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ABSTRACT

ON SOME PROBLEMS IN A THEORY OF THERMALLY AND MECHANICALLY INTERACTING CONTINUOUS MEDIA

By

Yong Mok Lee

Thesis

Using a linearized theory of thermally and mechanically interacting mixture of linear elastic solid and viscous fluid, we derive a fundamental relation in an integral form called a reciprocity relation. This reciprocity relation relates the solution of one initial-boundary value problem with a given set of initial and boundary data to the solution of a second initial-boundary value problem corresponding to a different initial and boundary data for a given interacting mixture. From this general integral relation we derive reciprocity relations for a heat-conducting linear elastic solid, and for a heat-conducting viscous fluid.

In this theory of interacting continua we pose and solve an initial-boundary value problem for the mixture of linear elastic solid and viscous fluid. We consider the mixture to occupy a half-space and its motion to be restricted to one space dimension. We prescribe a step function temperature on the face of the half-space where the face is constrained rigidly against motion. With the

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aid of the Laplace transform and the contour integration, a real integral representation for the displacement of the solid constituent is obtained as one of the principal results of this analysis. In addition, early time series expansions of the other field variables are given.

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By

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TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

LIST OF FIGURES

INTRODUCTION

Using a linearized theory of thermally and mechanically interacting mixture of linear elastic solid and viscous fluid, we derive a fundamental relation in an integral form called a reciprocity relation. This reciprocity relation relates the solution of one initial-boundary value problem with a given set of initial and boundary data to the solution of a second initial-boundary value problem corresponding to a different initial and boundary data for a given interacting mixture. From this general integral relation, we derive reciprocity relations for a heat-conducting linear elastic solid, and for a heat-conducting viscous fluid.

In this theory of interacting continua we pose and solve an initial-boundary value problem for the mixture of linear elastic solid and viscous fluid. We consider the mixture to occupy a half-space and its motion to be restricted to one space dimension. We prescribe a step function temperature on the face of the half-space where the face is constrained rigidly against motion. With the aid of the Laplace transform and the contour integration, a real integral representation for the displacement of the solid constituent is obtained as one of the principal results of this analysis. In addition, early time series expansions of the other field variables are given.

Chapter I includes a historical survey of early works and the various descriptions on mixture theory.

Chapter II presents modern mixture theories based on mathematically sound concepts of continuum mechanics. In Chapter III we derive the general integral reciprocity relation for a linearized version of an interacting mixture and in Chapter IV we pose and solve a basic one-dimensional problem using the linearized theory.

CHAPTER I. HISTORY

1.1. Darcy's Law

The theoretical description of the dynamics of situations in which one substance interpenetrates another has been a matter of interest to mathematicians, physicists and engineers for many years. The case in which a fluid permeates a solid is appropriate to a wide range of problems such as soil mechanics, petroleum engineering, water purification, industrial filtration, ceramic engineering, diffusion problems, absorption of oils by plastics and the reentry ablation process for spacecraft. A survey of earlier works on this subject up to 1959 is given by Scheidegger [1]*. An early work on this subject was the study of fluid flow through a porous solid with the assumption that the solid is undeformable. Intuitively, "pores" are void spaces which must be distributed more or less frequently through the solid if the latter is to be called "porous." Extreme small voids in a solid are called "molecular interstices," very large ones are called "caverns." "Pores" are void spaces intermediate between caverns and molecular interstices; the limitation of their size is therefore intuitive and rather indefinite.

Darcy [2] performed an experiment concerning the flow through a homogeneous porous solid.** A homogeneous filter bed of height h is bounded by horizontal plane areas of

*Number(s) after name(s) refer to the list of references to be found at the end of this paper.

**This experiment was originally performed by Darcy in 1856.

equal size A. These areas are congruent so that corresponding points could be connected by vertical straight lines. The filter bed is percolated by an incompressible liquid. If open manometer tubes are attached at the upper and lower boundaries of the filter bed, the liquid rises to the heights h_2 and h_1 respectively above an arbitrary datum level. By varying the various quantities involved, one can deduce the following relationship:

$$Q = \frac{-KA(h_2 - h_1)}{h} \tag{1.1}$$

Here, Q is the total volume of fluid percolating in unit time, and K is a constant depending on the properties of the fluid and of the porous solid. The relationship (1.1) is known as Darcy's law. Darcy's law can be restated in terms of the pressure p and the density r of the liquid. At the upper boundary of the bed, whose height is denoted by z_2, the pressure is $p_2 = rg(h_2 - z_2)$, and at the lower boundary, whose height is denoted by z_1, the pressure is $p_1 = rg(h_1 - z_1)$. Here g is the gravitational constant. Inserting this statement into (1.1), one obtains

$$Q = -KA((p_2 - p_1)/(rgh) + 1)$$

or, upon introduction of a new constant K',

$$Q = -K'A(p_2 - p_1 + rgh)/h \tag{1.2}$$

A constant of the type K', however, is not very satisfactory because one would like to separate the influence of the porous solid from that of the liquid. By 1933, the empirical relationship

$$K' = k/\mu \qquad (1.3)$$

was generally accepted where μ is the viscosity of the fluid and k the "permeability" of the porous solid. Physically, permeability measurements are very simple. The experiments are performed whereby in a certain system a pressure drop and a flow rate are measured. The solution of Darcy's law corresponding to the geometry of the system and to the fluid employed is calculated, and a comparison between the calculated and the experimentally found results immediately yields the only unknown quantity k. Darcy's law (1.2), when accounting for the separation of the general constant into "permeability" and "viscosity," is expressible as follows:

$$q \equiv Q/A = -(k/\mu)(p_2 - p_1 + rgh)/h \qquad (1.4)$$

If the solid is isotropic and if we consider h as an infinitesimal, then the expression (1.4) naturally extends to a vector form of Darcy's law:

$$\mathfrak{q} = -(k/\mu)(\text{grad } p - r\mathfrak{g}) \qquad (1.5)$$

where $\mathfrak{g}$ is a vector in the direction of gravity.

Engineering uses of Darcy's law are limited to flows exhibiting small pressure differentials and to constant viscosities and permeabilities. However, for liquids at high velocities or for gases, relation (1.1) is no longer valid. Further if k and μ are variable then this law must be modified.

The validity of Darcy's law has been tested on many occasions, and has been shown that it is valid for a wide domain of flows. For liquids, it is valid for arbitrary small pressure differentials. It has also been used to measure flow rates by determining the pressure drop across a fixed porous solid. For liquids at high velocities and for gases at very low and at very high velocities, Darcy's law becomes invalid.

For given boundary conditions Darcy's law (1.5) is by itself not sufficient to determine the flow pattern in a porous solid because it contains three unknowns (q, p, r). Two further equations are therefore required for the complete specification of a problem. One is the connection between r and p of the fluid:

$$r = r(p) \tag{1.6}$$

and the other a continuity equation, viz.:

$$- P \frac{\partial r}{\partial t} = \text{div } (rq) \tag{1.7}$$

where t is the time and P is the porosity defined by the fraction of void to the total volume of the porous solid. A great variety of methods for the measurement of the porosity are described by Scheidegger[1]. The physical conditions of flow for which solutions might be sought are (i) steady state flow, (ii) gravity flow with a free surface, and (i) unsteady state flow. Of these, steady state flow solutions for incompressible fluids are most easily obtained; they are simply represented by solutions of Laplace's equation. Except for a few other special cases, Darcy's law leads to nonlinear differential equations.

As an application of Darcy's law we will consider the steady state flow of an incompressible fluid. With the help of the equations (1.5) and (1.7), one may obtain

$$P \frac{\partial r}{\partial t} = \operatorname{div} \left((rk/\mu)(\operatorname{grad} p - rg) \right) \tag{1.8}$$

Due to the steady state condition, incompressibility and the porous solid being homogeneous, one has:

$$\nabla^2 p = 0 \tag{1.9}$$

As an example of a steady state solution we give the solution for two-dimensional radial flow of an incompressible fluid into a well which is completely penetrating the fluid-bearing medium. Assuming that the well is a cylinder of radius R_0, with pressure p_0, and that the pressure at distance R_1 from the well is p_1, the required solution follows easily by considering equations (1.5) and (1.9) as

$$Q = \frac{2\pi k}{\mu \log (R_1/R_0)} (p_1 - p_0)$$

where Q is the total discharge per unit time.

A major limitation in this theory is due to the assumption that the solid is rigid. In most applications this is simply not true. To incorporate the effects that a deformable solid imposes upon the flow, it is necessary to develop some connection between the stresses and the corresponding strains of both the fluid and the solid. We will consider the case of soil consolidation.

1.2. Biot's Work

A soil under load does not assume an instantaneous deflection under that load, but settles gradually at a variable rate according to the load variation as in clays and sands saturated with water. A simple mechanism to explain this phenomenon was proposed by Terzaghi [3] by assuming that the grains constituting the soil are bound together by molecular forces and constitute a porous material with elastic properties while the voids of the elastic skeleton are filled with water. A load applied to this system will produce a gradual settlement, depending on the rate at which the water is being squeezed out of the voids. Terzaghi applied these concepts to the analysis of the settlement of a column of soil under a constant load and prevented from lateral expansion. The remarkable success of this theory in predicting the settlement for many types of soils has led to the extension to the three-dimensional case and the establishment of equations valid for an arbitrary load variable with time. We will review extensive work done by Maurice A. Biot in this field.

Biot [4] assumed the following basic properties of the soil: (1) isotropy of the material, (2) reversibility of stress-strain relations under final equilibrium conditions, (3) linearity of stress-strain relations, (4) small strains, (5) the water contained in the pores is incompressible, (6) the water may contain air bubbles. (7) the water flows through the porous skeleton according to Darcy's law. We refer the points in this continuous medium to a rectangular cartesian system, x_i, $i = 1,2,3$. Consider a small cubic element of the consolidating soil, its sides being parallel with the coordinate axes. This element is taken to be large enough compared to the size of the pores so that it may be treated as homogeneous, and at the same time small enough compared to the scale of the macroscopic phenomena in which we are interested, so that it may be considered as infinitesimal in the mathematical treatment. Physically the stresses of the soil are composed of two parts; one which is caused by the hydrostatic pressure of the water filling the pores, the other caused by the average stress in the skeleton. They must satisfy the well-known equilibrium conditions of a stress field. Let σ_{ij} denote the stress components and let x_i denote axes of the cartesian system.

By σ_{ij} we shall mean the jth stress component of the skeleton acting on the face x_i constant. Then according to the equilibrium for the infinitesimal element of volume we have

$$\sigma_{ij,j} = 0^{*} \tag{1.10}$$

and

$$\sigma_{ij} = \sigma_{ji}^{*}. \tag{1.11}$$

Denoting by u_i the component of the displacement in the x_i direction, and assuming the strain to be small, the values of the strain components are

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.12}$$

In order to describe completely the macroscopic condition of the soil, an additional variable giving the amount of water in the pores is considered. The increment of water volume per unit volume of soil is called the variation in water content and is denoted by θ, and the increment of water pressure is denoted by σ. Let us consider a cubic element of soil. The water pressure in the pores may be considered as uniform throughout, provided either the size of the element is small enough or, if this is not the case, provided the changes occur at sufficiently slow rate to render the pressure differences negligible. Since it is assumed that the changes in the soil occur by reversible processes, the macroscopic condition of the soil must be a definite function of the stresses and the water pressure, i.e.; the seven variables e_{ij}, θ must be definite functions of the variables σ_{ij} and σ. Furthermore if the

*All subscripts run over values 1,2,3, and, when repeated indicate a sum on the index over 1,2,3. The notation $f_{i,j}$ denotes differentiation with respect to the jth independent variable, i.e., $\partial f_i/\partial x_j$.

strains and the variations in water content are assumed to be small quantities, the relation between two sets of variables may be taken as linear. Consider the case where $\sigma = 0$. The six components of strain are then functions only of the six stress components σ_{ij}. Assuming the soil to have isotropic properties, these relations reduce to the well known expressions of Hookes' law for an isotropic elastic body in the theory of elasticity

$$e_{ij} = \frac{1}{2G} (\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{\ell\ell}\delta_{ij})^{**} \tag{1.13}$$

where the constants G, ν may be interpreted, respectively, as the shear modulus and Poisson's ratio for the solid skeleton.

The effect of the water pressure σ is now introduced. By reason of the assumed isotropy of the soil, this effect is limited to a dependence upon the three strain components e_{11}, e_{22}, e_{33} and such dependence is uniform in each direction. Hence taking into account the influence of σ, the relations (1.13) become

$$e_{ij} = \frac{1}{2G} (\sigma_{ij} - \frac{\nu}{(1+\nu)} \sigma_{kk}\delta_{ij}) + \frac{\sigma}{3H} \delta_{ij} \tag{1.14}$$

where H is an additional physical constant which plays the role of a bulk modulus. These relations express the six strain components of the soil as a function of the stresses in the soil and the pressure of the water in the pores.

** The Kronecker delta, denoted by δ_{ij}, is defined as 1 if $i = j$ and 0 if $i \neq j$.

To derive the dependence of the increment of water content θ on these same variables Biot considers the general relation

$$\theta = a_1\sigma_{11} + a_2\sigma_{22} + a_3\sigma_{33} + a_4\sigma_{12} + a_5\sigma_{23} + a_6\sigma_{13} + a_7\sigma , \tag{1.15}$$

and argues that because of the isotropy of the material a change in sign of σ_{12}, σ_{23}, σ_{13} cannot affect the water content. Therefore $a_4 = a_5 = a_6 = 0$ and the effect of the shear stress components on θ vanishes. Furthermore, all three directions x_1, x_2, x_3 must have equivalent properties so that $a_1 = a_2 = a_3$. Relation (1.15) may be written in the form

$$\theta = \frac{1}{3H_1}\sigma_{\ell\ell} + \frac{\sigma}{R} \tag{1.16}$$

where H_1 are R are two new physical constants.

To this point in the derivation Biot has used assumptions (1), (3), (4). He now uses (2) to show that the five constants can be reduced to four. This assumption, i.e., the existence of a potential energy, means that the work done to bring the soil from the initial state to its final state of strain and water content is independent of the way by which the final state is reached and is a definite function of e_{ij}, and θ only. The potential energy of the soil per unit volume is

$$U = \frac{1}{2}(\sigma_{ij}e_{ij} + \sigma\,\theta) \tag{1.17}$$

As a result of some elementary manipulations, Biot shows that $H = H_1$, and we may write the equation (1.16) as

$$\theta = \frac{1}{3H}\,\sigma_{kk} + \frac{\sigma}{R} \tag{1.18}$$

Relations (1.14) and (1.18) are the fundamental relations describing completely in first approximation the properties of the soil, for strain and water content, under equilibrium conditions. They contain four distinct physical constants G, ν, H and R. Solving equation (1.14) with respect to the stresses, then substituting into the equilibrium conditions (1.10), one obtains

$$G\nabla^2 u_i + \frac{G}{1-2\nu}\frac{\partial e}{\partial x_i} - \alpha\frac{\partial \sigma}{\partial x_i} = 0 \tag{1.19}$$

with

$$\alpha = \frac{2(1+\nu)}{3(1-2\nu)}\frac{G}{H} \tag{1.20}$$

There are three equations with four unknown u_i, σ. In order to have a complete system, one more equation is needed. This equation is derived from Darcy's law governing the flow of water in a porous medium. An elementary cube of soil is considered and the volume of water flowing per second per unit area through the face of the cube perpendicular to the x_i-axis is denoted by v_i. According to Darcy's law these three components of the rate of flow are related to the water pressure by the relations

$$v_i = -\frac{k}{\mu}\frac{\partial \sigma}{\partial x_i} \tag{1.21}$$

where the physical constant k is the coefficient of permeability of the soil, and μ is the viscosity of the water. Since the water is assumed to be incompressible, one obtains

$$\frac{\partial \theta}{\partial t} = - \frac{\partial v_i}{\partial x_i}. \tag{1.22}$$

From equations (1.17), (1.20) and (1.21) one obtains that

$$\frac{k}{\mu} \nabla^2 \sigma = \alpha \frac{\partial e}{\partial t} + \frac{1}{Q} \frac{\partial \sigma}{\partial t} \tag{1.23}$$

where

$$\frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H}$$

The four differential equations (1.18) and (1.22) are the basic equations satisfied by the four unknown u_i, σ.

In a paper by Biot and Willis [5], methods of measurement for the four distinct physical constants G, ν, H and R are described and the physical interpretation of the constants in various alternate forms is also discussed.

In a later work by Biot [6] the stress-strain relations which are valid for the case of an elastic porous medium with nonuniform porosity, i.e., for which the porosity varies from point to point are derived and these relations lead to the six equations for the six components of the unknown displacement vector fields u for solid component, U for fluid component. The stress-strain relations are

$$\sigma_{ij} = 2\eta\, e_{ij} + \delta_{ij}(\lambda e - \alpha M \zeta) \tag{1.24a}$$

$$p_f = -\alpha Me + M\xi \qquad (1.24b)$$

where the coefficients M, λ, η, ξ and p_f of equations (1.24a) and (1.24b) are equivalent to the constants Q, $2G\nu/(1-\nu)$, G, θ and σ of the equations of (1.19) and (1.23). The six equations for the six components of the unknown vector fields u and f are

$$2\frac{\partial}{\partial x_j}(\eta\, e_{ij}) + \frac{\partial}{\partial x_i}(\lambda e - \alpha M\xi) = 0 \qquad (1.25a)$$

$$\frac{\partial f}{\partial t} = (k/\mu)\ \mathrm{grad}\ (\alpha Me - M\xi) = 0 \qquad (1.25b)$$

We will consider a uniform porosity case, i.e., let us consider a particular case where the coefficients η, λ, α, M and k/μ are constants. In this case equations (1.25) become

$$\eta\,\nabla^2 u + (\eta+\lambda)\ \mathrm{grad}\ e - \alpha M\ \mathrm{grad}\ \xi = 0 \qquad (1.26a)$$

$$\frac{\partial f}{\partial t} = (k/\mu)M\alpha\ \mathrm{grad}\ e - (kM/\mu)\ \mathrm{grad}\ \xi . \qquad (1.26b)$$

These equations can be written in the form of equations (1.19) and (1.23) by the application of the divergence operator to the equation (1.26b). With the aid of the general Papkovich-Boussinesq solution for Lame's equations of the theory of elasticity, the general solutions for the equations (1.26) are obtained as

$$u = \mathrm{grad}\ (\psi_0 + r\cdot\psi_1) - 2\frac{2\eta+\lambda}{\eta+\lambda}\psi_1 - \frac{\alpha M}{2\eta+\lambda}\ \mathrm{grad}\ \phi \qquad (1.27a)$$

$$f = \mathrm{grad}\ \phi - \frac{2k\alpha M\eta}{\mu(\eta+\lambda)}\int^t \mathrm{grad\ div}\ \psi_1 dt \qquad (1.27b)$$

where ψ_0 and ψ_1 are solutions of Laplace's equation, r is the position vector and ϕ satisfies the diffusion equation

$$\frac{\partial \phi}{\partial t} = (k/\mu)\, M\nabla^2\phi\ .$$

Utilizing equations (1.24a) and (1.24b), and considering the dynamical case, Biot [7] established equations for acoustic propagation in the elastic isotropic porous solid containing a viscous fluid by adding suitable inertia terms in the previous theory, and discussed the propagation of three kinds of body waves. For simplicity of notation we will use a new set of coefficients which are related with the coefficients of equations (1.24):

$$N = \eta,\ A = \lambda + M(\alpha-P)^2,\ B = P(\alpha-P)M,\ C = P^2M \qquad (1.28)$$

where P denotes porosity. With the vector notation

$$u = (u_x,\ u_y,\ u_z)$$

$$U = (U_x,\ U_y,\ U_z)$$

$$N\nabla^2 u + \mathrm{grad}((A+N)e + B\epsilon) = \frac{\partial^2}{\partial t^2}\,(\rho_{11}u + \rho_{12}U) \qquad (1.29a)$$

$$\mathrm{grad}(Be + C\epsilon) = \frac{\partial^2}{\partial t^2}\,(\rho_{12}u + \rho_{22}U) \qquad (1.29b)$$

where div u = e, div U = ϵ, and ρ_{11}, ρ_{12}, ρ_{22} are the mass coefficients which account for the fact that the relative fluid flow through the pores is not uniform. Applying the divergence operation to equations (1.29), one obtains

$$\nabla^2((A+2N)e + B\epsilon) = \frac{\partial^2}{\partial t^2}(\rho_{11}e + \rho_{12}\epsilon) \tag{1.30a}$$

$$\nabla^2(Be + C\epsilon) = \frac{\partial^2}{\partial t^2}(\rho_{12}e + \rho_{22}\epsilon) \tag{1.30b}$$

These two equations govern the propagation of dilatational waves which involve coupled motion in the fluid and the solid. Similarly, applying the curl operation to equations (1.24) one obtains

$$\frac{\partial^2}{\partial t^2}(\rho_{11}\omega + \rho_{12}\Omega) = N\nabla\omega \tag{1.31a}$$

$$\frac{\partial^2}{\partial t^2}(\rho_{12}\omega + \rho_{22}\Omega) = 0 \tag{1.31b}$$

where

$$\text{curl } u = \omega, \quad \text{curl } U = \Omega .$$

These equations govern the propagation of pure rotational waves. But there is only one type of rotational wave because equations (1.31) reduce to

$$N\nabla^2\omega = \rho_{11}\left(1 - \frac{\rho_{12}^2}{\rho_{11}\rho_{22}}\right)\frac{\partial^2\omega}{\partial t^2} \tag{1.32a}$$

$$\Omega = -\frac{\rho_{12}}{\rho_{22}}\,\omega \tag{1.32b}$$

An additional result found in [7] is that there is possibly a wave such that no relative motion occurs between the fluid and solid when a certain relation is satisfied between the elastic and dynamic constants.

Another type of mixture theory was considered by Truesdell and Toupin [8]. In this theory the concept of superimposed continua is introduced; i.e., it is assumed that the neighborhood of each point of a material is occupied by all members of the mixture. We define the density of the mixture to be the sum of the individual densities of each constituent. The velocity of the mixture is defined by the requirement that the mass flow of the mixture is the sum of the individual mass flows. Then the position of each particle of the mixture is defined by an integration of the velocity of the mixture; but such particles, in general, bear no simple relation to the particles of the constituents.

The main results of the work of Truesdell and Toupin are the following.

(a) The mass of the mixture satisfies an equation of continuity if the mass supply of the mixture is zero. This equation of continuity is precisely that found for ordinary continuum mechanics.

(b) Let the total stress of the mixture be defined as the sum of partial stresses plus the stresses arising from diffusion. Then a necessary and sufficient condition that Cauchy's first law holds for the mixture is that momentum supplied by unbalanced inertial forces of the several constituents plus momentum supplied through the creation of constituent diffusing masses shall add up to zero.

(c) We define the internal energy of the mixture is the sum of the internal energies of the constituents plus the kinetic energies of diffusion. This definition leads to the fact that the energy supplied by an excess internal energy rate, plus the energy supplied by the work of the excess inertial forces against diffusion, plus the energy due to mass supply, must add up to zero for the mixture.

Truesdell and Toupin's work on the mixture theory was incomplete in spite of the above results. However, their work inspired a number of researchers who have since made these theories more complete. For example, Adkins [9], [10], [11], Green and Adkins [12] among others have given discussions concerning nonlinear constitutive equations. Kelly [13] has extended this work to include electromagnetic effects while others have accounted for chemically reacting mixtures.

Recently, conceptually more simplified theories have been developed. The basic equations of mass and momentum balance in these theories are equivalent to those proposed by Truesdell and Toupin. These theories have led to the formulation of linearized equations governing thermomechanical disturbances. Since the present work is within the framework of these theories, these theories will be reviewed.

CHAPTER II. THEORY OF INTERACTING CONTINUA

2.1. Nonlinear Theory

For simplicity, attention is confined to two constituent continua in the theory [14]. We consider a mixture of two continua s_1 and s_2 which are in relative motion to each other. We will agree to call s_1 a solid and s_2 a fluid. We assume that each point within the mixture is occupied simultaneously by s_1 and s_2, and refer the motion of the continua to a fixed system of rectangular cartesian axes. The position of a typical particle of s_1 at time τ is denoted by $x_i(\tau)$, where

$$x_i(\tau) = x_i(X_1, X_2, X_3, \tau) \qquad (-\infty < \tau \leq t), \tag{2.1}$$

X_A is a reference position of the particle, and lower and upper case Latin indices take the values 1,2,3. We use the notation

$$x_i = x_i(t) \tag{2.2}$$

and can express (2.1) in the alternative form

$$x_i(\tau) = x_i(x_1, x_2, x_3, t, \tau) \tag{2.3}$$

where $$\left|\frac{\partial x_i(\tau)}{\partial X_A}\right| > 0, \quad \left|\frac{\partial x_i(\tau)}{\partial x_j}\right| > 0. \tag{2.4}$$

Similarly, for a typical particle of s_2, we have

$$y_i(\tau) = y_i(Y_1, Y_2, Y_3, \tau), \quad y_i(t) = y_i \qquad (-\infty < \tau \leq t) \tag{2.5}$$

or

$$y_i(\tau) = y_i(y_1, y_2, y_3, t, \tau) \tag{2.6}$$

together with

$$\left|\frac{\partial y_i(\tau)}{\partial Y_A}\right| > 0 \qquad \left|\frac{\partial y_i(\tau)}{\partial y_j}\right| > 0 \, . \tag{2.7}$$

We assume that the particles under consideration occupy the same position at time t so that

$$y_i = x_i \tag{2.8}$$

Velocity vectors at the point $x_i = y_i$ in s_1 and s_2 at time t are

$$u_i = \frac{\overset{(1)}{D}\, x_i}{Dt}, \qquad v_i = \frac{\overset{(2)}{D}\, y_i}{Dt} \tag{2.9}$$

where $D^{(1)}/Dt$ denotes differention with respect to t holding X_j fixed in continuum s_1 and $D^{(2)}/Dt$ denotes a similar operator for s_2, holding Y_j fixed. These operators may also be written as

$$\frac{D^{(1)}}{Dt} = \frac{\partial}{\partial t} + u_m \frac{\partial}{\partial x_m}, \quad \frac{D^{(2)}}{Dt} = \frac{\partial}{\partial t} + v_m \frac{\partial}{\partial y_m} \, . \tag{2.10}$$

Acceleration vectors at time t are denoted by $\hat{\hat{f}}_i$ and $\hat{g}_i$, where

$$\hat{f}_i = \frac{D^{(1)}u_i}{Dt}, \quad \hat{g}_i = \frac{D^{(2)}v_i}{Dt}. \tag{2.11}$$

The densities of s_1 and s_2 at time t are, respectively, ρ_1 and ρ_2, and the rate of deformation tensors at time t are defined to be

$$2d_{ij} = u_{i,j} + u_{j,i}, \quad 2f_{ij} = v_{i,j} + v_{j,i} \tag{2.12}$$

where a comma denotes partial differentiation with respect to x_k or y_k. We also define a mean velocity w_i be the equation

$$w_i = \rho_1 u_i + \rho_2 v_i, \quad \rho = \rho_1 + \rho_2 \tag{2.13}$$

and put

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + w_m \frac{\partial}{\partial x_m}. \tag{2.14}$$

It then follows that

$$\rho_1 \frac{D^{(1)}}{Dt} + \rho_2 \frac{D^{(2)}}{Dt} = \rho \frac{D}{Dt}. \tag{2.15}$$

Let ∂B be an arbitrary fixed closed surface enclosing a volume B and let n_k be the outward unit normal to ∂B. Let U be the internal energy per unit mass of the mixture. The externally applied body forces per unit masses of s_1 and s_2 are denoted, respectively, by the vectors F_i and G_i. And these vectors are defined through their rate of work contributions $F_i u_i$ and $G_i v_i$ for arbitrary velocity fields u_i and v_i. The surface force vector t_i per unit area of ∂B is such that the scalar $t_i u_i$, for arbitrary u_i, is a rate of work per unit area of ∂B. And a similar definition can be made for the vector p_i associated with the velocity vector v_i. The scalar r is the heat supply function per unit mass of the two continua due to radiation from the external world and heat sources. The flux of heat across ∂B is denoted by a scalar h per unit area and unit time.

<u>Theorem (Green and Naghdi)</u>

Let us postulate an energy balance at time t in the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_V [(\rho_1+\rho_2)U + \tfrac{1}{2}\rho_1 u_i u_i + \tfrac{1}{2}\rho_2 v_i v_i]\,dV \\
&+ \int_A [n_k(\rho_1 u_k + \rho_2 v_k)U + \tfrac{1}{2}\rho_1 n_k u_k u_i u_i + \tfrac{1}{2}\rho_2 n_k v_k v_i v_i]\,dA \\
&= \int_V (\rho r + \rho_1 F_i u_i + \rho_2 G_i v_i)\,dV \\
&+ \int_A (t_i u_i + p_i v_i)\,dA - \int_A h\,dA
\end{aligned} \tag{2.16}$$

Then it follows that

$$\frac{D\rho}{Dt} + \rho w_{k,k} = 0 \tag{2.17}$$

which states that mass elements of the mixtures are conserved. The vectors t_i, p_i are defined with reference to an arbitrary surface A. When the surface at a point x_i is perpendicular to the x_k-axis, we denote the corresponding values by σ_{ki}, π_{ki} and refer to these as stresses. Then it also follows from (2.16) that

$$(\sigma_{ki} + \pi_{ki})_{,k} + \rho_1 F_i + \rho_2 G_i =$$

$$\frac{\partial}{\partial t}(\rho_1 u_i) + \frac{\partial}{\partial t}(\rho_2 v_i) + \frac{\partial}{\partial x_k}(\rho_1 u_i u_k + \rho_2 v_i v_k) \tag{2.18}$$

which is the equation of motion.

