Japan. J. Math.
Vol. 2, No. 2, 1976

On some operators for p-adic uniformly differentiable functions

By Katsumi SHIRATANI

(Received January 20, 1976)

§1. Introduction

The purpose of this paper is to give two theorems on the multiplicative theory of p-adic Fourier analysis. Namely we shall treat operators $\mathfrak{M}_\chi$ with Dirichlet characters χ on the p-adic Banach algebra of p-adic uniformly differentiable functions defined on the group U_p of principal units in the rational p-adic number field. Then we discuss several fundamental properties of $\mathfrak{M}_\chi$. Such operators $\mathfrak{M}_\chi$ for the analytic functions on U_p are already defined and investigated by Kubota-Leopoldt in order to introduce their p-adic L-functions [1].

Recently C. F. Woodcock [4], [5] gave an additive theory of p-adic Fourier analysis for p-adic Lipschitz functions. But we need the multiplicative theory to explain both the classical congruences of Kummer for the Bernoulli numbers and the modules of continuity of the p-adic L-functions clearly, and hence to extend our previous results [3] about the Bernoulli numbers to the ones for more general p-adic functions.

§2. Preliminaries

Let Q_p be the rational p-adic number field and Z_p the ring of all rational p-adic integers. The ring of rational integers is denoted by Z. We set $q=p$ for the prime $p>2$ and $q=4$ for $p=2$, and we use the normalized exponential valuation ν_p of Q_p such that $\nu_p(p)=1$ and extend it to a valuation on the completion Ω_p of the algebraic closure of Q_p. The ring of integers in Ω_p is written by O_{Ω_p}.

We define a norm for Ω_p-valued continuous functions f defined on Z_p by

$$(1)\qquad \nu(f)=\min_{x\in Z_p}\nu_p(f(x)).$$

Then we know that all such functions form a p-adic Banach algebra $\mathscr{C}(Z_p, \Omega_p)$ under the pointwise operations and the above norm.

Now, we set

$$U_p=\{u \in Q_p^* ; \nu_p(u-1) \geqq \nu_p(q)\}. \tag{2}$$

Then U_p is the group of all principal units of Q_p ($p>2$) or its subgroup of index 2 ($p=2$).

The correspondence

$$u \in U_p \longleftrightarrow x=\frac{\log u}{\log(1+q)} \in Z_p \tag{3}$$

gives an isomorphism of the multiplicative group U_p and the additive group Z_p algebraically and topologically. Hence there is an isomorphism between the p-adic Banach algebra $\mathscr{C}(U_p, \Omega_p)$ under the norm $\nu(f)=\min_{u \in U_p} \nu_p(f(u))$ for $f \in \mathscr{C}(U_p, \Omega_p)$ and the p-adic Banach algebra $\mathscr{C}(Z_p, \Omega_p)$.

By a Lipschitz function $f \in \mathscr{C}(U_p, \Omega_p)$ we mean a function for which there exists a real constant R such that $\nu_p\left(\frac{f(u)-f(v)}{u-v}\right) \geqq R$ holds for all u, $v \in U_p$, $u \neq v$. If we set

$$R(f)=\inf_{\substack{u,v \in U_p \\ u \neq v}} \nu_p\left(\frac{f(u)-f(v)}{u-v}\right) \tag{4}$$

and define a norm $V(f)$ for a Lipschitz function f by

$$V(f)=\min(\nu(f), R(f)), \tag{5}$$

then all Ω_p-valued Lipschitz functions defined on U_p under the pointwise operations and the norm V form a p-adic Banach algebra $\mathscr{L}ip\,(U_p, \Omega_p)$.

If, for a given function $f: U_p \to \Omega_p$, there exists a continuous function $\phi_f(u, v): U_p \times U_p \to \Omega_p$ such that we have for all $u, v \in U_p$, $u \neq v$

$$\phi_f(u, v)=\frac{f(u)-f(v)}{u-v}, \tag{6}$$

then we call f a uniformly differentiable function.

In this case we have

$$R(f)=\inf_{\substack{u,v \in U_p \\ u \neq v}} \nu_p(\phi_f(u, v))=\nu(\phi_f) \tag{7}$$

and all uniformly differentiable functions make a subalgebra of the algebra $\mathscr{L}ip\,(U_p, \Omega_p)$. It is denoted by $\mathscr{U}\mathscr{D}(U_p, \Omega_p)$.

