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On some perturbations of the total variation image inpainting method. Part II: relaxation and dual variational formulation

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Abstract

In our note we discuss some strongly elliptic modifications of the total variation inpainting model formulated in the space $\mathrm{BV}(\Omega)$ and investigate the corresponding dual variational problems.

Remarkable features are the uniqueness of the dual solution and the uniqueness of the absolutely continuous part $\nabla^a u$ of the gradient of BV-solutions u on the whole domain. Additionally, any BV-minimizer u automatically satisfies the inequality $0 \leq u \leq 1$, which means that u measures the intensity of the grey level.

Outside of the damaged region we even have the uniqueness of BV-solutions, whereas on the damaged domain the L^2-deviation $\|u - v\|_{L^2}$ of different solutions is governed by the the total variation of the singular part $\nabla^s(u - v)$ of the vector measure $\nabla(u - v)$.

Moreover, the dual solution is related to the BV-solutions through an equation of stress strain type.

1 Introduction

In this note we continue the analysis of some perturbations of the total variation image inpainting method started in [BF2]. There is a variety of different image inpainting techniques (see, e.g., [ACS], [BHS], [BCMS], [CKS], [CS], [PSS], [Sh] and the references quoted therein), one of them being the variational approach, where the reconstructed image is found as a minimizer of the functional

$$J[u] = \int_\Omega \Psi\left(|\nabla u|\right) \,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 \,\mathrm{d}x \,. \tag{1.1}$$

Here and in what follows Ω denotes a bounded Lipschitz domain in $\mathbb{R}^2$, and D is a $\mathcal{L}^2$-measurable subset of Ω with the property

$$0 < \mathcal{L}^2(D) < \mathcal{L}^2(\Omega) \,, \tag{1.2}$$

where $\mathcal{L}^2$ stands for Lebesgue's measure in the plane.

We further assume that $\mathrm{Int}(D) \neq \emptyset$, where $\mathrm{Int}(D)$ is the set of interior points of D. On $\Omega - D$ we are given a measurable function f with values in $[0, 1]$, $f(x)$ measuring the intensity of the grey level at $x \in \Omega - D$.

Now one wants to restore the missing part $D \to [0, 1]$ of the black and white image by minimizing the functional J from (1.1) for suitable choices of densities Ψ and with free parameter $\lambda > 0$.

A very popular choice is $\Psi\left(|\nabla u|\right) := |\nabla u|$, i.e. one considers the total variation (TV) inpainting model. In this case the correct space for the functional J is the class $\mathrm{BV}(\Omega)$ consisting of all functions $u \in L^1(\Omega)$ whose distributional gradient is a vector valued Radon measure on Ω with finite total variation $\int_\Omega |\nabla u|$. For details concerning the space $\mathrm{BV}(\Omega)$ we refer to [Gi] or [AFP].

In our paper we are going to replace the unpleasant quantity $\int_\Omega |\nabla u|$ through a strictly convex functional $\int_\Omega F\left(\nabla u\right)$ of the vector measure ∇u. For this case it is possible to give a rather complete picture of the set of solutions of the problem

$$\int_\Omega F\left(\nabla u\right) + \frac{\lambda}{2}\int_{\Omega - D} (u-f)^2 \, \mathrm{d}x \to \min \quad \text{in } \mathrm{BV}(\Omega)\,.$$

Moreover, we can pass to the dual variational problem for which we show unique solvability and establish some regularity properties of the maximizer.

Let us fix our assumptions and notation: suppose that $F\colon\ \mathbb{R}^2 \to [0,\infty)$ is of class C^2 satisfying

$$F(0) = 0\,, \quad DF(0) = 0\,, \quad F(-p) = F(p)\,, \tag{1.3}$$
$$0 < D^2F(p)(q,q) \le \nu_1 \left(1+|p|\right)^{-1} |q|^2\,, \tag{1.4}$$
$$|DF(p)| \le \nu_2\,, \tag{1.5}$$
$$F(p) \ge \nu_3|p| - \nu_4 \tag{1.6}$$

with constants ν_1, ν_2, $\nu_3 > 0$, $\nu_4 \in \mathbb{R}$, for all p, $q \in \mathbb{R}^2$, $q \neq 0$. Note that (1.5) together with $F(0) = 0$ gives the validity of $F(p) \le \nu_2|p|$ for all $p \in \mathbb{R}^2$.

It is often convenient to replace (1.4) by the stronger condition of μ-ellipticity

$$\nu_0 \left(1+|p|\right)^{-\mu} |q|^2 \le D^2F(p)\left(q,q\right) \le \nu_1 \left(1+|p|\right)^{-1} |q|^2 \tag{1.4$_\mu$}$$

with $\nu_0 > 0$ and exponent $\mu > 1$. As it is outlined in [Bi], Remark 4.2, p. 97, the inequality (1.6) then follows from (1.3), (1.4$_\mu$) and (1.5).

In the TV-case, the density just depends on the modulus of ∇u, which also motivates the study of integrands F being of the special form

$$F(p) = \Phi\left(|p|\right), \quad p \in \mathbb{R}^2\,, \tag{1.7}$$

with $\Phi\colon\ [0,\infty) \to [0,\infty)$ of class C^2. In order to have (1.3) - (1.6) we then require (with suitable constants $\nu_i > 0$)

$$\Phi(0) = 0 = \Phi'(0)\,, \tag{1.3*}$$

$$0 < \min\left\{\frac{\Phi'(t)}{t}\,, \Phi''(t)\right\}\,, \quad \max\left\{\frac{\Phi'(t)}{t}\,, \Phi''(t)\right\} \le \nu_1 \frac{1}{1+t}\,, \tag{1.4*}$$

$$\Phi(t) \ge \nu_3 t - \nu_4 \tag{1.6*}$$

for all $t \ge 0$. Note that $0 \le \Phi'(t) \le \nu_2$ directly follows from the second inequality in (1.4*), and (1.4_μ) is implied by the requirement

$$\nu_0 \,(1+t)^{-\mu} \le \min\left\{\frac{\Phi'(t)}{t}\,, \Phi''(t)\right\}\,. \tag{1.4^*_μ}$$

In the paper [BF1] we constructed examples of densities satisfying all these conditions: for $\mu > 1$ let

$$\begin{aligned} F_\mu(p) &:= \Phi_\mu\left(|p|\right), \quad p \in \mathbb{R}^2\,, && (1.8)\\ \Phi_\mu(t) &:= \int_0^t \int_0^s (1+r)^{-\mu}\, \mathrm{d}r\, \mathrm{d}s, \quad t \ge 0\,, && (1.9) \end{aligned}$$

where in (1.9) the integrand $(1+r)^{-\mu}$ can be replaced by $(\varepsilon + r)^{-\mu}$ or $(\varepsilon + r^2)^{-\mu/2}$ for some parameter $\varepsilon > 0$. We have the explicit formulas

$$\begin{aligned} \Phi_\mu(t) &= \frac{t}{\mu-1} + \frac{1}{\mu-1}\frac{1}{\mu-2}(t+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}\,, \quad \mu \neq 2\,,\\ \Phi_2(t) &= t - \ln\left(1+t\right)\,, \quad t \ge 0\,, \end{aligned}$$

and the energy density F_μ from (1.8) approximates the TV-density in the sense that

$$\lim_{\mu\to\infty} \left[(\mu-1)\, F_\mu(p)\right] = |p|\,, \quad p \in \mathbb{R}^2\,.$$

Assuming that F satisfies (1.3) - (1.6) we next look at the variational problem

$$I[u] := \int_\Omega F(\nabla u)\, \mathrm{d}x + \frac{\lambda}{2}\int_{\Omega - D} (f-u)^2\, \mathrm{d}x \to \min\,, \tag{1.10}$$

which due to the linear growth of F has to be formulated in the Sobolev space $W^1_1(\Omega)$ (see, e.g., [Ad] for definitions) but in general is not solvable in this non-reflexive class.