Now let the mixture be composed of an elastic solid and a non-Newtonian viscous fluid. The kinematic quantities entering into the theory for the solid are the velocity $\mathbf{u}$, a strain tensor e, a rate of deformation tensor d and a vorticity tensor Γ; and, for the fluid, the velocity $\mathbf{v}$, a rate of deformation tensor f and a vorticity tensor Λ. In addition to the body forces previously defined we have for the solid constituent the following mechanical quantities: a partial stress tensor σ for the solid and a similar tensor π for the fluid. Due to the interaction of the two constituents, the theory gives rise to a diffusive resistance vector ω. The thermodynamic quantities, referring to the mixture as a whole, are the temperature T, the specific entropy S, the specific

Helmholtz free energy A, the heat flux vector q and the heat supply function r. For such a mixture the constitutive equations are given [15] in canonical form as

$$A = \tilde{A}(e, \rho_2, T) \tag{2.19a}$$

$$S = \tilde{S}(e, \rho_2, T) \tag{2.19b}$$

$$q = \tilde{q}(e, u-v, \operatorname{grad} T, \rho_2, T) \tag{2.19c}$$

$$\begin{aligned} \omega &= \tilde{\omega}^{s}(\operatorname{grad} T, \operatorname{grad} \rho_2, e, \rho_2, T) \\ &\quad + \tilde{\omega}^{e}(e, d, f, \Gamma-\Lambda, u-v, \rho_2, T) \end{aligned} \tag{2.19d}$$

$$\sigma = \tilde{\sigma}(e, d, f, \Gamma-\Lambda, u-v, \rho_2, T) \tag{2.19e}$$

$$\pi = \tilde{\pi}(e, d, f, \Gamma-\Lambda, u-v, \rho_2, T) \tag{2.19f}$$

Further restrictions upon equations (2.19) arise from a general principle of invariance under superposed rigid body motions and a material invariance associated with the assumed isotropy of the solid constituent [16,17]. It has been shown that in order to arrive at a determinate linearized theory it is sufficient to adjoin to the linearized forms of the field equations (2.18) a system of linearized forms of the constitutive equations (2.19). We now examine the linearized theory more closely.

2.2. Linearized Theory

Assume that the mixture undergoes a disturbance in which: (a) the material points of the solid constituent are displaced by only small amounts from their positions in an equilibrium state of the mixture in which the densities of the solid and fluid constituents and the temperature have the uniform values $\bar{\rho}_1$, $\bar{\rho}_2$, and $\bar{T}$, respectively and (b) the speed of the fluid constituent is small. We refer the points with respect to a fixed system of rectangular cartesian coordinates.

We consider a thermodynamic process in which the motions of the solid and fluid constituents of the mixture and the temperature field T each admit power series representations in terms of a positive real number ϵ. We choose ϵ to be a measure of the extent to which the mixture departs from some reference state. As our reference configuration we take the equilibrium state of the mixture in which

$$x = X = Y, \quad T = \overline{T} \tag{2.20}$$

and

$$F = G = 0, \quad r = 0 \tag{2.21}$$

If $a(x,t)$ is a field quantity, we denote its ϵ-expansion by $a_\epsilon(x,t)$; thus

$$a_\epsilon(x,t) = \sum_{n=1}^{\infty} \epsilon^n a_\epsilon^{(n)}(x,t). \tag{2.22}$$

We assume the following ϵ-expansions

$$x = X + w_\epsilon(x,t), \quad y = Y + z_\epsilon(x,t), \quad T = \overline{T} + \theta_\epsilon(x,t). \tag{2.23}$$

From equations (2.9) ϵ-expansions for the velocity vectors $\mathbf{u}$ are

$$u = u_\epsilon(x,t), \quad v = v_\epsilon(x,t). \tag{2.24}$$

We suppose that each of the expansions (2.23) and (2.24) is absolutely convergent with an interval of convergence $0 \le \epsilon < \epsilon_o$. The linearization of equations (2.9) to (2.15) is obtained by replacing each of the variables by the ϵ-expansions given in (2.22) to (2.24). Without further details it follows that the linearized displacement-strain relations for the solid are

$$u_i = \frac{\partial w_i}{\partial t}, \quad e_{ij} = w_{(i,j)}{}^*. \tag{2.25}$$

From (2.17) the individual continuity equations become

$$\rho_1 = \bar{\rho}_1(1 - e_{pp}), \quad \frac{\partial \rho_2}{\partial t} + \bar{\rho}_2 f_{pp} = 0. \tag{2.26}$$

The vorticity components of the two constituents are given in terms of the velocity components by

$$\Gamma_{ij} = u_{[i,j]}, \quad \Lambda_{ij} = v_{[i,j]}. \tag{2.27}$$

The application of the principle of invariance under superposed rigid body motions of the mixture to equation (2.18) leads to the following equations of motion:

$$\sigma_{pi,p} - \omega_i + \rho_1 F_i = \rho_1 \hat{\hat{f}}_i, \quad \pi_{pi,p} + \omega_i + \rho_2 G_i = \rho_2 \hat{\hat{g}}_i \tag{2.28}$$

On entering the ϵ-expansions for the various field

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*We use the notation $A_{(i,j)} = \frac{1}{2}(A_{i,j} + A_{j,i})$, $A_{[i,j]} = \frac{1}{2}(A_{i,j} - A_{j,i})$.

quantities into equations (2.16), (2.17), (2.26) and (2.28) and equating the coefficients of ϵ, we obtain the linearized field equations for the mixture as follows.

Equations of motion:

$$\sigma_{pi,p} - \omega_i + \bar{\rho}_1 F_i = \bar{\rho}_1 \frac{\partial u_i}{\partial t} \quad \text{in } B^{o*}, \ t \geq 0 \tag{2.29}$$

$$\pi_{pi,p} + \omega_i + \bar{\rho}_2 G_i = \bar{\rho}_2 \frac{\partial v_i}{\partial t} \quad \text{in } B^{o}, \ t \geq 0 \tag{2.30}$$

Energy equation:

$$\begin{aligned} & -\bar{\rho}(\bar{T}\frac{\partial S}{\partial t} + \bar{S}\frac{\partial T}{\partial t} + \frac{\partial A}{\partial t}) + \bar{\omega}_p(u_p - v_p) + \bar{\sigma}_{(pq)} d_{pq} \\ & + \bar{\pi}_{(pq)} f_{pq} + \bar{\sigma}_{[pq]}(\Gamma_{qp} - \Lambda_{qp}) - q_{p,p} + \bar{\rho} r = 0, \\ & \qquad \text{in } B, \ t \geq 0. \end{aligned} \tag{2.31}$$

In (2.31), $\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2$ is the total initial density of the mixture and $\bar{\omega}_i$, $\bar{\sigma}_{ij}$, $\bar{\pi}_{ij}$ are the diffusive resistance and the partial stress tensors in the equilibrium state. The linearized constitutive equations obtained from (2.19) are

$$\bar{\rho} A = \bar{\rho}\,\bar{A} + \alpha_1 e_{pp} + \alpha_2(\rho_2 - \bar{\rho}_2) + \alpha_3(T - \bar{T}) \tag{2.32}$$

$$\bar{\rho} S = -(\alpha_3 + \alpha_9 e_{pp} + \alpha_{10}(\rho_2 - \bar{\rho}_2) + \alpha_7(T - \bar{T})) \tag{2.33}$$

$$q_i = -kT_{,i} - K'(u_i - v_i) \tag{2.34}$$

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*See notations used in Theorem on Page 31.

$$w_i = -\frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 e_{pp,i} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \rho_{2,i} + \alpha(u_i - v_i)$$

$$+ a'' \epsilon_{ipq}(\Gamma_{pq} - \Lambda_{pq}) \tag{2.35}$$

$$\sigma_{(ij)} = (\alpha_1 - (\frac{\bar{\rho}_1}{\bar{\rho}} \alpha_1 - \alpha_3) e_{pp} + (\frac{\bar{\rho}_1}{\bar{\rho}} \alpha_1 + \alpha_6)(\rho_2 - \bar{\rho}_2) + \alpha_9 (T - \bar{T})^{*}$$

$$+ \lambda_1 d_{pp} + \lambda_3 f_{pp}) \delta_{ij} + 2(\alpha_1 + \alpha_5) e_{ij}$$

$$+ 2\mu_1 d_{ij} + 2\mu_3 f_{ij} \tag{2.36}$$

$$\pi_{(ij)} = (-\bar{\rho}_2 \alpha_2 + \bar{\rho}_2 (\frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 - \alpha_8) e_{pp} - (\frac{\bar{\rho}+\bar{\rho}_2}{\bar{\rho}} \alpha_2 + \bar{\rho}_2 \alpha_6)(\rho_2 - \bar{\rho}_2)$$

$$- \bar{\rho}_2 \alpha_{10} (T - \bar{T}) + \lambda_4 d_{pp} + \lambda f_{pp}) \delta_{ij} + 2\mu_4 d_{ij} + 2\mu f_{ij} \tag{2.37}$$

$$\sigma_{[ij]} = -\pi_{[ij]} = D\epsilon_{ijp}(u_p - v_p) - D''(\Gamma_{ij} - \Lambda_{ij}) \tag{2.38}$$

We note that there are total of 24 constants $\alpha_1, \ldots, \alpha_{10}$, $\lambda_1, \lambda, \lambda_3, \lambda_4, \mu_1, \mu, \mu_3, \mu_4$; $\alpha, a'', D, D'', k, K'$ which have to be determined by an experiment for the mixture. The entropy-production inequality [14] imposes restrictions upon the constitutive equations which, in the linearized theory, require the material constants to satisfy the following inequalities:

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*We use the notation $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$, $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$.

$$3\lambda_1 + 2\mu_1 \geq 0, \quad \mu_1 \geq 0, \quad 3\lambda + 2\mu \geq 0, \mu \geq 0, (\mu_3 + \mu_4)^2 \leq 4\mu_1\mu,$$

$$(3\lambda_3 + 2\mu_3 + 3\lambda_4 + 2\mu_4)^2 \leq 4(3\lambda_1 + 2\mu_1)(3\lambda + 2\mu) \qquad (2.39)$$

$$\alpha \geq 0, \; D'' \geq 0, (a'' - D)^2 \leq 4\alpha D'', k \geq 0, K'^2 \leq 4\bar{T}\alpha k$$

This linearized theory is well posed in the sense that the number of the field and constitutive equations equals the number of field quantities to be determined. We would expect that the boundary conditions for the initial-boundary value problems for the mixture are similar to the classical boundary conditions for the elasticity problem in which the stresses, strain, and displacements are sought. We recall that the classical boundary conditions are: (a) the forces may be given on the surface of the body, (b) the displacements may be given on the surface of the body, (c) the forces may be given on some portions of the body surface, while the displacements are given on the other portions. Indeed, the proper form of the initial and boundary conditions which should be adjoined to the field and constitutive equations of the linearized theory of interacting continua so that the sufficiently smooth solutions of the field and constitutive equations are determined uniquely are quite similar to the classical boundary conditions of the elasticity except that we have the temperature terms and we have to specify the boundary conditions to each component of the mixture. These conditions are specified in the following theorem.

Theorem (Atkin, Chadwick, Steel) [18]

Let B be a bounded regular region of three-dimensional Euclidean space occupied by a mixture of an elastic solid and a viscous fluid undergoing a disturbance of small amplitude during the time interval $t \geq 0$. We denote by ∂B the boundary and by B^0 the interior of B. We use notation ∂B_1, ∂B_2 and $\partial\bar{B}_1$, $\partial\bar{B}_2$ for arbitrary subsets of ∂B and their complements with respect to ∂B and n refers to the unit outward normal vector field on ∂B. Suppose that the constants $\lambda_1, \mu_1, \lambda, \lambda_3, \mu_3, \lambda_4, \mu_4, \alpha, \mu, a'', D, D''$, k and K' satisfy the conditions (2.39) and that $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ and α_8 satisfy the inequalities

$$\alpha_1+\alpha_5 \geq 0,\ \frac{2}{\rho}\alpha_2+\alpha_6 \geq 0, \alpha_7 \leq 0,\ \alpha_1\left(\frac{2\bar{\rho}_2}{\bar{\rho}}-\frac{1}{3}\right)+\alpha_4+\frac{2}{3}\alpha_5 \geq 0,$$

$$\left(\frac{1}{\rho}\alpha_1-\frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2+\alpha_8\right)^2 \leq \left(\alpha_1\left(\frac{2\bar{\rho}_2}{\bar{\rho}}-\frac{1}{3}\right)+\alpha_4+\frac{2}{3}\alpha_5\right)\left(\frac{2}{\rho}\alpha_6+\alpha_6\right) \tag{2.40}$$

Then there exists at most one set of functions v_i, ρ_2 of class C^1 and w_i, T of class C^2 which satisfy equations (2.25), (2.26), (2.27), (2.29), (2.30), (2.31), (2.34) to (2.38) and the subsidiary conditions

$$w_i = \hat{w}_i,\ u_i = \hat{u}_i,\ v_i = \hat{v}_i,\ \rho_2 = \bar{\rho}_2 + \hat{\rho}_2,\ T = \bar{T}+\hat{T}$$

$$\text{on } B \text{ at } t = 0, \tag{2.41}$$

$$u_i - v_i = R_i, \ (\sigma_{pi} + \pi_{pi})n_p = \Sigma_i \ \text{ on } \ \partial B_1 ,$$
$$u_i = U_i, \ v_i = V_i \ \text{ on } \ \partial \overline{B}_1 \ \text{ for } \ t \geq 0 , \tag{2.42}$$

$$T = \overline{T} + \Theta \text{ on } \partial B_2, \ q_p n_p = F \text{ on } \partial \overline{B}_2 \text{ for } t \geq 0 , \tag{2.43}$$

where $\hat{w}_i$, $\hat{u}_i$, $\hat{v}_i$, $\hat{\rho}_2$, $\hat{T}$, R_i, Σ_i, U_i, V_i, Θ, F and F_i, G_i, r are prescribed functions on the appropriate domains and $\overline{\rho}_1$, $\overline{\rho}_2$, $\overline{T}$ are given, strictly positive, constants.

It is well known that for the dynamical motions of linear isotropic elastic solid, the displacement vector $\mathbf{w}$ may be represented as a sum of two components representing motions of dilatational and rotational types, i.e.,

$$\mathbf{w} = \text{grad } \varphi + \text{curl } \psi ,$$

where φ and ψ satisfy the wave equations in which appear the speeds of propagation of dilatational and rotational body waves respectively. This representation is known to be complete in the sense that every sufficiently smooth solution $\mathbf{w}$ of the equation of motion of linear isotropic elastic solid is expressible in the stated form where the scalar and vector functions φ, ψ satisfy the above mentioned wave equations and in addition, div $\psi = 0$.

For the motions of the interacting continua of an elastic solid and a viscous fluid, Atkin [18] has established a decomposition of its motions into components representing motions of dilational and rotational types. Each part of the decomposition is somewhat simpler in form than the original system of the differential equations (2.26), (2.29) to (2.31) and (2.35) to (2.38), but considerably more complicated than the wave equation. The main merit of this new formulation for the motions of the interacting continua is that it allows the investigation of the propagation of small amplitude plane waves in a non-heat conducting mixture of an isotropic solid and an inviscid fluid. We define new material constants by the following combinations of the material constants.

$$K_1 = \alpha_4 + \frac{2}{3}\alpha_5 + \alpha_1\left(2\,\frac{\bar{\rho}_2}{\bar{\rho}} - \frac{1}{3}\right), \quad K_2 = \bar{\rho}_2^{\,2}\left(\alpha_6 + \frac{2}{\bar{\rho}}\,\alpha_2\right),$$

$$K_3 = -\bar{\rho}_2\left(\alpha_8 + \frac{1}{\bar{\rho}}\,\alpha_1 - \frac{\bar{\rho}_1}{\bar{\rho}}\,\alpha_2\right), \quad G_1 = \alpha_1 + \alpha_5 \tag{2.44}$$

$$B_1 = -\bar{T}\alpha_9, \quad B_2 = \bar{\rho}_2\bar{T}\alpha_{10}, \quad \beta = D - a'', \quad c_d = -\frac{\bar{T}\alpha_7}{\bar{\rho}}$$

Introducing the vector differential operator

$$\mathbb{L}[\xi, \eta] = \xi \text{ grad div} - \eta \text{ curl}^2 \tag{2.45}$$

and supposing that body forces and heat sources are absent, the governing equations take the form

$$\dot{\rho}_2 + \overline{\rho}_2 \operatorname{div} \mathbf{v} = 0^{*} \tag{2.46a}$$

$$\mathbb{L}[\lambda_1 + 2\mu_1,\ \mu_1 + \tfrac{1}{8}D'']\dot{\mathbf{w}} + \mathbb{L}[\lambda_3 + 2\mu_3,\ \mu_3 - \tfrac{1}{8}D'']\mathbf{v}$$

$$- (\alpha + \beta \operatorname{curl})(\dot{\mathbf{w}} - \mathbf{v}) + \mathbb{L}[K_1 + \tfrac{4}{3}G_1, G_1]\mathbf{w}$$

$$- \operatorname{grad}(K_3\rho_2/\overline{\rho}_2 + B_1\theta/\overline{T}) = \overline{\rho}_1\ddot{\mathbf{w}} \tag{2.46b}$$

$$\mathbb{L}[\lambda_4 + 2\mu_4,\ \mu_4 - \tfrac{1}{8}D'']\dot{\mathbf{w}} + \mathbb{L}[\lambda_2 + 2\mu_2,\ \mu_2 + \tfrac{1}{8}D'']\mathbf{v}$$

$$+ (\alpha + \beta \operatorname{curl})(\dot{\mathbf{w}} - \mathbf{v}) + \mathbb{L}[K_3, 0]\mathbf{w}$$

$$- \operatorname{grad}(K_2\ \rho_2/\overline{\rho}_2 + B_2\theta/\overline{T}) = \overline{\rho}_2\dot{\mathbf{v}} \tag{2.46c}$$

$$\overline{\rho}\, c_d\, \dot{\theta} + \operatorname{div}(B_1\dot{\mathbf{w}} + B_2\mathbf{v} - K'(\dot{\mathbf{w}} - \mathbf{v})) = k\, \nabla^2\theta \tag{2.46d}$$

Equations (2.46) contain twenty material constants of which nine, the λ's, μ's and D'', may be regarded as viscosity coefficients, the three K's as bulk moduli, G_1 as a shear modulus, B_1 and B_2 as products of bulk moduli. Of the remaining five constants, K' is associated with the transfer of heat in the mixture due to the relative motion of its constituents, k is the thermal conductivity, c_d is the specific heat at constant deformation, α, β arise from the interaction of the two constituents through the diffusive resistance and antisymmetric parts of the partial stress tensors.

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*A dot above a variable means partial derivative of the variable with respect to time.

Suppose now that there are scalar functions $\varphi_1(x,t)$ and $\varphi_2(x,t)$ which satisfy the equations

$$(\lambda_1 + 2\mu_1)\nabla^2\dot{\varphi}_1 + (K_1 + \frac{4}{3}G_1)\nabla^2\varphi_1 + (\lambda_3 + 2\mu_3)\nabla^2\varphi_2 - \alpha(\dot{\varphi}_1 - \varphi_2)$$
$$-K_3(\rho_2/\overline{\rho}_2 - 1) - B_1\theta/\overline{T} - \overline{\rho}_1\ddot{\varphi}_1 = 0 \tag{2.47a}$$

$$(\lambda_4 + 2\mu_4)\nabla^2\dot{\varphi}_1 + K_3\nabla^2\varphi_1 + (\lambda_2 + 2\mu_2)\nabla^2\varphi_2 + \alpha(\dot{\varphi}_1 - \varphi_2)$$
$$-K_2(\rho_2/\overline{\rho}_2 - 1) - B_2\theta/\overline{T} - \overline{\rho}_2\dot{\varphi}_2 = 0 \tag{2.47b}$$

$$k\nabla^2\theta - (B_1 - K')\nabla^2\dot{\varphi}_1 - (B_2 + K')\nabla^2\varphi_2 - \overline{\rho}\,c_d\,\dot{\theta} = 0 \tag{2.47c}$$

and

$$\dot{\rho}_2 + \overline{\rho}_2\nabla^2\varphi_2 = 0 \tag{2.48}$$

and vector functions $\psi_1(x,t)$ and $\psi_2(x,t)$ satisfying the equations

$$(\mu_1 + \tfrac{1}{2}D'')\nabla^2\dot{\psi}_1 + G_1\nabla^2\psi_1 + (\mu_3 - \tfrac{1}{2}D'')\nabla^2\psi_2$$
$$-(\alpha + \beta\ \mathrm{curl})(\dot{\psi}_1 - \psi_2) - \overline{\rho}_1\ddot{\psi}_1 = 0 \tag{2.49a}$$

$$(\mu_4 - \tfrac{1}{2}D'')\nabla^2\dot{\psi}_1 + (\mu_2 + \tfrac{1}{2}D'')\nabla^2\psi_2$$
$$+(\alpha + \beta\ \mathrm{curl})(\dot{\psi}_1 - \psi_2) - \overline{\rho}_2\dot{\psi}_2 = 0 \tag{2.49b}$$

If the functions $\varphi_i(x,t)$, $\psi_i(x,t)$ are sufficiently differentiable, then $w(x,t)$, $v(x,t)$, given by

$$w = \text{grad } \varphi_1 + \text{curl } \psi_1, \quad v = \text{grad } \varphi_2 + \text{curl } \psi_2, \tag{2.50}$$

$\rho_2(x,t)$, and $\theta(x,t)$ constitute a solution of (2.46). Now the converse question: if the functions $\rho_2(x,t)$, $w(x,t)$, $v(x,t)$, $\theta(x,t)$ satisfy equations (2.46), are there scalar and vector functions $\varphi_i(x,t)$ satisfying equations (2.47, 2.48, 2.49) such that (2.50) hold? The answer is given affirmatively, and the representation (2.50) is complete. These results are given in the following theorem:

Theorem (Atkin)
Let $\rho_2(x,t)$ and $\theta(x,t)$ be scalar functions which are twice continuously differentiable on B and let $w(x,t)$ and $v(x,t)$ be vector functions whose fourth and third partial derivatives respectively are Hölder continuous on B, the four functions together satisfying equations (2.46) in B. Then there exist scalar and vector functions $\varphi_i(x,t)$, $\psi_i(x,t)$ which satisfy equations (2.47),(2.48), (2.49) such that $w(x,t)$ and $v(x,t)$ admit the representations (2.50) in B. Moreover, it is possible to choose the vector functions $\psi_i(x,t)$ so that the dilatational conditions

$$\text{div } \psi_1 = 0$$

are satisfied in B.

With the aid of this theorem, the propagation of small amplitude waves in a non-heat-conducting mixture of an isotropic elastic solid and an inviscid fluid was studied by Atkin [19]. Equating to zero the viscosity coefficients λ_j, μ_j (j=1, 3, 4), D" and the thermal conductivity k, then differentiating each term of equations (2.47a) and (2.47b) with respect to t and eliminating ρ_2 and θ by means of (2.47c) and (2.48), one obtains

$$c_1^2 \nabla^2 \dot{\varphi}_1 + c_3^2 \nabla^2 \varphi_2 - a(1-f)(\ddot{\varphi}_1 - \dot{\varphi}_2) = \dddot{\varphi}_1 \tag{2.51a}$$

$$c_4^2 \nabla^2 \dot{\varphi}_1 + c_2^2 \nabla^2 \varphi_2 + af(\ddot{\varphi}_1 - \dot{\varphi}_2) = \ddot{\varphi}_2 \ . \tag{2.51b}$$

By the same process from equations (2.49a) and (2.49b), one obtains

$$\nu^2 \nabla^2 \psi_1 - a(1-f)(\dot{\psi}_1 - \psi_2) = \ddot{\psi}_1 \tag{2.51c}$$

$$af(\dot{\psi}_1 - \psi_2) = \dot{\psi}_2 \tag{2.51d}$$

where $f = \overline{\rho}_1 / \overline{\rho}_2$ is the fractional contribution of the solid constituent to the mass of the mixture and

$$\begin{aligned} c_1^2 &= (K_1 + 4G_1/3 + B_1^2 / \overline{\rho}\, \overline{T} c_d) / \overline{\rho}_1 \ , \\ c_2^2 &= (K_2 + B_2^2 / \overline{\rho}\, \overline{T} c_d) / \overline{\rho}_2 \\ c_3^2 &= (K_3 + B_1 B_2 / \overline{\rho}\, \overline{T} c_d) / \overline{\rho}_1 , \quad c_4^2 = (K_3 + B_1 B_2 / \overline{\rho}\, \overline{T} c_d) / \overline{\rho}_2 \\ \nu^2 &= G_1 / \overline{\rho}_1 \ . \end{aligned} \tag{2.51e}$$

Equations (2.51a) and (2.51b) govern the propagation of small amplitude waves of dilatational type, that is, motions of the mixture in which the vectors w and v are irrotational, and equations (2.51c) and (2.51d) describe motions of rotational type in which these vectors are solenoidal. Thus by the theorem (Atkin) all sufficiently regular motions of the mixture can be decomposed into dilatational and rotational components. From equations (2.51a) and (2.51b) one may obtain

$$((v_{P1}^2 \nabla^2 - \frac{\partial^2}{\partial t^2})(v_{P2}^2 \nabla^2 - \frac{\partial^2}{\partial t^2})$$

$$- a \frac{\partial}{\partial t}(v_{P3}^2 \nabla^2 - \frac{\partial^2}{\partial t^2}))(\varphi_1, \varphi_2) = 0 \qquad (2.52)$$

where

$$v_{P1} = \frac{1}{\sqrt{2}}(c_1^2 + c_2^2 + ((c_1^2 - c_2^2)^2 + 4c_3^2c_4^2)^{\frac{1}{2}})^{\frac{1}{2}},$$

$$v_{P2} = \frac{1}{\sqrt{2}}(c_1^2 + c_2^2 - ((c_1^2 - c_2^2)^2 + 4c_3^2c_4^2)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$v_{P3} = (f(c_1^2 + c_3^2) + (1-f)(c_2^2 + c_2^4))^{\frac{1}{2}}$$

It has been shown that if v_{P1} and v_{P2} are real, v_{P3} is also real and the three dilatational wave speeds satisfy the inequalities

$$v_{P2} \leq v_{P3} \leq v_{P1} \qquad (2.53)$$

The form of equation (2.52) suggests that at high frequencies there are two modes of dilatational wave propagation associated with the speeds v_{p1}, v_{p2}, while at low frequencies there is only one mode of wave propagation, associated with the speed v_{p3}, the second mode being a diffused disturbance.

From equations (2.51c) and (2.51d) one may obtain

$$\left(\frac{\partial}{\partial t}\left(\nu^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right) + a\left(f\nu^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\right)\psi_1 = 0 \qquad (2.54)$$

The form of equation (2.54) suggests that rotational disturbances of the mixture comprise a single mode which has a wave-like character at all frequencies, the speed of propagation being of $f^{\frac{1}{2}}\nu$ in the limit when frequencies approach to zero and ν in the limit when frequencies approach to infinity.

So far we have reviewed the recent developments of the interacting continua of an isotropic elastic solid and a viscous fluid. Due to the complication of the system, comparatively little progress has so far been made concerning the application of the linearized theory to particular physical situations, or the properties and understandings of the character of the system of the partial differential equations.

2.3. Summary of the Equations and Other Formulations

A. Fully Coupled Mixture Theory

At this point we summarize the pertinent equations which govern the motion of a thermally and mechanically interacting continuous mixture according to the theory expounded by Atkin, Chadwick and Steel [18]. We call this problem by the name "fully coupled mixture theory."

strain-displacement equations

$$e_{ij} = w_{(i,j)} \tag{2.25}$$

rate of deformation-velocity equations

$$d_{ij} = u_{(i,j)}, \quad f_{ij} = v_{(i,j)} \tag{2.12}$$

vorticity-velocity equations

$$\Gamma_{ij} = u_{[i,j]}, \quad \Lambda_{ij} = v_{[i,j]} \tag{2.27}$$

equations of motion

$$\sigma_{ij,i} - \omega_j + \bar{\rho}_1 F_j = \bar{\rho}_1 \frac{\partial u_j}{\partial t}, \tag{2.29}$$

$$\pi_{ij,i} + \omega_j + \bar{\rho}_2 G_j = \bar{\rho}_2 \frac{\partial v_j}{\partial t} \tag{2.30}$$

continuity equations

$$\rho_1 = \bar{\rho}_1 (1 - e_{kk}), \quad \frac{\partial \rho_2}{\partial t} + \bar{\rho}_2 f_{kk} = 0. \tag{2.26}$$

energy equations

$$\alpha_7 \frac{\partial T}{\partial t} + \alpha_9 d_{mm} - \bar{\rho}_2 \alpha_{10} f_{mm} + \frac{\bar{\rho}}{\bar{T}} r$$

$$+ \frac{k}{\bar{T}} T,_{mm} + \frac{K}{\bar{T}}(u_{m,m} - v_{m,m}) = 0 \; * \qquad (2.31)$$

constitutive equations

$$\sigma_{(ij)} = [\alpha_1 - (\frac{\bar{\rho}_1}{\bar{\rho}} \alpha_1 - \alpha_4) e_{kk} + (\frac{\alpha_1}{\bar{\rho}} + \alpha_8)(\rho_2 - \bar{\rho}_2)$$

$$+ \alpha_9 (T - \bar{T}) + \lambda_1 d_{kk} + \lambda_3 f_{kk}] \delta_{ij}$$

$$+ 2(\alpha_1 + \alpha_5) e_{ij} + 2\mu_1 d_{ij} + 2\mu_3 f_{ij} , \qquad (2.35)$$

$$\pi_{(ij)} = [-\bar{\rho}_2 \alpha_2 + \bar{\rho}_2 (\frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 - \alpha_8) e_{kk} - \bar{\rho}_2 \alpha_{10} (T - \bar{T})$$

$$- (\frac{\bar{\rho} + \bar{\rho}_2}{\bar{\rho}} \alpha_2 + \bar{\rho}_2 \alpha_6)(\rho_2 - \bar{\rho}_2) + \lambda_4 d_{kk} + \lambda f_{kk}] \delta_{ij}$$

$$+ 2\mu_4 d_{ij} + 2\mu f_{ij} , \qquad (2.36)$$

$$\omega_i = -\frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 e_{kk,i} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \, \rho_{2,i} + \alpha (u_i - v_i) \qquad (2.37)$$

$$+ a'' \, \epsilon_{ipq} (\Gamma_{pq} - \Lambda_{pq})$$

$$\sigma_{[ij]} = -\pi_{[ij]} = D \, \epsilon_{ijp} (u_p - v_p) - D'' (\Gamma_{ij} - \Lambda_{ij}) \qquad (2.38)$$

The complete initial-boundary value problem is specified by the above equations and: the initial conditions

---

*Obtained from (2.31) by using (2.32) to (2.38) in (2.31). Usually the form so obtained is called heat equation.

