm}(p)-a_{jm}(q)](p_j-q_j)\geqslant|p-q|^2/m. \tag{13.5}$$

Hence, by the first part of this proof, there exists a $u_m\in\mathcal{K}_\varphi^K$, $K=K_0$, satisfying

$$\int_\Omega a_{jm}(u_{mx})(v-u_m)_{x_j}dx=0 \quad for\ v-u_m\in H_0^1(\Omega). \tag{13.6}$$

After a selection of a subsequence, it can be supposed that

$$u_0=\lim u_m \text{ exists weakly in } H^1(\Omega). \tag{13.7}$$

Then $u_0\in\mathcal{K}_\varphi^K$, $K=K_0$. *By* (13.5) and (13.6),

$$\int_\Omega a_{jm}(v)(v-u_m)_{x_j}dx\geqslant 0 \quad \text{for } v-u_m\in\mathcal{K}_0.$$

It is clear from (13.4) and (13.7) that a term-by-term integration of the last relation is permitted,

$$\int_\Omega a_j(v)(v-u_0)_{x_j}dx\geqslant 0 \quad \text{for } v-u_0\in\mathcal{K}_0.$$

This is equivalent to

$$\int_\Omega a_j(u_0)(v-u_0)_{x_j}dx\geqslant 0 \quad \text{for } v-u_0\in\mathcal{K}_0;$$

cf. Lemma 2.3. Since this, in turn, is equivalent to (13.3), the existence proof is complete. The convexity of the set of solutions follows from Corollary 2.1.

The proof of the last theorem suggest the following

THEOREM 13.2. *Let $f(p)=f(p_1,\dots,p_n)$ be a real-valued, convex function for all p. Let Ω be a bounded, convex set in E^n and $\varphi(x)$ satisfy a bounded slope condition on $\partial\Omega$ with constant K_0. Then in the class $\mathcal{K}_\varphi$ of uniformly Lipschitz functions on Ω satisfying $u(x)=\varphi(x)$ on $\partial\Omega$, there exists at least one $u=u_0(x)$ such that*

$$\min \int_\Omega f(u_x)\,dx = \int_\Omega f(u_{0x})\,dx \tag{13.8}$$

(and $u_0 \in \mathcal{K}_\varphi^K$, $K=K_0$). The set of solutions $u_0 \in \mathcal{K}_\varphi$ of (13.8) is convex. The function $u_0 \in \mathcal{K}_\varphi$ is unique if $f(p)$ is strictly convex.

When, in addition,

$$f(p)\in C^2 \quad \text{and} \quad f_{p_ip_j}\xi_i\xi_j>0, \tag{13.9}$$

Stampacchia [12], p. 395, has shown that any minimizing function $u(x)\in C^1(\Omega)\cap H^2(\Omega)$ satisfies $\lambda(u)\leqslant K_0$. This a priori estimate can be obtained from Lemma 11.2 for any minimizing $u(x)\in\mathcal{K}_\varphi$, without the assumption $u\in C^1(\Omega)\cap H^2(\Omega)$. In fact, a minimizing function satisfies the Euler equation

$$\int_\Omega f_{p_j}(u_x)\eta_{x_j}\,dx = 0 \quad \text{for } \eta\in H_0^1(\Omega.$$

Miranda [10] uses the a priori bound $\lambda(u)\leqslant K_0$ to prove Theorem 13.2 for strictly convex $f(p)\in C^2$. We shall use it in a similar way.

Proof. Assume first that (13.9) holds. By lower semi-continuity (cf. the arguments leading to (13.12) below), $\min I[u]$, where

$$I[u]=\int_\Omega f(u_x)\,dx,$$

is attained on the set of functions $\mathcal{K}_\varphi^K$, which is not empty for $K\geqslant K_0$, and is compact under uniform convergence. The set $\mathcal{K}_\varphi^K$ is convex and $I[u]$ is strictly convex, so that the minimizing function $u_0(x)$ is unique and, by Lemma 11.2, $\lambda(u_0)\leqslant K_0$, independent of K. Thus $u_0(x)$ minimizes $I[u]$, $u\in\mathcal{K}_\varphi$.