Now, the dual group of U_p in the sense of Pontrjagin is isomorphic to the p-adic torus $T_p=\varinjlim_n Z/p^n Z$ and if we take the functions $\phi_\chi(u)=\chi(u)$ on

U_p corresponding to the elements χ of T_p, then we obtain, from the additive theory of Woodcock and by the isomorphism (3), a Fourier series expansion of $f \in \mathcal{U}\mathcal{D}(U_p, \Omega_p)$

$$f(u)=\sum_{\chi} M_0(f\phi_{\bar{\chi}})\phi_{\chi}(u). \tag{8}$$

Herein we denote by $M_0(f\phi_{\bar{\chi}})$, simply written by $M_{\bar{\chi}}(f)$, the integral

$$\begin{aligned} M_0(f\phi_{\bar{\chi}}) &= I_0(f((1+q)^x)\phi_{\bar{\chi}}((1+q)^x)) \\ &= \lim_{\rho\to\infty} \frac{1}{p^{\rho}} \sum_{i=0}^{p^{\rho}-1} f((1+q)^i)\phi_{\bar{\chi}}((1+q)^i), \end{aligned} \tag{9}$$

and the sum over χ means the limit of partial sums over all elements χ in the cyclic subgroup $T_p^{(n)}$ of order p^n of T_p under letting n tend to the infinity. Finally $\bar{\chi}$ denotes the inverse character χ^{-1} of χ.

In the above we note that χ can be regarded as a Dirichlet character of the second kind, namely such character of the conductor p-power, that satisfies $\chi(x)=\chi(\langle x\rangle)$ for the canonical decomposition $x=\omega(x)\langle x\rangle$ of $x \in Z$, $(x, p)=1$ with $\langle x\rangle \in U_p$, where $\omega(x)=\lim_{\rho\to\infty} x^{p^{\rho}}$ for $p>2$ or $\omega(x)=\pm 1 \equiv x \pmod 4$ for $p=2$. We call any other Dirichlet character to be not of the second kind, namely it does not satisfy these conditions. Furthermore we see at once $\phi_{\chi}(u) \in \mathcal{U}\mathcal{D}(U_p, \Omega_p)$ and $\phi'_{\chi}(u)=0$.

§3. The operator $\mathfrak{M}_{\chi}$

Let χ be a primitive Dirichlet character with conductor $\mathfrak{f}_{\chi}=\mathfrak{f}$.

For any $f \in \mathcal{U}\mathcal{D}(U_p, \Omega_p)$ and any non-negative rational integer ρ we set

$$\mathfrak{M}_{\chi}^{(\rho)}(f)=\frac{1}{\mathfrak{f}p^{\rho}} \sum_{x=1}^{\mathfrak{f}p^{\rho}}{}^{*} \chi(x)f(\langle x\rangle), \tag{10}$$

where $*$ denotes to take sum over all integers prime to p in the given range.

First we give an elementary lemma as follows.

LEMMA. *If $f \in \mathcal{C}(U_p, \Omega_p)$, then we have*

$$\lim_{\rho\to\infty} \sum_{i=1}^{\mathfrak{f}p^{\rho}}{}^{*} \chi(i)f(\langle i\rangle)=0.$$

PROOF. Take a p-power p^{μ} arbitrarily and fix it. By continuity of f there is a bound σ_0 such that $\langle i\rangle \equiv \langle i_0\rangle \pmod{p^{\sigma}}$ with $\sigma \geqq \sigma_0$ yields $f(\langle i\rangle) \equiv f(\langle i_0\rangle) \pmod{p^{\mu}}$. Hence, for ρ sufficiently large we see

$$\sum_{i=1}^{\mathfrak{f}p^{\rho}}{}^{*} \chi(i)f(\langle i\rangle) \equiv \sum_{i_0=1}^{\mathfrak{f}p^{\sigma}}{}^{*} \chi(i_0)f(\langle i_0\rangle)N_{\sigma} \pmod{p^{\mu}},$$

where $N_\sigma = \# \{i \in Z\,;\, 1 \leqq i \leqq \mathfrak{f}p^\rho,\, (i,p)=1,\, i \equiv i_0 \pmod{\mathfrak{f}p^\sigma}\}$.