(2.41), the boundary conditions (2.42), (2.45), and the material inequalities (2.39), (2.40). The problem is solved if one can obtain at each place x_i and $t > 0$ the functions w_i, v_i, ρ_1, ρ_2 and T.

B. The Mixture Theory of Green and Steel.

In a series of papers by Green and Steel [20], Green and Naghdi [14], Steel [21], a more tractable initial-boundary value problem than the fully coupled mixture theory of section 2.3 A has been presented. The major difference between the linearized version of the theory presented in [15] and that of A lies in the constitutive relations. Green and Steel's relations follow from A if one sets equal to zero

$$\lambda_2,\ \lambda_3,\ \lambda_4,\ \mu_1,\ \mu_3,\ \mu_4, \qquad (2.55)$$

$$a'',\ D,\ D''.$$

C. Uncoupled Theory.

By the term uncoupled theory we shall mean the initial-boundary value problem as specified in section 2.3A using the constitutive equations of Green and Steel presented in section 2.3B and in addition, neglecting the time rate of change of the dilatational effects of the solid and fluid components in the energy equation. If this is done, then the heat equation is uncoupled from the equations of motion and the temperature may be treated as a known function of space and time.

2.4. Single Constituent Theories.

So that we can make a comparison of the mixture theories presented in the previous sections with the classical theories of elasticity and viscous fluids we record here the changes that must be effected. Our purpose is two fold: it allows us to draw upon the vast literature available in the classical theories of single constituent continuous media and it allows us to give meaningful intrepretation to the mechanical and thermal material properties used in the mixture theory of subsections 2.3.

A. Linear Thermoelasticity

If the fluid component is not present then the body may be interpreted as a linear elastic solid undergoing thermal deformation in which the variation in temperature is small [22],[25],[24]. One seeks to obtain the components of displacement w_i and the temperature T which satisfy equations (2.25), (2.26), (2.29), in the absence of the fluid component and, with certain modifications of the heat and constitutive equations, satisfy (2.31) and (2.35). These modifications are formally equivalent to employing (2.55) and setting

$$\alpha, \alpha_1, \alpha_2, \alpha_6, \alpha_8, \alpha_{10}, \lambda, \mu, K' \tag{2.56}$$

equal to zero.

Equation (2.36) is then interpreted as the classical linear elastic stress-strain law if we identify

$$\alpha_5 = \mu_E, \alpha_4 = \lambda_E, \alpha_9 = -\gamma K_E \tag{2.57}$$

where μ_E, λ_E are the Lamé elastic constants, γ is the coefficient of linear thermal expansion and K_E is the isothermal bulk modulus, $K_E = \frac{1}{3}(2\mu_E + 3\lambda_E)$. From (2.31) the modified heat equation becomes

$$kT,_{mm} - \bar{\rho} c_e \frac{\partial T}{\partial t} - \gamma \bar{T} K_E \frac{\partial}{\partial t} e_{mm} + \bar{\rho} r = 0 \tag{2.58}$$

where $\bar{\rho}$ is the total density of the body, c_e the specific heat at constant strain. These coefficients are related to α_7 by

$$\alpha_7 = -\frac{\bar{\rho} c_e}{\bar{T}}. \tag{2.59}$$

The material inequalities (2.39) and (2.40) simplify to

$$k \geq 0, \ \mu_E \geq 0, \ c_e \geq 0, \ 2\mu_E + 3\lambda_E = 3K_E \geq 0. \tag{2.60}$$

A properly posed initial-boundary value problem of elasticity consists of finding w_i and T of class C^2 which satisfy the modified equations and the following initial and boundary data:

$$
\begin{aligned}
&w_i = \hat{w}_i,\ u_i = \hat{u}_i,\ T = \overline{T} + \hat{T} \quad \text{on } B \text{ at } t=0,\\
&\sigma_{ij}n_j = \Sigma_i \quad \text{on } \partial B_1,\\
&w_i = W_i \quad \text{on } \partial\overline{B}_1, \text{ for } t \geq 0, \qquad (2.61)\\
&T = \overline{T} + \Theta \quad \text{on } \partial B_2,\\
&q_k n_k = F \quad \text{on } \partial\overline{B}_2, \text{ for } t \geq 0
\end{aligned}
$$

where $\hat{w}_i$, $\hat{u}_i$, $\hat{T}$, Σ_i, W_i, Θ, F, and F_i are prescribed functions on the appropriate domains.

We close this subsection with the remark that if the term

$$-\gamma\, \overline{T}\, K_E \frac{\partial}{\partial t} e_{mm} \qquad (2.62)$$

is ignored in (2.58), then the resulting thermoelastic theory is known as the classical uncoupled thermoelastic theory. [22], [23]

B. Linear Viscous Fluid

If the solid component is not present then the body may be interpreted as a fluid undergoing thermal deformations in which the variation in temperature is small. One seeks to obtain the components of velocity v_i and the temperature T which satisfy equations (2.12), (2.27), (2.30), (2.26), in the absence of the solid component, and with certain modifications of the heat and constitutive equations (2.31), (2.36). These modifications are formally equivalent to employing (2.55) and setting

$$\alpha,\ \alpha_1,\ \alpha_4,\ \alpha_5,\ \alpha_8,\ \alpha_9 \tag{2.63}$$

equal to zero.

Equation (2.37) is then concerned with the unsteady linearized compressible flow about a state of rest of a heat-conducting viscous fluid, if we identify $\lambda,\ \mu,\ \bar\rho\,\alpha_2,\ \bar\rho^2\,(\frac{2}{\rho}\,\alpha_2 + \alpha_6),\ \bar\rho\,\alpha_{10}$ of which λ and μ are the coefficients of viscosity of the fluid, $\bar\rho\,\alpha_2$ is the pressure of the fluid in the rest state, $\bar\rho^2\,(\frac{2}{\rho}\,\alpha_2 + \alpha_6)$ is the isothermal bulk modulus and $\bar\rho\,\alpha_{10}$ is the product of the volume coefficient of thermal expansion and the isothermal bulk modulus.

From (2.31) the modified heat equation becomes

$$\bar\rho\, c_v \frac{\partial T}{\partial t} + \gamma K_T \bar T f_{pp} + kT,_{pp} + \bar\rho r = 0 \tag{2.64}$$

where c_v is the specific heat at constant volume, γ is the volume coefficient of thermal expansion, K_T is the isothermal bulk modulus, and k is the thermal conductivity. The coefficient c_v is related to α_7 by

$$c_v = -\frac{\bar T}{\bar\rho}\,\alpha_7 \ . \tag{2.65}$$

The material inequalities (2.39) and (2.40) simplify to

$3\lambda + 2\mu \geq 0$, $\mu \geq 0$, $k \geq 0$, $\frac{2}{\rho}\alpha_2 + \alpha_6 \geq 0$, $\alpha_7 \leq 0$

$K_T \geq 0$ and $C_v \geq 0$.

A properly posed initial-boundary value problem is then to find v_i, ρ of class C^1 and T of class C^2 which satisfy the modified equations and the following initial and boundary data:

$$
\begin{aligned}
& v_i = \hat{v}_i, \ \rho = \bar{\rho} + \hat{\rho}, \ T = \bar{T} + \hat{T} \ \text{ on } B \text{ at } t = 0 \\
& \pi_{pi} n_p = \pi_i \ \text{ on } \partial B_1, \\
& v_i = V_i \ \text{ on } \partial \bar{B}_1, \ \text{for } t \geq 0 \\
& T = \bar{T} + \Theta \ \text{ on } \partial B_2, \\
& q_k n_k = F \ \text{ on } \partial \bar{B}_2, \ \text{for } t \geq 0
\end{aligned}
\tag{2.66}
$$

where $\hat{v}_i$, $\hat{\rho}$, π_i, V_i, Θ, F G_i and r are prescribed functions on the appropriate domains.

CHAPTER III. RECIPROCITY THEOREMS

3.1. Introduction

Prior to considering the reciprocity relation in the mixture theory, we will review the well known reciprocity theorem of elasticity.

Suppose that an elastic body is subjected to two systems of body and surface forces. The work that would be done by the first system's body and surface forces in acting through the displacements due to the second system's forces is equal to the work that would be done by the second system's body and surface forces in acting through the displacements due to the first system of forces. Mathematically this is incorporated in the Betti-Rayleigh reciprocal theorem.

Theorem (Betti-Rayleigh) [25]

Consider two equilibrium states of an elastic body: one with displacements u_i due to the body forces F_i and surface forces T_i, and the other with displacements u_i' due to body forces F_i' and surface forces T_i'. Then it follows that

$$\int_{\partial B} T_i u_i' \, ds + \int_B F_i u_i' \, dv = \int_{\partial B} T_i' u_i \, ds + \int_B F_i' u_i \, dv .$$

A generalization of the reciprocity relation to dynamic problems is given as follows.

Theorem (Fung) [26]

Consider two problems where the applied body force and the surface tractions and displacements are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses such that the body force is $X_i^{(j)}(x_k;t)$, the specified surface traction is $f_i^{(j)}(x_k;t)$ on ∂B_1, and the specified displacement is $g_i^{(j)}(x_k;t)$ on $\partial \overline{B}_1$ where $j = 1,2$. Assuming that the action starts at $t > 0$ in each case, we have

$$\int_B \int_0^t X_i^{(1)}(x,t-y)u_i^{(2)}(x,y)\ dy\ dv$$

$$+ \int_{\partial B_1} \int_0^t f_i^{(1)}(x,t-y)u_i^{(2)}(x,y)\ dy\ ds$$

$$+ \int_{\partial \overline{B}_1} \int_0^t \sigma_{ij}^{(1)}(x,t-y)g_i^{(2)}(x,y)n_j\ dy\ ds$$

$$= \int_B \int_0^t X_i^{(2)}(x,t-y)u_i^{(1)}(x,y)\ dy\ dv$$

$$+ \int_{\partial B_1} \int_0^t f_i^{(2)}(x,t-y)u_i^{(1)}(x,y)\ dy\ ds$$

$$+ \int_{\partial \overline{B}} \int_0^t \sigma_{ij}^{(2)}(x,t-y)g_i^{(1)}(x,y)n_j\ dy\ ds$$

As an illustration of this theorem consider the following problems. By problem 1 let us mean the displacement and stress field in an infinite region that results due to the body force system

$$X_i^{(1)} = F_i^{(1)}\delta(p_1)\ \delta(t)$$

where p_i is a fixed point in the medium and $F_i^{(1)}$ refers to a force magnitude. For problem 2 let us find the displacement and stress field in the infinite region due to the traveling impulsive force system

$$X_i^{(2)} = F_i^{(2)} \delta(t - \frac{x_1}{U}) \delta(x_2) \delta(x_3) .$$

Then substituting into the reciprocal theorem and using the properties of the Dirâc delta function we find that

$$F_i^{(1)} u_i^{(2)} (p_1, t) \tag{3.1}$$

$$= F_i^{(2)} \iiint \delta(x_2) \delta(x_3) \; dx_1 \; dx_2 \; dx_3 \int_0^t \delta(y - \frac{x_1}{U}) u_i^{(1)} (x_1, x_2, x_3, t-y) \, dy$$

$$= F_i^{(2)} \int_{-\infty}^{x_1/U} u_i^{(1)} (x_1, 0, 0, t - \frac{x_1}{U}) \; dx_1 .$$

From this relation $u_i^{(2)} (p_1, t)$ can be found when $u_i^{(1)} (x_1, 0, 0, t - \frac{x_1}{U})$ is known. When $F_1^{(1)} = 1$ and $F_2^{(1)} = F_3^{(1)} = 0$, Payton [27] has determined $u_i^{(1)}$ by applying the Laplace and Hankel transforms to the elastic equations of motion. The solution is

$$u_1^{(1)} (x, t) = \frac{t}{4\pi R^2} G(x_1, R, t), \quad u_2^{(1)} (x, t) = \frac{x_1 \; x_2 \; t}{4\pi R^4} F(R, t)$$

$$u_3^{(1)} (x, t) = \frac{x_3 \; x_1 \; t}{4\pi R^4} F(R, t) \tag{3.2}$$

where

$$F(R,t) = \frac{3}{R}H(t-R/c_1) + \frac{1}{c_1}\delta(t-R/c_1) - \frac{3}{R}H(t-R/c_2) - \frac{1}{c_2}\delta(t-R/c_2)$$

$$G(x_1,R,t) = \left(\frac{3x^2}{R^2} - 1\right)\frac{1}{R}H(t-R/c_1) + \frac{x^2}{c_1R^2}\delta(t-R/c_1)$$

$$- \left(\frac{3x^2}{R^2} - 1\right)\frac{1}{R}H(t-R/c_2) - \left(\frac{x^2}{R^2} - 1\right)\frac{1}{c_2}\delta(t-R/c_2)$$

$R = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$; c_1 and c_2 are the speeds of the propagation of dilatational and equivoluminal waves.

Payton used the equations (3.1) and (3.2) to compute the $u_i^{(2)}$.

3.2. Reciprocal Relations for Mechanically and Thermally Interacting Mixture

We will investigate a reciprocity relation for the interacting continua of an elastic solid and linear viscous fluid using the theory derived in section 2.

We put

$$x_i = X_i + w_i, \quad \rho_2 = \bar{\rho}_2 + \eta, \quad T = \bar{T} + \theta \tag{3.3}$$

where X_i is a reference position at time $t = 0$, x_i is a position at time t, ρ_2 is the density of the fluid component at (x_i, t), T is the temperature at (x_i, t), $\bar{T}$ is the initial temperature; then from (2.26)

$$\frac{\partial \eta}{\partial t} + \bar{\rho}_2 \frac{\partial v_i}{\partial x_i} = 0 \tag{3.4}$$

all quantities now being regarded as functions of x_i and t. Since initially the medium is in equilibrium under zero total applied force it follows that

$$\alpha_1 = \bar{\rho}_2\alpha_2. \tag{3.5}$$

Let us consider the problems in which the body forces $F_i(X_i,t)$, $G_i(X_i,t)$, the specified surface tractions f_i, g_i, and the specified velocities u_i, v_i are given functions of time and space, respectively, for solid and fluid which starts its action at $t > 0$, with the initial conditions

$$w_i = \frac{\partial w_i}{\partial t} = 0\,,\quad v_i = 0, \eta = 0\,,\quad \theta = 0 \quad \text{for} \quad t \leq 0\,, \tag{3.6}$$

Let the Laplace transform of a function $u(x_k,t)$ be written as $\tilde{u}(x_k,P)$ where

$$\tilde{u}(x_k,P) = \int_0^\infty e^{-Pt}\, u(x_k,t)\,dt\,.$$

We apply the Laplace transform with respect to the time t to every dependent variable. From (2.29), (2.30), (2.35), (2.36), (2.37), (2.55), (3.4) and (3.6) we obtain

$$\tilde{\sigma}_{pi,p} - \tilde{\omega}_i + \bar{\rho}_1\tilde{F}_i = \bar{\rho}_1 P\tilde{u}_i \tag{3.7a}$$

$$\tilde{\pi}_{pi,p} + \tilde{\omega}_i + \bar{\rho}_2\tilde{G}_i = \bar{\rho}_2 P\tilde{v}_i \tag{3.7b}$$

$$\tilde{\omega}_i = \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}}\,\tilde{\eta}_{,i} - \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}}\,\tilde{e}_{mm,i} + \alpha(P\tilde{w}_i - \tilde{v}_i) \tag{3.7c}$$

$$\tilde{\sigma}_{ik} = \frac{\alpha_1}{P}\,\delta ik + \left(\alpha_4 - \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}}\right)\tilde{e}_{mm}\delta_{ik} + 2(\alpha_1+\alpha_5)\tilde{e}_{ik} + \left(\alpha_8 + \frac{\alpha_1}{\bar{\rho}}\right)\tilde{\eta}\,\delta_{ik}$$
$$+ \alpha_9\,\tilde{\theta}\,\delta_{ik} \tag{3.7d}$$

To complete the problem let us consider the boundary ∂B as the sum of the disjoint sets ∂B_1 and $\partial \overline{B}_1$ or as the sum of the sets (disjoint) ∂B_2 and $\partial \overline{B}_2$. We specify for $t \geq 0$,

$$u_i - v_i = R_i \quad \text{on} \quad \partial B_1 \tag{3.8a}$$

$$(\sigma_{ij} + \pi_{ij})n_i = \Sigma_i \quad \text{on} \quad \partial B_1 \tag{3.8b}$$

$$u_i = U_i, v_i = V_i \quad \text{on} \quad \partial \overline{B}_1 \tag{3.8c}$$

$$T = \overline{T} + \Theta \quad \text{on} \quad \partial B_2 \tag{3.8d}$$

$$q_p n_p = F \quad \text{on} \quad \partial \overline{B}_2 \tag{3.8e}$$

To aid in the computations we introduce the following combinations of material constants:

$$\beta_2 = \alpha_4 - \frac{\overline{\rho}_1 \alpha_1}{\overline{\rho}}, \quad \beta_1 = \alpha_8 + \frac{\alpha_1}{\rho} \tag{3.9a}$$

$$\beta_3 = \alpha_1 + \alpha_5, \quad \gamma_1 = \overline{\rho}_2 \alpha_6 + \frac{\overline{\rho}_2 + \overline{\rho}}{\overline{\rho}} \alpha_2, \tag{3.9b}$$

$$\gamma_2 = \overline{\rho}_2 \left(-\alpha_8 + \frac{\overline{\rho}_1 \alpha_2}{\overline{\rho}}\right) \tag{3.9c}$$

Now consider two problems specified by equations (2.26), (2.29), (2.30), (2.31), (2.35), (2.36), (2.37), (3.6) and (3.8). We identify one problem by a superscript one on all field variables and a second problem by a superscript two on all field variables. For example $u_i^{(1)}$, $v_i^{(1)}$, $\theta^{(1)}$ will satisfy (3.6), (3.7) and (3.8) for $R_i^{(1)}$, $\Sigma_i^{(1)}$, $U_i^{(1)}$, $V_i^{(1)}$, $\Theta^{(1)}$, $F^{(1)}$, while $u_i^{(2)}$, $v_i^{(2)}$, $\theta^{(2)}$ are solutions of (3.6), (3.7), (3.8) for different $R_i^{(2)}$, $\Sigma_i^{(2)}$, $U_i^{(2)}$, $V_i^{(2)}$, $\Theta^{(2)}$, $F^{(2)}$.

To derive the reciprocal theorem we begin with the equations of motion and the constitutive equations to which we have applied the Laplace transform and used the initial conditions as specified in equation (3.6). For the sake of convenience these are

$$\tilde{\sigma}^{(j)}_{pi,p} - \tilde{\omega}^{(j)}_{i} + \bar{\rho}_1 \tilde{F}^{(j)}_{i} = \bar{\rho}_1 p \tilde{u}^{(j)}_{i} \tag{3.10a}$$

$$\tilde{\pi}^{(j)}_{pi,p} + \tilde{\omega}^{(j)}_{i} + \bar{\rho}_2 G^{(j)}_{i} = \bar{\rho}_2 p \tilde{v}^{(j)}_{i} \tag{3.10b}$$

$$\tilde{\omega}^{(j)}_{i} = \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \, \tilde{\eta}^{(j)}_{,i} - \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \, \tilde{e}^{(j)}_{mm,i} + \alpha (p \tilde{w}^{(j)}_{i} - \tilde{v}^{(j)}_{i}) \tag{3.10c}$$

$$\tilde{\sigma}^{(j)}_{ik} = \frac{\alpha_1}{p} \delta_{ik} + \beta_2 \tilde{e}^{(j)}_{mm} \delta_{ik} + 2\beta_3 \tilde{e}^{(j)}_{ik} + \beta_1 \tilde{\eta}^{(j)} \delta_{ik} + \alpha_9 \tilde{\theta}^{(j)} \delta_{ik} \tag{3.10d}$$

$$\tilde{\pi}^{(j)}_{ik} = [- \frac{\bar{\rho}_2}{p} \alpha_2 - \gamma_1 \tilde{\eta}^{(j)} + \gamma_2 \tilde{e}^{(j)}_{mm} - \bar{\rho}_2 \alpha_{10} \tilde{\theta}^{(j)}] \delta_{ik} + \lambda \tilde{f}^{(j)}_{rr} \delta_{ik}$$
$$+ 2\mu \tilde{f}^{(j)}_{ik} \tag{3.10e}$$

for $j = 1,2$.

Multiplying equation (3.10a) for $j = 1$ by $\tilde{u}^{(2)}_{i}$ and again for $j = 2$ by $\tilde{u}^{(1)}_{i}$, subtracting these two results and then integrating over the region B, we obtain,

$$\int_B \tilde{u}^{(2)}_{i} \, \tilde{\sigma}^{(1)}_{pi,p} \, dv + \int_B \tilde{u}^{(2)}_{i} \, \bar{\rho}_1 \, \tilde{F}^{(1)}_{i} dv$$
$$- \int_B \tilde{u}^{(2)}_{i} \, \tilde{\omega}^{(1)}_{i} \, dv \tag{3.11}$$
$$= \int_B \tilde{u}^{(1)}_{i} \, \tilde{\sigma}^{(2)}_{pi,p} \, dv + \int_B \tilde{u}^{(1)}_{i} \, \bar{\rho}_1 \, \tilde{F}^{(2)}_{i} dv$$
$$- \int_B \tilde{u}^{(1)}_{i} \, \tilde{\omega}^{(2)}_{i} \, dv$$

where dv is the element of volume of B.

Consider the first integral on the left and let us apply the divergence theorem to it. In this way we obtain

$$\int_B \tilde{u}_i^{(2)} \tilde{\sigma}_{pi,p}^{(1)} dv = \int_{\partial B} \tilde{u}_i^{(2)} \tilde{\sigma}_{pi}^{(1)} n_p ds - \int_B \tilde{u}_{i,p}^{(2)} \tilde{\sigma}_{pi}^{(1)} dv \qquad (3.12)$$

where ds is the element of surface area of ∂B. We note that the first integral on the ri t of (3.11) can be manipulated in the same way and the re lt is the same as (3.12) with the superscripts interchanged.

Consider now the third integral on the left. From the constitutive equation (3.10c) we may write

$$\int_B \tilde{u}_i^{(2)} \tilde{\omega}_i^{(1)} dv = \int_B P\tilde{w}_i^{(2)} \alpha(P\tilde{w}_i^{(1)} - \tilde{v}_i^{(1)}) dv$$

$$+ \int_{\partial B} P\tilde{w}^{(2)} [-\frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(1)}] n_i ds$$

$$- \int_B P\tilde{w}_{i,i}^{(2)} [-\frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(1)}] dv \qquad (3.13)$$

A similar result is obtained after the superscripts are exchanged. Using the constitutive equation (3.10d), we have for the second integral on the right of equation (3.12)

$$\int_B \tilde{u}_{i,p}^{(2)} \tilde{\sigma}_{pi}^{(1)} dv = \int_B P\tilde{w}_{i,p}^{(2)} \tilde{\sigma}_{pi}^{(1)} dv \qquad (3.14)$$

$$= \int_B P\tilde{e}_{ip}^{(2)} \tilde{\sigma}_{pi}^{(1)} dv$$

$$= \int P\tilde{e}_{ip}^{(2)} [\beta_2 \tilde{e}_{mm}^{(1)} \delta_{pi} + 2\beta_3 \tilde{e}_{pi}^{(1)}] dv$$

$$+ \int_B P\tilde{e}_{ip}^{(2)} [\frac{\alpha_1}{P} \delta_{pi} + \beta_1 \tilde{\eta}^{(1)} \delta_{pi} + \alpha_9 \tilde{\theta}^{(1)} \delta_{pi}] dv$$

A similar result is obtained for the middle integral on the right of equation (3.12) after the superscripts are interchanged. When the equations (3.12), (3.13), (3.14) along with the equations after the superscripts are interchanged are substituted into equation (3.11), we have

$$\int_B \tilde{u}_i^{(2)} \bar{\rho}_1 \tilde{F}_i^{(1)} dv + \int_{\partial B} \tilde{u}_i^{(2)} \tilde{\sigma}_{pi}^{(1)} n_p ds + \int_B P\alpha\, \tilde{w}_i^{(2)} \tilde{v}_i^{(1)} dv \quad (3.15)$$

$$- \int_{\partial B} P\tilde{w}_i^{(2)} [- \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(1)}] n_i ds + \int_B P\tilde{e}_{mm}^{(2)} \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(1)} dv$$

$$- \int_B P\tilde{e}_{ip}^{(2)} [\frac{\alpha_1}{P} \delta_{pi} + \beta_1 \tilde{\eta}^{(1)} \delta_{pi} + \alpha_9 \tilde{\theta}^{(1)} \delta_{pi}] dv$$

$$= \int_B \tilde{u}_i^{(1)} \bar{\rho}_1 \tilde{F}_i^{(2)} dv + \int_{\partial B} \tilde{u}_i^{(1)} \tilde{\sigma}_{pi}^{(2)} n_p ds + \int_B P\alpha \tilde{w}_i^{(1)} \tilde{v}_i^{(2)} dv$$

$$- \int_{\partial B} P\tilde{w}_i^{(1)} [- \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(2)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(2)}] n_i ds + \int_B P\tilde{e}_{mm}^{(1)} \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(2)} dv$$

$$- \int_B P\tilde{e}_{ip}^{(1)} [\frac{\alpha_1}{P} \delta_{pi} + \beta_1 \tilde{\eta}^{(2)} \delta_{pi} + \alpha_9 \tilde{\theta}^{(2)} \delta_{pi}] dv$$

By the same process used to derive (3.15) from (3.10a), we find that from (3.10b)

$$\int_B \tilde{v}_i^{(2)} \bar{\rho}_2 \tilde{G}_i^{(1)} dv + \int_{\partial B} \tilde{v}_i^{(2)} \tilde{\pi}_{pi}^{(1)} n_p ds + \int_B P\alpha\, \tilde{v}_i^{(2)} \tilde{w}_i^{(1)} dv$$

$$+ \int_{\partial B} \tilde{v}_i^{(2)} [- \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(1)}] n_i ds + \int_B \tilde{v}_{i,i}^{(2)} \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} dv$$

$$- \int_B \tilde{v}_{i,p}^{(2)} [(- \frac{\bar{\rho}_2}{P} \alpha_2 + \gamma_2 \tilde{e}_{mm}^{(1)} - \bar{\rho}_2 \alpha_{10} \tilde{\theta}^{(1)}) \delta_{pi}] dv$$

$$= \int_B \tilde{v}_i^{(1)} \bar{\rho}_2 \tilde{G}_i^{(2)} dv + \int_{\partial B} \tilde{v}_i^{(1)} \tilde{\pi}_{pi}^{(2)} n_p ds + \int_B P\alpha \tilde{v}_i^{(1)} \tilde{w}_i^{(2)} dv$$

$$+ \int_{\partial B} \tilde{v}_i^{(1)} [- \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(2)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(2)}] n_i ds + \int_B \tilde{v}_{i,i}^{(1)} \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(2)} dv$$

$$- \int_B \tilde{v}_{i,p}^{(1)} [- \frac{\bar{\rho}_2}{P} \alpha_2 + \gamma_2 \tilde{e}_{mm}^{(2)} - \bar{\rho}_2 \alpha_{10} \tilde{\theta}^{(2)} \delta_{pi}] dv . \qquad (3.16)$$

Before we add (3.15) and (3.16), we notice that the following volume integrals

$$\int_B \tilde{v}_{i,i}^{(2)} \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} dv + \int_B P \tilde{\eta}^{(1)} \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}} \tilde{e}_{mm}^{(2)} dv$$

$$- \int_B \tilde{v}_{i,p}^{(2)} (- \frac{\tilde{\rho}_2}{P} \alpha_2 + \gamma_2 \tilde{e}_{mm}^{(1)} - \tilde{\rho}_2 \alpha_{10} \tilde{\theta}^{(1)}) \delta_{pi} dv$$

$$- \int_B P \tilde{e}_{ip}^{(2)} [\frac{\alpha_1}{P} \delta_{pi} + \beta_1 \tilde{\eta}^{(1)} \delta_{pi} + \alpha_9 \tilde{\theta}^{(1)} \delta_{pi}] dv$$

may be recast by using (3.7) and (3.9) as

$$\int_B P \tilde{\eta}^{(2)} [(-\alpha_8 + \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}} - \frac{\alpha_1}{\bar{\rho}}) \tilde{e}_{mm}^{(1)} - \frac{\alpha_2}{P} - \alpha_{10} \tilde{\theta}^{(1)}] dv \qquad (3.17)$$

$$+ \int_B P \tilde{e}_{mm}^{(2)} [(-\alpha_8 + \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}} - \frac{\alpha_1}{\bar{\rho}}) \tilde{\eta}^{(1)} - \frac{\alpha_1}{P} - \alpha_9 \tilde{\theta}^{(1)}] dv .$$