However we could show in [BF2], Theorem 1.3 and 1.4:

Theorem 1.1. *Let (1.2) hold and suppose that we have (1.3), (1.4_μ) with $1 < \mu < 2$ and (1.5), where the validity of $F(-p) = F(p)$ is not required.*

i) Then the problem (1.10) admits a unique solution $u \in W^1_1(\Omega)$.

ii) The function u satisfies $u(x) \in [0, 1]$ (i.e. $u(x)$ measures the intensity of the grey level at $x \in \Omega$).

iii) The quantity $\sigma := DF(\nabla u)$ is continuous in the interior of Ω. Moreover, there is an open subset Ω_0 of Ω such $\mathcal{H}^{\varepsilon}(\Omega - \Omega_0) = 0$ for any $\varepsilon > 0$ and $u \in C^{1,\alpha}(\Omega_0)$ for all $\alpha < 1$, where $\mathcal{H}^s$ is the s-dimensional Hausdorff-measure. In particular we have $\mathrm{Int}(D) \subset \Omega_0$.

Theorem 1.1 can be understood in the sense that for μ-elliptic integrands with exponent $1 < \mu < 2$ our variational problem is uniquely solvable in the framework of the classical Sobolev space $W_1^1(\Omega)$, the solution satisfies the natural constraint $0 \le u(x) \le 1$ and in addition has a high degree of regularity.

As outlined in [Bi], p. 132, Theorem 1.1 can not be expected to hold at least for $\mu > 3$ (we conjecture that $\mu = 2$ in general is the best possible choice in the presence of a data term $\int |u - f|^2 \,\mathrm{d}x$), thus the question arises how to deal with problem (1.10) in general, e.g. for exponents $\mu > 2$, if we have a μ-elliptic density.

A very natural approach is to use the notion of a convex function of a measure as done in e.g. [AG], [DT] or [GMS1], [GMS2] by letting for $w \in \mathrm{BV}(\Omega)$

$$K[w] := \int_\Omega F\left(\nabla^a w\right) \mathrm{d}x + \int_\Omega F^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) \mathrm{d}|\nabla^s w| + \frac{\lambda}{2}\int_{\Omega - D} (w - f)^2 \,\mathrm{d}x\,. \tag{1.11}$$

Due to the embedding $\mathrm{BV}(\Omega) \hookrightarrow L^2(\Omega)$ (for the 2D case) the third integral is well defined. For vector valued Radon measures ρ we let $\rho^a(\rho^s)$ denote the regular (singular) part of ρ w.r.t. to the measure $\mathcal{L}^2$, and F^∞ is the recession function of F, i.e.

$$F^\infty(p) := \lim_{t\to\infty} \frac{1}{t}\, F\left(tp\right), \quad p \in \mathbb{R}^2\,.$$

If F satisfies in addition (1.7), then it holds

$$F^\infty(p) = \Phi^\infty |p|\,, \quad \Phi^\infty := \lim_{t\to\infty} \frac{1}{t}\, \Phi(t)\,,$$

and for the particular case

$$F(p) := (\mu - 1)\, \Phi_\mu\left(|p|\right)$$

with Φ_μ defined in (1.9) we obtain $F^\infty(p) = |p|$.

Now we can state our first theorem which shows solvability in $\mathrm{BV}(\Omega)$ and gives the uniqueness of the absolutely continuous part $\nabla^a u$ of the gradient of BV-solutions on the whole domain and additionally the uniqueness of BV-solutions outside of the damaged region.

Theorem 1.2. *Let D satisfy (1.2) and let (1.3), (1.4), (1.5), (1.6) hold for F, which in particular is true for $F_\mu(p) = \Phi_\mu\left(|p|\right)$ with Φ_μ from (1.9) with $\mu > 1$.*

Then it holds:

i) The problem $K \to \min$ *in* $\mathrm{BV}(\Omega)$, K *defined in (1.11), admits at least one solution and each solution satisfies the inequality* $0 \le u(x) \le 1$ *a.e.*

ii) Suppose that u *and* $\widetilde{u}$ *are* K*-minimizing in* $\mathrm{BV}(\Omega)$*. Then*

$$u = \widetilde{u} \quad a.e.\ on\ \Omega - D \qquad and \qquad \nabla^a u = \nabla^a \widetilde{u} \quad a.e.\ on\ \Omega\,.$$

If F *is of the form (1.7) (with* Φ *satisfying* (1.3^*)*,* (1.4^*) *and* (1.6^*)*), then we deduce*

$$|\nabla^s u|\,(\Omega) = |\nabla^s \widetilde{u}|\,(\Omega)\,.$$

iii) If I *is defined according to (1.10), then*

$$\inf_{W_1^1(\Omega)} I = \inf_{\mathrm{BV}(\Omega)} K\,.$$

iv) Let $\mathcal{M}$ *denote the set of all* $L^1(\Omega)$*-cluster points of* I*-minimizing sequences from the space* $W_1^1(\Omega)$*. Then* $\mathcal{M}$ *coincides with the set of all* $\mathrm{BV}(\Omega)$*-minimizers of the functional* K*.*

v) For any $u \in \mathcal{M}$ *there exists an open subset* G^u *of* $G := \mathrm{Int}(D)$ *such that* $u \in C^{1,\alpha}(G^u)$ *for any* $\alpha \in (0,1)$ *and* $\mathcal{L}^2(G - G^u) = 0$*.*

From $ii)$ of Theorem 1.2 we deduce the uniqueness in the case of W_1^1-solvability and in the general case an estimate for the L^2-deviation $\|u - v\|_{L^2}$ of different solutions on the damaged domain in terms of $\nabla^s(u - v)$, precisely:

Corollary 1.1. *i) If there exists* $u \in \mathcal{M}$ *such that* $u \in W_1^1(\Omega)$*, then we must have* $\mathcal{M} = \{u\}$*.*

ii) Suppose that $\overline{D} \subset \Omega$*. Then it holds for* u*,* $v \in \mathcal{M}$

$$\|u - v\|_{L^2(\Omega)} = \|u - v\|_{L^2(D)} \le \frac{1}{2\sqrt{\pi}}\,|\nabla^s(u - v)|\,(\overline{D})\,.$$

In particular, the constant on the r.h.s. is not depending on the free parameter λ*.*

To justify $i)$ of Corollary 1.1 we consider u, $v \in \mathcal{M}$ and assume $u \in W_1^1(\Omega)$. From $K[u] = K[v]$ we get with Theorem 1.2 $ii)$: $0 = \int_\Omega F^\infty\left(\frac{\nabla^s v}{|\nabla^s v|}\right) d\,|\nabla^s v|$, hence $\nabla^s v = 0$ and thereby $v \in W_1^1(\Omega)$ together with $\nabla u = \nabla v$. But then $u = v$ on Ω since $u = v$ on $\Omega - D$.

For $ii)$ we just observe that due to $\overline{D} \subset \Omega$ and $u - v = 0$ on $\Omega - D$ the function $u - v$ has compact support, thus we can apply the Sobolev-Poincarè inequality Theorem 1.28, p. 24, from [Gi] with optimal constant stated on p. 151 in [GT].

Remark 1.1. *From iii) and iv) in Theorem 1.2 we see that the minimization of* K *in* $\mathrm{BV}(\Omega)$ *is a natural extension of the original problem (1.10) being in general unsolvable in* $W_1^1(\Omega)$*. Moreover, we clearly have* $I = K$ *on* $W_1^1(\Omega)$*.*

Remark 1.2. *Since minimizers automatically satisfy* $0 \le u(x) \le 1$ *a.e., they can be interpreted as measures for the intensity of the grey level.*

Remark 1.3. *According to Theorem 1.2 ii) we have "uniqueness on* $\Omega - D$*", and the measures* ∇u*,* $\nabla \tilde{u}$ *may only differ in their singular parts.*

For the μ-elliptic case we can construct K-minimizers which show a nice behaviour in the interior of the inpainting region.