Thus we see

$$\sum_{i=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(i) f(\langle i \rangle) \equiv \sum_{i_0=1}^{\mathfrak{f}p^\sigma}{}^{*}\, \chi(i_0) f(\langle i_0 \rangle) p^{\rho-\sigma} \pmod{p^\mu},$$

from which we have for ρ sufficiently large

$$\sum_{i=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(i) f(\langle i \rangle) \equiv 0 \pmod{p^\mu}.$$

This means our assertion.

Now, from the definition we see readily

$$\begin{aligned} \mathfrak{M}_\chi^{(\rho+1)}(f) &= \frac{1}{\mathfrak{f}p^{\rho+1}} \sum_{x=1}^{\mathfrak{f}p^{\rho+1}}{}^{*}\, \chi(x) f(\langle x \rangle) \\ &= \frac{1}{\mathfrak{f}p^{\rho+1}} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*} \sum_{y=0}^{p-1} \chi(x_\rho)\{f(\langle x_\rho \rangle + \varpi(x_\rho)\mathfrak{f}p^\rho y) - f(\langle x_\rho \rangle)\} \\ &\quad + \frac{1}{\mathfrak{f}p^\rho} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(x_\rho) f(\langle x_\rho \rangle), \end{aligned} \tag{11}$$

where ϖ means also the inverse character ω^{-1} of ω.

Therefore we have

$$\begin{aligned} &\mathfrak{M}_\chi^{(\rho+1)}(f) - \mathfrak{M}_\chi^{(\rho)}(f) \\ &= \frac{1}{\mathfrak{f}p^{\rho+1}} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*} \sum_{y=0}^{p-1} \chi(x_\rho)\varpi(x_\rho)\mathfrak{f}p^\rho y \frac{f(\langle x_\rho \rangle + \varpi(x_\rho)\mathfrak{f}p^\rho y) - f(\langle x_\rho \rangle)}{\varpi(x_\rho)\mathfrak{f}p^\rho y} \\ &= \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(x_\rho) \sum_{y=0}^{p-1} \varpi(x_\rho) y \frac{f(\langle x_\rho \rangle + \varpi(x_\rho)\mathfrak{f}p^\rho y) - f(\langle x_\rho \rangle)}{\varpi(x_\rho)\mathfrak{f}p^\rho y}, \end{aligned} \tag{12}$$

and we see

$$\begin{aligned} &\mathfrak{M}_\chi^{(\rho+1)}(f) - \mathfrak{M}_\chi^{(\rho)}(f) \\ &= \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(x_\rho) \sum_{y=0}^{p-1} \varpi(x_\rho) y \phi_f(\langle x_\rho \rangle + \varpi(x_\rho)\mathfrak{f}p^\rho y, \langle x_\rho \rangle) \\ &= \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(x_\rho) \sum_{y=0}^{p-1} \varpi(x_\rho) y \phi_f(\langle x_\rho \rangle, \langle x_\rho \rangle) \\ &\quad + \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\, \chi(x_\rho) \sum_{y=0}^{p-1} \varpi(x_\rho) y \{\phi_f(\langle x_\rho \rangle + \varpi(x_\rho)\mathfrak{f}p^\rho y, \langle x_\rho \rangle) - \phi_f(\langle x_\rho \rangle, \langle x_\rho \rangle)\}. \end{aligned} \tag{13}$$

Because we have $\phi_f(u,u)$, $\phi_f(u+\varpi(x)\mathfrak{f}p^\rho y, u) \in \mathscr{C}(U_p, \Omega_p)$ we conclude by Lemma and uniform continuity of $\phi_f(u,v)$ that $\mathfrak{M}_\chi^{(\rho+1)}(f) - \mathfrak{M}_\chi^{(\rho)}(f) \to 0$ as $\rho \to \infty$.

Furthermore we have

$$\nu_p(\mathfrak{M}_\chi^{(\rho+1)}(f)-\mathfrak{M}_\chi^{(\rho)}(f))\geqq R(f)-1 \tag{14}$$

for any ρ. It follows from this that the operator $\mathfrak{M}_\chi$ defined by

$$\mathfrak{M}_\chi(f)=\lim_{\rho\to\infty}\mathfrak{M}_\chi^{(\rho)}(f)$$

is a bounded operator, and hence a continuous operator on $\mathscr{U}\mathscr{D}(U_p, \Omega_p)$.