With the aid of (3.17) the sum of (3.15) and (3.16) can be written as

$$(3.18)$$

$$\int_B \tilde{v}_i^{(2)} \bar{\rho}_2 \tilde{G}_i^{(1)} dv + \int_B \tilde{u}_i^{(2)} \bar{\rho}_1 \tilde{F}_i^{(1)} dv + \int_{\partial B} \tilde{v}_i^{(2)} \tilde{\pi}_{pi}^{(1)} n_p ds + \int_{\partial B} \tilde{u}_i^{(2)} \tilde{\sigma}_{pi}^{(1)} n_p ds$$

$$+ \int_{\partial B} (\tilde{v}_i^{(2)} - P \tilde{w}_i^{(2)}) (- \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(1)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(1)}) n_i ds$$

$$+ \int_{\partial B} (\frac{\bar{\rho}_2 \alpha_2}{P} \tilde{v}_m^{(2)} - \alpha_1 \tilde{w}_m^{(2)}) n_m ds + \int_B \tilde{\theta}^{(1)} (\bar{\rho}_2 \alpha_{10} \tilde{f}_{rr}^{(2)} - \alpha_9 P \tilde{e}_{mm}^{(2)}) dv$$

$$= \int_B \tilde{v}_i^{(1)} \bar{\rho}_2 \tilde{G}_i^{(2)} dv + \int_B \tilde{u}_i^{(1)} \bar{\rho}_1 \tilde{F}_i^{(2)} dv + \int_{\partial B} \tilde{v}_i^{(1)} \tilde{\pi}_{pi}^{(2)} n_p ds$$

$$+ \int_{\partial B} \tilde{u}_i^{(1)} \tilde{\sigma}_{pi}^{(2)} n_p ds$$

$$+ \int_{\partial B} (\tilde{v}_i^{(1)} - P\tilde{w}_i^{(1)}) \left(- \frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 \tilde{e}_{mm}^{(2)} + \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2 \tilde{\eta}^{(2)}\right) n_i ds$$

$$+ \int_{\partial B} \left(\frac{\bar{\rho}_2 \alpha_2}{P} \tilde{v}_m^{(1)} - \alpha_1 \tilde{w}_m^{(1)} \right) n_m ds + \int_B \tilde{\theta}^{(2)} (\bar{\rho}_2 \alpha_{10} \tilde{f}_{rr}^{(1)} - \alpha_9 P \tilde{e}_{mm}^{(1)}) dv.$$

Let the flux of heat and heat supply function be incorporated into (3.18). The energy equation (2.31) reduces to

$$\alpha_7 \frac{\partial \theta}{\partial t} + \alpha_9 d_{pp} - \bar{\rho}_2 \alpha_{10} f_{pp} - \frac{1}{\bar{T}} q_{p,p} + \frac{\bar{\rho}}{\bar{T}} r = 0 . \qquad (3.19)$$

Application of Laplace transform to (3.19) leads to

$$P\alpha_7 \tilde{\theta} + P\alpha_9 \tilde{e}_{mm} - \bar{\rho}_2 \alpha_{10} \tilde{f}_{pp} - \frac{1}{\bar{T}} \tilde{q}_{p,p} + \frac{\bar{\rho}}{\bar{T}} r = 0 . \qquad (3.20)$$

Consider the last integral on the left in (3.18). Using (3.20) it becomes

$$\int_B \tilde{\theta}^{(1)} \left(P\alpha_7 \tilde{\theta}^{(2)} - \frac{1}{\bar{T}} \tilde{q}_{p,p}^{(2)} + \frac{\bar{\rho}}{\bar{T}} \tilde{r}^{(2)}\right) dv$$

and if we now apply the divergence theorem to the $\tilde{q}_{p,p}^{(2)} \tilde{\theta}^{(1)}$ term we may write (using (2.34) and (2.39))

$$\int_B P\alpha_7 \tilde{\theta}^{(1)} \tilde{\theta}^{(2)} dv + \frac{\bar{\rho}}{\bar{T}} \int_B \tilde{\theta}^{(1)} \tilde{r}^{(2)} dv - \frac{1}{\bar{T}} \int_{\partial B} \tilde{\theta}^{(1)} \tilde{q}_p^{(2)} n_p ds$$

$$- \frac{k}{\bar{T}} \int_B \tilde{\theta}_{,p}^{(1)} \tilde{\theta}_{,p}^{(2)} dv - \frac{K'}{\bar{T}} \int_B \tilde{\theta}_{,p}^{(1)} (\tilde{u}_p^{(2)} - \tilde{v}_p^{(2)}) dv . \qquad (3.21)$$

We note that a similar expression can be obtained for the last integral of (3.18) by interchanging indices.

Substituting (3.21) into (3.18) and employing the transformed boundary conditions (3.8) leads to the general reciprocal theorem in the Laplace transformed state. Before giving this expression we introduce one additional condition. We set

$$\Sigma_i^{(j)} = f_i^{(j)} + g_i^{(j)} \tag{3.22a}$$

and require

$$\sigma_{pi} n_p = f_i^{(j)} \tag{3.22b}$$

$$\pi_{pi} n_p = g_i^{(j)} \tag{3.22c}$$

on the boundary, ∂B_1. This introduction somewhat simplifies the notation but it must be recognized that only the total stress vector Σ_i is specified on ∂B_1.

Thus, by (3.8), (3.18), (3.21) and (3.22) we have

$$\int_B \tilde{v}_i^{(2)} \bar{\rho}_2 \tilde{G}_i^{(1)} dv + \int_B \tilde{u}_i^{(2)} \bar{\rho}_1 \tilde{F}_i^{(1)} dv + \int_{\partial B_1} \tilde{v}_i^{(2)} \tilde{g}_i^{(1)} ds + \int_{\partial B_1} \tilde{u}_i^{(2)} \tilde{f}_i^{(1)} ds$$

$$+ \int_{\partial \bar{B}_1} \tilde{V}_i^{(2)} \tilde{\pi}_{pi}^{(1)} n_p ds + \int_{\partial \bar{B}_1} \tilde{U}_i^{(2)} \tilde{\sigma}_{pi}^{(1)} n_p ds$$

$$+ \int_{\partial B_1} (-\tilde{R}_i^{(2)}) [-\frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 (\frac{\bar{\rho}_1}{p} - \tilde{\rho}_1^{(1)}) / \bar{\rho}_1 + \bar{\rho}_1 \alpha_2 (\tilde{\rho}_2^{(1)} - \frac{\bar{\rho}_2}{p}) / \bar{\rho}] n_i ds$$

$$+ \int_{\partial \bar{B}_1} (\tilde{V}_i^{(2)} - p\tilde{W}_i^{(2)}) [-\frac{\bar{\rho}_2}{\bar{\rho}} \alpha_1 (\frac{\bar{\rho}_1}{p} - \tilde{\rho}_1^{(1)}) / \bar{\rho}_1 + \bar{\rho}_1 \alpha_2 (\tilde{\rho}_2^{(1)} - \frac{\bar{\rho}_2}{p}) / \bar{\rho}] n_i ds$$

$$+\int_{\partial B_1}\left(\frac{\bar\rho_2\alpha_2}{P}\right)(-\tilde R_m^{(2)})n_m ds + \int_{\partial\bar B_1}\left(\frac{\bar\rho_2\alpha_2}{P}\tilde V_m^{(2)} - \alpha_1\tilde W_m^{(2)}\right)n_m\,ds$$

$$+\frac{\bar\rho}{\bar T}\int_B \tilde\theta^{(1)}\tilde r^{(2)}dv - \frac{1}{\bar T}\int_{\partial B_2}\tilde\Theta^{(1)}\tilde q_p^{(2)}n_p ds - \frac{1}{\bar T}\int_{\partial\bar B_2}\tilde\theta^{(1)}\tilde F^{(2)}ds$$

$$-\frac{K'}{\bar T}\int_B \tilde\theta^{(1)}_{,p}(\tilde u_p^{(2)} - \tilde v_p^{(2)})dv$$

$$=\int_B \tilde v_i^{(1)}\bar\rho_2\tilde G_i^{(2)}dv + \int_B \tilde u_i^{(1)}\bar\rho_1\tilde F_i^{(2)}dv + \int_{\partial B_1}\tilde v_i^{(1)}\tilde g_i^{(2)}ds + \int_{\partial B_1}\tilde u_i^{(1)}\tilde f_i^{(2)}ds$$

$$+\int_{\partial\bar B_1}\tilde V_i^{(1)}\tilde\pi_{pi}^{(2)}n_p ds + \int_{\partial\bar B_1}U_i^{(1)}\tilde\sigma_{pi}^{(2)}n_p ds$$

$$+\int_{\partial B_1}(-\tilde R_i^{(1)})\left[-\frac{\bar\rho_2}{\bar\rho}\alpha_1\left(\frac{\bar\rho_1}{P}-\tilde\rho_1^{(2)}\right)/\bar\rho_1 + \bar\rho_1\alpha_2\left(\tilde\rho_2^{(2)}-\frac{\bar\rho_2}{P}\right)/\bar\rho\right]n_i ds$$

$$+\int_{\partial\bar B_1}(\tilde V_i^{(1)} - P\tilde W_i^{(1)})\left[-\frac{\bar\rho_2}{\bar\rho}\alpha_1\left(\frac{\bar\rho_1}{P}-\tilde\rho_1^{(2)}\right)/\bar\rho_1 + \bar\rho_1\alpha_2\left(\tilde\rho_2^{(2)}-\frac{\bar\rho_2}{P}\right)/\bar\rho\right]n_i ds$$

$$+\int_{\partial B_1}\left(\frac{\bar\rho_2\alpha_2}{P}\right)(-\tilde R_m^{(1)})n_m ds + \int_{\partial\bar B_1}\left(\frac{\bar\rho_2\alpha_2}{P}\tilde V_m^{(1)} - \alpha_1\tilde W_m^{(1)}\right)n_m ds$$

$$+\frac{\rho}{\bar T}\int_B \tilde\theta^{(2)}\tilde r^{(1)}dv - \frac{1}{\bar T}\int_{\partial B_2}\tilde\Theta^{(2)}\tilde q_p^{(1)}n_p ds - \frac{1}{\bar T}\int_{\partial\bar B_2}\tilde\theta^{(2)}\tilde F^{(1)}ds$$

$$-\frac{K'}{\bar T}\int_B \tilde\theta^{(2)}_{,p}(\tilde u_p^{(1)} - \tilde v_p^{(1)})dv \qquad (3.23)$$

Since the inverse transform of the product of two functions is the convolution of the inverses, we obtain:

<u>Theorem (Reciprocity Relation for Heat Conducting mixture)</u>.

Let B be a bounded regular region of three-dimensional Euclidean space occupied by a mixture of elastic solid and a Newtonian viscous fluid undergoing a disturbance of small amplitude during the time interval $t \geq 0$. We denote by ∂B the boundary and by B^{O} the interior of B and introduce regions $\mathcal{B}$, $\mathcal{B}^{\circ}$ of space-time defined by

$$\mathcal{B} = \{(p,t): p \in B,\ t \geq 0\},\ \mathcal{B}^{\circ} = \{(p,t): p \in B^{O},\ t \geq 0\}.$$

We use notation ∂B_1, ∂B_2 and $\partial \overline{B}_1$, $\partial \overline{B}_2$ for arbitrary subsets of ∂B and their complements with respect to ∂B and n refers to the unit outward normal vector field on ∂B. Suppose that the constants λ, μ, α, k and K' satisfy $3\lambda + 2\mu \geq 0$, $\mu \geq 0$, $\alpha \geq 0$, $k \geq 0$, $K'^2 \leq 4\bar{T}\alpha K$, and that α_1, α_2, α_4, α_5, α_6, α_7 and α_8 satisfy the inequalities

$$\alpha_1 + \alpha_5 \geq 0,\ \frac{2}{\bar{\rho}}\alpha_2 + \alpha_6 \geq 0,\ \alpha_7 \leq 0,\ \alpha_1\left(\frac{2\bar{\rho}_2}{\bar{\rho}} - \frac{1}{3}\right) + \alpha_4 + \frac{2}{3}\alpha_5 \geq 0,\ \left(\frac{\lambda}{\bar{\rho}}\alpha_1 - \frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2 + \alpha_8\right)^2 \leq \left(\alpha_1\left(\frac{2\bar{\rho}_2}{\bar{\rho}} - \frac{1}{3}\right) + \alpha_4 + \frac{2}{3}\alpha_5\right)\left(\frac{2}{\bar{\rho}}\alpha_2 + \alpha_6\right).$$

Let the mixture of elastic solid and viscous fluid be subjected to two systems which are distinguished by superscripts in parentheses. Let the functions $v_i^{(j)}$, $\rho_2^{(j)}$, $u_i^{(j)}$ of class C^1 and $T^{(j)}$ of class C^2 on $\mathcal{B}$ which satisfy equations (2.26), (2.29), (2.30), (2.31) on $\mathcal{B}^{\circ}$, equations (2.25), (2.27) and (2.34) to (2.38) on $\mathcal{B}$ and the subsidiary conditions

$w_i = 0,\ u_i = 0,\ v_i = 0,\ \eta = 0,\ \theta = 0$ on B at $t = 0$,

$u_i^{(j)} - v_i^{(j)} = R_i^{(j)},\ \sigma_{pi}^{(j)} n_p = f_i^{(j)},\ \pi_{pi}^{(j)} n_p = g_i^{(j)}$,

$\sigma_{pi}^{(j)} n_p + \pi_{pi}^{(j)} n_p = \Sigma_i^{(j)*}$ on ∂B_1 for $t \geq 0$

$u_i = U_i^{(j)},\ v_i = V_i^{(j)}$ on $\partial \overline{B}_1$ for $t \geq 0$,

$T^{(j)} = \overline{T} + \Theta^{(j)}$ on ∂B_2, $q_p^{(j)} n_p = F^{(j)}$ on $\partial \overline{B}_2$ for $t \geq 0$,

where $R_i^{(j)}, f_i^{(j)}, g_i^{(j)}, U_i^{(j)}, V_i^{(j)}, \Theta^{(j)}, F^{(j)}, F_i^{(j)}, G_i^{(j)}$ and $r^{(j)}$

are prescribed functions on the appropriate domains and $\overline{\rho}_1, \overline{\rho}_2, \overline{T}$ are given, positive, constants. Then the work that would be done by the first system in acting through the velocities of the second system and the work that would be done by the second system in acting through the velocities of the first system satisfy

$$\overline{\rho}_2 \int_B \int_0^t v_i^{(2)}(x,t-y)\, G_i^{(1)}(x,y)\, dy\, dv + \overline{\rho}_1 \int_B \int_0^t u_i^{(2)}(x,t-y)\, F_i^{(1)}(x,y)\, dy\, dv$$

$$+ \int_{\partial B_1} \int_0^t v_i^{(2)}(x,t-y)\, g_i^{(1)}(x,y)\, dy\, ds + \int_{\partial B_1} \int_0^t u_i^{(2)}(x,t-y)\, f_i^{(1)}(x,y)\, dy\, ds$$

$$+ \int_{\partial \overline{B}_1} \int_0^t V_i^{(2)}(x,t-y)\, \pi_{pi}^{(1)}(x,y)\, n_p\, dy\, ds + \int_{\partial \overline{B}_1} \int_0^t U_i^{(2)}(x,t-y)\, \sigma_{pi}^{(1)}(x,y)\, n_p\, dy\, ds$$

$$- \int_{\partial B_1} \int_0^t R_i^{(2)}(x,t-y) \Big[\frac{-\overline{\rho}_2}{\overline{\rho}_1 \overline{\rho}} \alpha_1 (\overline{\rho}_1 - \rho_1^{(1)}(x,y))$$

$$+ \frac{\overline{\rho}_1 \alpha_2}{\overline{\rho}} (\rho_2^{(1)}(x,y) - \overline{\rho}_2) \Big] n_i\, dy\, ds +$$

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*Attention is called to the fact that only Σ_i is known on ∂B_1 and hence only the sum $f_i + g_i$ is given.

$$+ \int_{\partial \bar{B}_1} \int_0^t (V_i^{(2)}(x,t-y) - U_i^{(2)}(x,t-y))\Big[-\frac{\rho_2}{\bar{\rho}_1 \bar{\rho}}\alpha_1(\bar{\rho}_1 - \rho_1^{(1)}(x,y))$$

$$+ \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}}(\rho_2^{(1)}(x,y) - \bar{\rho}_2)\Big]n_i \, dy \, ds$$

$$- \int_{\partial B_1} \int_0^t \bar{\rho}_2 \alpha_2 R_m^{(2)}(x,t-y) n_m \, dy \, ds + \int_{\partial \bar{B}_1} \int_0^t [\bar{\rho}_2 \alpha_2 V_m^{(2)}(x,t-y)$$

$$- \alpha_1 W_m^{(2)}(x,t-y)] n_m \, dy \, ds$$

$$+ \frac{\bar{\rho}}{T} \int_B \int_0^t r^{(2)}(x,t-y)\,\theta^{(1)}(x,y) \, dy \, dv$$

$$- \frac{1}{T} \int_{\partial B_2} \int_0^t [q_p^{(2)}(x,t-y) \; \Theta^{(1)}(x,y)] n_p \, dy \, ds$$

$$- \frac{1}{T} \int_{\partial \bar{B}_2} \int_0^t F^{(2)}(x,t-y)\,\theta^{(1)}(x,y) \, dy \, ds - \frac{K'}{T} \int_B \int_0^t [u_p^{(2)}(x,t-y)$$

$$- v_p^{(2)}(x,t-y)]\,\theta^{(1)}(x,y) \, dy \, dv$$

$$= \bar{\rho}_2 \int_B \int_0^t v_i^{(1)}(x,t-y)\, G_i^{(2)}(x,y) \, dy \, dv$$

$$+ \bar{\rho}_1 \int_B \int_0^t u_i^{(1)}(x,t-y)\, F_i^{(2)}(x,y) \, dy \, dv$$

$$+ \int_{\partial B_1} \int_0^t v_i^{(1)}(x,t-y)\, g_i^{(2)}(x,y) \, dy \, ds$$

$$+ \int_{\partial B_i} \int_0^t u_i^{(1)}(x,t-y)\, f_i^{(2)}(x,y) \, dy \, ds$$

$$+ \int_{\partial \bar{B}_1} \int_0^t v_i^{(1)}(x,t-y)\,\pi_{pi}^{(2)}(x,y)\,n_p \; dy \; ds$$

$$+ \int_{\partial \bar{B}_1} \int_0^t U_i^{(1)}(x,t-y) \;\; \sigma_{pi}^{(2)}(x,y)\,n_p \; dy \; ds$$

$$- \int_{\partial \bar{B}_1} \int_0^t R_i^{(1)}(x,t-y[- \frac{\bar{\rho}_2}{\bar{\rho}_1\bar{\rho}} \, \alpha_1 (\bar{\rho}_1 - \rho_1^{(2)}(x,y))$$

$$+ \frac{\bar{\rho}_1}{\bar{\rho}} \;\; \alpha_2 \;\; (\rho_2^{(2)}(x,y) - \bar{\rho})] \; n_i \; dy \; ds$$

$$+ \int_{\partial \bar{B}_1} \int_0^t [(V_i^{(1)}(x,t-y) - U_i^{(1)}(x,t-y)][- \frac{\bar{\rho}_2}{\bar{\rho}_1\bar{\rho}} \, \alpha_1 (\bar{\rho}_1 - \rho_1^{(2)}(x,y))$$

$$+ \frac{\rho_1}{\bar{\rho}} \, \alpha_2 (\rho_2^{(2)}(x,y) - \bar{\rho}_2)] \; n_i \; dy \; ds$$

$$- \int_{\partial B_1} \int_0^t \bar{\rho}_2 \alpha_2 R_m^{(1)}(x,t-y) \;\; n_m \; dy \; ds$$

$$+ \int_{\partial \bar{B}_1} \int_0^t [\bar{\rho}_2 \alpha_2 V_m^{(1)}(x,t-y) \; - \; \alpha_1 W_m^{(1)}(x,t-y)] \; n_m \; dy \; ds$$

$$+ \frac{\bar{\rho}}{T} \int_B \int_0^t r^{(1)}(x,t-y)\,\theta^{(2)}(x,y)\,dy \; dv$$

$$- \frac{1}{T} \int_{\partial B_2} \int_0^t q_p^{(1)}(x,t-y)\,\Theta^{(2)}(x,y) \;\; n_p \; dy \; ds$$

$$- \frac{1}{T} \int_{\partial \bar{B}_2} \int_0^t F^{(1)}(x,t-y)\,\theta^{(2)}(x,y)\,dy \; ds \; - \; \frac{K'}{T} \int_B \int_0^t (u_p^{(1)}(x,t-y)$$

$$- v_p^{(1)}(x,t-y))\,\theta_{,p}^{(2)}(x,y) \;\; dy \; dv \qquad (3.24)$$

3.3. Special Cases

A. Reciprocity Relation for Heat Conducting Mixture of Linear Elastic Solid and Non-Newtonian Viscous Fluid

Let B be a bounded regular region of three-dimensional Euclidean space occupied by a mixture of elastic solid and a non-Newtonian viscous fluid undergoing a disturbance of small amplitude during the time interval $t \geq 0$. When the viscous fluid is non-Newtonian, the restriction (2.55) does not hold in general. Let the mixture of elastic solid and non-Newtonian viscous fluid be subjected to two systems which are distinguished by superscripts in parentheses, where each of these systems has same initial and boundary conditions stated in the previous theorem. Then the reciprocity relation is again an integral relation obtained by adding the following equality to (3.24).

$$- a'' \int_B \int_0^t u_i^{(2)}(x,t-y)\,\epsilon_{ipq}(\Gamma_{pq}^{(1)}(x,y) - \Lambda_{pq}^{(1)}(x,y))\,dy\,dv$$

$$- 2 \int_B \int_0^t d_{ip}^{(2)}(x,t-y)(\lambda_3 f_{pp}^{(1)}(x,y)\,\delta_{pi} + 2\,\mu_3 f_{pi}^{(1)}(x,y))\,dy\,dv$$

$$+ a'' \int_B \int_0^t v_i^{(2)}(x,t-y)\,\epsilon_{ipq}(\Gamma_{pq}^{(1)}(x,y) - \Lambda_{pq}^{(1)}(x,y))\,dy\,dv$$

$$- \int_B \int_0^t v_{i,p}^{(2)}(x,t-y)(\lambda_4 d_{pp}^{(1)}(x,y)\,\delta_{pi} + 2\mu_4 d_{pi}^{(1)}(x,y))\,dy\,dv$$

$$= - a'' \int_B \int_0^t u_i^{(1)}(x,t-y)\,\epsilon_{ipq}(\Gamma_{pq}^{(2)}(x,y) - \Lambda_{pq}^{(2)}(x,y))\,dy\,dv$$

$$- 2 \int_B \int_0^t d_{ip}^{(1)}(x,t-y)(\lambda_3 f_{pp}^{(2)}(x,y)\delta_{pi} + 2\mu_3 f_{pi}^{(2)}(x,y))\, dy\, dv$$

$$+ a' \int_B \int_0^t v_i^{(1)}(x,t-y)\epsilon_{ipq}(\Gamma^{(2)}(x,y) - \Lambda_{pq}^{(2)}(x,y))\, dy\, dv$$

$$- \int_B \int_0^t v_{i,p}^{(1)}(x,t-y)(\lambda_4 d_{pp}^{(2)}(x,y)\delta_{pi} + 2\mu_4 d_{pi}^{(2)}(x,y))\, dy\, dv \tag{3.25}$$

B. Reciprocity Relation for Mixture of Linear Elastic Solid and Newtonian Viscous Fluid in Isothermal Process

Let B be a bounded regular region of three-dimensional Euclidean space occupied by a non-heat conducting mixture of elastic solid and a Newtonian viscous fluid undergoing a disturbance of small amplitude during the time interval $t \geq 0$. Let the mixture be subjected to two systems which are distinguished by superscripts in parentheses, and let the process be isothermal. Moreover, except the temperature terms, we let each of these systems have same initial and boundary conditions stated in the previous theorem. Then the reciprocity relation is given as following:

$$\bar{\rho}_2 \int_B \int_0^t v_i^{(2)}(x,t-y) G_i^{(1)}(x,y)\, dy\, dv$$

$$+ \bar{\rho}_1 \int_B \int_0^t u_i^{(2)}(x,t-y)\; F_i^{(1)}(x,y)\, dy\, dv$$

$$+ \int_{\partial B_1} \int_0^t v_i^{(2)}(x,t-y) g_i^{(1)}(x,y)\, dy\, ds$$

$$+ \int_{\partial B_1} \int_0^t u_i^{(2)}(x,t-y)\; f_i^{(1)}(x,y)\, dy\, ds +$$

$$+ \int_{\partial\overline{B}_1} \int_0^t v_i^{(2)}(x,t-y)\pi_{pi}^{(1)}(x,y)n_p \, dy \, ds$$

$$+ \int_{\partial\overline{B}_1} \int_0^t u_i^{(2)}(x,t-y) \ \sigma_{pi}^{(1)}(x,y)n_p \, dy \, ds$$

$$- \int_{\partial B_1} \int_0^t R_i^{(2)}(x,t-y)\Big[\frac{-\overline{\rho}_2}{\overline{\rho}_1\overline{\rho}} \alpha_1(\overline{\rho}_1 - \rho_1^{(1)}(x,y))$$

$$+ \frac{\overline{\rho}_1\alpha_2}{\overline{\rho}} (\rho_2^{(1)}(x,y) - \overline{\rho}_2)\Big]n_i \, dy \, ds$$

$$+ \int_{\partial\overline{B}_1} \int_0^t (v_i^{(2)}(x,t-y) - u_i^{(2)}(x,t-y))\Big[- \frac{\overline{\rho}_2}{\overline{\rho}_1\overline{\rho}} \alpha_1(\overline{\rho}_1 - \rho_1^{(1)}(x,y)$$

$$+ \frac{\overline{\rho}_1\alpha_2}{\overline{\rho}} (\rho_2^{(1)}(x,y) - \overline{\rho}_2)\Big]n_i \, dy \, ds$$

$$- \int_{\partial B_1} \int_0^t \overline{\rho}_2\alpha_2 \, R_m^{(2)}(x,t-y)n_m \, dy \, ds + \int_{\partial\overline{B}_1} \int_0^t (\overline{\rho}_2\alpha_2 \, V_m^{(2)}(x,t-y)$$

$$- \alpha_1 W_m^{(2)}(x,t-y))n_m \, dy \, ds$$

$$= \overline{\rho}_2 \int_B \int_0^t v_i^{(1)}(x,t-y)G_i^{(2)}(x,y) \, dy \, dv$$

$$+ \overline{\rho}_1 \int_B \int_0^t u_i^{(1)}(x,t-y) \ F_i^{(2)}(x,y) \, dy \, dv$$

$$+ \int_{\partial B_1} \int_0^t v_i^{(1)}(x,t-y)g_i^{(2)}(x,y) \, dy \, ds + \int_{\partial B_1} \int_0^t u_i^{(1)}(x,t-y)f_i^{(2)}(x,y) \, dy \, ds$$

$$+ \int_{\partial\overline{B}_1} \int_0^t v_i^{(1)}(x,t-y)\pi_{pi}^{(2)}(x,y)n_p \, dy \, ds$$

$$+ \int_{\partial\overline{B}_1} \int_0^t u_i^{(1)}(x,t-y) \ \sigma_{pi}^{(2)}(x,y)n_p \, dy \, ds$$

$$- \int_{\partial B_1} \int_0^t R_i^{(1)}(x,t-y)\Big[-\frac{\bar{\rho}_2}{\bar{\rho}_1 \bar{\rho}} \alpha_1(\bar{\rho}_1 - \rho_1^{(2)}(x,y))$$

$$+ \frac{1}{\rho} \alpha_2(\rho_2^{(2)}(x,y) - \bar{\rho}_2)\Big] n_i \, dy \, ds$$

$$+ \int_{\partial \bar{B}_1} \int_0^t (v_i^{(1)}(x,t-y) - U_i^{(1)}(x,t-y))\Big[-\frac{\bar{\rho}_2}{\bar{\rho}_1 \bar{\rho}} \alpha_1(\bar{\rho}_1 - \rho_1^{(2)}(x,y))$$

$$+ \frac{\bar{\rho}_1}{\bar{\rho}} \alpha_2(\rho_2^{(2)}(x,y) - \bar{\rho}_2)\Big] n_i \, dy \, ds$$

$$- \int_{\partial B_1} \int_0^t \bar{\rho}_2 \alpha_2 R_m^{(1)}(x,t-y) n_m \, dy \, ds + \int_{\partial \bar{B}_1} \int_0^t [\bar{\rho}_2 \alpha_2 v_m^{(1)}(x,t-y)$$

$$-\alpha_1 W_m^{(1)}(x,t-y)] n_m \, dy \, ds \tag{3.28}$$

C. <u>Reciprocity Relation for Heat Conducting Mixture of Linear Elastic Solid and Newtonian Viscous Fluid Occupying Infinite Region</u>

Let three-dimensional Euclidean space be occupied by a mixture of elastic solid and a Newtonian viscous fluid undergoing a disturbance of small amplitude during the time interval $t \geq 0$. Since there is no boundary in this infinite region case, regularity conditions will be considered in the place of boundary conditions. Let the mixture be subjected to two systems which are distinguished by superscripts in parentheses, where each of these systems has same initial conditions stated in the previous theorem and satisfies the following regularity conditions:

$\sigma_{ij}^{(i)}, \pi_{ij}^{(i)}, \overline{w}^{(i)}, \overline{v}^{(i)}, \theta^{(i)} \rightarrow 0$ as $|\overline{x}| \rightarrow \infty$ for $i = 1,2$.