Theorem 1.3. *Suppose that* D *satisfies (1.2) and let (1.3),* (1.4_μ) *and (1.5) hold for the density* F*, in particular we may choose* $F = F_\mu$ *defined according to (1.8) and (1.9).*

i) In case $\mu < 3$ *we can find a* K*-minimizer* $u \in \mathrm{BV}(\Omega)$ *such that* $u \in C^{1,\alpha}(\mathrm{Int}(D))$ *for any* $0 < \alpha < 1$*.*

ii) If $\mu = 3$*, then there exists a* K*-minimizer* $u \in \mathrm{BV}(\Omega)$ *such that* $u \in W^1_{1,\mathrm{loc}}(\mathrm{Int}(D))$*. We even have* $\int_C |\nabla u| \ln\left(1 + |\nabla u|^2\right) \mathrm{d}x < \infty$ *for each compact subset* C *of* $\mathrm{Int}(D)$*.*

In the theory of perfect plasticity (see, e.g. [FS]) the underlying variational problem is formulated on the non-reflexive space $\mathrm{BD}(\Omega)$ of functions having bounded deformation and for the same reasons as outlined above one has to pass to a suitable relaxed version of the original problem. But in plasticity there is a natural alternative: if one looks at the dual variational problem, then it turns out that there is unique maximizer, namely the stress tensor σ.

In analogy to this mechanical point of view we now will also consider the problem dual to "$K \to \min$ in $\mathrm{BV}(\Omega)$", whose solution σ in the widest sense equals $DF(\nabla^a u)$, if u is a K-minimizer from $\mathrm{BV}(\Omega)$.

Let us note that this analogy is not only formal, although the given data f are not related to any mechanical quantities, at least we do not know such an interpretation.

Nevertheless, the dual solution σ is a part of the regularized problem in the sense that now the gradient of the grey level produces a stress tensor obeying the constitutive law (stress-strain relation) of the particular regularization. Here the data term $\int_{\Omega-D}(u-f)^2\,\mathrm{d}x$ in fact plays the role of a volume force for the given regularization. For instance, the constitutive law of the TV regularization allows sharp edges in correspondence to perfect plasticity or perfectly plastic fluids, where we may have jumps of the tangential velocities of different layers. It would be interesting to know, if the dual solution is a significant quantity in image analysis and if the mechanical point of view might lead to new results.

Let F satisfy (1.3) - (1.6) (of course partially the following considerations work under much weaker requirements). We define the Lagrangian

$$l(v,\tau) := \int_\Omega \left[\tau \cdot \nabla v - F^*(\tau)\right] \mathrm{d}x + \frac{\lambda}{2}\int_{\Omega-D} (v-f)^2\, \mathrm{d}x$$

for $v \in W_1^1(\Omega)$ and $\tau \in L^\infty(\Omega, \mathbb{R}^2)$. Here F^* is the function conjugate to F, i.e.

$$F^*(q) := \sup_{p \in \mathbb{R}^2} [p \cdot q - F(p)] \,, \quad q \in \mathbb{R}^2 \,.$$

According to [ET], Proposition 2.1, p. 271, it holds

$$\int_\Omega F(p) \, \mathrm{d}x = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^2)} \int_\Omega [\tau \cdot p - F^*(\tau)] \, \mathrm{d}x$$

for functions $p \in L^1(\Omega, \mathbb{R}^2)$, and this leads to the representation

$$I[u] = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^2)} \ell(u, \tau), \quad u \in W_1^1(\Omega) \,, \tag{1.12}$$

for the functional I from (1.10). We now introduce the dual functional

$$R : \, L^\infty\left(\Omega, \mathbb{R}^2\right) \longrightarrow [-\infty, \infty] \,, \quad R[\tau] := \inf_{v \in W_1^1(\Omega)} l\,(v, \tau) \,.$$

Our main result on the dual variational problem is

Theorem 1.4. *Suppose that we have (1.2) - (1.6) for the data. Then it holds:*

i) The dual problem

$$R \to \max \quad \textit{in } L^\infty(\Omega, \mathbb{R}^2)$$

admits at least one solution. Moreover, the inf - sup *relation*

$$\inf_{v \in W_1^1(\Omega)} I[v] = \sup_{\sigma \in L^\infty(\Omega, \mathbb{R}^2)} R[\sigma] \,,$$

I from (1.10), is satisfied.

ii) We have uniqueness if the conjugate function F^* is strictly convex on the set $\{p \in \mathbb{R}^2 : F^*(p) < \infty\}$.

This in particular is true for $F = F_\mu$ with F_μ from (1.8) and (1.9). More generally, we can look at F of the type (1.7) with Φ of class C^3 satisfying (1.3), (1.4*) and (1.6*).*

iii) If the condition for uniqueness holds, then the unique maximizer satisfies

$$\sigma \in W^1_{2,\mathrm{loc}}(\mathrm{Int}(D), \mathbb{R}^2) \qquad \textit{as well as} \qquad \sigma(x) = DF\left(\nabla^a u(x)\right) \quad \textit{a.e. on } \mathrm{Int}(D) \,,$$

where u is any K-minimizer from the space $\mathrm{BV}(\Omega)$.

In addition, σ is Hölder continuous on an open subset of $\mathrm{Int}(D)$ with full measure for any exponent $\alpha \in (0,1)$.

iv) Suppose that the condition for uniqueness is satisfied and that in addition we have (1.4_μ) *with* $1 < \mu < 3$. *Then the dual solution is of class* $C^{0,\alpha}\left(\mathrm{Int}(D), \mathbb{R}^2\right)$ *for any* $\alpha < 1$.

If the observed intensity $f\colon \Omega - D \to [0,1]$ has a certain degree of regularity, then for $u \in \mathcal{M}$ the possible singular part of the measure ∇u is supported on $\overline{D}$, moreover, the maximal solution from Theorem 1.4 *ii*) is in the space $W^1_{2,\mathrm{loc}}\left(\Omega - \overline{D}, \mathbb{R}^2\right)$, more precisely it holds:

Theorem 1.5. *Let (1.2) - (1.6) be satisfied and assume that* $f \in W^1_{2,\mathrm{loc}}\left(\Omega - \overline{D}\right)$. *Then we have:*

i) The unique restriction v *of any* $u \in \mathcal{M}$ *to the set* $\Omega - \overline{D}$ *belongs to the class* $W^1_{2,\mathrm{loc}}(\Omega - \overline{D})$.

ii) If the condition for uniqueness stated in Theorem 1.4 ii) is satisfied, then the unique R-maximizer σ *is an element of the space* $W^1_{2,\mathrm{loc}}\left(\Omega - \overline{D}, \mathbb{R}^2\right)$. *It holds*

$$\operatorname{div} \sigma = \lambda(v - f) \quad \textit{a.e. on } \Omega - \overline{D}\,.$$

Moreover, we have the duality relation

$$\sigma = DF(\nabla v) \quad \textit{a.e. on } \Omega - \overline{D}\,.$$

Our paper is organized as follows: in Section 2 we give the proof of Theorem 1.2 studying the relaxed version of the original problem (1.10). The dual problem is investigated in Section 3, i.e. we present the proof of Theorem 1.4. The discussion of Theorem 1.3 is the subject of Section 4, and in Section 5 we finally establish Theorem 1.5.