§ 4. Fundamental properties of $\mathfrak{M}_\chi$

For $c\in Z$, $(c, \mathfrak{f})=1$ and $f\in\mathscr{U}\mathscr{D}(U_p, \Omega_p)$ we define $f^c(u)=f(\langle c\rangle u)$. Then we see $f^c\in\mathscr{U}\mathscr{D}(U_p, \Omega_p)$. By the condition $cx=x_\rho+\mathfrak{f}p^\rho r_\rho(x_\rho)$ with $1\leqq x_\rho\leqq\mathfrak{f}p^\rho$, $1\leqq x\leqq\mathfrak{f}p^\rho$ the numbers $r_\rho(x_\rho)\in Z$ depending on c are well determined. We have easily

$$\begin{aligned}\mathfrak{M}_\chi^{(\rho)}(f^c)&=\bar{\chi}(c)\frac{1}{\mathfrak{f}p^\rho}\sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\chi(x_\rho)f(\langle x_\rho\rangle+\bar{\omega}(x_\rho)\mathfrak{f}p^\rho r_\rho(x_\rho))\\&=\bar{\chi}(c)\mathfrak{M}_\chi^{(\rho)}(f)\\&\quad+\bar{\chi}(c)\frac{1}{\mathfrak{f}p^\rho}\sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\chi(x_\rho)\mathfrak{f}p^\rho\bar{\omega}(x_\rho)r_\rho(x_\rho)\phi_f(\langle x_\rho\rangle+\bar{\omega}(x_\rho)\mathfrak{f}p^\rho r_\rho(x_\rho), \langle x_\rho\rangle).\end{aligned} \tag{15}$$

Therefore we obtain from Lemma as before

$$\mathfrak{M}_\chi(f^c)=\bar{\chi}(c)\mathfrak{M}_\chi(f)+\bar{\chi}(c)\lim_{\rho\to\infty}\sum_{x_\rho=1}^{\mathfrak{f}p^\rho}{}^{*}\chi(x_\rho)\bar{\omega}(x_\rho)r_\rho(x_\rho)f'(\langle x_\rho\rangle). \tag{16}$$

In the case $p=2$ we readily see

$$\begin{aligned}&\lim_{\rho\to\infty}\sum_{x_\rho=1}^{\mathfrak{f}2^\rho}{}^{*}\chi(x_\rho)\bar{\omega}(x_\rho)r_\rho(x_\rho)f'(\langle x_\rho\rangle)\\&\quad=(1+\chi(-1))\lim_{\rho\to\infty}\sum_{x_\rho=1}^{\mathfrak{f}2^{\rho-1}}{}^{*}\chi(x_\rho)\bar{\omega}(x_\rho)r_\rho(x_\rho)f'(\langle x_\rho\rangle).\end{aligned} \tag{17}$$

Consequently we obtain to $f\in\mathscr{U}\mathscr{D}(U_p, \Omega_p)$ such that $f'(u)\in O_{\Omega_p}$ for any $u\in U_p$

$$\chi(c)\mathfrak{M}_\chi(f^c)\equiv\mathfrak{M}_\chi(f)\ (\mathrm{mod}\ p^{-1}q). \tag{18}$$

If $\mathfrak{f}$ is a p-power, then χ can be extended to a function on Z_p naturally and $\chi(c)$ for $c\in Z_p$, $(c, \mathfrak{f})=1$ is well determined. By the same argument as above, but using $cx=x_\rho+\mathfrak{f}p^\rho r_\rho(x_\rho)$ with $x, x_\rho\in Z$, $r_\rho(x_\rho)\in Z_p$, we conclude the congruence (18) for any such c. Thus, take $c=\zeta_{p-1}$ a primitive $(p-1)$-th root of unity for $p>2$ and notice $\mathfrak{M}_\chi(f^c)=\mathfrak{M}_\chi(f)$, then we have $(1-\chi(\zeta_{p-1}))\mathfrak{M}_\chi(f)\equiv 0\ (\mathrm{mod}\ p^0)$.