Then the reciprocity theorem (3.24) reduces to:

$$\overline{\rho}_2 \int_B \int_0^t v_i^{(2)}(x,t-y)\,G_i^{(1)}(x,y)\;dy\;dv$$

$$+\ \overline{\rho}_1 \int_B \int_0^t u_i^{(2)}(x,t-y)\;F_i^{(1)}(x,y)\;dy\;dv$$

$$+\ \frac{\overline{\rho}}{\overline{T}} \int_B \int_0^t r^{(2)}(x,t-y)\,\theta^{(1)}(x,y)\,dy\;dv - \frac{K'}{\overline{T}} \int_B \int_0^t [u_p^{(2)}(x,t-y)$$

$$-\ v_p^{(2)}(x,t-y)]\,\theta_{,p}^{(1)}(x,y)\;dy\;dv$$

$$=\ \overline{\rho}_2 \int_B \int_0^t v_i^{(1)}(x,t-y)\,G_i^{(2)}(x,y)\;dy\;dv$$

$$+\ \overline{\rho}_1 \int_B \int_0^t u_i^{(1)}(x,t-y)\;F_i^{(2)}(x,y)\;dy\;dv$$

$$+\ \frac{\overline{\rho}}{\overline{T}} \int_B \int_0^t r^{(1)}(x,t-y)\,\theta^{(2)}(x,y)\,dy\;dv - \frac{K'}{\overline{T}} \int_B \int_0^t [\;u_p^{(1)}(x,t-y)$$

$$-\ v_p^{(1)}(x,t-y)]\,\theta_{,p}^{(2)}(x,y)\;dy\;dv \qquad (3.27)$$

where the integration on B is over the entire three-dimensional space, $|x| < \infty$, $|y| < \infty$, $|z| < \infty$.

D. Reciprocity Relation for Heat Conducting Elastic Solid

Let the elastic solid be subjected to two systems which are distinguished by superscripts in parentheses. Let the functions $w_i^{(j)}$ and $T^{(j)}$ be of class C^2 on B and the subsidiary conditions are

$w_i = 0,\ \theta = 0$ on B' at $t = 0$,

$\sigma_{pi} n_p = f_i^{(j)}$ on B_1, $u_i = W_i^{(j)}$ on ∂B_1 for $t \geq 0$,

$T^{(j)} = \bar{T} + \Theta^{(j)}$ on ∂B_2, $q_p^{(j)} n_p = F^{(j)}$ on $\partial \bar{B}_2$ for $t \geq 0$,

where $F_i^{(j)}$, $r^{(j)}$, $f_i^{(j)}$, $W_i^{(j)}$, $\Theta^{(j)}$ and $F^{(j)}$ are prescribed functions on the appropriate domains, and $\bar{\rho}$, $\bar{T}$ are given positive constants. Then by the result of Section 2.4 Part A, it follows that the work that would be done by the first system in acting through the displacement of the second system and the work that would be done by the second system in acting through the displacement of the first system satisfy the following relation.

$$\begin{aligned}
&\bar{\rho} \int_B \int_0^t w_i^{(2)}(x,t-y)\, F_i^{(1)}(x,y)\, dy\, dv \\
&+ \int_{\partial B_1} \int_0^t w_i^{(2)}(x,t-y)\, f_i^{(1)}(x,y)\, dy\, ds \\
&+ \int_{\partial \bar{B}_1} \int_0^t W_i^{(2)}(x,t-y)\, \sigma_{pi}^{(1)}(x,y)\, n_p\, dy\, ds \\
&+ \frac{\bar{\rho}}{\bar{T}} \int_B \int_0^t r^{(2)}(x,t-y)\, \theta^{(1)}(x,y)\, dy\, dv \\
&- \frac{1}{\bar{T}} \int_{\partial B_2} \int_0^t q_p^{(2)}(x,t-y)\, \Theta^{(1)}(x,y)\, n_p\, dy\, ds \\
&- \frac{1}{\bar{T}} \int_{\partial \bar{B}_2} \int_0^t F^{(2)}(x,t-y)\, \theta^{(1)}(x,y)\, dy\, ds
\end{aligned}$$

$$= \bar{\rho} \int_B \int_0^t w_i^{(1)}(x,t-y)\; F_i^{(2)}(x,y)\; dy\; dv$$

$$+ \int_{\partial B_1} \int_0^t w_i^{(1)}(x,t-y)\; g_i^{(2)}(x,y)\; dy\; ds$$

$$+ \int_{\partial \bar{B}_1} \int_0^t W_i^{(1)}(x,t-y)\, \sigma_{pi}^{(2)}(x,y) n_p\; dy\; ds$$

$$+ \frac{\bar{\rho}}{\bar{T}} \int_B \int_0^t r^{(1)}(x,t-y)\, \theta^{(2)}(x,y)\; dy\; dv$$

$$- \frac{1}{\bar{T}} \int_{\partial B_2} \int_0^t q_p^{(1)}(x,t-y)\, \Theta^{(2)}(x,y) n_p\; dy\; ds$$

$$- \frac{1}{\bar{T}} \int_{\partial \bar{B}_2} \int_0^t F^{(1)}(x,t-y)\, \theta^{(2)}(x,y)\; dy\; ds \qquad (3.28)$$

We remark that this result agrees with the well known reciprocity relation for elastic solid in isothermal case.

E. Reciprocity Relation for Heat Conducting Viscous Fluid

Let the viscous fluid be subjected to two systems which are distinguished by superscripts in parentheses. Let the functions $v_i^{(j)}$ be of class C^1 and $T^{(j)}$ be of class C^2 on B and the subsidiary conditions are

$v_i = 0$, $\eta = 0$, $\theta = 0$ on B at $t = 0$,

$\tau_{pi} n_p = g_i^{(j)}$ on ∂B_1, $v_i = V_i^{(j)}$ on $\partial \bar{B}_1$ for $t \geq 0$,

$T^{(j)} = \bar{T} + \Theta^{(j)}$ on ∂B_2, $q_p^{(j)} n_p = F^{(j)}$ on $\partial \bar{B}_2$ for $t \geq 0$,

where $g_i^{(j)}$, $V_i^{(j)}$, $\Theta^{(j)}$, $F^{(j)}$, $G_i^{(j)}$ and $r^{(j)}$

are prescribed functions on the appropriate domains and $\bar{\rho}$, $\bar{T}$

are given, positive, constants. Then by the result of Section 2.4 Part B, it follows that the work that would be done by the first system in acting through the velocities of the second system and the work that would be done by the second system in acting through the velocities of the first system satisfy the following relation:

$$\bar{\rho} \int_B \int_0^t v_i^{(2)}(x,t-y)\,G_i^{(1)}(x,y)\;dy\;dv$$

$$+ \int_{\partial B_1} \int_0^t v_i^{(2)}(x,t-y)\,(g_i^{(1)}(x,y) + \bar{\rho}\,\alpha_2 n_i)\;dy\;ds$$

$$+ \int_{\partial \bar{B}_1} \int_0 v_i^{(2)}(x,t-y)\,(\pi_{pi}^{(1)}(x,y)\,n_p + \bar{\rho}\,\alpha_2 n_i)\;dy\;ds$$

$$+ \frac{\bar{\rho}}{\bar{T}} \int_B \int_0^t r^{(2)}(x,t-y)\,\theta^{(1)}(x,y)\;dy\;dv$$

$$- \frac{1}{\bar{T}} \int_{\partial B_2} \int_0 q_p^{(2)}(x,t-y)\,\theta^{(1)}(x,y)\,n_p\;dy\;ds$$

$$- \frac{1}{\bar{T}} \int_{\partial \bar{B}_2} \int_0^t F^{(2)}(x,t-y)\,\theta^{(1)}(x,y)\;dy\;ds$$

$$= \bar{\rho} \int_B \int_0^t v_i^{(1)}(x,t-y)\,G_i^{(2)}(x,y)\;dy\;dv$$

$$+ \int_{\partial B_i} \int_0^t v_i^{(1)}(x,t-y)\,(g_i^{(2)}(x,y) + \bar{\rho}\,\alpha_2 n_i)\;dy\;ds$$

$$+ \int_{\partial \overline{B}_1} \int_0^t v_i^{(1)}(x,t-y)\,(\pi_{pi}^{(2)}(x,y)\,n_p + \overline{\rho}\,\alpha_2 n_i)\; dy\; ds$$

$$+ \frac{\overline{\rho}}{\overline{T}} \int_B \int_0^t r^{(1)}(x,t-y)\,\theta^{(2)}(x,y)\; dy\; dv$$

$$- \frac{1}{\overline{T}} \int_{\partial B_2} \int_0^t q_p^{(1)}(x,t-y)\,\Theta^{(2)}(x,y)\,n_p\; dy\; ds$$

$$- \frac{1}{\overline{T}} \int_{\partial \overline{B}_2} \int_0^t F^{(1)}(x,t-y)\,\theta^{(2)}(x,y)\; dy\; ds \qquad (3.29)$$

F. <u>An Application of Reciprocity Relation in Mixture Theory</u>

Suppose that infinite three-dimensional Euclidean space is occupied by a mixture of elastic solid and Newtonian viscous fluid undergoing an isothermal disturbance of small amplitude during the time interval $t \geq 0$. Let the mixture be subjected to two systems which are distinguished by superscripts in parentheses with the following specified body force systems:

$$F_i^{(1)} = a_i^{(1)}\,\delta(p_1)\,\delta(t)$$
$$G_i^{(1)} = 0$$

where p_1 is a fixed point in the region and $a_i^{(1)}$ refers to a force magnitude,

$$F_i^{(2)} = 0$$
$$G_i^{(2)} = a_i^{(2)}\,\delta(p_1)\,\delta(t)$$

where p_1 is the same fixed point in the region and $a_i^{(2)}$

refers to a force magnitude. Let us assume that the velocity fields of these preliminary problems are known, that is, $u_i^{(1)}$, $v_i^{(1)}$, $u_i^{(2)}$, and $v_i^{(2)}$ are known. Suppose that the mixture is subjected to an arbitrary body force system

$$F_i^{(3)} \quad \text{and} \quad G_i^{(3)} .$$

Then the velocity fields $u_i^{(3)}$ and $v_i^{(3)}$ at the point p_1 at time t are given by the following integrals as a result of (3.28):

$$\bar{\rho}_1 a_i^{(1)} u_i^{(3)}(p_1,t) = \bar{\rho}_2 \int_B \int_0^t v_i^{(1)}(x,t-y) G_i^{(3)}(x,y)\, dy\, dv$$

$$+ \bar{\rho}_1 \int_B \int_0^t u_i^{(1)}(x,t-y) F_i^{(3)}(x,y)\, dy\, dv ,$$

$$\bar{\rho} a_i^{(2)} v_i^{(3)}(p_1,t) = \bar{\rho}_2 \int_B \int_0^t v_i^{(2)}(x,t-y) G_i^{(3)}(x,y)\, dy\, dv$$

$$+ \bar{\rho}_1 \int_B \int_0^t u_i^{(2)}(x,t-y) F_i^{(3)}(x,y)\, dy\, dv . \qquad (3.30)$$

CHAPTER IV.

A FUNDAMENTAL ONE-DIMENSIONAL INITIAL-BOUNDARY VALUE PROBLEM

4.1. Introduction

In linear thermoelasticity, for one dimensional model, a series of papers by Danilovskaya [28], Sternberg and Chakravorty [29], and Muki and Breuer [30] have answered some of the basic questions concerning the effects of the inertia terms in the elastic equations of motion and the effect of the mechanical coupling term in the Fourier heat conduction equation.

Recently, Martin [31] studied the initial-boundary value problem corresponding to Danilovskaya's in the framework of the linearized interacting mixture theory. In [31], [32] a mixture of linear elastic solid and viscous fluid occupying a half-space undergoing deformation due to a transient temperature on the boundary was considered. Method of the solution was that a parameter occuring in the diffusive resistance vector was used as the basis for a perturbation procedure in the equations of motion.

Here, we consider the same problem but use a different method of solution. Let the mixture occupy a half-space and let its motion be restricted to one space dimension. We prescribe a step function temperature on the face of the half-space where the face is constrained rigidly against motion.

4.2. Formulation of the Problem

Departing from the indicial notation, we describe the cartesian coordinates as (x,y,z) and consider the mixture of an elastic solid and a viscous fluid as occupying the region $x \geq 0$ [31]. We assume that the mixture is subjected to a time-dependent temperature field of the form

$$T = T(x,t) \tag{4.1}$$

and is restricted to uniaxial motion so that the displacement vector of the elastic solid has components

$$w_i = (w(x,t),\ 0,\ 0) \tag{4.2}$$

and the fluid velocity vector is

$$v_i = (v(x,t),\ 0,\ 0). \tag{4.3}$$

It is convenient to define a nondimensional temperature field in place of T in (4.1) by

$$\theta(x,t) = \frac{T(x,t) - \bar{T}}{\bar{T}} \tag{4.4}$$

where $\bar{T}$ denotes the temperature of the equilibrium state. Substitution of (4.2) and (4.3) into equations (2.12), (2.25) and (2.26) shows that we may write

$$e_x = \frac{\partial w}{\partial x}, \quad f_x = \frac{\partial v}{\partial x}, \quad d_x = \frac{\partial e_x}{\partial t} = \frac{\partial^2 w}{\partial t\, \partial x}, \tag{4.5}$$

$$\rho_1(x,t) = \bar{\rho}_1\left(1 - \frac{\partial w}{\partial x}\right), \tag{4.6}$$

$$\frac{\partial\rho_2}{\partial t}(x,t) + \bar{\rho}_2 \frac{\partial v}{\partial x} = 0, \tag{4.7}$$

as the only non-vanishing kinematic relations, and we note that all other strain, rate of deformation and vorticity components are identically equal to zero. The constitutive equations (2.35) to (2.37) under the restriction (2.55) become

$$\omega_x = \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}} \frac{\partial^2 w}{\partial x^2} + \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}} \frac{\partial\rho_2}{\partial x} + \alpha\left(\frac{\partial w}{\partial t} - v\right), \omega_y = \omega_z = 0 \tag{4.8}$$

$$\sigma_x(x,t) = \left(2(\alpha_1+\alpha_5) - \left(\frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} - \alpha_4\right)\right) \frac{\partial w}{\partial x}$$

$$+ \alpha_9\bar{T}\theta + \left(\frac{\alpha_1}{\bar{\rho}} + \alpha_8\right)\eta + \alpha_1 , \tag{4.9}$$

$$\sigma_y(x,t) = \sigma_z(x,t) =$$

$$-\left(\frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} - \alpha_4\right)\frac{\partial w}{\partial x} + \alpha_9\bar{T}\theta + \left(\frac{\alpha_1}{\bar{\rho}} + \alpha_8\right)\eta + \alpha_1 , \tag{4.10}$$

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0, \tag{4.11}$$

$$\pi_x(x,t) = (2\mu+\lambda)\frac{\partial v}{\partial x} + \bar{\rho}_2\left(\frac{\bar{\rho}_1\alpha_2}{\bar{\rho}} - \alpha_8\right)\frac{\partial w}{\partial x} - \bar{\rho}_2\alpha_{10}\bar{T}\theta$$

$$-\left(\frac{\bar{\rho}+\bar{\rho}_2}{\bar{\rho}}\alpha_2 + \bar{\rho}_2\alpha_6\right)\eta - \bar{\rho}_2\alpha_2 , \tag{4.12}$$

$$\pi_y(x,t) = \pi_z(x,t) = \lambda\frac{\partial v}{\partial x} + \bar{\rho}_2\left(\frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2 - \alpha_8\right)\frac{\partial w}{\partial x} - \bar{\rho}_2\alpha_{10}\bar{T}\theta$$

$$-\left(\frac{\bar{\rho}+\bar{\rho}_2}{\bar{\rho}}\alpha_2 + \bar{\rho}_2\alpha_6\right)\eta - \bar{\rho}_2\alpha_2 , \tag{4.13}$$

$$\pi_{xy} = \pi_{xz} = \pi_{yz} = 0. \tag{4.14}$$

The equations of motion (2.29), (2.30) and the energy equation (2.31), under the constraints (4.2) to (4.4), become

$$\frac{\partial\sigma_x}{\partial x} - \omega_x = \bar{\rho}_1 \frac{\partial^2 w}{\partial t^2}, \tag{4.15a}$$

$$\frac{\partial\pi_x}{\partial x} + \omega_x = \bar{\rho}_2 \frac{\partial v}{\partial t}, \tag{4.15b}$$

$$\alpha_7\bar{T}\frac{\partial\theta}{\partial t} + k\frac{\partial^2\theta}{\partial x^2} + \left(\frac{\alpha_9\bar{T}+K'}{\bar{T}}\right)\frac{\partial^2 w}{\partial x\partial t} - \left(\frac{\bar{\rho}_2\alpha_{10}\bar{T}+K'}{\bar{T}}\right)\frac{\partial v}{\partial x} = 0. \tag{4.16}$$

To complete our formulation of the initial-boundary value problem we prescribe that for $t \leq 0$

$$w(x,t) = \frac{\partial w}{\partial t}(x,t) = v(x,t) = 0,$$
$$\rho_2 = \bar{\rho}_2, \quad \theta_{(x,t)} = 0, \tag{4.17}$$

In addition, we require that on the boundary $x = 0$,

$$\theta(0,t) = h(t)^*, \ \frac{\partial w}{\partial t}(0,t) = 0, \ v(0,t) = 0, \tag{4.18}$$

while as $x\to\infty$, we stipulate that

$\theta(x,t), w(x,t), \rho_2(x,t), v(x,t), \sigma_x(x,t), \pi_x(x,t), \sigma_y(x,t)$ and $\pi_y(x,t)$ approach to zero. (4.19)

At this point, we introduce dimensionless variables. For this purpose we use the notation introduced in (3.9)

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* $h(t)$ is the Heaviside unit step-function defined to be zero for $t < 0$ and one for $t > 0$.

wherein the relation (3.5) is used. A direct substitution of (3.9) into (4.9) yields

$$\sigma_x(x,t) = (\beta_2 + 2\beta_3)\frac{\partial w}{\partial x} + \alpha_9\overline{T}\theta + \beta_1\eta + \alpha_1 \tag{4.20}$$

which, if the material were elastic, would lead us to expect $\beta_2 + 2\beta_3$ to play the role of the Lamé constants $(\lambda_E + 2\mu_E)$ while $\alpha_9\overline{T}$ would play the role of $(2\mu_E + 3\lambda_E)\alpha_E$ where α_E is the coefficient of linear thermal expansion of an elastic material. With this in mind we choose a velocity c_1,

$$c_1^2 = \frac{\beta_2 + 2\beta_3}{\overline{\rho}_1} \tag{4.21}$$

which would be the irrotational velocity of sound if the material were elastic. Since $\alpha_7 \leq 0$ by (2.40), we define

$$\omega^2 = \frac{-k}{\alpha_7\overline{T}} \quad . \tag{4.22}$$

By a dimensional analysis we have that c_1 is a velocity while (4.22) has dimensions of length squared per unit time. Thus, if we take

$$a = \frac{\omega^2}{c_1}\ , \ t_0 = \frac{\omega^2}{c_1^2} \tag{4.23}$$

then a dimensionless x-coordinate and time are given by

$$\zeta = \frac{x}{a} = \frac{c_1 x}{\omega^2}\ , \ \tau = \frac{t}{t_0} = \frac{c_1^2 t}{\omega^2} \quad . \tag{4.24}$$

Proceeding further, we introduce non-dimensional partial stresses, solid displacement, fluid velocity, densities and diffusive force by

$$\hat{\sigma}_x = \frac{\sigma_x}{\beta_2 + 2\beta_3}\ ,\ \hat{w} = \frac{w}{a}\ ,\ \hat{\sigma}_0 = \frac{\alpha_1}{\beta_2 + 2\beta_3}\ ,\quad \hat{\sigma}_y = \frac{\sigma_y}{\beta_2 + 2\beta_3}\ ,$$

$$\hat{\pi}_x = \frac{\pi_x}{\beta_2 + 2\beta_3}\ ,\ \hat{v} = \frac{v\,t_0}{a}\ ,\ \hat{\pi}_y = \frac{\pi_y}{\beta_2 + 2\beta_3}\ ,$$

$$\eta_2 = \frac{\rho_2 - \bar{\rho}_2}{\bar{\rho}_2}\ ,\ \eta_1 = \frac{\rho_1 - \bar{\rho}_1}{\bar{\rho}_1}\ ,\ \hat{\omega} = \frac{a\,\omega_x}{\beta_2 + 2\beta_3}\ . \tag{4.25}$$

In addition, the following quantities are conveniently grouped:

$$s^2 = \frac{2\mu + \lambda}{t_0(\beta_2 + 2\beta_3)}\ ,\ d_1 = \frac{\alpha_9 \bar{T}}{\beta_2 + 2\beta_3}\ ,\ d_2 = \frac{\bar{\rho}_2 \alpha_{10} \bar{T}}{\beta_2 + 2\beta_3}$$

$$f = \frac{\bar{\rho}_2}{\bar{\rho}}\ ,\ \delta_1 = \frac{\gamma_2}{\beta_2 + 2\beta_3} - f\hat{\sigma}_0 = \frac{-\bar{\rho}_2}{\beta_2 + 2\beta_3}\left(\beta_1 - \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}}\right)$$

$$\delta_2 = \frac{f\alpha_2}{c_1^2} - \frac{\bar{\rho}_2 \gamma_1}{\bar{\rho}_1 c_1^2}\ ,\ \tilde{\epsilon}_1 = \frac{\alpha_9 \bar{T} + K'}{\alpha_7 \bar{T}^2}\ ,\ \tilde{\epsilon}_2 = \frac{\bar{\rho}_2 \alpha_{10} \bar{T} + K'}{\alpha_7 \bar{T}^2}\ . \tag{4.26}$$

Incorporating all of these changes leads us the following summary:

Constitutive equations

$$\hat{\sigma}_x(\zeta,\tau) = \hat{\sigma}_o + \frac{\partial \hat{w}(\zeta,\tau)}{\partial \zeta} + d_1 \theta(\zeta,\tau) + [(1-f)\hat{\sigma}_o - \delta_1]\eta_2(\zeta,\tau) \tag{4.27a}$$

$$\hat{\pi}_x(\zeta,\tau) = -\hat{\sigma}_o + s^2 \frac{\partial \hat{v}(\zeta,\tau)}{\partial \zeta} + (\delta_1 + f\hat{\sigma}_o)\frac{\partial \hat{w}(\zeta,\tau)}{\partial \zeta} - d_2\theta(\zeta,\tau)$$

$$+(\delta_2 - (1-f)\hat{\sigma}_o)\eta_2(\zeta,\tau), \qquad (4.27b)$$

$$\hat{\omega}(\zeta,\tau) = -f\hat{\sigma}_o \frac{\partial^2 w}{\partial \zeta^2}(\zeta,\tau) + (1-f)\hat{\sigma}_o \frac{\partial \eta_2}{\partial \zeta}(\zeta,\tau)$$

$$+ \frac{\alpha t_o}{\bar{\rho}_1}\left[\frac{\partial \hat{w}}{\partial \tau}(\zeta,\tau) - v(\zeta,\tau)\right], \qquad (4.27c)$$

$$\hat{\sigma}_y(\zeta,\tau) = \frac{\beta_2}{\beta_2 + 2\beta_3} \frac{\partial \hat{w}}{\partial \zeta}(\zeta,\tau) + d_1\theta(\zeta,\tau)$$

$$+[(1-f)\hat{\sigma}_o - \delta_1]\eta_2(\zeta,\tau) + \hat{\sigma}_o, \qquad (4.27d)$$

$$\hat{\sigma}_z(\zeta,\tau) = \hat{\sigma}_y(\zeta,\tau), \qquad (4.27e)$$

$$\hat{\pi}_y(\zeta,\tau) = \frac{\lambda}{t_o(\beta_2 + 2\beta_3)} \frac{\partial \hat{v}}{\partial \zeta} + (f\hat{\sigma}_o + \delta_1)\frac{\partial w}{\partial \zeta}(\zeta,\tau) - d_2\theta(\zeta,\tau)$$

$$+[\delta_2 - (1-f)\hat{\sigma}_o]\eta_2(\zeta,\tau) - \hat{\sigma}_o, \qquad (4.27f)$$

$$\hat{\pi}_z(\zeta,\tau) = \hat{\pi}_y(\zeta,\tau). \qquad (4.27g)$$

Equations of motion

$$(1+f\hat{\sigma}_o)\frac{\partial^2 \hat{w}}{\partial \zeta^2} - \frac{\alpha t_o}{\bar{\rho}_1}\left(\frac{\partial \hat{w}}{\partial \tau} - \hat{v}\right) + d_1\frac{\partial \theta}{\partial \zeta} - \delta_1\frac{\partial \eta_2}{\partial \zeta} = \frac{\partial^2 \hat{w}}{\partial \tau^2}, \qquad (4.28)$$

$$\delta_1 \frac{\partial^2 \hat{w}}{\partial \zeta^2} + s^2 \frac{\partial^2 \hat{v}}{\partial \zeta^2} + \frac{\alpha t_o}{\bar{\rho}_1}\left(\frac{\partial \hat{w}}{\partial \tau} - \hat{v}\right) - d_2\frac{\partial \theta}{\partial \zeta} + \delta_2\frac{\partial \eta_2}{\partial \zeta} = \frac{\bar{\rho}_2}{\bar{\rho}_1}\frac{\partial \hat{v}}{\partial \tau}, \qquad (4.29)$$

$$\frac{\partial^2 \theta}{\partial \zeta^2} - \frac{\partial \theta}{\partial \tau} - \tilde{\epsilon}_1 \frac{\partial^2 \hat{w}}{\partial \zeta \partial \tau} + \tilde{\epsilon}_2 \frac{\partial \hat{v}}{\partial \zeta} = 0, \qquad (4.30)$$

$$\frac{\partial \eta_2}{\partial \tau} + \frac{\partial \hat{v}}{\partial \zeta} = 0. \qquad (4.31)$$

Initial Conditions

For $\tau \leq 0$, we take

$$\hat{w}(\zeta,\tau) = \frac{\partial \hat{w}}{\partial \tau}(\zeta,\tau) = \hat{v}(\zeta,\tau) = \eta_2(\zeta,\tau) = \theta(\zeta,\tau) = 0. \qquad (4.32)$$

Boundary conditions

At $\zeta = 0$, we take

$$\theta(0,\tau) = h(\tau), \ \frac{\partial \hat{w}}{\partial \tau}(0,\tau) = 0, \ \hat{v}(0,\tau) = 0. \qquad (4.33)$$

As $\zeta \to \infty$, we take

$$\theta(\zeta,\tau), \ \hat{w}(\zeta,\tau), \ \hat{v}(\zeta,\tau), \ \eta_2(\zeta,\tau), \hat{\sigma}_x(\zeta,\tau),$$
$$\text{and} \ \hat{\pi}_x(\zeta,\tau) \to 0. \qquad (4.34)$$

For this initial-boundary value problem specified by equations (4.27) to (4.34), we assume that the initial stress of the solid constituent is zero, i.e.,

$$\hat{\sigma}_o = 0, \qquad (4.35)$$

that the mechanical coupling terms in the heat conduction equation (4.30) can be neglected

$$\tilde{\epsilon}_1 = 0, \ \tilde{\epsilon}_2 = 0, \qquad (4.36)$$

and that the constants of the mixture satisfy

$$s > r >> -\delta_2 > \delta_1^2 >> t > 0, \ -\delta_2 - \delta_1^2 >> t > 0, 1 >> -\delta_2, \qquad (4.37)$$

where $t = \frac{\alpha t_0}{\bar{\rho}_1}$ and $r = \frac{\bar{\rho}_2}{\bar{\rho}_1}$. (4.38)

The assumption of (4.36) is analogous to neglecting the effect of the mechanical coupling term in the Fourier heat conduction equation in linear thermoelasticity theory [22], [23] and is justified by the conditions (a) and (b) of Section 2.2.

Various material constants which appear in (4.37) have to be determined experimentally for the mixture. The restrictions on the material constants, (4.37), are the results of (2.39), (2.40), and their interpretations in Section 2.4, Single Constituent Theory. For additional references consult [33] to [36].

With the aid of (4.35) to (4.38), the equations (4.28) to (4.31) are written

$$\frac{\partial^2 \hat{w}}{\partial \zeta^2} - t\,\frac{\partial \hat{w}}{\partial \tau} + t\hat{v} + d_1\,\frac{\partial \theta}{\partial \zeta} - \delta_1\,\frac{\partial \eta_2}{\partial \zeta} = \frac{\partial^2 \hat{w}}{\partial \tau^2}, \tag{4.39}$$

$$\delta_1\,\frac{\partial^2 \hat{w}}{\partial \zeta^2} + t\,\frac{\partial \hat{w}}{\partial \tau} + s^2\,\frac{\partial^2 \hat{v}}{\partial \zeta^2} - t\hat{v} - d_2\,\frac{\partial \theta}{\partial \zeta} + \delta_2\,\frac{\partial \hat{\eta}_2}{\partial \zeta} = r\,\frac{\partial \hat{v}}{\partial \tau}, \tag{4.40}$$

$$\frac{\partial^2 \theta}{\partial \zeta^2} - \frac{\partial \theta}{\partial \tau} = 0\,, \tag{4.41}$$

$$\frac{\partial \hat{\eta}_2}{\partial \tau} + \frac{\partial \hat{v}}{\partial \zeta} = 0\,. \tag{4.42}$$

4.3. Solution by Integral Transforms

We denote the Laplace transform of a function $F(\zeta, \tau)$ with respect to τ by $\bar{F}(\zeta, p)$, where

$$\bar{F}(\zeta, p) = \int_0^\infty F(\zeta, \tau) e^{-p\tau} \, d\tau , \qquad (4.43)$$

$$2\pi i \, F(\zeta, \tau) = \int_{c-i\infty}^{c+i\infty} \bar{F}(\zeta, p) e^{p\tau} \, dp. \qquad (4.44)$$

In (4.44) c is a positive real number such that the path of integration is any vertical line to the right of all singularities of $\bar{F}(\zeta, p)$ in p-plane.