When $f(p)$ does not satisfy (13.9), let $f_1(p)$, $f_2(p)$, ... be a sequence of functions satisfying (13.9) for all p and $f_m(p)\to f(p)$ uniformly on every bounded p-set, as $m\to\infty$. It can also be supposed that for any $R>0$, there is an $m(R)$ such that

$$f(p)\leqslant f_m(p) \quad \text{for} \quad |p|\leqslant R, \quad m\geqslant m(R). \tag{13.10}$$

Let $u=u_m(x)$ be the unique minimizing function for

$$\int_\Omega f_m(u_x)\,dx$$

in the class $\mathcal{K}_\varphi$. Then $\lambda(u_m) \leq K_0$. After a selection of a subsequence, it can be supposed that $u_0 = \lim u_m$ exists weakly in $H^1(\Omega)$ and uniformly on $\overline{\Omega}$. Then, by a well-known theorem, there exist numbers $N(m)$ and λ_{mj}, $j = m, \ldots, N(m)$, such that

$$\lambda_{mj} \geq 0 \quad \text{and} \quad \sum_{j=m}^{N(m)} \lambda_{mj} = 1,$$

$$\sum_{j=m}^{N(m)} \lambda_{mj} u_j \to u_0 \text{ in } H^1(\Omega),\ m \to \infty.$$

Consequently, as $m \to \infty$ through a suitable subsequence,

$$\sum_{j=m}^{N(m)} \lambda_{mj} u_{jx} \to u_{0x} \quad \text{almost erverywhere.} \tag{13.11}$$

Let $v \in \mathcal{K}_\varphi$. Then, by the convexity of f,

$$\int_\Omega f\left(\sum_{j=m}^{N(m)} \lambda_{mj} u_{xj}\right) dx \leq \sum_{j=m}^{N(m)} \lambda_{mj} \int_\Omega f(u_{jx})\, dx.$$

For large m, (13.10) shows that this is at most

$$\sum_{j=m}^{N(m)} \lambda_{mj} \int_\Omega f_j(u_{jx})\, dx \leq \sum_{j=m}^{N(m)} \lambda_{mj} \int_\Omega f_j(v_x)\, dx,$$

where the last inequality follows from the minimizing property of u_j. Thus

$$\int_\Omega f\left(\sum_{j=m}^{N(m)} \lambda_{mj} u_{jx}\right) dx \leq \sum_{j=m}^{N(m)} \lambda_{mj} \int_\Omega f_j(v_x)\, dx.$$

By (13.11) and $\lambda(u_j) \leq K_0$, we get

$$\int_\Omega f(u_{0x})\, dx \leq \int_\Omega f(v_x)\, dx; \tag{13.12}$$

i.e., $u_0(x)$ minimizes $I[u]$, $u \in \mathcal{K}_\varphi$.

This proves existence in Theorem 13.2. The convexity of the set of solutions follows from the convexity of $I[u]$, and uniqueness from the fact that $I[u]$ is strictly convex if $f(p)$ is strictly convex.

14. Regularity of solutions. The existence theorems of the last two sections were obtained without the use of the regularity theorems of De Giorgi and their extensions. If we use these results, we can show that additional conditions on the given data $a(p)$, Ω, $\varphi(x)$, $F[u]$ imply more smoothness for the solutions. The first two theorems of this section deal with the homogeneous case of Section 13 and the last two with the case of Section 12.

Before stating the results, we recall some definitions. A function $u(x)$ defined on

Ω [or $\bar{\Omega}$] is said to be of class $C^{m,\lambda}(\Omega)$ [or $C^{m,\lambda}(\bar{\Omega})$], where $m=0,1,\dots$ and $0<\lambda\leqslant 1$, if it is of class $C^m(\Omega)$ [or $C^m(\bar{\Omega})$] and its mth order partial derivatives are uniformly Hölder continuous of order λ on compacts in Ω [or on $\bar{\Omega}$]. The boundary $\partial\Omega$ of Ω is said to be of class C^m [or $C^{m,\lambda}$] if, for every $x_0\in\partial\Omega$, the subset of $\partial\Omega$ in some neighborhood of x_0 has a parametric representation $x=x(t_1,\dots,t_{n-1})$, where $x(t_1,\dots,t_{n-1})\in C^m$ [or $C^{m,\lambda}$] on $|t_1|^2+\dots+|t_{n-1}|^2<1$ and the rank of the Jacobian matrix of $x_1,\dots,x_n$ with respect to $t_1,\dots,t_{n-1}$ is $n-1$. In this case, a function $\varphi(x)$, $x\in\partial\Omega$, is said to be of class $C^m(\partial\Omega)$ [or $C^{m,\lambda}(\partial\Omega)$] if, in terms of local coordinates $t_1,\dots,t_{n-1}$, the function $\varphi(x)=\varphi(x(t_1,\dots,t_{n-1}))$ is of class C^m [or $C^{m,\lambda}$].