If χ is not of the second kind, then we have $\chi(\zeta_{p-1})\neq 1$ and $1-\chi(\zeta_{p-1})$ is a unit and we see $\mathfrak{M}_\chi(f)\equiv 0 \pmod{p^0}$. In the case $p=2$, if χ is not of the second kind, then $\chi(-1)=-1$ and $\mathfrak{M}_\chi(f)=0$ hold. In either case we have $\mathfrak{M}_\chi(f)\equiv 0 \pmod{p^{-1}q}$ for χ not of the second kind.

Next, if $\mathfrak{f}=\mathfrak{f}_0 p^\mu$, $(\mathfrak{f}_0, p)=1$, $\mathfrak{f}_0>1$, then the canonical decomposition $\chi=\chi_0\chi_1$ holds with $\mathfrak{f}_{\chi_0}=\mathfrak{f}_0$, $\mathfrak{f}_{\chi_1}=p^\mu$.

When we set $x=x_0\mathfrak{f}_0+x_1p^{\mu+\rho}+\mathfrak{f}p^\rho r_\rho(x_0, x_1)$, $1\leqq x_0\leqq p^{\mu+\rho}$, $0\leqq x_1\leqq \mathfrak{f}_0-1$, $1\leqq x\leqq \mathfrak{f}p^\rho$, the numbers $r_\rho(x_0, x_1)\in Z$ are also well determined. The condition $(x, p)=1$ is equivalent to $(x_0, p)=1$. Then we have similarly as above

$$\langle x\rangle=\langle x_0\mathfrak{f}_0\rangle+\varpi(x_0\mathfrak{f}_0)\ (x_1p^{\mu+\rho}+\mathfrak{f}p^\rho r_\rho(x_0, x_1)), \tag{19}$$

and

$$\begin{aligned}
\mathfrak{M}_\chi^{(\rho)}(f)&=\frac{1}{\mathfrak{f}p^\rho}\sum_{x_0=1}^{p^{\mu+\rho}}{}^{*}\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\chi_1(x_0\mathfrak{f}_0)f(\langle x_0\mathfrak{f}_0\rangle+\varpi(x_0\mathfrak{f}_0)(x_1p^{\mu+\rho}+\mathfrak{f}p^\rho r_\rho(x_0, x_0)))\\
&=\frac{1}{\mathfrak{f}p^\rho}\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\sum_{x_0=1}^{p^{\mu+\rho}}{}^{*}\chi_1(x_0\mathfrak{f}_0)f(\langle x_0\mathfrak{f}_0\rangle+p^{\mu+\rho}\varpi(x_0\mathfrak{f}_0)(x_1+\mathfrak{f}_0r_\rho(x_0, x_1)))\\
&=\frac{1}{\mathfrak{f}p^\rho}\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\sum_{x_0=1}^{p^{\mu+\rho}}{}^{*}\chi_1(x_0\mathfrak{f}_0)f(\langle x_0\mathfrak{f}_0\rangle)\\
&\quad+\frac{1}{\mathfrak{f}_0}\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\sum_{x_0=0}^{p^{\mu+\rho}}{}^{*}\chi_1(x_0\mathfrak{f}_0)\varpi(x_0\mathfrak{f}_0)(x_1+\mathfrak{f}_0r_\rho(x_0, x_1))\\
&\qquad\qquad\times\phi_f(\langle x_0\mathfrak{f}_0\rangle+p^{\mu+\rho}\varpi(x_0\mathfrak{f}_0)(x_1+\mathfrak{f}_0r_\rho(x_0, x_1)), \langle x_0\mathfrak{f}_0\rangle),\\
&=\frac{1}{\mathfrak{f}_0}\sum_{x_0=1}^{p^{\mu+\rho}}{}^{*}\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\chi_1(x_0\mathfrak{f}_0)\varpi(x_0\mathfrak{f}_0)(x_1+\mathfrak{f}_0r_\rho(x_0, x_1))\\
&\qquad\qquad\times\phi_f(\langle x_0\mathfrak{f}_0\rangle+p^{\mu+\rho}\varpi(x_0\mathfrak{f}_0)(x_1+\mathfrak{f}_0r_\rho(x_0, x_1)), \langle x_0\mathfrak{f}_0\rangle).
\end{aligned}\tag{20}$$