2 Minimization in BV. Proof of Theorem 1.2

We start with some auxiliary results assuming from now on the validity of our hypotheses (1.2) - (1.6).

Lemma 2.1. *Let* $u \in \mathrm{BV}(\Omega)$ *be given. Then there exists a sequence* $u_n \in C^\infty(\Omega) \cap W^1_1(\Omega)$ *such that (as* $n \to \infty$*)*

$$\left.\begin{aligned} u_n &\to u \textit{ in } L^1(\Omega), \\ \int_\Omega \sqrt{1 + |\nabla u_n|^2}\,\mathrm{d}x &\to \textstyle\int_\Omega \sqrt{1 + |\nabla u|^2}, \\ \int_\Omega \nabla u_n\,\mathrm{d}x &\to \textstyle\int_\Omega \nabla u\,. \end{aligned}\right\} \tag{2.1}$$

Proof. See [AG], Proposition 2.3. Here the quantity $\int_\Omega \sqrt{1 + |\nabla u|^2}$ is defined according to [AG], Definition 2.1, or [DT], p. 675. □

Lemma 2.2. *For $u \in \mathrm{BV}(\Omega)$ let*

$$\widetilde{K}[u] := \int_\Omega F\left(\nabla^a u\right) \mathrm{d}x + \int_\Omega F^\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) \mathrm{d}\left|\nabla^s u\right| .$$

i) Suppose that u_n, $u \in \mathrm{BV}(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$. Then it holds:

$$\widetilde{K}[u] \le \liminf_{n\to\infty} \widetilde{K}[u_n] . \tag{2.2}$$

ii) If we know in addition

$$\int_\Omega \sqrt{1+|\nabla u_n|^2} \longrightarrow \int_\Omega \sqrt{1+|\nabla u|^2} ,$$

then it follows

$$\lim_{n\to\infty} \widetilde{K}[u_n] = \widetilde{K}[u] . \tag{2.3}$$

Proof.

i) From $u_n \to u$ in $L^1_{(\mathrm{loc})}(\Omega)$ we immediately get $\overline{K}[u] \le \liminf_{n\to\infty} \overline{K}[u_n]$, where $\overline{K}$ is the relaxed functional defined in [AFP], formula (5.60), p. 298 (compare also Remark 5.46 on p. 303 in this reference).

However, as it was shown in [GS], we have $\widetilde{K} = \overline{K}$ on $\mathrm{BV}(\Omega)$ (see also [AFP], Theorem 5.4.7, p. 304), thus (2.2) follows.

ii) Now assume in addition to $u_n \to u$ in $L^1(\Omega)$ that

$$\lim_{n\to\infty} \int_\Omega \sqrt{1+|\nabla u_n|^2} = \int_\Omega \sqrt{1+|\nabla u|^2}$$

holds. Then from Proposition 3.13, p. 125, in [AFP] we deduce that (u_n) weakly* converges to u in $\mathrm{BV}(\Omega)$, so that in particular ([AFP], Definition 3.11, p. 124)

$$\int_\Omega \phi \,\mathrm{d}(\nabla u_n) \to \int_\Omega \phi \,\mathrm{d}(\nabla u)$$

for $\phi \in C^0_0(\Omega)$. Thus we may quote [AG], Proposition 2.2, to get our claim (2.3). For further details we also refer to [BS], Theorem 2.4 and Remark 2.5. □

Let $u_n \in \mathrm{BV}(\Omega)$ denote a K-minimizing sequence. Then we follow the arguments used in the proof of Theorem 2 in [BF3] to see that $\widetilde{K}\left[\max\left(u_n, 0\right)\right] \le \widetilde{K}[u_n]$. Alternatively we observe that this inequality is immediate for $u \in W^1_1(\Omega)$ and extends via approximation quoting Lemma 2.1 to the BV-case.

At the same time we get from $0 \leq f(x) \leq 1$ a.e. on $\Omega - D$ that $|\max(u,0) - f| \leq |u - f|$ on $\Omega - D$ for any $u \in \mathrm{BV}(\Omega)$, thus

$$K\left[\max(u_n, 0)\right] \leq K[u_n]$$

for the functional K defined in (1.11). We may therefore assume w.l.o.g. that $u_n \geq 0$. By considering $\min(u_n, 1)$ and using analogous arguments we can replace our minimizing sequence through a minimizing sequence with the property

$$0 \leq u_n \leq 1 \quad \text{a.e. on } \Omega\,. \tag{2.4}$$

From assumption (1.6) it is clear that

$$\sup_n \int_\Omega |\nabla u_n| < \infty\,. \tag{2.5}$$

Putting together (2.4) and (2.5) and quoting the BV-compactness theorem (see [AFP], Theorem 3.23, p. 132) we find $u \in \mathrm{BV}(\Omega)$ such that for a subsequence $u_n \to u$ in $L^1(\Omega)$ and a.e. From (2.2) it follows

$$\widetilde{K}(u) \leq \liminf_{n\to\infty} \widetilde{K}[u_n]\,,$$

and Lebesgue's theorem on dominated convergence (recall (2.4)) finally shows

$$K[u] \leq \liminf_{n\to\infty} K[u_n]\,.$$

Thus $u \in \mathrm{BV}(\Omega)$ is K-minimizing with the additional property

$$0 \leq u(x) \leq 1\,. \tag{2.6}$$

Let $w \in \mathrm{BV}(\Omega)$ denote any K-minimizer. With $\Psi(t) := \min(1, t)$, $t \in \mathbb{R}$, we observe (by the minimality of w)

$$K[w] \leq K\left[\Psi(w)\right]\,. \tag{2.7}$$

As mentioned before we have for all $v \in \mathrm{BV}(\Omega)$

$$\widetilde{K}\left[\Psi\left(v\right)\right] \leq \widetilde{K}[v]\,, \qquad \int_{\Omega-D} \left(\Psi(v) - f\right)^2 \mathrm{d}x \leq \int_{\Omega-D} \left(v - f\right)^2 \mathrm{d}x\,, \tag{2.8}$$

and if we apply (2.8) to $v = w$ we get in combination with (2.7)

$$\widetilde{K}\left[\Psi(w)\right] = \widetilde{K}[w]\,, \tag{2.9}$$

$$\int_{\Omega-D} \left(\Psi(w) - f\right)^2 \mathrm{d}x = \int_{\Omega-D} (w - f)^2 \,\mathrm{d}x\,. \tag{2.10}$$

The identity (2.9) may then be used as done in [BF3] starting with (26) from this paper leading to the result that $\nabla\left(\Psi(w)\right) = \nabla w$. From (2.10) we infer $w = \Psi(w)$ a.e. on $\Omega - D$, and by Proposition 3.2, p. 118, in [AFP] (recall (1.2)) we get that $w = \Psi(w)$ a.e. on Ω, hence $w \leq 1$ a.e. on Ω, and in the same manner $w \geq 0$. This proves the validity of (2.6) for arbitrary K-minimizers, part *i*) of Theorem 1.2 is established.

For proving *ii*) we notice that $\int_\Omega F(\nabla^a w)\,\mathrm{d}x$, $\int_{\Omega-D}(w-f)^2\,\mathrm{d}x$ are the strictly convex parts of the functional K, whereas $\int_\Omega F^\infty(\nabla^s w/|\nabla^s w|)\,\mathrm{d}|\nabla^s w|$ is just convex and takes the value $c|\nabla^s w|(\Omega)$ if F has the form (1.7), c denoting the number $\lim_{t\to\infty}\frac{1}{t}\Phi(t)$.

In order to establish *iii*) we let

$$\alpha := \inf_{\mathrm{BV}(\Omega)} K\,, \quad \beta := \inf_{W_1^1(\Omega)} I$$

and observe that due to $I = K$ on $W_1^1(\Omega)$ we have $\alpha \leq \beta$.