THEOREM 14.1. *Let $a(p)\in C^1(E^n)$ satisfy*

$$\frac{\partial a_j}{\partial p_j}\xi_j\xi_i>0 \quad \textit{for } 0\neq\xi\in E^n, \tag{14.1}$$

Ω a bounded open convex set, and $\varphi(x)$ a function on $\partial\Omega$ satisfying a bounded slope condition. Then the unique solution $u(x)\in\mathcal{K}_\varphi$ of (13.3) supplied by Theorem 13.1 has the following properties:

(i) *$u(x)\in H^2(\Omega_0)$ for every open $\Omega_0, \bar{\Omega}_0\subset\Omega$;*

(ii) *$u(x)$ satisfies*

$$\frac{\partial a_j(u_x)}{\partial p_i}\frac{\partial^2 u}{\partial x_i\partial x_j}=0 \tag{14.2}$$

almost everywhere on Ω;

(iii) *$u(x)\in C^{1,\lambda}(\Omega)$ for every λ, $0<\lambda<1$.*

If, in addition, $\partial\Omega\in C^{1,1}$, then $u(x)\in H^2(\Omega)\cap C^{1,\lambda}(\bar{\Omega})$ for every λ, $0<\lambda<1$.

Proof. Condition (14.1) implies (13.2), so that Theorem 13.1, including its uniqueness assertion, is applicable. Since $u(x)\in\mathcal{K}_\varphi$, $u_x(x)$ is bounded. Consequently, (14.1) implies the existence of positive constants ν, ν_1 such that

$$\frac{\partial a_j(u_x)}{\partial p_i}\xi_j\xi_i\geqslant\nu|\xi|^2 \quad \text{for } x\in\bar{\Omega},\ \xi\in E^n, \tag{14.3}$$

$$\left|\frac{\partial a_j(u_x)}{\partial p_i}\right|\leqslant\nu_1 \quad \text{for } x\in\bar{\Omega} \tag{14.4}$$

The arguments in the proof of the first part of Theorem 3.1 in [14] give properties (i), (ii).

Let Ω_0 be on open sphere, $\bar{\Omega}_0\subset\Omega$ and $h=1,\dots,n$ fixed. If, in (13.3), η is replaced by $\partial\zeta/\partial x_h$ for $\zeta\in C_0^\infty(\Omega_0)$, then $v=\partial u/\partial x_h$ satisfies

$$\int_\Omega\frac{\partial a_j(u_x)}{\partial p_i}\frac{\partial v}{\partial x_i}\frac{\partial\zeta}{\partial x_j}dx=0 \quad \text{for } \zeta\in C_0^\infty(\Omega_0). \tag{14.5}$$

Hence, De Giorgi's theorem implies that there is some λ, $0<\lambda<1$, such that v satisfies a uniformly Hölder condition of order λ on compacts in Ω_0. Thus, $u \in C^{1,\lambda}(\Omega)$.

In particular, $u_x(x) \in C^0(\Omega)$ and so, the coefficients in (14.2) are continuous. Consequently, $u \in C^{1,\lambda}(\Omega)$ for every λ, $0<\lambda<1$; cf. [2]. This gives (iii). Note that the conditions $\partial\Omega \in C^{1,1}$ and $\varphi(x)$ satisfies a bounded slope condition imply that $\varphi(x) \in C^{1,1}(\partial\Omega)$ and is the trace of a function $\Psi(x) \in C^{1,1}(E^n)$; see Corollary 4.2 and the Remark following it in [6]. Hence, the proofs of the last parts of Theorems 3.1 and 3.2 in [14] give the last part of Theorem 14.1.

THEOREM 14.2. *Let the conditions of the first part of Theorem 14.1 hold. If, in addition,* $a(p) \in C^{m,\lambda}(E^n)$ *for some* $m \geq 1$ *and* $0<\lambda<1$, *then* $u(x) \in C^{m+1,\lambda}(\Omega)$. *Moreover, if* $\partial\Omega \in C^{m+1,\lambda}$ *and* $\varphi(x) \in C^{m+1,\lambda}(\partial\Omega)$, *then* $u(x) \in C^{m+1,\lambda}(\bar{\Omega})$.

This is a consequence of Theorem 14.1 and the usual boot-strap arguments involving Schauder estimates; cf. Theorem 3.3 in [14].