Hence we have

$$\mathfrak{M}_\chi(f)=\lim_{\rho\to\infty}\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\sum_{x_0=1}^{p^{\mu+\rho}}{}^{*}\chi_1(x_0\mathfrak{f}_0)r_\rho(x_0, x_1)f'(\langle x_0\mathfrak{f}_0\rangle). \tag{21}$$

In the case $p=2$ we see moreover

$$\mathfrak{M}_\chi(f)=\lim_{\rho\to\infty}(1+\chi(-1))\sum_{x_1=0}^{\mathfrak{f}_0-1}\chi_0(x_1p^{\mu+\rho})\sum_{x_0=1}^{2^{\mu+\rho-1}}{}^{*}\chi_1(x_0\mathfrak{f}_0)\varpi(x_0\mathfrak{f}_0)r_\rho(x_0, x_1)f'(\langle x_0\mathfrak{f}_0\rangle). \tag{22}$$

Consequently, if $f'(u)\in O_{\Omega_p}$ for any $u\in U_p$, then we obtain

$$\mathfrak{M}_\chi(f)\equiv 0 \pmod{p^{-1}q}. \tag{23}$$

For a character χ of the second kind and $p>2$ we set $x=\zeta_{p-1}^\alpha(1+p)^\beta+p^\rho R_\rho(\alpha, \beta)$, $0\leqq\alpha\leqq p-2$, $0\leqq\beta\leqq p^{\rho-1}-1$, $1\leqq x\leqq p^\rho$ and determine $R_\rho(\alpha, \beta)\in Z_p$.

Then we have

$$\mathfrak{M}_\chi^{(\rho)}(f)=\frac{p-1}{p^\rho}\sum_{\beta=0}^{p^{\rho-1}-1}\chi((1+p)^\beta)f((1+p)^\beta)$$
$$(24)\qquad +\sum_{\alpha=0}^{p-2}\sum_{\beta=0}^{p^{\rho-1}-1}\zeta_{p-1}^{-\alpha}\chi((1+p)^\beta)R_\rho(\alpha,\beta)\phi_f((1+p)^\beta+p^\rho\zeta_{p-1}^{-\alpha}R_\rho(\alpha,\beta),(1+p)^\beta).$$

Hence we have also

$$(25)\qquad \mathfrak{M}_\chi(f)=\frac{p-1}{p}I_0(\chi((1+p)^x)f((1+p)^x))$$
$$+\lim_{\rho\to\infty}\sum_{\alpha=0}^{p-2}\sum_{\beta=0}^{p^{\rho-1}-1}\zeta_{p-1}^{-\alpha}\chi((1+p)^\beta)R_\rho(\alpha,\beta)f'((1+p)^\beta).$$

Thus, for $f\in\mathcal{U}\mathcal{D}(U_p,\Omega_p)$ such that $f'(u)\in O_{\Omega_p}$ for any $u\in U_p$ we obtain

$$(26)\qquad \mathfrak{M}_\chi(f)\equiv\frac{p-1}{p}M_\chi(f)\pmod{p^0}.$$

Similarly in the case $p=2$ we have

$$(27)\qquad \mathfrak{M}_\chi(f)=\frac{1}{2}I_0(\chi(5^x)f(5^x))$$
$$+\lim_{\rho\to\infty}(1+\chi(-1))\sum_{\beta=0}^{2^{\rho-2}-1}\chi(5^\beta)R_\rho(0,\beta)f'(5^\beta).$$

Namely, we have for $f'(u)\in O_{\Omega_p}$ for any $u\in U_p$

$$(28)\qquad \mathfrak{M}_\chi(f)\equiv\frac{1}{2}M_\chi(f)\pmod 2.$$

In the both cases we obtain finally

$$(29)\qquad \mathfrak{M}_\chi(f)\equiv\frac{p-1}{p}M_\chi(f)\pmod{p^{-1}q}.$$

We summarize our results in the following

THEOREM 1. *Let f be any function in $\mathcal{U}\mathcal{D}(U_p,\Omega_p)$ such that $f'(u)\in O_{\Omega_p}$ for each $u\in U_p$. Then we have the congruences:*

(30) $\mathfrak{M}_\chi(f)\equiv 0\pmod{p^{-1}q}$ *if χ is not of the second kind,*

(31) $\mathfrak{M}_\chi(f)\equiv\frac{p-1}{p}M_\chi(f)\pmod{p^{-1}q}$ *if χ is of the second kind.*

For any $c \in Z$, $(c, f)=1$ *it holds that*

$$\chi(c)\mathfrak{M}_{\chi}(f^{c}) \equiv \mathfrak{M}_{\chi}(f) \pmod{p^{-1}q}. \tag{32}$$

Furthermore, when the function $f(u)$ is multiplicative, i.e. $f(uv) = f(u)f(v)$ for any $u, v \in U_p$, we have the following Theorem 2.