The solution to the thermal problem, equations (4.41), (4.32) to (4.34), in the (ζ, p) plane is

$$\bar{\theta}(\zeta, p) = \frac{1}{p} \exp(-p^{\frac{1}{2}}\zeta). \qquad (4.45)$$

Now transforming the equations of motion (4.39), (4.40) and the equation of continuity (4.42), then combining these equations with the solution to the thermal problem (4.45) gives

$$(D^2 - pt - p^2)\bar{w} + (\frac{\delta_1}{p} D^2 + t)\bar{v} = \frac{d_1}{p^{\frac{1}{2}}} \exp(-p^{\frac{1}{2}}\zeta) \qquad (4.46)$$

$$(\delta_1 D^2 + pt)\bar{w} + [(s^2 - \frac{\delta_2}{p}) D^2 - t - pr]\bar{v} = -\frac{d_2}{p^{\frac{1}{2}}} \exp(-p^{\frac{1}{2}}\zeta) \qquad (4.47)$$

where the differential operator is defined by

$$D = \frac{d}{d\zeta} .$$

Due to equations (4.32) to (4.34), the transformed boundary conditions and the regularity conditions are

$$\bar{w}(0, p) = 0, \ \bar{v}(0, p) = 0 \qquad (4.48)$$

and as $\zeta \to \infty$

$$\bar{w}(\zeta,p),\ \bar{v}(\zeta,p) \to 0. \tag{4.49}$$

Solving for the displacement of the solid constituent and the velocity of the fluid constituent from equations (4.46) and (4.47) for the homogeneous solutions only which conform with (4.49), we have

$$\bar{w}_h(\zeta,p) = B(p)\ \exp\left(-\frac{p^{\frac{1}{2}}}{2f_1}(g_1+g_2)\zeta\right) + D(p)\ \exp\left(-\frac{p^{\frac{1}{2}}}{2f_1}(g_1-g_2)\zeta\right), \tag{4.50}$$

$$\bar{v}_h = [p\{(t+p)(ps^2-\delta_2)+t\ \delta_1\}-\tfrac{1}{4}p(g_1+g_2)^2]B(p)\exp\left(-\frac{p^{\frac{1}{2}}}{2f_1}(g_1+g_2)\zeta\right) +[p\{(t+p)(ps^2-\delta_2)+t\ \delta_1\}-\tfrac{1}{4}p(g_1-g_2)^2]D(p) \exp\left(-\frac{p^{\frac{1}{2}}}{2f_1}(g_1-g_2)\zeta\right), \tag{4.51}$$

where $B(p)$ and $D(p)$ are integration constants, and

$$g_1^2(p) = (ps^2-\delta_2)(p+t)+(pr+t)+2t\ \delta_1 + 2p^{\frac{1}{2}}(ps^2-\delta_1^2-\delta_2)^{\frac{1}{2}}(pr+t+tr)^{\frac{1}{2}}, \tag{4.52}$$

$$g_2^2(p) = (ps^2-\delta_2)(p+t)+(pr+t)+2t\ \delta_1 - 2p^{\frac{1}{2}}(ps^2-\delta_1^2-\delta_2)^{\frac{1}{2}}(pr+t+tr)^{\frac{1}{2}}, \tag{4.53}$$

$$f_1^2(p) = ps^2-\delta_1^2-\delta_2\ . \tag{4.54}$$

Here we have assumed that

$$\mathrm{Re}\left(\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta\right) > 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{p^{1/2}}{f_1}(g_1-g_2)\zeta\right) > 0 \qquad (4.55)$$

for p such that $\mathrm{Re}(p) > M > 0$, and we will show this to be so later. Let us indicate particular solutions to the equations (4.46) and (4.47) by a subscript "p". Then we have

$$\bar{w}_p(\zeta,p) = \frac{A_1}{A_2}\exp(-p^{1/2}\zeta) \qquad (4.56)$$

$$\bar{v}_p(\zeta,p) =$$

$$\left[\frac{p\{\delta_1^2+\delta_2-ps^2+t\delta_1+(ps^2-\delta_2)(p+t)\}\frac{A_1}{A_2}}{p(ts^2+r\delta_1)+t(\delta_1-\delta_2)} + \right.$$

$$\left.\frac{\dfrac{ps^2d_1-\delta_2d_1+\delta_1d_2}{p^{1/2}}}{p(ts^2+r\delta_1)+t(\delta_1-\delta_2)}\right]\exp(-p^{1/2}\zeta) \qquad (4.57)$$

where

$$A_1 = d_1(p(s^2-r)-\delta_2-t)+d_2(\delta_1+t), \qquad (4.58)$$

$$A_2 = pp^{1/2}(ps^2-\delta_1^2-\delta_2-t-pr-(t+p)(ps^2-\delta_2)-2t\delta_1+p^2r+pt+ptr). \qquad (4.59)$$

The entire solutions are

$$\bar{w}(\zeta,p) = \bar{w}_h(\zeta,p) + \bar{w}_p(\zeta,p), \qquad (4.60)$$

$$\bar{v}(\zeta,p) = \bar{v}_h(\zeta,p) + \bar{v}_p(\zeta,p). \qquad (4.61)$$

To determine $B(p)$ and $D(p)$ we substitute equations (4.60) and (4.61) into the boundary conditions (4.48). Then

$$B(p) + D(p) + \frac{A_1(p)}{A_2(p)} = 0 \qquad (4.62)$$

$$B(p)p[(p+t)(ps^2-\delta_2)+t\delta_1-\frac{(g_1+g_2)^2}{4}] +$$

$$D(p)p[(p+t)(ps^2-\delta_2)+t\delta_1-\frac{(g_1-g_2)^2}{4}]$$

$$+ [p(\delta_1^2+\delta_2-ps^2+t\delta_1+(ps^2-\delta_3)(p+t))\frac{A_1}{A_2}$$

$$+ \frac{(ps^2-\delta_2)d_1}{p^{1/2}} + \frac{\delta_1 d_2}{p^{1/2}}] = 0 . \tag{4.63}$$

Solving for the unknowns $B(p)$ and $D(p)$ from equations (4.62) and (4.63) simultaneously, we have

$$B(p) = \frac{A_1}{g_1 g_2 A_2}[\delta_1^2+\delta_2-ps^2+\frac{1}{2}\{(ps^2-\delta_2)(p+t)+(pr+t)+2t\delta_1\}]$$

$$+ \frac{\{(ps^2-\delta_2)d_1 + \delta_1 d_2\}}{pp^{1/2}g_1 g_2} - \frac{A_1}{2A_2} , \tag{4.64}$$

$$D(p) = \frac{-A_1}{g_1 g_2 A_2}[\delta_1^2+\delta_2-ps^2+\frac{1}{2}\{(ps^2-\delta_2)(p+t)+(pr+t)+2t\delta_1\}]$$

$$- \frac{\{(ps^2-\delta_2)d_1 + \delta_1 d_2\}}{pp^{1/2}g_1 g_2} - \frac{A_1}{2A_2} . \tag{4.65}$$

The Laplace transformed displacement of the solid constituent and velocity of the fluid constituent are now explicitly given by equations (4.50), (4.51), (4.56), (4.57), (4.60), (4.61), (4.64) and (4.65). These quantities constitute the complete solution, in the transform plane, to the initial-boundary value problem posed by equations (4.32) to (4.34), (4.39) to (4.42). Stresses of each constituent may be expressed immediately in terms of the transformed displacement of the solid constituent and velocity of the fluid constituent by means of the constitutive equations (4.27a) to (4.27g). Now that the Laplace transformed displacements of the solid constituent and velocities of the fluid constituent are in their simplest form we proceed to invert these expressions.

4.4. Inversion

A. Location of Zeros of $g_1^2(p)g_2^2(p)$

As a first step toward the inversion of $\bar{w}(\zeta,p)$, we examine the multiple valued functions appearing in equations (4.52) and (4.53). Set

$$\epsilon_1 = \frac{-\delta_1^2 - \delta_2}{s^2} \tag{4.66}$$

$$\epsilon_2 = \frac{t + tr}{r} \quad . \tag{4.67}$$

Then equations (4.52) and (4.53) become

$$g_1(p)=(ps^2+\delta_2)(p+t)+(pr+t)+2t\delta_1+2r^{1/2}sp^{1/2}(p+\epsilon_1)^{1/2}(p+\epsilon_2)^{1/2}, \tag{4.68}$$

$$g_2(p)=(ps^2-\delta_2)(p+t)+(pr+t)+2t\delta_1-2r^{1/2}sp^{1/2}(p+\epsilon_1)^{1/2}(p+\epsilon_2)^{1/2}. \tag{4.69}$$

We take the domain of definition of $p^{1/2}$ as the entire p-plane cut along the negative real axis, and we choose a branch of $p^{1/2}$ through the requirement that

$$p^{1/2} = \sqrt{\ell} \quad \text{for} \quad p = \ell > 0.^{*} \tag{4.70}$$

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*$\sqrt{\ell}$ refers to the positive root for real positive ℓ.

We take the domain of definition of $(p+\epsilon_1)^{1/2}$ as the entire p-plane cut from $-\epsilon_1$ to $-\infty$ along the negative real axis, and we choose a branch of $(p+\epsilon_1)^{1/2}$ through the requirement that

$$(p+\epsilon_1)^{1/2} = \sqrt{\ell+\epsilon_1} \quad \text{for } p = \ell > -\epsilon_1 \,^{**} \tag{4.71}$$

We take the domain of definition of $(p+\epsilon_2)^{1/2}$ as the entire p-plane cut from $-\epsilon_2$ to $-\infty$ along the negtive real axis, and we choose a branch of $(p+\epsilon_2)^{1/2}$ through the requirement that

$$(p+\epsilon_2)^{1/2} = \sqrt{\ell+\epsilon_2} \quad \text{for } p = \ell > -\epsilon_2 \,^{**} \tag{4.72}$$

Now we will choose a domain in which $g_1(p)$ and $g_2(p)$ are single valued. In view of (4.70), (4.71) and (4.72), we find that $g_1(p)$ does not have branch points but $g_2(p)$ does. To see this we note that all possible branch points of $g_1(p)$ or $g_2(p)$ are the zeros of $g_1^2(p)g_2^2(p)$ which is a fourth degree polynomial,

$$g_1^2(p)g_2^2(p) = s^4[p^4 + p^3(\frac{2(ts^2-r-\delta_2)}{s^2})+p^2[\frac{2t(-1+2\delta_1-\delta_2-2r)}{s^2} + \frac{[(ts^2+r-\delta_2)^2+4r(\delta_1^2+\delta_2)]}{s^4}]$$
$$+p[(1+2\delta_1-\delta_2)(ts^2+r-\delta_2)+(2+2r)(\delta_1^2+\delta_2)]\frac{2t}{s^4}$$
$$+\frac{t^2}{s^4}(1+2\delta_1-\delta_2)^2]. \tag{4.73}$$

** Due to (4.66), (4.67) and (4.37), we have $0 < \epsilon_2 < \epsilon_1$.

We now use (4.37), i.e. $0<t\ll 1$, and expand (4.73) to terms of order t. Then (4.73) may be written

$$g_1^2(p)g_2^2(p) = s^4\left[p^2+\frac{2}{s^2}(-r-\delta_2+o(t))p+\frac{4r(\delta_1^2+\delta_2)+(r-\delta_2)^2}{s^4}\right]$$

$$\left[p^2+\frac{2t[(1+2\delta_1-\delta_2)(r-\delta_2)+(2+2r)(\delta_1^2+\delta_2)+o(t)]}{4r(\delta_1^2+\delta_2)+(r-\delta_2)^2}\,p\right.$$

$$\left.+\frac{t^2(1+2\delta_1-\delta_2)^2}{4r(\delta_1^2+\delta_2)+(r-\delta_2)^2}\right]$$

and the zeros are easily found to be p_1, p_2, p_1^*, p_2^* where

$$p_1 = \frac{r+\delta_2}{s^2}+o(t) + 2i\left[\frac{\sqrt{r\delta_1^2}}{s^2}+o(t)\right] \tag{4.74}$$

$$p_2 = \frac{-t[(1+2\delta_1-\delta_2)(r-\delta_2)+(2+2r)(\delta_1^2+\delta_2)+o(t)]}{4r(\delta_1^2+\delta_2)+(r-\delta_2)^2} \tag{4.75}$$

$$+2ti\frac{\sqrt{(-\delta_1^2-\delta_2)\{r^2-4r\delta_1+2r^2\delta_1-2\delta_1\delta_2(1-r)+\delta_1^2+\delta_2^2+2r\delta_2+r^2\delta_1^2+o(t)\}}}{4r(\delta_1^2+\delta_2)+(r-\delta_2)^2},$$

and p_i^* is the conjugate of p_i.+ Here we note that the expressions under the square root signs are positive due to (4.37).

B. Determination of Branches for $g_1(p)$ and $g_2(p)$

All the zeros of $g_1^2(p)g_2^2(p)$ are the zeros of $g_2(p)$ because, due to (4.63), (4.70) to (4.72), (4.74) and (4.75),

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+ For computational purposes it is desirable to have p_i, p_i^* in a series expansion of t. See Appendix 1.

we find that

$$\mathrm{Re}\ g_1^2(p_1) = \frac{4r}{s^2}(r+\delta_2-2\delta_1^2) + o(t) \tag{4.76}$$

$$\mathrm{Im}\ g_1^2(p_2) > \frac{1}{2}\, st\sqrt{\epsilon_1} > 0. \tag{4.77}$$

Since $g_1(p)$ never vanishes on the entire p-plane with negative real-axis being deleted due to (4.70) to (4.72), $g_1(p)$ does not have branch points. Moreover, we find from equations (4.73), (4.74) and (4.75)

$$g_2(p) = \frac{s^2}{g_1(p)}\,(p-p_1)^{1/2}(p-p_1^*)^{1/2}(p-p_2)^{1/2}(p-p_2^*)^{1/2}, \tag{4.78}$$

and this shows that p_1, p_1^*, p_2, p_2^* are branch points of order one for $g_2(p)$. We define the domain of the $g_1(p)$ to be the entire p-plane with negative-real axis being deleted, and choose the branch $g_1(p)$ by the requirement that

$$g_1(p) = \sqrt{(\ell s^2-\delta_2)(\ell+t)+(\ell r+t)+2t\delta_1 + \sqrt{2r}\, s\sqrt{\ell}\sqrt{\ell+\epsilon_1}\sqrt{\ell+\epsilon_2}}$$
$$\text{for } p = \ell > 0 \tag{4.79}$$

We define a domain in which $g_2(p)$ is single valued such that the domain is the entire p-plane cut along the following curves;

(a) the negative real axis

(b) the line joining p_1 and p_1^*, that is, the locus of points p such that $\mathrm{Re}(p) = \mathrm{Re}(p_1)$ and $\mathrm{Im}(p_1^*) \le \mathrm{Im}(p) \le \mathrm{Im}(p_1)$

(c) the curve joining p_2 and p_2^*, that is, the locus of points p along

(1) the line such that $Re(p) = Re(p_2)$ and $Im(p_2) \leq Im(p) \leq Re(p_2)$,

(2) the portion of circular arc whose radius is equal to $-\sqrt{2}\, Re(p_2)$ and whose argument lies between $\frac{-3\pi}{4}$ and $\frac{3\pi}{4}$,

(3) the line such that $Re(p) = Re(p_2)$ and $Re(p_2) \leq Im(p) \leq Im(p_2^*)$.

We choose the single-valued branch of $g_2(p)$ by the requirement that

$$g_2(p) = \sqrt{(\ell s^2 - \delta_2)(\ell + t) + (\ell r + t) + 2t\delta_1 - \sqrt{2r}\, s\sqrt{\ell}\sqrt{\ell+\epsilon_1}\sqrt{\ell+\epsilon_2}} \quad \text{for} \quad p = \ell > 0. \tag{4.80}$$

Due to (4.79) and (4.80), we find

$$g_1(p) = ps\left(1 + \frac{r^{1/2}}{p^{1/2}s} + O\left(\frac{1}{p}\right)\right) \quad \text{as} \quad p \to \infty \tag{4.81}$$

$$g_2(p) = ps\left(1 - \frac{r^{1/2}}{p^{1/2}s} + O\left(\frac{1}{p}\right)\right) \quad \text{as} \quad p \to \infty \tag{4.82}$$

We find that $-\epsilon_2 < Re(p_2) < Im(p_2^*) < 0$ with the aid of (4.37), (4.38), (4.67) and (4.75) as in the Figure 1.

C. Formulation of $\bar{w}(\zeta,p)$ in Convolution Form

It is expedient to define $\bar{w}^{(1)}(\zeta,p)$, $\bar{w}^{(2)}(\zeta,p)$, $w^{(1)}(\zeta,\tau)$ and $w^{(2)}(\zeta,\tau)$ such that

$$\bar{w}(\zeta,p) = \bar{w}^{(1)}(\zeta,p)\bar{w}^{(2)}(\zeta,p), \tag{4.83}$$

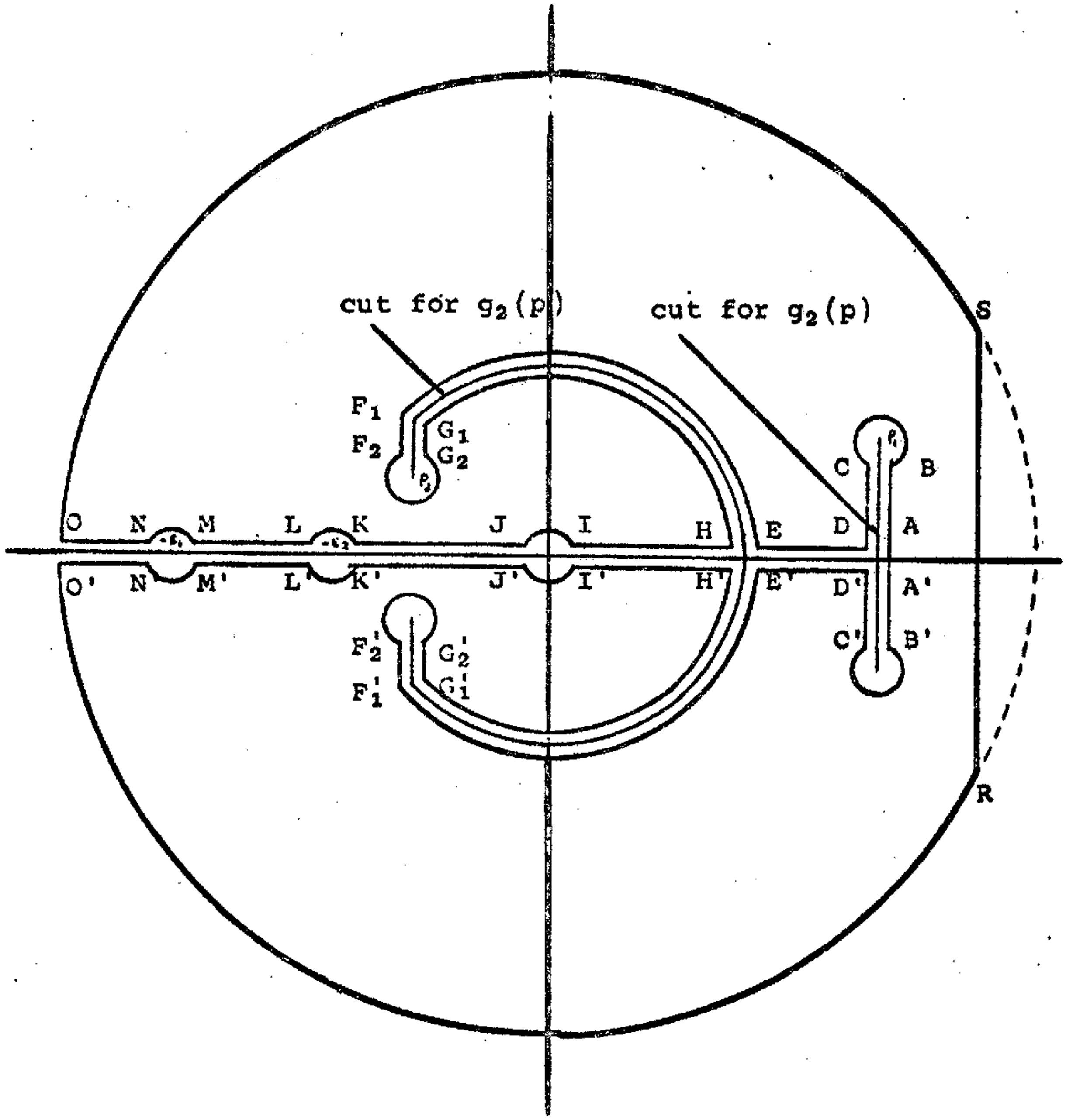

Figure 1. The p-plane.

$$w^{(i)}(\zeta,\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{w}^{(i)}(\zeta,p)\exp(p\tau)\,dp \quad \text{with } i = 1,2 \tag{4.84}$$

so that by the convolution theorem

$$w(\zeta,\tau) = \int_0^\tau w^{(1)}(\zeta,u)w^{(2)}(\zeta,\tau-u)\,du. \tag{4.85}$$

From (4.50), (4.56), (4.60), (4.64) and (4.65) set

$$\bar{w}^{(1)}(\zeta,p) = \frac{1}{p^{1/2}(p+b)} \tag{4.86}$$

$$\begin{aligned}\bar{w}^{(2)}(\zeta,p) = {} & \Big[\frac{A_1}{(r-s^2)p(p-a)}\,\frac{1}{g_1g_2}\{\delta_1^2+\delta_2-ps^2+\frac{1}{2}((ps^2-\delta_2)(p+t) \\ & +(pr+t)+2t\delta_1)\} + \frac{(p+b)\{(ps^2-\delta_2)d_1+\delta_1d_2\}}{pg_1g_2} \\ & - \frac{A_1}{2(r-s^2)p(p-a)}\Big]\exp\{-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta\} \\ & -\Big[\frac{A_1}{(r-s^2)p(p-a)}\,\frac{1}{g_1g_2}\{\delta_1^2+\delta_2-ps^2+\frac{1}{2}((ps^2-\delta_2)(p+t) \\ & +(pr+t)+2t\delta_1)\} + \frac{(p+b)\{(ps^2-\delta_2)d_1+\delta_1d_2\}}{pg_1g_2} \\ & + \frac{A_1}{2(r-s^2)p(p-a)}\Big]\exp\{-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta\} \\ & + \frac{A_1}{(r-s^2)p(p-a)}\exp(-p^{1/2}\zeta)\end{aligned} \tag{4.87}$$

with a and b in (4.86) and (4.87) defined from (4.59) such that

$$A_2(p) = (r-s^2)pp^{1/2}(p-a)(p+b). \tag{4.88}$$

Here a and $-b$ are zeros of

$$ps^2-\delta_1^2-\delta_2-t-pr-(p+t)(ps^2-\delta_2)-2t\delta_1+pt+ptr = 0$$

and they are

$$a = \frac{1}{2(r-s^2)}\{r+ts^2-s^2-\delta_2-t-tr-[(s^2-r-ts^2+\delta_2+t+tr)^2 \\ -4(r-s^2)(-\delta_1^2-\delta_2-t+t\delta_2-2t\delta_1)]^{1/2}\} \tag{4.89}$$

$$b = \frac{1}{2(r-s^2)}\{r+ts^2-s^2-\delta_2-t-tr+[(s^2-r-ts^2+\delta_2+t+tr)^2 \\ -4(r-s^2)(-\delta_1^2-\delta_2-t+t\delta_2-2t\delta_1)]^{1/2}\} \tag{4.90}$$

and we find that $a > 0$, $b > 0$ by the relation (4.37).

The inversion of $\bar{w}^{(1)}(\zeta,p)$ of (4.86) is found with the aid of the table [37]

$$w^{(1)}(\zeta,\tau) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\tau}} \exp(-b(\tau-z^2))\, dz. \tag{4.91}$$

To obtain a real integral representation of $w^{(2)}(\zeta,\tau)$ for $\tau > \zeta$, consider the integral

$$\frac{1}{2\pi i} \int \bar{w}^{(2)}(\zeta,p)\exp(p\tau)\, dp \tag{4.92}$$

evaluated along the closed contour shown in Figure 1.

D. Inversion of $\bar{w}^{(2)}(\zeta,p)$ by Contour Integration

We express $\bar{w}^{(2)}(\zeta,p)$ in a simpler form from (4.87);

$$\bar{w}^{(2)}(\zeta,p) = \left(\frac{A_3(p)}{(r-s^2)p(p-a)}\frac{1}{g_1g_2} - \frac{A_1}{2(r-s^2)p(p-a)}\right)\exp\left(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta\right) \\ -\left(\frac{A_3(p)}{(r-s^2)p(p-a)}\frac{1}{g_1g_2} + \frac{A_1}{2(r-s^2)p(p-a)}\right)\exp\left(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta\right) \\ + \frac{A_1}{(r-s^2)p(p-a)}\exp(-p^{1/2}\zeta) \tag{4.93}$$

where

$$
\begin{aligned}
A_3(p) = {} & \tfrac{1}{2}d_1s^2(r-s^2)p^3+(s^2(\tfrac{1}{2}rd_1+d_1\delta_2-\tfrac{1}{2}\delta_1d_2) \\
& + r(\delta_1d_2-\tfrac{1}{2}d_1\delta_2-\tfrac{1}{2}d_1r)+ts^2(-\tfrac{1}{2}d_1s^2+\tfrac{1}{2}d_1+\tfrac{1}{2}d_1r+\tfrac{1}{2}d_2))p^2 \\
& + (r(-d_1\delta_1^2-\tfrac{1}{2}d_1\delta_2-\tfrac{1}{2}d_2\delta_1)+\tfrac{1}{2}d_2\delta_1\delta_2-\tfrac{1}{2}d_1\delta_2^2+t(s^2(\tfrac{1}{2}d_1-d_2 \\
& + d_1\delta_2-d_1\delta_1-\tfrac{1}{2}d_2\delta_1)-\tfrac{1}{2}d_1r\delta_2-d_1r-d_1r\delta_1+\delta_1d_2+r\delta_1d_2-d_1\delta_2 \\
& + \tfrac{1}{2}d_1\delta_2-\tfrac{1}{2}d_2\delta_2+\tfrac{1}{2}rd_2)+t^2(\tfrac{1}{2}d_2s^2-\tfrac{1}{2}d_1s^2))p+\tfrac{1}{2}t((\delta_1d_2 \\
& -d_1\delta_2)(\delta_2-1-2\delta_1)+2(d_2-d_1)(\delta_1^2+\delta_2)) \\
& + \tfrac{1}{2}t^2(d_2-d_1)(-\delta_2+1+2\delta_1).
\end{aligned} \tag{4.94}
$$

With the aid of (4.81) and (4.82) we find that

$$
\left.
\begin{aligned}
& \exp(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta+p\tau) = O(\exp(p(\tau-\zeta)) \text{ as } p \to \infty, \\
& \exp(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta+p\tau) = O(\exp(p\tau-\frac{r^{1/2}p^{1/2}}{s}))
\end{aligned}
\right\} \tag{4.95}
$$

$$
\text{as} \quad p \to \infty, \tag{4.96}
$$

and consequently as $p \to \infty$

$$
\bar{w}^{(2)}(\zeta,p)\exp(p\tau) = O(\frac{1}{p}\exp(p(\tau-\zeta))+O(\frac{1}{p}\exp(p\tau-\frac{r^{1/2}\zeta p^{1/2}}{s}))
$$

$$
+ O(\frac{1}{p}\exp(p\tau-p^{1/2}\zeta)). \tag{4.97}
$$

From (4.97) we conclude that the contribution to the integral (4.92) from the large circular portion of the contour goes to zero as the radius tends to infinity when $\tau > \zeta$. From (4.73) and (4.93) we see that the contributions from the small circles around the branch points of $g_2(p)$ go to zero as the radii tend to zero. Let us consider the contribution to the integral from the small circle around the origin as the radius tend to zero. We may express $\bar{w}^{(2)}(\zeta,p)$ of (4.93) as

$$\bar{w}^{(2)}(\zeta,p) = \frac{A_3(p)}{(r-s^2)p(p-a)}\frac{1}{g_1g_2}[\exp(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta)$$

$$-\exp(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta)]$$

$$-\frac{A_1}{2(r-s^2)p(p-a)}[\exp(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta)$$

$$+\exp(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta)-2\exp(-p^{1/2}\zeta)]. \quad (4.98)$$

We note that

$$\exp(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta)-\exp(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta)$$

$$= -\frac{p^{1/2}}{f_1}g_2\zeta + O(p) \quad \text{as} \quad p \to 0, \quad (4.99)$$

$$\exp(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta)+\exp(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta)-2\exp(-p^{1/2}\zeta)$$

$$= 2p^{1/2}\zeta(1-\frac{g_1}{2f_1})+O(p) \quad \text{as} \quad p \to 0, \quad (4.100)$$

Combining (4.98), (4.99) and (4.100), the contribution to the integral (4.92) along the small circle around the origin goes to zero as the radius tends to zero. Also, from (4.93) we see that the contribution from the small circle around $p = -\epsilon_2$ goes to zero as the radius tends to zero because the integrand is bounded around $p = -\epsilon_2$. These considerations, with the aid of (4.89), (4.93) and Cauchy's integral theorem, lead to

$$\lim \left(\frac{1}{2\pi i} \int \bar{w}^{(2)}(\zeta,p) \exp(p\tau)dp\right)$$

$$\equiv w^{(2)}(\zeta,\tau) + \frac{1}{2\pi i} \lim\Big(\int_{ON} + \int_{NM} + \int_{ML} + \int_{KJ} + \int_{HG_1} + \int_{G_1G_2} + \int_{F_2F_1}$$

$$+ \int_{F_1E} + \int_{DC} + \int_{BA} + \int_{N'O'} + \int_{M'N'} + \int_{L'M'} + \int_{J'K'}$$

$$+ \int_{G_1'H'} + \int_{G_2'G_1'} + \int_{F_1'F_2'} + \int_{E'F_1'} + \int_{C'D'} + \int_{A'B'}\Big)$$

$$= \left(\frac{A_3(a)}{(r-s^2)ag_1(a)g_2(a)} - \frac{A_1(a)}{2a(r-s^2)}\right)\exp\left(-\frac{a^{1/2}}{2f_1(a)}(g_1(a)+g_2(a))\zeta+a\tau\right)$$

$$-\left(\frac{A_3(a)}{(r-s^2)ag_1(a)g_2(a)} + \frac{A_1(a)}{2a(r-s^2)}\right)\exp\left(-\frac{a^{1/2}}{2f_1(a)}(g_1(a)-g_2(a))\zeta+a\tau\right)$$

$$+ \frac{A_1(a)}{a(r-s^2)} \exp(-a^{1/2}\zeta+a\tau) \tag{4.101}$$

in which the integrand of the integrals in the brackets is $\bar{w}^{(2)}(\zeta,p)\exp(p\tau)$ and "lim" refer to the limit process such that the large radius tends to infinity and the small radii tend to zero. The values of $p^{1/2}$, $f_1(p)$, $g_1(p)$, $g_2(p)$, which are needed to evaluate the integrals in the parenthesis are to be determined consistently with the construction of the Riemann sheet described in (4.70), (4.71), (4.72), (4.79) and (4.80).