THEOREM 14.3. *Let* $a(p) \in C^1(E^n)$, $F[u]$, Ω *and* φ *satisfy the conditions of Theorem 12.1. Then a solution* $u(x) \in \mathcal{K}_\varphi$ *of (12.6) has the properties:*

(i) $u(x) \in H^2(\Omega_0)$ *for every open* Ω_0, $\bar{\Omega}_0 \subset \Omega$;

(ii) $u(x)$ *satisfies*

$$\frac{\partial a_j(u_x)}{\partial p_i} \frac{\partial^2 u}{\partial x_i \partial x_j} + F[u](x) = 0 \tag{14.6}$$

almost everywhere on Ω;

(iii) $u(x) \in C^{1,\lambda}(\Omega)$ *for every* λ, $0<\lambda<1$.

If, in addition, $\partial\Omega \in C^{1,1}$, *then* $u \in H^2(\Omega) \cap C^{1,\lambda}(\bar{\Omega})$ *for every* $\lambda, 0<\lambda<1$.

Since (A4) and $u(x) \in \mathcal{K}_\varphi$ imply that $F[u](x) \in L^\infty(\Omega)$, the proof follows from those of Theorems 3.1 and 3.2 in [12] and from the remarks above in the proof of Theorem 14.1.

THEOREM 14.4. *Let the conditions of the first part of Theorem 14.3 hold. In addition, assume that* $a(p) \in C^{m,\lambda}(E^n)$ *for some* $m \geq 1$ *and* $0<\lambda<1$ *and that*

$$v(x) \in C^{r+1,\lambda}(\bar{\Omega}) \quad \Rightarrow \quad F[v](x) \in C^{r,\lambda}(\bar{\Omega}) \tag{14.7}$$

for $r=0,1,\ldots,m-1$. *Then a solution* $u(x) \in \mathcal{K}_\varphi$ *of (12.6) is of class* $C^{m+1,\lambda}(\Omega)$. *Moreover, if* $\partial\Omega \in C^{m+1,\lambda}$ *and* $\varphi(x) \in C^{m+1,\lambda}(\partial\Omega)$, *then* $u(x) \in C^{m+1,\lambda}(\bar{\Omega})$.

The proof is similar to that of Theorem 14.2. One can obtain analogous theorems by replacing Schauder estimates by L^α estimates and (14.7) by an assumption of the type

$$v(x) \in H^{r+1}(\Omega) \quad \Rightarrow \quad F[v](x) \in H^r(\Omega).$$

References

[1]. BONNESSEN, T. & FENCHEL, W., *Theorie der konvexen Körper.* Ergeb. Math. (Berlin) 1934.

[2]. BERS, L., JOHN, F. & SCHECTER, M., *Partial differential equations.* (New York) 1964, Part II, Chapter 5.

[3]. BROWDER, F. E., Non-linear elliptic boundary value problems II. *Trans. Amer. Math. Soc.*, 117 (1965), 530–550.

[4]. DE GIORGI, E., Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino*, 3 (1957), 25–43.

[5]. GILBARG, D., Boundary value problems for nonlinear elliptic equations in n variables. *Symposium on Nonlinear Problems*, Madison (Wisconsin) 1962.

[6]. HARTMAN, P., On the bounded slope condition. To appear, *Pacific J. Math.*

[7]. LADYZHENSKAIA, O. A. & URAL'TSEVA, N. N., Quasi-linear elliptic equations and variational problems with many independent variables. *Uspehi Mat. Nauk*, 16 (1961), 19–92; translated in *Russian Math. Surveys*, 16 (1961), 17–91.

[8]. LERAY, J. & LIONS, J. L., Quelques résultats de Visik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty–Browder. *Bull. Soc. Math. France*, 93 (1965), 97–107.

[9]. LEWY, H., Über die Methode der Differenzengleichungen zur Lösung von Variations- und Randwertproblemen. *Math. Ann.*, 98 (1928), 107–124.

[10]. MINTY, G. J., Monotone (non-linear) operators in Hilbert space. *Duke Math. J.*, 29 (1962), 341–346.

[11]. MIRANDA, M., Un teorema di esistenza e unicità per il problema dell'area minima in n variabili. *Ann. Scuola Norm. Sup. Pisa*, 19 (1965), 233–249.

[12]. RADO, T., *On the problem of Plateau.* Ergeb. Math. (Berlin) 1933.

[13]. SERRIN, T., Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111 (1964), 247–302.

[14]. STAMPACCHIA, G., On some regular multiple integral problems in the calculus of variations. *Comm. Pure Appl. Math.*, 16 (1963), 383–421.

[15]. STAMPACCHIA, G., Formes bilinéaires coercitives sur les ensembles convexes. *C. R. Acad. Sci. Paris*, 258 (1964), 4413–4416.

Received May 19, 1965, in revised form June 30, 1965