From the definition we have

$$\begin{aligned} M_{\chi}(f^{1+p}) &= M_{\chi}(f) + \lim_{\rho\to\infty} \frac{1}{p^{\rho}}\{f((1+p)^{p^{\rho}}) - f(1)\} \\ &= M_{\chi}(f) + \lim_{\rho\to\infty} \frac{f((1+p)^{p^{\rho}}) - f(1)}{(1+p)^{p^{\rho}}-1}\,\frac{(1+p)^{p^{\rho}}-1}{p^{\rho}}. \end{aligned} \tag{33}$$

Therefore we have

$$M_{\chi}(f^{1+p}) = M_{\chi}(f) + f'(1)\log(1+p). \tag{34}$$

By virtue of the multiplicative property $M_{\chi}(f^{1+p}) = \chi(1+p)f(1+p)M_{\chi}(f)$ it follows that

$$(1-\chi(1+p)f(1+p))M_{\chi}(f) = -f'(1)\log(1+p). \tag{35}$$

In the case $p=2$ quite similarly it also holds:

$$(1-\chi(1+q)f(1+q))M_{\chi}(f) = -f'(1)\log(1+q). \tag{36}$$

Thus we obtain

THEOREM 2. *For any multiplicative function* $f \in \mathcal{UD}(U_p, \Omega_p)$ *such that* $f(u), f'(u) \in O_{\Omega_p}$ *for each* $u \in U_p$ *we have*

$$(1-\chi(1+q)f(1+q))\mathfrak{M}_{\chi}(f) \equiv 0 \pmod{p^{-1}q}. \tag{37}$$

This is a generalization of determination of the denominators of the Bernoulli numbers.

§ 5. Examples

1) As usual we define a linear difference operator Δ for any sequence $\{a_m\}$ in Ω_p by $\Delta\, a_m = a_{m+1} - a_m$.

We take the function $f(u) = \frac{1}{q^k}\Delta^k \frac{1}{m} u^m$ for $m \geqq 1$, where k denotes an arbitrarily fixed non-negative rational integer. Then the formulas (30), (31) in Theorem 1 reduce simply to the known congruences of Kummer.

In fact we see

(38) $$f'(u)=\frac{1}{q^k}\Delta^k u^{m-1}=\frac{1}{q^k}u^{m-1}(u-1)^k\in Z_p$$

and

(39) $$\mathfrak{M}_\chi(f)=\frac{1}{q^k}\Delta^k\mathfrak{M}_\chi\left(\frac{1}{m}u^m\right)=\frac{1}{q^k}\Delta^k\frac{1}{m}D^m_{\chi\omega^{-m}},$$

where we put $D^m_{\chi\omega^{-m}}=-(1-\chi\omega^{-m}(p)p^{m-1})B^m_{\chi\omega^{-m}}$ with the m-th generalized Bernoulli number $B^m_{\chi\omega^{-m}}$ [3].

2) If we take $f(u)=-\frac{1}{1-s}u^{1-s}$, $s\in Z_p$, $s\neq 1$, then Theorem 1 means a determination of the exact modules of continuity of p-adic L-functions of Kubota-Leopoldt. Namely, we see

(40) $$f'(u)=-\frac{1}{1-s}\frac{d}{du}(e^{(1-s)\log u})=-u^{1-s}u^{-1}=-u^{-s}\in Z_p,$$

and

(41) $$\mathfrak{M}_\chi(f)=-\frac{1}{1-s}\mathfrak{M}_\chi(u^{1-s})=L_p(s,\chi)$$

by the definition of the p-adic L-functions.