If u from $\mathrm{BV}(\Omega)$ is such that $K[u] = \alpha$, then we choose u_n according to Lemma 2.1. Since $0 \leq u \leq 1$ it is easy to check that during the construction of these functions the inequality $0 \leq \ldots \leq 1$ is preserved. By Lemma 2.2 we have $\widetilde{K}[u_n] \longrightarrow \widetilde{K}[u]$, and by dominated convergence it holds

$$\int_{\Omega-D}(u_n-f)^2\,\mathrm{d}x \longrightarrow \int_{\Omega-D}(u-f)^2\,\mathrm{d}x\,,$$

thus

$$\beta \leq I[u_n] = K[u_n] \longrightarrow K[u]\,,$$

so that $\beta \leq \alpha$.

Finally we establish *iv*). Let $u \in \mathcal{M}$ and consider a I-minimizing sequence $u_n \in W_1^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$. Clearly $u \in \mathrm{BV}(\Omega)$ and after passing to a subsequence we may assume that $u_n \to u$ a.e. on Ω, thus

$$\int_{\Omega-D}(u-f)^2\,\mathrm{d}x \leq \liminf_{n\to\infty}\int_{\Omega-D}(u_n-f)^2\,\mathrm{d}x$$

by Fatou's lemma, whereas (2.2) gives

$$\widetilde{K}[u] \leq \liminf_{n\to\infty}\widetilde{K}[u_n]\,.$$

This shows (using *iii*) in the last equality)

$$K[u] \leq \liminf_{n\to\infty} K[u_n] = \lim_{n\to\infty} I[u_n] = \inf_{W_1^1(\Omega)} I = \inf_{\mathrm{BV}(\Omega)} K\,,$$

so that u is K-minimizing.

Let $w \in \mathrm{BV}(\Omega)$ denote any K-minimizer. By *i)* we know $0 \le w \le 1$, and we apply Lemma 2.1 to find a sequence w_m with (2.1) and w.l.o.g. $0 \le w_m \le 1$ (check the construction carried out in the proof of Lemma 2.1 presented in [AG]). From Lemma 2.2 (+ dominated convergence) it follows (again usin *iii)*)

$$K\left[w_m\right] \to K\left[w\right] = \inf_{W_1^1(\Omega)} I\,,$$

and since $I\left[w_m\right] = K\left[w_m\right]$ we see that (w_m) is a I-minimizing sequence such that $w_m \to w$ in $L^1(\Omega)$, thus $w \in \mathcal{M}$ by definition of this set.

Altogether the proof of Theorem 1.2 is complete, since our final claim *v)* directly follows from Theorem 1.1 in [AG]: in fact, if $u \in \mathcal{M}$, then according to *iv)* the function u is locally $\widetilde{K}$ minimizing on the open set G and we can quote Theorem 1.1 of [AG] which is applicable due to our assumption (1.4). □

3 The dual problem. Proof of Theorem 1.4

Let (1.2) - (1.6) hold throughout this section. We first wish to remark that part *i)* of Theorem 1.4 can be deduced from [ET], we also refer to [FS], Theorem 1.2.1. We here prefer to give a more direct proof based on a sequence of regularized problems for the reason that this sequence might be of interest for numerical computations, since its solutions are smooth with rather strong convergence properties. Moreover, we will exploit this sequence in the next section.

Lemma 3.1. *Let $\delta \in (0,1]$ and consider the problem*

$$I_\delta[u] := \int_\Omega F_\delta(\nabla u)\,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega - D} (u-f)^2\,\mathrm{d}x \to \min \qquad \text{in } W_2^1(\Omega)\,, \tag{3.1}$$

where $F_\delta(p) := \frac{\delta}{2}|p|^2 + F(p)$.

Then there exists a unique solution u_δ of (3.1) and u_δ satisfies

i) $0 \le u_\delta \le 1$ *on* Ω,

ii) $u_\delta \in W^2_{2,\mathrm{loc}}(\Omega) \cap C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$.

Proof of Lemma 3.1. For $\delta \in (0,1]$ being fixed we consider an I_δ-minimizing sequence w_n. As shown in Section 2 we can assume w.l.o.g. that $0 \le w_n \le 1$.

Since clearly $\sup_n \int_\Omega |\nabla w_n|^2\,\mathrm{d}x \le c(\delta) < \infty$, we have compactness in $W_2^1(\Omega)$, thus $w_n \rightharpoondown: \overline{u}$ in $W_2^1(\Omega)$ for a subsequence.

At the same time we can also arrange $w_n \to \overline{u}$ a.e. (for a further subsequence), thus $0 \le \overline{u} \le 1$ and $\overline{u}$ solves (3.1) by lower semicontinuity of I_δ w.r.t. weak convergence in $W^1_2(\Omega)$.

Suppose that $\hat{u} \in W^1_2(\Omega)$ is another solution of (3.1). By strict convexity it follows $\nabla\overline{u} = \nabla\hat{u}$ a.e. on Ω and $\overline{u} = \hat{u}$ a.e. on $\Omega - D$, thus $\overline{u} = \hat{u}$ by (1.2).

For *ii*) we use the technique of difference quotients to get $\nabla u_\delta \in W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^2)$, the second statement follows from elliptic regularity theory. □

From the definition of the sequence u_δ and the properties of F we immediately deduce

$$\delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x \le c < \infty\,, \qquad \int_\Omega |\nabla u_\delta| \,\mathrm{d}x \le c < \infty\,. \tag{3.2}$$

Let

$$\tau_\delta := DF\left(\nabla u_\delta\right)\,, \quad \sigma_\delta := DF_\delta\left(\nabla u_\delta\right) = \delta\nabla u_\delta + \tau_\delta\,. \tag{3.3}$$

From (3.2) it follows

$$\delta\nabla u_\delta \to 0 \qquad \text{in } L^2(\Omega, \mathbb{R}^2) \text{ as } \delta \to 0\,, \tag{3.4}$$

moreover we have

$$\sup_\delta \|\tau_\delta\|_{L^\infty(\Omega)} < \infty\,. \tag{3.5}$$

From (3.3) - (3.5) we get (after passing to a suitable sequence $\delta \to 0$)

$$\sigma_\delta \rightharpoonup: \sigma \quad \text{in } L^2(\Omega, \mathbb{R}^2)\,, \qquad \tau_\delta \overset{*}{\rightharpoonup}: \tau \quad \text{in } L^\infty(\Omega, \mathbb{R}^2)\,, \tag{3.6}$$

and (3.6) in combination with (3.4) yields $\sigma = \tau$.

Recalling Lemma 3.1 *i*) and using (3.2) we may also assume

$$u_\delta \to: \overline{u} \quad \text{in } L^1(\Omega) \text{ and a.e.} \tag{3.7}$$

for a function $\overline{u} \in \mathrm{BV}(\Omega)$ such that $\overline{u}(x) \in [0,1]$. We emphasize that (3.6) and (3.7) hold for a particular sequence $\delta \to 0$.