D-1. Evaluation of $g_i(p)$ along the Contour

For this purpose it is expedient to introduce new functions

$$z_i(p) = g_i^2(p) \qquad i = 1,2 \tag{4.102}$$

$$G(Z_i) = Z_i^{1/2} \qquad i = 1,2 \tag{4.103}$$

where we define a cut for $G(Z_i)$ to be the negative real axis on the Z_i-plane such that

$$G(\ell) = \sqrt{\ell} \qquad \text{for} \qquad \ell > 0$$

and (4.103a)

$$G(-\ell) = i\sqrt{\ell} \qquad \text{for} \qquad \ell > 0 \; .$$

Then we shall choose proper signs along the contour in Fig. 1 in the expression

$$g_i(p) = \pm\, G(Z_i)$$

so that $g_i(p)$ for $i = 1,2$ are consistent with the Riemann sheet described in (4.70) to (4.72), (4.79) and (4.80). Due to (4.79) and (4.80) we may utilize the Schwarz reflection principle, and in this case we have

$$g_i(p^*) = g_i^*(p), \tag{4.104}$$

and since the contour in Fig. 1 is symmetric with respect to the real line, we determine the value of $g_i(p)$ along the contour which lies in the upper half-plane only and utilize (4.104) for the lower half-plane. Considering the mapping of the contour in the half-plane of Fig. 1 into the Z_i-plane, and then into the G-plane under the restriction of (4.37), we find that

$$g_1(p) = G(Z_1(p)) \text{ along } HG_1G_2,\ F_2F_1E,\ DC,\ BA,\ JK,\ LM \tag{4.105}$$

$$g_1(p) = -\,G(Z_1(p)) \quad \text{along NO} \tag{4.106}$$

$$g_2(p) = G(Z_2(p)) \quad \text{along } HG_1G_2,\ BA,\ JK,\ LM \tag{4.107}$$

$$g_2(p) = -\,G(Z_2(p)) \text{ along } F_2F_1E,\ DC,\ NO. \tag{4.108}$$

We are now ready to evaluate the integrals in the parenthesis of (4.101).

D-2. Integration along ON-N'O' for $\tau \gtrsim \zeta$ and $\tau < \zeta^*$

Along ON, $p = -\ell$ with $\epsilon_1 < \ell < \infty$, and by (4.106) we find that

$$g_1 = -\sqrt{\frac{X + \sqrt{X^2 + Y^2}}{2}} + i\sqrt{\frac{-X + \sqrt{X^2 + Y^2}}{2}} \qquad (4.109)$$

where

$$X = s^2\ell^2 - (r+ts^3-\delta_2)\ell + t + 2t\delta_1 - t\delta_2$$

$$Y = 2r^{1/2}s\ell^{1/2}(\ell-\epsilon_2)^{1/2}(\ell-\epsilon_1)^{1/2}.$$

The contribution to the integral of (4.92) along the contour ON-N'O' is then

$$\frac{1}{\pi}\int_{\epsilon_1}^{\infty} \left(\frac{-A_1(-\ell)}{(r-s^2)\ell(\ell+a)}\sin(\sqrt{\ell}\zeta) + \left(\frac{A_3(-\ell)}{(r-s^2)\ell(\ell+a)|g_1|^2}\right.\right.$$

$$\left.\left. - \frac{A_1(-\ell)}{2(r-s^2)\ell(\ell+a)}\right)\sin\left(\frac{\sqrt{\ell}}{s\sqrt{\ell-\epsilon_1}}(\text{Im } g_1)\zeta\right)\right) \exp(-\ell\tau)\, d\ell \qquad (4.110)$$

for $\tau \gtrsim \zeta$ and $\tau < \zeta$.

D-3. Integration along the Arc NM-M'N' for $\tau \gtrsim \zeta$ and $\tau < \zeta^*$

We next consider the contribution to the integral of (4.92) along the arcs NM-M'N'. Near $p = -\epsilon_1$, we have that $p^{1/2}(g_1-g_2)$ and

---

*For $\tau < \zeta$, see Section D-7.

$\dfrac{A_3(p)}{(r-s^2)p(p-a)g_1g_2} + \dfrac{A_1(p)}{2(r-s^2)p(p-a)}$ are analytic due to the fact that along ML, $p = -\ell$ and

$$g_1 = i\sqrt{-s^2\ell^2+(r+ts^2-\delta_2)\ell-(t+2t\delta_1-t\delta_2)+2s\sqrt{r}\,\sqrt{\ell}\sqrt{\ell-\epsilon_2}\sqrt{\epsilon_1-\ell}}, \quad (4.111)$$

$$g_2 = g_1^*$$

and along ON,

$g_2 = g_1^*$ where g_1 is given by (4.109).

Relations (4.52), (4.53), (4.54), (4.66) and (4.67) lead to

$$g_1^2 - g_2^2 = 4r^{1/2}f_1p^{1/2}(p+\epsilon_2)^{1/2}$$

or

$$\frac{g_1+g_2}{2f_1} = \frac{2r^{1/2}p^{1/2}(p+\epsilon_2)^{1/2}}{g_1-g_2}, \quad (4.112)$$

and this last relation shows that the contribution to the integral (4.92) from the summand of the integrand, i.e.,

$$\left(\frac{A_3(p)}{(r-s^2)p(p-a)g_1g_2} - \frac{A_1(p)}{2(r-s^2)p(p-a)}\right)\exp\left(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta+p\tau\right)$$

$$+\frac{A_1(p)}{(r-s^2)p(p-a)}\exp(-p^{1/2}\zeta+p\tau),$$

vanishes as the radius of the arc NM-M'N' approaches to zero. But the contribution to the integral (4.92) from the summand of the integrand, i.e.,

$$\left(\frac{A_3(p)}{(r-s^2)p(p-a)g_1g_2} - \frac{A_1(p)}{2(r-s^2)p(p-a)}\right)\exp\left(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\zeta+p\tau\right)$$

is not readily established because $f_1(p)$ has a branch point at $p = -\epsilon_1$. We consider the mapping:

$$u = (p+\epsilon_1)^{1/2} \quad (4.113)$$

which maps the contour in Fig. 2 onto the semi-circle C' in Fig. 3 with the direction clockwise in both cases. We define

$$I_R = \frac{1}{2\pi i}\int_{C'} I(u)\,\exp\left(-\frac{v(u)}{2u} + (u^2 - \epsilon_1)\tau\right) du \qquad (4.114)$$

where

$$I(u) = \frac{2uA_3(u^2-\epsilon_1)}{(r-s^2)(u^2-\epsilon_1)(u^2-\epsilon_1-a)g_1(u^2-\epsilon_1)g_2(u^2-\epsilon_1)} - \frac{uA_1(u^2-\epsilon_1)}{(r-s^2)(u^2-\epsilon_1)(u^2-\epsilon_1-a)}, \qquad (4.115)$$

$$v(u) = (u^2 - \epsilon_1)^{1/2}(g_1(u^2-\epsilon_1)-g_2(u^2-\epsilon_1)). \qquad (4.116)$$

Now we consider the branch of $(p + \epsilon_1)^{1/2}$ such that

$$(p+\epsilon_1)^{1/2} = -\sqrt{\ell+\epsilon_1} \quad \text{for} \quad p = \ell > -\epsilon_1, \qquad (4.117)$$

with the cut from $-\epsilon_1$ to $-\infty$ along the negative real axis. We consider the mapping

$$u = (p + \epsilon_1)^{1/2}. \qquad (4.118)$$

If we were to go over the circular arc twice in Fig. 2, then we see that the mapping (4.118) will map the circular arc into the semi-circle C" shown in Fig. 3. As we did in (4.114), we define

$$I_L = \frac{1}{2\pi i}\int_{C''} I(u)\exp\left(-\frac{v(u)}{2u} + (u^2-\epsilon_1)\tau\right) du. \qquad (4.119)$$

If we add (4.114) and (4.119), then by the residue theorem

$$I_R + I_L = \text{Residue}\left(I(u)\exp\left(-\frac{v(u)}{2u} + (u^2-\epsilon_1)\tau\right)\right) \qquad (4.120)$$

where $u = 0$ is the only singularity enclosed by the contours

C'-C". Consider Now I_L as given in (4.119) where the branch defined by (4.116) is used. Due to (4.118), (4.119), we see that the integration I_L of (4.119) vanishes as the radius of the arc NM-M'N' approaches zero.

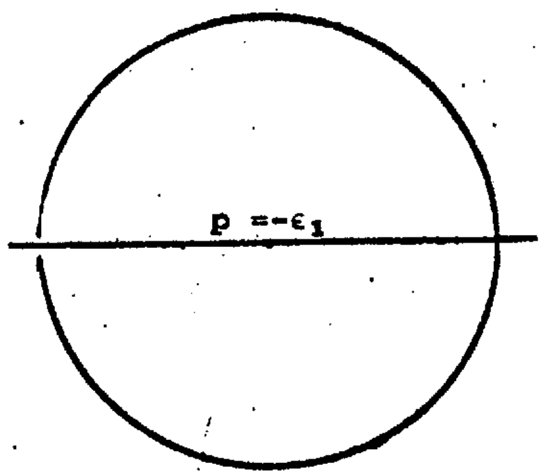

Fig. 2

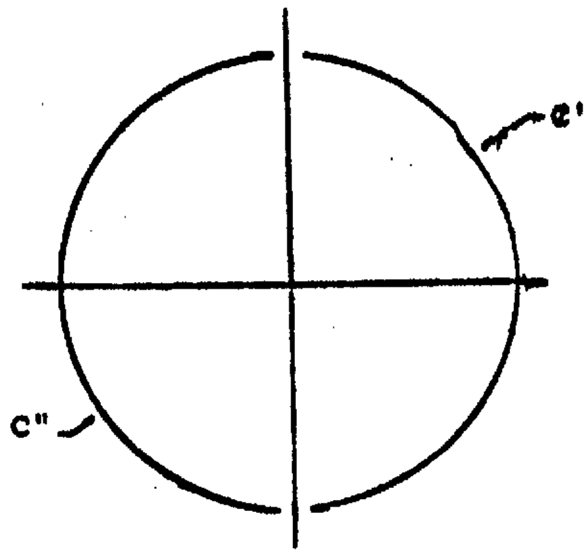

Fig. 3

D-4. Integration along KJ-J'K' for $\tau \geq \zeta$

Along KJ, $p = -\ell$ with $0 < \ell < \epsilon_2$, and by (4.105) we find that

$$g_1 = \sqrt{\frac{X + \sqrt{X^2 + Y^2}}{2}} + i\sqrt{\frac{-X + \sqrt{X^2 + Y^2}}{2}} \qquad (4.121)$$

where

$$X = s^2\ell^2 - (r + ts^2 - \delta_2)\ell + t + 2t\delta_1 - t\delta_2$$

$$Y = 2r^{1/2}s\ell^{1/2}(\epsilon_1 - \ell)^{1/2}(\epsilon_2 - \ell)^{1/2}.$$

For $\tau \gtrsim \zeta$, the contribution to the integral of (4.92) along the contour KJ-J'K' is found to be

$$\frac{1}{\pi}\int_0^{\epsilon_2}(\sin(\frac{\sqrt{\ell}}{s\sqrt{\epsilon_1-\ell}}(\mathrm{Re}\ g_1)\zeta)(\frac{A_1(-\ell)}{2(r-s^2)\ell(\ell+a)}-\frac{A_3(-\ell)}{(r-s^2)\ell(\ell+a)|g_1|^2})$$

$$-\sin(\sqrt{\ell}\zeta)\frac{A_1(-\ell)}{(r-s^2)\ell(\ell+a)})\exp(-\ell\tau)\,d\ell \qquad (4.122)$$

where $A_1(p)$, $A_3(p)$ and "a" are given by (4.58), (4.94) and (4.89).

D-5. Integration along ML-L'M' for $\tau \gtrsim \zeta$ and $\tau < \zeta$*

Along ML, $p = -\ell$ with $\epsilon_2 < \ell < \epsilon_1$, and with the aid of (4.105) and (4.107) we find that the contribution to the integral of (4.92) along the countour ML-L'M' is

$$\frac{1}{\pi}\int_{\epsilon_2}^{\epsilon_1}\frac{A_1(-\ell)}{(s^2-r)\ell(\ell+a)}\sin(\sqrt{\ell}\ \zeta)\ d\ell \qquad (4.123)$$

for $\tau \gtrsim \zeta$ and $\tau < \zeta$.

D-6. Integration along DC-BA-A'B'-C'D' and HG_1-G_1G_2-F_2F_1-F_1E-$E'F_1'$-$F_1'F_2'$-$G_2'G_1'$-$G_1'H_1'$ for $\tau \gtrsim \zeta$

For $\tau \gtrsim \zeta$, the utilization of (4.104), (4.105) and (4.108) lead that the contribution to the integral (4.92) along the contour DC-BA-A'B'-C'D' and HG_1-G_1G_2-F_2F_1-F_1E-$E'F_1'$-$F_1'F_2'$-$G_2'G_1'$-$G_1'H_1'$ vanishes. (4.124)

---

*For $\tau < \zeta$, see this section D-7.

D-7. Modifications on Integrations for $\tau < \zeta$

When $\tau < \zeta$, the contribution to the integral of (4.92) from the summand of the integrand,

$$\left(\frac{A_3(p)}{(r-s^2)p(p-a)g_1g_2} - \frac{A_1(p)}{2(r-s^2)p(p-a)}\right)\exp\left(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta+p\tau\right) \tag{4.125}$$

along the Bromwich contour vanishes because we let the contour for this summand be closed by the right arc of the circle which is shown as a dotted curve in Fig. 1. Then the contribution from this circular arc to the integral of (4.125) approaches to zero as the radius gets large. Hence for $\tau < \zeta$, some modifications should be made on (4.122) and (4.124), but the results in (4.110), (4.120), (4.123) are not affected.

D-8. Integration along DC-BA-A'B'-C'D' and HG_1-G_1G_2-F_2F_1-F_1E-$E'F_1'$-$F_1'F_2'$-$G_2'G_1'$-$G_1'H_1'$ for $\tau < \zeta$

Along DC, $p = \text{Re}\, p_1 + i\ell$ with $0 \leq \ell \leq \text{Im}\, p_1$ and we let

$$\begin{aligned} p^{1/2} &= a_1 + ia_2 \\ (p+\epsilon_2)^{1/2} &= b_1 + ib_2 \\ (p+\epsilon_1)^{1/2} &= c_1 + ic_2 . \end{aligned} \tag{4.126}$$

With the aid of (4.52), (4.53), (4.102) and (4.103), we have

$$Z_1 = X_1 + iY_1 \tag{4.127}$$

where

$$X_1 = s^2(\mathrm{Re}^2 p_1-\ell^2)+r+ts^2-\delta_2)\mathrm{Re}\ p_1+t+2t\delta_1-t\delta_2$$
$$+\ 2r^{1/2}s(c_1(a_1b_1-a_2b_2)-c_2(a_1b_2+a_2b_1))$$

$$Y_1 = 2\ell s^2\mathrm{Re}\ p_1+\ell(r+ts^2-\delta_2)+2r^{1/2}s(c_1(a_1b_2-a_2b_1)$$
$$+c_2(a_1b_1-a_2b_2))$$

and

$$Z_2 = X_2 + iY_2 \tag{4.128}$$

where

$$X_2 = s^2(\mathrm{Re}^2 p_1-\ell^2)+(r+ts^2-\delta_2)\mathrm{Re}\ p_1+t+2t\delta_1-t\delta_2$$
$$-\ 2r^{1/2}s(c_1(a_1b_1-a_2b_2)-c_2(a_1b_2\ \ a_2b_1))$$

$$Y_2 = 2\ell s^2\mathrm{Re}\ p_1+\ell(r+ts^2-\delta_2)-2r^{1/2}s(c_1(a_1b_2-a_2b_1)$$
$$+\ c_2(a_1b_1-a_2b_2)).$$

From (4.105) and (4.108), we have that

$$g_1 = G(Z_1)$$
$$g_2 = -G(Z_2)$$

and the contribution to the integral of (4.92) along DC-BA -A'B'-C'D' when $\tau < \zeta$, is

$$\frac{1}{\pi}\int_0^{\mathrm{Im}\ p_1}\mathrm{Re}[\frac{\exp(-\frac{p^{1/2}}{2f_1}\ g_1\zeta+p\tau)}{(s^2-r)p(p-a)}\{A_1(p)\sinh(\frac{p^{1/2}}{2f_1}\ g_2\zeta)$$
$$-\ \frac{2A_3(p)}{g_1g_2}\cosh(\frac{p^{1/2}}{2f_1}\ g_2\zeta)\}]d\ell. \tag{4.129}$$

Along F_2F_1, $p = \mathrm{Re}\ p_2 + i\ell$ with $\mathrm{Im}\ p_2 \le \ell \le 2^{1/2}\mathrm{Re}\ p_2$, and by (4.105) and (4.108)

$$g_1 = G(Z_1)$$

$$g_2 = -G(Z_2)$$

where Z_1, Z_2 are given by (4.127) and 4.128), and the contribution to the integral of (4.92) along $G_1G_2-F_2F_1-F_1'F_2'-G_2'G_1'$ for $\tau < \zeta$ is

$$\frac{1}{\pi}\int_{\mathrm{Im}\ p_2}^{\sqrt{2}\mathrm{Re}\ p_2} \mathrm{Re}\Big[\frac{\exp(-\frac{p^{1/2}}{2f_1}g_1\zeta+p\tau)}{(s^2-r)p(p-a)}(A_1(p)\sinh(\frac{p^{1/2}}{2f_1}\ g_2\zeta)$$

$$-\frac{2A_3(p)}{g_1g_2}\cosh(\frac{p^{1/2}}{2f_1}\ g_2\zeta))\Big]\ d\ell\ . \qquad (4.130)$$

Along G_1H, $p = 2^{1/2}\mathrm{Re}\ p_2\exp(i\theta)$ with $0 \leq \theta \leq 3\pi/4$, and by (4.105) and (4.107)

$$g_1 = G(Z_1)$$

$$g_2 = G(Z_2)$$

where Z_1, Z_2 are given by (4.127) and 4.128), and the contribution to the integral of (4.92) along $HG_1-F_1E-E'F_1'-G_1'H'$ for $\tau < \zeta$ is

$$\frac{2^{1/2}(\mathrm{Re}\ p_2)}{\pi}\int_{3\pi/4}^{0}\{\sin\theta[\mathrm{Im}\{\frac{\exp(-\frac{p^{1/2}}{2f_1}\ g_1\zeta+p\tau)}{(r-s^2)p(p-a)}\ [-A_1(p)\sinh(\frac{p^{1/2}}{2f_1}g_2\zeta)$$

$$-\frac{2A_3(p)}{g_1g_2}\cosh(\frac{p^{1/2}}{2f_1}\ g_2\zeta)]\}]$$

$$+\cos\theta[\mathrm{Re}\{\frac{\exp(-\frac{p^{1/2}}{2f_1}\ g_1\zeta+p\tau)}{(r-s^2)p(p-a)}(A_1(p)\sinh(\frac{p^{1/2}}{2f_1}\ g_2\zeta)$$

$$+\frac{2A_3(p)}{g_1g_2}\cosh(\frac{p^{1/2}}{2f_1}\ g_2\zeta))]\}d\theta \qquad (4.131)$$

D-9. Integration along KJ-J'K' for $\tau < \zeta$

The contribution to the integral of (4.92) along KJ-J'K' for $\tau < \zeta$ is easily deduced from (4.122) as

$$\frac{1}{\pi}\int_{\epsilon_3}^{0} \exp(-\ell\tau)\sin(\sqrt{\ell}\zeta)\ \frac{A_1(-\ell)}{(r-s^2)\ell(\ell+a)}\ d\ell. \tag{4.132}$$

D-10. $w^{(2)}(\zeta,\tau)$ Obtained by Inversion of $\bar{w}^{(2)}(\zeta,p)$

We consider two cases, i.e., $\tau \geq \zeta$ and $\tau < \zeta$. For $\tau \geq \zeta$, we have that from (4.101)

$$w^{(2)}(\zeta,\tau) = \frac{1}{a(r-s^2)}[(\frac{A_3(a)}{g_1(a)g_2(a)} - \frac{A_1(a)}{2})\exp\{-\frac{a^{1/2}}{2f_1(a)}(g_1(a)$$

$$+ g_2(a))\zeta+a\tau\} -(\frac{A_3(a)}{g_1(a)g_2(a)} + \frac{A_1(a)}{2})\exp\{-\frac{a^{1/2}}{2f_1(a)}(g_1(a)$$

$$- g_2(a))\zeta+a\tau\}+A_1(a)\exp(-a^{1/2}\zeta+a\tau)] - I_1(\zeta,\tau) \tag{4.133}$$

where $I_1(\zeta,\tau) = \frac{1}{2\pi i}\lim(\int_{ON} + \int_{NM} + \int_{ML} + \int_{KJ}$

$$+ \int_{N'O'} + \int_{M'N'} + \int_{L'M'} + \int_{J'K'})$$

whose integrals are evaluated in (4.110), (4.120), (4.122) and (4.123). For $\tau < \zeta$, we have that from (4.101) and the vanishing of the integration of (4.125)

$$w^{(2)}(\zeta,\tau) = \frac{1}{a(r-s^2)}(A_1(a)\exp(-a^{1/2}\zeta+a\tau)-(\frac{A_3(a)}{g_1(a)g_2(a)}$$

$$+ \frac{A_1(a)}{2})\exp(-\frac{a^{1/2}}{2f_1(a)}(g_1(a)-g_2(a))\zeta+a\tau))-I_2(\zeta,\tau) \tag{4.134}$$

where

$$I_2(\zeta,\tau) = \frac{1}{2\pi i} \lim\Big(\int_{ON} + \int_{NM} + \int_{ML} + \int_{KJ} + \int_{HG_1} + \int_{G_1G_2} + \int_{F_2F_1}$$
$$+ \int_{F_1E} + \int_{DC} + \int_{BA} + \int_{N'O'} + \int_{M'N'} + \int_{L'M'} + \int_{J'K'} + \int_{G_1'H'}$$
$$+ \int_{G_2'G_1'} + \int_{F_1'F_2'} + \int_{E'F_1'} + \int_{C'D'} + \int_{A'B'} \Big)$$

Those integrals are evaluated in (4.110), (4.120), (4.123), (4.129), (4.132), (4.130) and (4.131).

E. Inversion of $\bar{w}(\zeta,p)$ in Real Integral Form

Combining (4.133), (4.134), (4.85) and (4.86) leads to the complete description of the displacement field of the solid component $\hat{w}(\zeta,\tau)$

$$\hat{w}(\zeta,\tau) = \frac{2}{\sqrt{\pi}} \int_0^{\tau} \int_0^{\sqrt{u}} \exp(-b(u-z^2))\, dz\, \Phi^{(2)}(\zeta,\tau-u)\, du. \quad (4.135)$$

From (4.133), (4.134) and (4.135) we see that $\hat{w}(\zeta,\tau)$ satisfies the boundary and regularity conditions specified by (4.33) and (4.34).

Since the material constants have to be determined by experiment for the mixture and such an experiment has not yet been devised, we shall not attempt any further investigation about the behavior of the displacement field of the solid component at this point, even though we have the exact solution given by (4.135) which may be evaluated numerically by computers.

4.5. Early Time Solutions

The numerical evaluation of (4.135) does not seem to be an easy task. One way to avoid this difficulty is to represent $\bar{w}(\zeta,p)$ in a power series with respect to $\frac{1}{p}$ for sufficiently large p, and then invert the resulting expression term by term. This procedure leads to an early time solution for $\hat{w}(\zeta,\tau)$.