Thus we can know immediately:

(42) $$L_p(s,\chi)\equiv 0 \pmod{p^{-1}q} \qquad \text{for } \chi \text{ not of the second kind,}$$

(43) $$L_p(s,\chi)\equiv -\frac{u_p}{1-\chi(1+q)(1+q)^{1-s}} \pmod{p^{-1}q}$$

with a constant $u_p=-\frac{p-1}{p}\log(1+q)$ for χ of the second kind.

But these facts are already well known [2], [3].

3) When we select $f(u)$ as $f(u)=\phi_{\chi_2}(u)=\chi_2(u)$ with any Dirichlet character χ_2 of the second kind, we have $f'(u)=0$. Therefore all the congruences in Theorems 1, 2 are automatically equalities as can be seen in the preceding section. Thus we have

(44) $$\mathfrak{M}_\chi(\phi_{\chi_2})=0 \qquad \text{for } \chi \text{ not of the second kind,}$$

(45) $$\mathfrak{M}_\chi(\phi_{\chi_2})=\frac{p-1}{p}M_\chi(\phi_{\chi_2}) \qquad \text{for } \chi \text{ of the second kind,}$$

(46) $$M_\chi(\phi_{\chi_2})=I_0(\phi_{\chi\chi_2})=\begin{cases}0 & \text{for } \chi\neq\bar{\chi}_2,\\ 1 & \text{for } \chi=\bar{\chi}_2.\end{cases}$$

4) Let χ be a Dirichlet character of the second kind. Multiplying $\bar{\chi}(u)$ to the both hand sides in (25) and summing over the elements in $T_p^{(n)}$ we have in the case $p\neq 2$

$$\sum_{\chi\in T_p^{(n)}} \mathfrak{M}_\chi(f)\bar{\chi}(u)=\frac{p-1}{p}\sum_{\chi\in T_p^{(n)}} I_0(f((1+p)^x)\phi_\chi((1+p)^x))\phi_{\bar{\chi}}(u)$$
$$+\lim_{\rho\to\infty}\sum_{\alpha=0}^{p-2}\zeta_{p-1}^{-\alpha}\sum_{\substack{0\leq\beta\leq p^\rho-1-1\\(1+p)^\beta=u}} p^n R_\rho(\alpha,\beta)f'((1+p)^\beta).$$

Consequently we have

$$\sum_{\chi\in T_p} \mathfrak{M}_\chi(f)\phi_{\bar{\chi}}(u)=\frac{p-1}{p}\sum_{\chi\in T_p} I_0(f((1+p)^x)\phi_\chi((1+p)^x))\phi_{\bar{\chi}}(u). \tag{47}$$

Note here that we regard $\psi(x)=\phi_\chi((1+p)^x)$ as a character of the additive group Z_p. Hence the sum

$$\sum_{\chi\in T_p} I_0(f((1+p)^x)\phi_\chi((1+p)^x))\phi_{\bar{\chi}}(u)$$

is a Fourier series expansion of the function $g(x)=f((1+p)^x)$.

By making use of Woodcock's theory [4] we obtain

$$\sum_{\chi\in T_p} \mathfrak{M}_\chi(f)\phi_{\bar{\chi}}(u)=\frac{p-1}{p}f(u). \tag{48}$$

In the case $p=2$ we have quite the same.

In particular, if we take $f(u)=-\dfrac{1}{1-s}u^{1-s}$, then we conclude from the above

$$-\frac{1}{1-s}u^{1-s}=\frac{p}{p-1}\sum_{\chi\in T_p} L_p(s,\chi)\phi_{\bar{\chi}}(u). \tag{49}$$

References

[1] T. Kubota and H. W. Leopoldt, Eine p-adische Theorie der Zetawerte: I, J. Reine Angew. Math., **214/215** (1964), 328–339.

[2] J.-P. Serre, Formes modulaires et fonctions zêta p-adiques, Lecture Notes in Math., **350** (1973), 191–268.

[3] K. Shiratani, Kummer's congruence for generalized Bernoulli numbers and its application, Mem. Fac. Sci. Kyushu Univ. Ser. A, **26** (1972), 119–138.

[4] C. F. Woodcock, Fourier analysis for p-adic Lipschitz functions, J. London Math. Soc., **7** (1974), 681–693.

[5] C. F. Woodcock, An invariant p-adic integral on Z_p, J. London Math. Soc., 8 (1974), 731–734.

Department of Mathematics
Kyushu University
812 Fukuoka