Now we show that σ is a solution of the dual problem. From (3.1) it follows

$$\int_\Omega \tau_\delta \cdot \nabla\varphi \,\mathrm{d}x + \delta \int_\Omega \nabla u_\delta \cdot \nabla\varphi \,\mathrm{d}x + \lambda \int_{\Omega - D} (u_\delta - f)\,\varphi \,\mathrm{d}x = 0\,, \quad \varphi \in W^1_2(\Omega)\,, \tag{3.8}$$

and the identity $F\left(\nabla u_\delta\right) = \tau_\delta \cdot \nabla u_\delta - F^*\left(\tau_\delta\right)$ yields

$$I_\delta[u_\delta] = \frac{\delta}{2} \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x + \int_\Omega \left[\tau_\delta \cdot \nabla u_\delta - F^*\left(\tau_\delta\right)\right] \mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (u_\delta - f)^2 \,\mathrm{d}x\,.$$

We apply (3.8) with the choice $\varphi = u_\delta$ and obtain

$$\begin{aligned} I_\delta[u_\delta] &= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x + \int_\Omega \left(-F^*\left(\tau_\delta\right)\right) \mathrm{d}x \\ &\quad +\frac{\lambda}{2}\int_{\Omega-D} (u_\delta - f)^2 \,\mathrm{d}x - \lambda \int_{\Omega-D} (u_\delta - f)\, u_\delta \,\mathrm{d}x \\ &= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x + \int_\Omega \left(-F^*\left(\tau_\delta\right)\right) \mathrm{d}x \\ &\quad -\frac{\lambda}{2}\int_{\Omega-D} u_\delta^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega-D} f^2 \,\mathrm{d}x\,. \end{aligned} \tag{3.9}$$

Let $v \in W_1^1(\Omega)$. Then (recall (1.12) and the definition of R)

$$I[v] = \sup_{\kappa \in L^\infty(\Omega,\mathbb{R}^2)} l(v,\kappa) \geq l(v,\rho) \geq \inf_{w \in W_1^1(\Omega)} l(w,\rho) = R[\rho]$$

for any $\rho \in L^\infty(\Omega,\mathbb{R}^2)$ and in conclusion

$$\sup_{\rho \in L^\infty(\Omega,\mathbb{R}^2)} R[\rho] \leq \inf_{v \in W_1^1(\Omega)} I[v]\,.$$

Obviously $\inf_{v \in W_1^1(\Omega)} I[v] \leq I[u_\delta] \leq I_\delta[u_\delta]$ and with (3.9) we have shown

$$\begin{aligned} \sup_{\kappa \in L^\infty(\Omega,\mathbb{R}^2)} R[\kappa] &\leq \inf_{v \in W_1^1(\Omega)} I[v] \\ &\leq -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega-D} f^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_{\Omega-D} u_\delta^2 \,\mathrm{d}x \\ &\quad + \int_\Omega \left(-F^*(\tau_\delta)\right) \mathrm{d}x\,. \end{aligned} \tag{3.10}$$

The a.e. convergence $u_\delta \to \overline{u}$ (compare (3.7)) together with $u_\delta(x) \in [0,1]$ yields $\int_{\Omega-D} u_\delta^2 \,\mathrm{d}x \to \int_{\Omega-D} \overline{u}^2 \,\mathrm{d}x$, and (3.6) combined with the convexity of F^* shows

$$\limsup_{\delta \to 0} \int_\Omega \left(-F^*\left(\tau_\delta\right)\right) \mathrm{d}x \leq \int_\Omega \left(-F^*\left(\tau\right)\right) \mathrm{d}x\,.$$

Neglecting the term $-\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x$ for the moment we get from (3.10)

$$\sup_{L^\infty(\Omega,\mathbb{R}^2)} R \leq \inf_{W_1^1(\Omega)} I \leq \int_\Omega \left(-F^*\left(\tau\right)\right) \mathrm{d}x - \frac{\lambda}{2}\int_{\Omega-D} \overline{u}^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\Omega-D} f^2 \,\mathrm{d}x\,. \tag{3.11}$$

Passing to the limit $\delta \to 0$ in (3.8) we deduce (recall (3.4), (3.6) and (3.7))

$$\int_\Omega \tau \cdot \nabla\varphi \,\mathrm{d}x + \lambda \int_{\Omega-D} (\overline{u} - f)\,\varphi \,\mathrm{d}x = 0 \tag{3.12}$$

for any $\varphi \in W_2^1(\Omega)$. At the same time it holds

$$\begin{aligned} R[\tau] &:= \inf_{v \in W_1^1(\Omega)} l(v,\tau) \\ &= \int_\Omega (-F^*(\tau))\,\mathrm{d}x + \inf_{v \in W_1^1(\Omega)} \left[\int_\Omega \tau \cdot \nabla v \,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (v-f)^2 \,\mathrm{d}x \right] . \end{aligned} \tag{3.13}$$

Due to the boundedness of τ, $\overline{u}$ and f equation (3.12) extends to $\varphi \in W_1^1(\Omega)$ (note: $\|\varphi - \varphi_m\|_{W_1^1(\Omega)} \to 0$ for a suitable sequence $\varphi_m \in W_2^1(\Omega)$), thus we may write

$$\begin{aligned} &\inf_{v \in W_1^1(\Omega)} \left[\int_\Omega \tau \cdot \nabla v \,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (v-f)^2 \,\mathrm{d}x \right] \\ &= \inf_{v \in W_1^1(\Omega)} \left[-\lambda \int_{\Omega - D} (\overline{u} - f)\, v \,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} (v-f)^2 \,\mathrm{d}x \right] \\ &= \inf_{v \in W_1^1(\Omega)} \left\{ \frac{\lambda}{2} \int_{\Omega - D} (\overline{u} - v)^2 \,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} f^2 \,\mathrm{d}x - \frac{\lambda}{2} \int_{\Omega - D} \overline{u}^2 \,\mathrm{d}x \right\} . \end{aligned}$$

Finally we remark the validity of $\inf_{v \in W_1^1(\Omega)}\{\dots\} = \inf_{v \in L^2(\Omega)}\{\dots\}$, which follows from $\|v - v_m\|_{L^2(\Omega)} \to 0$ for $v \in L^2(\Omega)$ by choosing an appropriate sequence $v_m \in W_2^1(\Omega)$. But obviously $\{\dots\}$ becomes minimal for the choice $v = \overline{u}$ and (3.13) turns into

$$R[\tau] = \int_\Omega (-F^*(\tau))\,\mathrm{d}x + \frac{\lambda}{2} \int_{\Omega - D} f^2 \,\mathrm{d}x - \frac{\lambda}{2} \int_{\Omega - D} \overline{u}^2 \,\mathrm{d}x \,. \tag{3.14}$$

With (3.14) we infer from (3.11)

$$\sup_{L^\infty(\Omega,\mathbb{R}^2)} R \le \inf_{W_1^1(\Omega)} I \le R[\tau] \,,$$

thus τ is R-maximizing and the inf-sup relation holds which proves part $i)$ of Theorem 1.4.

Moreover we have shown (compare (3.10) and the inequality stated before (3.10)):

$$\delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x \to 0 \,, \tag{3.15}$$

$$(u_\delta) \text{ is an } I\text{-minimizing sequence} \tag{3.16}$$

at least for a suitable sequence $\delta_m \to 0$. From part $iv)$ of Theorem 1.2 and (3.16) we further deduce (see (3.7))

$$\overline{u} \text{ is } K\text{-minimizing in } \mathrm{BV}(\Omega) \,. \tag{3.17}$$

Let us now discuss the uniqueness problem: let $H_v : L^\infty(\Omega,\mathbb{R}^2) \to \mathbb{R}$,

$$H_v[\kappa] := \int_\Omega \left[\kappa \cdot \nabla v + \frac{\lambda}{2}\, \mathbf{1}_{\Omega - D}\, (v-f)^2 \right] \mathrm{d}x$$

for functions $v \in W_1^1(\Omega)$. This gives the representation

$$R[\kappa] = \int_\Omega \left(-F^*(\kappa)\right) \mathrm{d}x + \inf_{v \in W_1^1(\Omega)} H_v[\kappa]$$

and $\kappa \mapsto \inf_{v \in W_1^1(\Omega)} H_v[\kappa]$ is easily seen to be concave.