As $p \to \infty$, we have that

$$2r^{1/2}sp^{1/2}(p+\epsilon_1)^{1/2}$$

$$= 2pp^{1/2}r^{1/2}s[1+\frac{M_1}{p}+\frac{M_2}{p^2}+\frac{M_3}{p^3}+\frac{M_4}{p^4}+\frac{M_5}{p^5}+O(\frac{1}{p^6})] \qquad (4.136)$$

where

$$M_1 = \frac{s^2(t+tr)-r(\delta_1^2+\delta_2)}{2rs^2}$$

$$M_2 = \frac{-(r^2(\delta_1^2+\delta_2)^2+2rs^2(t+tr)(\delta_1^2+\delta_2)+s^4(t+tr)^2)}{8r^3s^4}$$

$$M_3 = \frac{1}{16}(-\frac{(\delta_1^2+\delta_2)^3}{s^6}+\frac{(\delta_1^2+\delta_3)(t+tr)^2}{r^2s^2}$$

$$-\frac{(\delta_1^2+\delta_2)^2(t+tr)}{rs^4}+\frac{(t+tr)^3}{r^3})$$

$$M_4 = \frac{-1}{128}(\frac{5(t+tr)^4}{r^4}+\frac{20(t+tr)^3(\delta_1^2+\delta_2)}{r^3s^2}-\frac{2(t+tr)^2(\delta_1^2+\delta_2)^2}{r^2s^4}$$

$$\frac{5(t+tr)(\delta_1^2+\delta_2)^3}{rs^6}+\frac{5(\delta_1^2+\delta_2)^4}{s^8}$$

$$M_5 = \frac{1}{256}\left(\frac{7(t+tr)^5}{r^5} + \frac{5(t+tr)^4(\delta_1^2+\delta_2)}{r^4s^2} - \frac{2(t+tr)^3(\delta_1^2+\delta_1)^2}{r^3s^4}\right.$$

$$\left. + \frac{2(t+tr)^2(\delta_1^2+\delta_2)^3}{r^2s^6} - \frac{5(t+tr)(\delta_1^2+\delta_2)^4}{rs^8} - \frac{7(\delta_1^2+\delta_2)^5}{s^{10}}\right).$$

With the aid of (4.68), (4.69), (4.79), (4.80) and (4.136), we have that as $p \to \infty$

$$g_1(p) = ps[1 + \frac{N_1}{p^{1/2}} + \frac{N_2}{p} + \frac{N_3}{pp^{1/2}} + \frac{N_4}{p^2} + \frac{N_5}{p^2p^{1/2}} + \frac{N_6}{p^4} + \frac{N_7}{p^4p^{1/2}}$$

$$+ \frac{N_8}{p^4} + \frac{N_9}{p^4p^{1/2}} + \frac{N_{10}}{p^5} + \frac{N_{11}}{p^5p^{1/2}} + O(\frac{1}{p^6})] \quad (4.137a)$$

$$g_2(p) = ps[1 - \frac{N_1}{p^{1/2}} + \frac{N_2}{p} - \frac{N_3}{pp^{1/2}} + \frac{N_4}{p^2} - \frac{N_5}{p^2p^{1/2}} + \frac{N_6}{p^3} - \frac{N_7}{p^3p^{1/2}}$$

$$+ \frac{N_8}{p^4} - \frac{N_9}{p^4p^{1/2}} + \frac{N_{10}}{p^5} - \frac{N_{11}}{p^5p^{1/2}} + O(\frac{1}{p^6})] \quad (4.137b)$$

where

$$N_1 = \frac{r^{1/2}}{s}$$

$$N_2 = \frac{(ts^2 - \delta_2)}{2s^2}$$

$$N_3 = \frac{M_1r(ts^2-\delta_2)}{2s^4}$$

$$N_4 = \frac{t(1+2\delta_1-\delta_2)}{2s^2} - N_1N_3 - \frac{N_2^2}{2}$$

$$N_5 = \frac{r^{1/2}M_2}{s} - N_1N_4 - N_2N_3$$

$$N_6 = -N_1N_5 - N_2N_4 - \frac{N_3^2}{2}$$

$$N_7 = \frac{r^{1/2}M_3}{s} - N_1N_6 - N_2N_5 - N_3N_4$$

$$N_8 = -N_1N_7 - N_2N_6 - N_3N_5 - \frac{N_4^2}{2}$$

$$N_9 = \frac{r^{1/2}M_4}{s} - N_1N_8 - N_2N_7 - N_3N_6 - N_4N_5$$

$$N_{10} = -N_1N_9 - N_2N_8 - N_3N_7 - N_4N_6 - \frac{N_5^2}{2}$$

$$N_{11} = \frac{r^{1/2}M_5}{s} - N_1N_{10} - N_2N_9 - N_3N_8 - N_4N_7 - N_5N_6 .$$

As $p \to \infty$, we have that $\frac{1}{f_1(p)}$ is, due to (4.54), (4.66),

$$\frac{1}{f_1(p)} = \frac{1}{p^{1/2}s}\Big(1 - \frac{1}{2}\frac{\epsilon_1}{p} + \frac{3}{8}\frac{\epsilon_1^2}{p^2} - \frac{3\cdot 5}{3!2^3}\frac{\epsilon_1^3}{p^3}$$

$$+ \frac{3\cdot 5\cdot 7}{4!2^4}\frac{\epsilon_1^4}{p^4} - \frac{3\cdot 5\cdot 7\cdot 9}{5!2^5}\frac{\epsilon_1^5}{p^5} + o\Big(\frac{1}{p^6}\Big)\Big). \tag{4.138}$$

Equations (4.137) and (4.138) lead to, as $p \to \infty$,

$$\exp\Big(-\frac{p^{1/2}}{2f_1}(g_1+g_2)\zeta\Big) = \exp\Big[-p\zeta-\Big(N_2-\frac{\epsilon_1}{2}\Big)\zeta\Big]\Big[1-\frac{\zeta}{p}\Big(N_4-\frac{1}{2}\epsilon_1N_2+\frac{3}{8}\epsilon_1^2\Big)$$

$$+\frac{\zeta}{p^2}\Big[-N_6+\frac{N_4\epsilon_1}{2}-\frac{3N_2\epsilon_1^2}{8}+\frac{3\cdot 5}{3!2^3}\epsilon_1^3+\frac{1}{2}\Big(N_4-\frac{\epsilon_1N_2}{2}+\frac{3}{8}\epsilon_1^2\Big)^2\zeta\Big]$$

$$+\frac{\zeta}{p^3}\Big[-N_8+\frac{N_6\epsilon_1}{2}-\frac{3N_4\epsilon_1^2}{8}+\frac{3\cdot 5}{3!2^3}N_2\epsilon_1^3-\frac{3\cdot 5\cdot 7}{4!2^4}\epsilon_1^4$$

$$+\Big(N_4-\frac{\epsilon_1N_2}{2}+\frac{3}{8}\epsilon_1^2\Big)\Big(N_6-\frac{N_4\epsilon_1}{2}+\frac{3}{8}N_2\epsilon_1^2-\frac{3\cdot 5}{3!2^3}\epsilon_1^3\Big)\zeta$$

$$-\frac{\big(N_4-\epsilon_1N_2+\frac{3}{8}\epsilon_1^2\big)^3}{3!}\zeta^2\Big] + o\Big(\frac{1}{p^4}\Big)\Big], \tag{4.139}$$

$$\exp\left(-\frac{p^{1/2}}{2f_1}(g_1-g_2)\ \zeta\right)=\exp[-p^{1/2}N_1\zeta]\,[1-\frac{(N_3-\epsilon_1N_1/2)\zeta}{p^{1/2}}$$

$$+\frac{(N_3-\epsilon_1N_1/2)^2\zeta^2}{2p}-\frac{[(\frac{3}{8}N_1\epsilon_1^2-\frac{1}{2}N_3\epsilon_1\ N_5)\zeta+(N_3-\frac{1}{2}N_1\epsilon_1)^3\zeta^3]}{pp^{1/2}}$$

$$+\frac{(N_3-\frac{1}{2}N_1\epsilon_1)(\frac{3}{8}N_1\epsilon_1^2-\frac{1}{2}N_3\epsilon_1+N_5)\zeta^2}{p^2}+O(\frac{1}{p^2p^{1/2}})]\,. \quad (4.140)$$

Let us consider $\frac{A_1(p)}{A_2(p)}$ given in (4.56), (4.57) and (4.58). As $p \rightarrow \infty$ we have

$$\frac{A_1(p)}{A_2(p)}=-\frac{d_1}{p^2p^{1/2}}[1+\frac{1}{p}(R_1-R_2)+\frac{1}{p^2}(R_2^2-R_3-R_1R_2)$$

$$+\frac{1}{p^3}(2R_2R_3-R_2^3+R_1R_2^2-R_1R_3)+\frac{1}{p^4}(R_3^2-3R_2^2R_3+R_2^4+2R_1R_2R_3-R_1R_2^3)$$

$$+\frac{1}{p^5}(-3R_2R_3^2+4R_2^3R_3-R_2^5+R_1R_3^2-3R_1R_2^2R_3+R_1R_2^4)+O(\frac{1}{p^6})] \quad (4.141)$$

where

$$R_1=\frac{\{d_2(t+\delta_1)/d_1\}-\delta_2-t}{s^2-r}$$

$$R_2=s^2-r+\delta_2+t(1+r-s^2)$$

$$R_3=-\delta_1^2-\delta_2-t(1-\delta_2+2\delta_1).$$

From (4.73) we immediately have

$$g_1^2(p)g_2^2(p)=p^4s^4(1+\frac{C_1}{p}+\frac{C_2}{p^2}+\frac{C_3}{p^3}+\frac{C_4}{p^4}) \quad (4.142)$$

where

$$C_1=\frac{2(ts^2-r-\delta_2)}{s^2}$$

$$C_2 = \frac{2t(-1+2\delta_1-\delta_2-2r)}{s^2} \quad \frac{(ts^2+r-\delta_2)^2+4r(\delta_1^2+\delta_2)}{s^4}$$

$$C_3 = \frac{2t}{s^4}[(1 + 2\delta_1-\delta_2)(ts^2+r-\delta_2)+(2+2r)(\delta_1^2 + \delta_2)]$$

$$C_4 = \frac{t^2}{s^4}(1 + 2\delta_1-\delta_2)^2.$$

With the aid of (4.142), we find the asymptotic expansion of $\frac{1}{g_1(p)g_2(p)}$ as $p \to \infty$

$$\frac{1}{g_1(p)g_2(p)} = \frac{1}{p^2s^2}(1- \frac{C_1}{2p} + \frac{1}{p^2}(- \frac{C_2}{2} + \frac{3}{8}C_1^2)$$

$$+ \frac{1}{p^3}(- \frac{C_3}{2} + \frac{3}{4}C_1C_2 - \frac{5}{2^4}C_1^3)+ \frac{1}{p^4}(- \frac{C_4}{2} + \frac{3}{8}(C_2^2+2C_1C_3)$$

$$- \frac{15}{2^4} C_1^2C_2 + \frac{3\cdot5\cdot7}{4!2^4} C_1^4) + O(\frac{1}{p^5})). \tag{4.143}$$

With the aid of (4.143), we find the asymptotic expansion of $B(p)$ and $D(p)$, given in equation (4.50), is

$$B(p) = \frac{d_1}{p^2p^{1/2}} (1 + \frac{S_1}{p} + \frac{S_2}{p^2} + \frac{S_3}{p^3} + O(\frac{1}{p^4})) \tag{4.144}$$

where

$$S_1 = \frac{\delta_1d_2-d_1\delta_2}{d_1s^2} - \frac{(ts^2+r-\delta_2-2s^2)}{2s^2} - \frac{C_1}{4}$$

$$S_2 = - \frac{C_2}{4} + \frac{3}{4^2}C_1^2- \frac{C_1}{4}(\frac{2(\delta_1d_2-d_1\delta_2)}{d_1s^2} - R_1+R_2- \frac{(ts^2+r-\delta_2-2s^2)}{s^2}$$

$$- \frac{(t(1+2\delta_1-\delta_2)+2(\delta_1^2+\delta_2)+(R_1-R_2)(ts^2+r-\delta_2-2s^2)}{2s^2}$$

$$S_3 = \frac{(-R_1+R_2)(t(1+2\delta_1-\delta_2)+2(\delta_1^2+\delta_2))}{2s^2}$$

$$+ \frac{(R_2^2+R_3+R_1R_2)(ts^2-\delta_2+r-2s^2)}{2s^2} - \frac{C_1}{4}(-R_2^2+R_3+R_1R_2$$

$$- \frac{(t(1+2\delta_1-\delta_2)+2(\delta_1^2+\delta_2))}{s^2} - \frac{(R_1-R_2)(ts^2-\delta_2+r-2s^2)}{s^2})$$

$$+ (\frac{-C_2}{4} + \frac{3}{4^2} C_1^2)(\frac{2(\delta_1 d_2-d_1\delta_2)}{d_1 s^2} - R_1+R_2 - \frac{(ts^2+r-\delta_2-2s^2)}{s^2}$$

$$+ (-\frac{C_3}{4} + \frac{3}{8}C_1C_2 - \frac{5}{2^5} C_1^3),$$

$$D(p) = \frac{d_1}{2p^3p^{1/2}} (D_0 + \frac{D_1}{p} + \frac{D_2}{p^2} + O(\frac{1}{p^3})) \qquad (4.145)$$

where

$$D_0 = 2R_1-2R_2 - \frac{2(\delta_1 d_2-d_1\delta_2)}{d_1 s^2} + \frac{(ts^2+r-\delta_2-2s^2)}{s^2} + \frac{C_1}{2}$$

$$D_1 = 2R_2^2-2R_3-2R_1R_2 + \frac{C_2}{2} - \frac{3}{8} C_1^2 + \frac{C_1}{2}(\frac{2(\delta_1 d_2-d_1\delta_2)}{d_1 s^2} - R_1+R_2$$

$$- \frac{(ts^2+r-\delta_2-2s^2)}{s^2}) + \frac{1}{s^2}(t(1+2\delta_1-\delta_2)+2(\delta_1^2+\delta_2)$$

$$+ (R_1-R_2)(ts^2-\delta_2+r-2s^2))$$

$$D_2 = 2(2R_2R_3 - R_2^3 + R_1R_2^2 - R_1R_3) + (R_1 - R_2)\frac{(t(1+2\delta_1-\delta_2)+2(\delta_1^2+\delta_2))}{s^2}$$

$$- \frac{(-R_2^2+R_3+r_1R_2)(ts^2+r-\delta_2-2s^2)}{s^2} + \frac{C_1}{2}(-R_2^2+R_3+R_1R_2$$

$$- \frac{(t(1+2\delta_1-\delta_2)+2(\delta_1^2+\delta_2))}{s^2} - \frac{(R_1-R_2)(ts^2+r-\delta_2-2s^2)}{s^2}$$

$$- (\frac{3}{8}C_1^2 - \frac{C_2}{2})(\frac{2(\delta_1 d_2 - d_1\delta_2)}{d_1 s^2} - R_1 + R_2 - \frac{(ts^2+r-\delta_2-2s^2)}{s^2}$$

$$- (-\frac{C_3}{2} + \frac{3}{4}C_1C_2 - \frac{5}{2^4}C_1^3).$$

Combining (4.50), (4.56), (4.60), (4.139), (4.140), (4.141), (4.144) and (4.145) and inverting term by term, we have that for early time

$$\hat{w}(\zeta,\tau) = d_1(-4\tau)^{3/2} i^3\mathrm{erfc}\,\frac{\zeta}{2\sqrt{\tau}} + (R_2-R_1)(4\tau)^{5/2} i^5\mathrm{erfc}\,\frac{\zeta}{2\sqrt{\tau}}$$

$$+(R_3+R_1R_2-R_2^2)(4\tau)^{7/2}\ i^7\mathrm{erfc}\,\frac{\zeta}{2\sqrt{\tau}} \ +$$

$$+\exp((-N_2+\frac{\epsilon_1}{2})\zeta)H(\tau-\zeta)(\frac{(\tau-\zeta)^{3/2}}{\Gamma(5/2)} + \frac{(\tau-\zeta)^{5/2}}{\Gamma(7/2)}(S_1$$

$$-\zeta(N_4 - \frac{N_2\epsilon_1}{2} + \frac{3}{8}\epsilon_1^2)) + \frac{(\tau-\zeta)^{7/2}}{\Gamma(9/2)}(S_2 - S_1\zeta(N_4 - \frac{\epsilon_1 N_2}{2} \quad \frac{3}{8}\epsilon_1^2)$$

$$+\zeta(-N_6 + \frac{N_4\epsilon_1}{2} - \frac{3}{8}N_2\epsilon_1^2 + \frac{5}{2^4}\epsilon_1^3 + \frac{(N_4-\epsilon_1N_2/2+\frac{3}{8}\epsilon_1^2)^2}{2}\zeta) +$$

$$\frac{D_0}{2}(4\tau)^{5/2}\ i^5\mathrm{erfc}\,\frac{N_1\zeta}{2\sqrt{\tau}} + \frac{D_0\zeta}{2}(\frac{N_1\epsilon_1}{2} - N_3)(4\tau)^3\ i^6\mathrm{erfc}\,\frac{N_1\zeta}{2\sqrt{\tau}}$$

$$\frac{(D_1 + \frac{D_0}{2}(N_3 - \frac{N_1\epsilon_1}{2})^2\zeta^2)}{2}(4\tau)^{7/2}\ i^7\mathrm{erfc}\,\frac{N_1\zeta}{2\sqrt{\tau}} + \cdots . \qquad (4.146)$$

Here erfc(x) is the complimentary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-m^2)\, dm,$$

and the repeated integrals of the complementary error function are defined by

$$i^0\operatorname{erfc}(x) = \operatorname{erfc}(x),$$

$$i^n\operatorname{erfc}(x) = \int_x^{\infty} i^{n-1} \operatorname{erfc}(t)\, dt, \quad n = 1,2,\cdots.$$

See [38] for example.

A similar procedure may be used to find an early time solution for $v(\zeta,\tau)$. We begin by finding the asymptotic expansions of the factors in equations (4.51), (4.57) as $p \to \infty$. The following factor is written as a series expansion in terms of $(\frac{1}{p})$

$$\frac{(p((t+p)(ps^2-\delta_2)+t\delta_1)-\frac{1}{4}p(g_1+g_2)^2)}{(p(ts^2+r\delta_1)+t(\delta_1-\delta_2))} = K_0 + \frac{K_1}{p} + \frac{K_2}{p^2} + O(\frac{1}{p^3}) \quad (4.147)$$

where

$$K_0 = \frac{t(\delta_1-\delta_2)-s^2(2N_4+N_2^2)}{ts^2+r\delta_1}$$

$$K_1 = \frac{-2s^2(N_6+N_2N_4)}{ts^2+r\delta_1} - \frac{t(\delta_1-\delta_2)(t(\delta_1-\delta_2)-s^2(2N_4+N_2^2))}{(ts^2+r\delta_1)^2}$$

$$K_2 = \frac{t^2(\delta_1-\delta_2)^2}{(ts^2+r\delta_1)^2}(t(\delta_1-\delta_2)-s^2(2N_4+N_2^2)) + \frac{2ts^2(\delta_1-\delta_2)}{(ts^2+r\delta_1)}(N_6+N_2N_4)$$

$$- s^2(2N_8 + 2N_2N_6 + N_4^2).$$

The following factor is written as a series expansion in terms of $(\frac{1}{p})$

$$\frac{(p((t+p)(ps^2-\delta_2)+t\delta_1)-\frac{1}{4}p(g_1-g_2)^2)}{(pts^2+r\delta_1)+t(\delta_1-\delta_2))} = L_0p^2+L_1p+L_2+O(\frac{1}{p}) \tag{4.148}$$

where

$$L_0 = \frac{s^2}{ts^2+r\delta_1}$$

$$L_1 = \frac{ts^2-\delta_2-s^2N_1^2}{ts^2+r\delta_1} - \frac{ts^2(\delta_1-\delta_2)}{(ts^2+r\delta_1)^2}$$

$$L_2 = \frac{t^2s^2(\delta_1-\delta_2)^2}{(ts^2+r\delta_1)^3} - \frac{t(\delta_1-\delta_2)(ts^2-\delta_2-s^2N_1^2)}{(ts^2+r\delta_1)^2} + \frac{t(\delta_1-\delta_2)-2s^2N_1N_3}{(ts^2+r\delta_1)} .$$

The following factor is written as a series expansion in terms of $(\frac{1}{p})$

$$\frac{p(\delta_1^2+\delta_2-ps^2+t\delta_1+(ps^2-\delta_2)(p+t))\frac{A_1(p)}{A_2(p)} + \frac{(ps^2-\delta_2)d_1}{p^{1/2}} + \frac{\delta_1 d_2}{p^{1/2}}}{p(ts^2+r\delta_1)+t(\delta_1-\delta_2)}$$

$$= \frac{-d_1}{pp^{1/2}(ts^2+r\delta_1)}(J_0+\frac{J_1}{p}+\frac{J_2}{p^2}+O(\frac{1}{p^3})) \tag{4.149}$$

where

$$J_0 = s^2(R_1-R_2)+ts^2-s^2-\frac{d_2}{d_1}\delta_1$$

$$J_1 = s^2(R_2^2-R_3-R_1R_2)+(R_1-R_2)+ts^2-\delta_2-s^2)+(\delta_1^2+\delta_2+t\delta_1-t\delta_2)$$

$$-(s^2(R_1-R_2)+ts^2-s^2-\frac{d_2}{d_1}\delta_1)\frac{t(\delta_1-\delta_2)}{(ts^2+r\delta_1)}$$

$$J_2 = \frac{t^2(\delta_1-\delta_2)^2}{(ts^2+r\delta_1)^2}(s^2(R_1-R_2)+ts^2-s^2-\frac{d_2}{d_1}\delta_1)$$

$$-\frac{t(\delta_1-\delta_2)}{(ts^2+r\delta_1)}(s^2(R_2^2-R_3-R_1R_2)+(R_1-R_2)(ts^2-\delta_2-s^2)$$

$$+(\delta_1^2+\delta_2+t\delta_1-t\delta_2))+s^2(2R_2R_3-R_2^3+R_1R_2^2-R_1R_3)$$

$$+(ts^2-\delta_2-s^2)(R_2^2-R_3-R_1R_2)+(R_1-R_2)(\delta_1^2+\delta_2+t(\delta_1-\delta_2)) .$$

With the aid of (4.51), (4.57), (4.61), (4.139), (4.140), (4.141), (4.145), (4.147), (4.148) and (4.149), and with the elementary inversion process, we have that for early time

$$v(\zeta,\tau) = d_1[\frac{-J_0}{(ts^2+r\delta_1)}(4\tau)^{1/2}\ i^1\mathrm{erfc}\frac{\zeta}{2\sqrt{\tau}}$$

$$-\frac{J_1}{(ts^2+r\delta_1)}(4\tau)^{3/2}i^3\mathrm{erfc}\ \frac{\zeta}{2\sqrt{\tau}}-\frac{J_2}{(ts^2+r\delta_1)}(4\tau)^{5/2}\ i^5\mathrm{erfc}\frac{\zeta}{2\sqrt{\tau}}+$$

$$+\exp\{(-N_2+\frac{1}{2}\epsilon_1)\zeta\}H(\tau-\zeta)[\frac{T_0(\tau-\zeta)^{3/2}}{\Gamma(5/2)}+\frac{T_1(\tau-\zeta)^{5/2}}{\Gamma(7/2)}$$

$$+\frac{T_2(\tau-\zeta)^{7/2}}{\Gamma(9/2)}+]\qquad+\frac{Z_0}{2}(4\tau)^{1/2}\ i^1\mathrm{erfc}\ \frac{N_1\zeta}{2\sqrt{\tau}}$$

$$\frac{Z_2}{2}(4\tau)^{3/2}\ i^3\mathrm{erfc}\ \frac{N_1\zeta}{2\sqrt{\tau}}+\cdots] , \qquad (4.150)$$

where

$$T_0 = K_0$$

$$T_1 = K_1 + K_0(S_1-\zeta(N_4-\frac{1}{2}\epsilon_1N_2+\frac{3}{8}\epsilon_1^2))$$

$$T_2 = K_2 + K_1\{[S_1 - \zeta(N_4 - \frac{1}{2}\epsilon_1 N_2 + \frac{3}{8}\epsilon_1^2)]\} + K_0[S_2 - S_1\zeta(N_4 - \frac{1}{2}\epsilon_1 N_2 + \frac{3}{8}\epsilon_1^2)$$

$$+(-N_6 + \frac{1}{2}N_4\epsilon_1 - \frac{3}{8}N_2\epsilon_1^2) + \frac{5}{2^4}\epsilon_1^3 + \frac{(N_4 - \frac{1}{2}N_2\epsilon_1 + \frac{3}{8}\epsilon_1^2)^2}{2}\zeta],$$

$$Z_0 = D_0 L_0 ,$$

$$Z_1 = -\zeta D_0 L_0 (N_3 - \frac{1}{2}\epsilon_1 N_1),$$

$$Z_2 = L_0 (D_1 + \frac{1}{2}D_0\zeta^2 (N_3 - \frac{1}{2}N_1\epsilon_1)^2) + D_0 L_1 .$$

We note that these early time solutions of $\hat{w}(\zeta,\tau)$ and $\hat{v}(\zeta,\tau)$ are in effect without any restrictions beyond the conditions of (2.40) and (2.41). And the early time solutions of all other field variables follow trivially from the equations (4.27), (4.146) and (4.150).

SUMMARY AND CONCLUSIONS

In this thesis, we have reviewed the major contributions to the development of a theory of mechanically and thermally interacting continuous media. Beginning with the work of Darcy and Terzaghi we have traced the work of Biot, Truesdell and Toupin and the recent work of Green, Naghdi, Steel, Atkins and Chadwick. As is usual, the theoretical development has preceded the number of applications and in this thesis we have attempted to utilize a linearized version of the mixture theory to derive results which are readily applicable to practical boundary value problems.

Our first result is in the form of an integral relation commonly known as a reciprocal theorem. It relates the solution of one problem to that of another problem each of which is due to different boundary and initial data. We have indicated how this theorem reduces to a theorem applicable to a single constituent and we have shown how one might use such a theorem. Indeed, we intend to explore its uses in future research much along the lines used in classical elasticity.

Our second major result consists of a solution of a fundamental initial boundary value problem using the linearized mixture theory. It is the first actual boundary value problem to be solved using a mixture theory. Due to its complexity the results are given in integral form only. Further development must await experimental evidence concerning the size of the material properties. Such experiments, incidentally, are a second possible line of future research and it is our intention to attempt to devise simple analytical models which will lead to estimates of the material constants. We will be guided by those methods used in single constituent theories.

The integral representation of the solution of the boundary value problem given in Chapter 4 is exact to terms of order t^2 but, due to the complexity of the integrands in the integrals, not much can be inferred about the displacement field. For this reason we have given the starting solution, i.e., the early time approximation. This solution may prove more useful as far as actual computation is concerned.

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APPENDIX

The location of the zeros of (4.73) may be given in a power series of t as follows,

$$p_1 = \frac{1}{2}\left(-\phi_0 - t\left(\frac{1}{4}\phi_0 E_1 - 1\right) - t^2\left(\frac{\phi_0}{2}\right)\left(\frac{E_1^2}{8} - \frac{1}{2}E_2\right) + O(t^3)\right)$$

$$+ \frac{1}{2}\left(\sqrt{4\psi_0 - \phi_0^2} + t\,\frac{(4k_1 + 2\psi_0 B_1 - \frac{1}{2}E_1\ \phi_0^2 - \phi_0\phi_1)}{2\sqrt{4\psi_0 - \phi_0^2}}\right.$$

$$+ t^2\left(\frac{(4k_2\ 2\psi_0\psi_2 - (\frac{1}{4}E_1\phi_0 + \frac{1}{2}\phi_1)^2 - \phi_0^2(\frac{1}{2}E_2 - E_1^2/8))}{2\sqrt{4\psi_0 - \phi_0^2}}\right.$$

$$\left.\left. + \frac{(4k_1 + 2\psi_0\psi_1 - \frac{1}{2}E_1\phi_0^2 - \phi_0\phi_1)^2}{8(4\psi_0 - \phi_0^2)^{3/2}}\right) + O(t^3)\right)$$

$$p_2 = -\frac{1}{4}\,\phi_0 t\left(\left(\frac{1}{2}E_1 - \frac{2}{\phi_0}\right) + \left(\frac{1}{2}E_2 - E_1^2/8\right)t + O(t^2)\right)$$

$$+ \frac{it}{2}\left(\sqrt{4(k_2 - \frac{1}{2}\psi_0 B_2) - \frac{1}{4}\phi_0\left(\frac{1}{2}E_1 - \frac{2}{\phi_0}\right)^2}\right.$$

$$\left. + \frac{4(k_3 - \frac{1}{2}\psi_0 B_3) - \frac{1}{4}\phi_0^2\left(\frac{1}{2}E_1 - \frac{2}{\phi_0}\right)\left(E_2 - \frac{1}{4}E_1^2\right)}{2\sqrt{4(k_2 - \frac{1}{2}\psi_0 B_2) - \frac{1}{4}\phi_0\left(\frac{1}{2}E_1 - \frac{2}{\phi_0}\right)^2}}\,t + O(t^2)\right).$$

and their conjugates p_1^* and p_2^*, where we used the following abbreviations

$$\phi_0 = -\frac{2(r + \delta_2)}{s^2}$$

$$\psi_0 = \frac{(r - \delta_2)^2 + 4r(\delta_1^2 + \delta_2)}{s^4}$$

$$\psi_1 = \frac{2(-1+2\ \delta_1 - 2\delta_2 - r)}{s^2}$$

$$\psi_2 = \frac{2}{s^4}((1+2\delta_1-\delta_2)(r-\delta_2)+(2+2r)(\delta_1^2+\delta_2))$$

$$\psi_3 = \frac{2}{s^3}(1+2\ \delta_1-\delta_2)$$

$$\psi_4 = \frac{1}{s^4}(1+2\delta_1-\delta_2)^2$$

$$H_0 = -\psi_0/36$$

$$H_1 = \phi_0\psi_2/12 - \psi_0\psi_1/18$$

$$H_2 = (\phi_0\psi_3+2\psi_2-4\psi_4)/12 - (\psi_1^2 + 2\psi_0)/36$$

$$G_0 = -\psi_0^3/108$$

$$G_1 = -\psi_0^2\psi_1/36 + \phi_0\psi_0\psi_2/24$$

$$G_2 = -(\psi_0^2 + \psi_0\psi_1^2)/36+(\phi_0\psi_1\psi_2+\psi_0(\phi_0\psi_3+2\psi_2-4\psi_4))/24$$
$$+(4\psi_0\psi_3-\phi_0^2\psi_3-\psi_2^2)/8$$

$$\Sigma_0^2 = \frac{((r-\delta_2)^2+4r(\delta_1^2+\delta_2))^3}{3^3 2^6 s^8}(\frac{4}{s^8}((1+2\delta_1-\delta_2)(r-\delta_2)$$

$$+(2+2r)(\delta_1^2+\delta_2))^2 \quad (\frac{12}{s^4}(r+\delta_2)^2 - \frac{4}{s^4}((r-\delta_2)^2+4r(\delta_1^2+\delta_2))$$

$$+ \frac{64r\delta_1^2}{s^{12}}((r-\delta_2)^2+4r(\delta_1+\delta_2))(1+2\delta_1-\delta_2)^2$$

$$\Sigma_1 = \frac{1}{2(-\Sigma_0)}(\frac{1}{2^3 3^3}\ \psi_0^3(2\phi_0\psi_4-2^3\psi_1\psi_4+\psi_2\psi_3)+\frac{1}{2^5 3^3}\ \psi_0^2(6\phi_0\psi_1\psi_4$$

$$+6\psi_1\psi_2^2-\phi_0\psi_2\psi_3-2\phi_0\psi_2^2+2^3\phi_0\psi_3\psi_4) - \frac{\psi_0}{2^6 3^3}(3^2\phi_0^2\psi_2\psi_4+\phi_0^2\psi_1\psi_3^2)$$

$$+ \frac{1}{2^6 3^3}(-3^2\phi_0\psi_1\psi_2^3 + 2\phi_0^3\psi_2^3))$$

$$k_1 = \frac{G_1\psi_0}{18G_0} - \frac{6(H_1 - H_0G_1/(3G_0))}{\psi_0} + \frac{\psi_1}{6}$$

$$k_2 = \frac{\psi_0}{6}\left(\frac{G_2}{3G_0} - \frac{(G_1^2 - \Sigma_0^2)}{9G_0^2}\right) - \frac{6}{\psi_0}\left(H_2 - H_1\left(\frac{G_1}{3G_0}\right) - H_0\left(\frac{G_2}{3G_0} - \frac{2(G_1 - \Sigma_0^2)}{9G_0^2}\right)\right) + \frac{\psi_2}{6}$$

$$E_1 = \frac{4}{\phi_0^2}(\phi_0 + 2k_1 - \psi_1)$$

$$E_2 = \frac{4}{\phi_0^2}(1 + 2k_2 - 1)$$

$$B_1 = \frac{2}{\phi_0} + \frac{2k_1}{\psi_0} - \frac{2\psi_2}{\phi_0\psi_0} - \frac{1}{2}E_1$$

$$B_2 = \frac{2k_2}{\psi_0} + \frac{4k_1}{\phi_0\psi_0} - \frac{2\psi_3}{\phi_0\psi_0} - \left(\frac{1}{2}E_1\left(\frac{2}{\phi_0} + \frac{2k_1}{\psi_0} - \frac{2\psi_2}{\phi_0\psi_0} - \frac{3}{4}E_1\right) + \frac{1}{2}E_2\right)$$