Suppose that F^* is strictly convex and assume that τ_1, τ_2 are R-maximizing but $\tau_1 \neq \tau_2$ on a set $S \subset \Omega$ with $\mathcal{L}^2(S) > 0$. Except of a set of points with zero measure we must have

$$F^*(\tau_i(x)) < \infty\,, \qquad i = 1, 2\,,$$

since otherwise $R[\tau_i] = -\infty$.

Let $\kappa := \frac{1}{2}(\tau_1 + \tau_2)$. Then on the set S it holds $F^*(\kappa) < \frac{1}{2}F^*(\tau_1) + \frac{1}{2}F^*(\tau_2)$ and "$\leq$" on $\Omega - S$, hence

$$\int_\Omega \left(-F^*(\kappa)\right) \mathrm{d}x > \frac{1}{2}\int_\Omega \left(-F^*(\tau_1)\right) \mathrm{d}x + \frac{1}{2}\int_\Omega \left(-F^*(\tau_2)\right) \mathrm{d}x$$

and in conclusion

$$R[\kappa] > \frac{1}{2}R[\tau_1] + \frac{1}{2}R[\tau_2] = \sup_{L^\infty(\Omega,\mathbb{R}^2)} R$$

which is a contradiction. Thus strict convexity of F^* yields uniqueness and the convergences (3.6) and (3.15) hold for any sequence $\delta \to 0$.

Let us now look at the case $F(p) = \Phi(|p|)$ with Φ satisfying (1.3*), (1.4*) and (1.6*). We want to prove the strict convexity of F^* in this particular case. To this purpose we first observe that $F^*(p) = \Phi^*(|p|)$ holds with $\Phi^*(t) := \sup_{s \geq 0}[st - \Phi(s)]$. From (1.4*) we deduce $\Phi''(t) > 0$ for all $t \geq 0$, thus Φ' strictly increases and the second inequality in (1.4*) shows the boundedness of Φ'. More precisely, there is a number $R > 0$ such that

$$\Phi'(t) \to R \qquad \text{as } t \to \infty$$

and

$$\Phi'([0, \infty)) = [0, R)\,.$$

Let $\Psi := (\Phi')^{-1} : [0, R) \to [0, \infty)$. Consider points $t \in [0, R)$. Then it holds

$$\Phi^*(t) = \Psi(t)t - \Phi(\Psi(t))$$

and from this formula we deduce by elementary calculations

$$\frac{d^2\Phi^*}{dt^2}(t) = \frac{1}{\Phi''(\Psi(t))} > 0\,.$$

This implies strict convexity of Φ^* on $[0, R)$ and thereby on $[0, R] \supset \mathrm{dom}\Phi^*$ with value $\Phi^*(R)$ in $[0, +\infty]$. But then F^* is strictly convex on the closed disk $\overline{B_R(0)} \supset \mathrm{dom}F^*$.

Here dom g denotes the set of all points for which the function g takes finite values.

Now we prove part *iii*) of Theorem 1.4: the weak differentiability of σ, i.e. our claim $\sigma \in W^1_{2,\mathrm{loc}}(\mathrm{Int}(D))$, follows along the lines of [Bi], proof of Theorem 2.10, since equation (10) on p. 19 of this reference with Ω being replaced by $\mathrm{Int}(D)$ is a consequence of (3.8) if we consider test functions φ supported in $\mathrm{Int}(D)$.

Next we define $\overline{u}$ according to (3.7) and recall (3.17). From Theorem 1.2 it follows that $\overline{u}$ is of class $C^{1,\alpha}$ on an open subset O of $\mathrm{Int}(D)$ with full measure. Exactly as in [Bi], proof of Theorem 2.24, we deduce from $\overline{u} \in C^{1,\alpha}(O)$ that $\nabla u_\delta \to \nabla \overline{u}$ a.e. on O (see [Bi], (28) on p. 30) as $\delta \to 0$ (for a suitable subsequence).

But then $DF(\nabla u_\delta) \to DF(\nabla \overline{u})$ a.e. on the set O, and since $\tau_\delta := DF(\nabla u_\delta) \rightharpoondown \sigma$ in $L^2(\Omega, \mathbb{R}^2)$ (recall (3.6)), we find $\sigma = DF(\nabla \overline{u})$ a.e. on O, which shows (Hölder-)continuity of σ on the set O. From our calculations it clearly follows that

$$\sigma = DF(\nabla^a \overline{u}) \qquad \text{a.e. on } \mathrm{Int}(D)\,,$$

and by Theorem 1.2 *ii*) we have established the desired duality relation.

Finally we pass to the proof of Theorem 1.4 *iv*): according to Theorem 1.3 *i*), whose proof will be given in Section 4, we find a K-minimizer being of class $C^{1,\alpha}(\mathrm{Int}(D))$, provided $\mu < 3$. But then our claim follows from the stress-strain relation stated in *iii*) of Theorem 1.4. □

4 Proof of Theorem 1.3

Let the inpainting region D satisfy (1.2) and consider a density F with (1.3), (1.4$_\mu$) and (1.5). We recall Lemma 3.1, define $\overline{u}$ according to (3.7), where in (3.7) a suitable sequence $\delta \to 0$ has to be considered. We know (compare (3.15))

$$\delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x \to 0\,, \qquad \delta \to 0\,,$$

moreover, by (3.17) $\overline{u}$ is K-minimizing, hence (see Theorem 1.2 *iv*)) an element of the set $\mathcal{M}$. From (3.1) we obtain letting $G := \mathrm{Int}(D)$

$$\int_G DF_\delta(\nabla u_\delta) \cdot \nabla \varphi \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \overset{\circ}{W}{}^1_2(G)\,. \tag{4.1}$$

If μ is equal to 3 we may copy Lemma 4.33 and Theorem 4.36 from [Bi] which means that we insert exactly the same test-functions into (4.1) choosing disks now compactly contained in G. This yields for such disks $B_r(x_0)$

$$\int_{B_r(x_0)} |\nabla u_\delta| \ln\left(1 + |\nabla u_\delta|^2\right) \,\mathrm{d}x \le c\,(B_r(x_0)) < \infty \tag{4.2}$$

with a local constant $c\,(B_r(x_0))$ independent of δ. From (4.2) we get the weak compactness of the sequence (∇u_δ) in $L^1_{\rm loc}\,(G,\mathbb{R}^2)$, and from $u_\delta \to \overline{u}$ in $L^1(\Omega)$ we find $\nabla\overline{u} \in L^1_{\rm loc}\,(G,\mathbb{R}^2)$ together with $\nabla u_\delta \rightharpoondown \nabla\overline{u}$ in $L^1_{\rm loc}\,(G,\mathbb{R}^2)$.

By lower semicontinuity we see that (4.2) extends to $\nabla\overline{u}$, thus the minimizer $\overline{u}$ has the integrability properties stated in *ii*) of Theorem 1.3.

If $\mu \in (1,3)$ we like to show that $\overline{u} \in C^{1,\alpha}(G)$ for any $0 < \alpha < 1$. But this can be done exactly as in Section 4.3.2 of [Bi] replacing u^* and Ω in this reference through $\overline{u}$ and G. We leave the details to the reader. □

5 Proof of Theorem 1.5

We work with our regularizing sequence (u_δ) from Lemma 3.1 and observe that (3.8) implies ($i = 1,\,2$)

$$0 = \int_{\Omega-\overline{D}} D^2F_\delta\,(\nabla u_\delta)\,(\partial_i\nabla u_\delta, \nabla\varphi)\,{\rm d}x - \lambda\int_{\Omega-\overline{D}}(u_\delta - f)\,\partial_i\varphi\,{\rm d}x \tag{5.1}$$

for $\varphi \in \overset{\circ}{W}{}^1_2\left(\Omega-\overline{D}\right)$. With $\eta \in C^1_0\,(B_{2r}(x_0))$, $0 \le \eta \le 1$, $\eta = 1$ on $B_r(x_0)$, $|\nabla\eta| \le c/r$, for a disk $B_{2r}(x_0) \Subset \Omega - \overline{D}$ we let $\varphi := \eta^2\partial_i u_\delta$ in (5.1) and obtain (from now summation w.r.t. $i = 1,\,2$)

$$\begin{aligned}
&\int_{\Omega-\overline{D}} D^2F_\delta\,(\nabla u_\delta)\,(\partial_i\nabla u_\delta, \partial_i\nabla u_\delta)\,\eta^2\,{\rm d}x\\
&\quad= -2\int_{\Omega-\overline{D}} D^2F_\delta\,(\nabla u_\delta)\,(\partial_i\nabla u_\delta\eta, \nabla\eta\partial_i u_\delta)\,{\rm d}x\\
&\qquad -\lambda\int_{\Omega-\overline{D}}\partial_i\,(u_\delta - f)\,\eta^2\partial_i u_\delta\,{\rm d}x =: -T_1 - T_2\,.
\end{aligned} \tag{5.2}$$

Using the Cauchy-Schwarz inequality for the bilinear form $D^2F_\delta(\nabla u_\delta)$ and then applying Young's inequality we find

$$\begin{aligned}
|T_1| \;\le\; & \frac{1}{2}\int_{\Omega-\overline{D}} D^2F_\delta\,(\nabla u_\delta)\,(\partial_i\nabla u_\delta, \partial_i\nabla u_\delta)\,\eta^2\,{\rm d}x\\
&+c\int_{\Omega-\overline{D}} D^2F_\delta\,(\nabla u_\delta)\,(\nabla\eta, \nabla\eta)\,|\nabla u_\delta|^2\,{\rm d}x\,,
\end{aligned}$$

whereas

$$-T_2 = -\lambda\int_{\Omega-\overline{D}}\eta^2\,|\nabla u_\delta|^2\,{\rm d}x - \lambda\int_{\Omega-\overline{D}}\eta^2\partial_i u_\delta\partial_i f\,{\rm d}x\,,$$

and from (5.2) it follows:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega-\overline{D}} D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\nabla u_\delta,\partial_i\nabla u_\delta\right)\eta^2\,\mathrm{d}x+\lambda\int_{\Omega-\overline{D}}\eta^2\left|\nabla u_\delta\right|^2\,\mathrm{d}x\\
&\quad\le\ c\left[\int_{\Omega-\overline{D}} D^2F_\delta\left(\nabla u_\delta\right)\left(\nabla\eta,\nabla\eta\right)\left|\nabla u_\delta\right|^2\,\mathrm{d}x+\int_{\Omega-\overline{D}}\eta^2\left|\nabla u_\delta\right|\left|\nabla f\right|\,\mathrm{d}x\right]. \qquad (5.3)
\end{aligned}$$

The second integral on the r.h.s. of (5.3) is handled with Young's inequality, for the discussion of the first one we use the second inequality in (1.4) as well as the bound (3.2). This gives

$$\int_{B_r(x_0)} D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\nabla u_\delta,\partial_i\nabla u_\delta\right)\,\mathrm{d}x+\int_{B_r(x_0)}\left|\nabla u_\delta\right|^2\,\mathrm{d}x\le c\left(B_r(x_0)\right)<\infty \qquad (5.4)$$

for a local constant independent of δ. Clearly, (5.4) implies part *i)* of Theorem 1.5 by recalling the convergence (3.7) (for a suitable sequence δ_m) of u_δ to a generalized minimizer $\overline{u}$ and the statement of Theorem 1.2 *ii)*.

Now assume that we are in the situation of Theorem 1.4 *ii)*. It holds (compare (3.3))

$$\begin{aligned}
\partial_i\sigma_\delta\cdot\partial_i\sigma_\delta &= D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\nabla u_\delta,\partial_i\sigma_\delta\right)\\
&\le \left(D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\nabla u_\delta,\partial_i\nabla u_\delta\right)\right)^{\frac{1}{2}}\left(D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\sigma_\delta,\partial_i\sigma_\delta\right)\right)^{\frac{1}{2}}\\
&\le c\left(D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\nabla u_\delta,\partial_i\nabla u_\delta\right)\right)^{\frac{1}{2}}\left|\nabla\sigma_\delta\right|,
\end{aligned}$$

where we used (1.4) in the last inequality. Hence

$$\left|\nabla\sigma_\delta\right|^2\le c\ D^2F_\delta\left(\nabla u_\delta\right)\left(\partial_i\nabla u_\delta,\partial_i\nabla u_\delta\right),$$

so that $\sigma_\delta\in W^1_{2,\mathrm{loc}}(\Omega-\overline{D},\mathbb{R}^2)$ uniformly w.r.t. δ. From (3.6) we then deduce $\sigma\in W^1_{2,\mathrm{loc}}(\Omega-\overline{D},\mathbb{R}^2)$.

The second statement of Theorem 1.5 *ii)* follows from (3.8) choosing $\varphi\in\overset{\circ}{W}{}^1_2(\Omega-\overline{D})$ and by passing to the limit using (3.6) as well as (3.7).

Finally, we wish to show the validity of $\sigma=DF(\nabla v)$ a.e. on $\Omega-\overline{D}$. By (3.17) $\overline{u}$ is K-minimizing, hence v is locally minimizing the functional

$$\int_{\Omega-\overline{D}}F\left(\nabla w\right)\,\mathrm{d}x+\frac{\lambda}{2}\int_{\Omega-\overline{D}}\left(w-f\right)^2\,\mathrm{d}x$$

among functions e.g. from $W^1_{1,\mathrm{loc}}(\Omega-\overline{D})$ which follows from Theorem 1.5 *i)*. We therefore have

$$\int_{\Omega-\overline{D}}DF\left(\nabla v\right)\cdot\nabla\varphi\,\mathrm{d}x+\lambda\int_{\Omega-\overline{D}}\left(v-f\right)\varphi\,\mathrm{d}x=0$$

for $\varphi \in C_0^1(\Omega - \overline{D})$, and in combination with (3.8) we get

$$\int_{\Omega-\overline{D}} (DF_\delta(\nabla u_\delta) - DF(\nabla v)) \cdot \nabla\varphi \,\mathrm{d}x + \lambda \int_{\Omega-\overline{D}} (u_\delta - v)\,\varphi \,\mathrm{d}x = 0\,. \tag{5.5}$$

Again by part i), $\varphi := \eta^2 (u_\delta - v)$ is admissible, where η is as above. Recalling (3.7) and the inequalities $0 \le u_\delta, v \le 1$ a.e. as well as (1.5), we deduce from (5.5) with Lebesgue's theorem on dominated convergence

$$\lim_{\delta\to 0} \int_{\Omega-\overline{D}} \eta^2 (DF(\nabla u_\delta) - DF(\nabla v)) \cdot (\nabla u_\delta - \nabla v)\,\mathrm{d}x = 0\,,$$

thus for a suitable sequence

$$(DF(\nabla u_\delta) - DF(\nabla v)) \cdot (\nabla u_\delta - \nabla v) \to 0$$

a.e. on $\Omega - \overline{D}$. As it is outlined in [Bi], p. 31, 32, this implies

$$DF(\nabla u_\delta) \to DF(\nabla v) \qquad \text{a.e. on } \Omega - \overline{D}\,. \tag{5.6}$$

At the same time (see (3.5) and (3.6))

$$\tau_\delta := DF(\nabla u_\delta) \rightharpoondown \sigma \qquad \text{in } L^2(\Omega, \mathbb{R}^2)\,, \tag{5.7}$$

and by combining (5.6) with (5.7) we deduce $\sigma = DF(\nabla v)$ a.e. on $\Omega - \overline{D}$. □

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