# On Some Problems of Elementary and Combinatorial Geometry (*).

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**Summary.** – *The author discusses various solved and unsolved geometrical problems all of which are of a combinatorial nature. Some are of metrical character and some are more number theoretic.*

Elementary geometry has been studied for thousands of years. Nevertheless, I hope to show in this article that the subject is full of easily stated but difficult, unsolved problems. Most of the questions which I discuss will be of a combinatorial nature. I certainly do not claim completeness but will mostly only discuss problems on which I worked myself, and will try to indicate the literature of related problems. To save space I usually do not give proofs.

**1.** – Let there be given $n$ distinct points $x_1, \dots, x_n$ in $k$-dimensional Euclidean space. Denote by $d(x_i, x_j)$ the distance from $x_i$ to $x_j$. Denote by $D_k(x_1, \dots, x_n)$ the number of distinct distances amongst $x_1, \dots, x_n$ and put

$$f_k(n) = \min_{x_1, \dots, x_n} D_k(x_1, \dots, x_n) .$$

Trivially $f_1(n) = n - 1$, but in the plane the situation becomes already very difficult. I proved

$$(n-1)^{\frac{1}{2}} - 1 < f_2(n) < c_1 n/(\log n)^{\frac{1}{2}} \tag{1}$$

and L. MOSER improved the lower bound to $n^{\frac{2}{3}}/29^{\frac{1}{3}} - 1$. It seems certain that $f_2(n) > n^{1-\varepsilon}$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$ and in fact probably $f_2(n) > c_2 n/(\log n)^{\frac{1}{2}}$. The upper bound in (1) is given by the lattice points in the plane.

Denote by $d_2(x_i)$ the number of distinct distances from $x_i$. Moser in fact proved

$$\max_{1 \leqslant i \leqslant n} d_2(x_i) \geqslant \frac{n^{\frac{2}{3}}}{2 \cdot 9^{\frac{1}{3}}} - 1 .$$

One is tempted to conjecture

$$\sum_{i=1}^{n} d_2(x_i) > c_3 n^2/(\log n)^{\frac{1}{2}}$$

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which would be a considerable strengthening of (1). I only showed

$$\sum_{i=1}^{n} d_2(x_i) > \tfrac{1}{2}n^{\frac{3}{2}} .$$

Assume now that the points $x_1, \ldots, x_n$ are the vertices of a convex polygon. I made three conjectures. My first conjecture was that in this case $f_2(n) = [n/2]$, equality, say, for the regular polygon. This conjecture was proved by ALTMAN. Next I conjectured

$$\max_{1 \leqslant i \leqslant n} d_2(x_i) \geqslant \left[\frac{n}{2}\right] .$$

As far as I know this is not yet settled. Finally I conjectured that every convex polygon always has a vertex which does not have three vertices equidistant from it. DANZER to my great surprise disproved this conjecture. In fact be showed that to every $k$ there is a convex polygon of $n_k$ vertices so that every vertex has $k$ other vertices equidistant from it. Danzer's example is not yet published. It would be of interest to determine or estimate the smallest possible value of $n_k$.

The lattice points $(u, v)$, $1 \leqslant u$, $v \leqslant n^{\frac{1}{2}}$ show that one can give $n$ points $x_1, \ldots, x_n$ in the plane so that to every $x_i$ there are $n^{c_4/\log\log n}$ others which are equidistan tfrom it. It is not impossible that this bound is essentially best possible; in other words, if $x_1, \ldots, x_n$ are any points in the plane then for at least one $x_i$ there are fewer than $n^{c_5/\log\log n}$ points $x_j$ equidistant from it. I can only prove this with $2n^{\frac{1}{2}}$, and would like to see this bound improved to $o(n^{\frac{1}{2}})$ and beyond.

It seemed likely to me that if $D_2(x_1, \ldots, x_n)$ is small, then many of the $x_i$ must lie on a line. More precisely: If no $k$ of the $x_i$ are on a line, then $D_2(x_1, \ldots, x_n) > \varepsilon_k n$.

SZEMERÉDI recently gave a surprisingly simple proof of this conjecture. In fact he shows that if no $k$ of the $x_j$'s are on a line then

$$\max_{1 \leqslant i \leqslant n} d_2(x_i) > \varepsilon_k n . \tag{2}$$

To prove (2), denote by $\beta_1^{(i)}, \ldots, \beta_{s_i}^{(i)}$ the distinct values of the numbers $d(x_i, x_j)$, $1 \leqslant j \leqslant n$, $j \neq i$ and assume that there are $\alpha_u^{(i)}$ values of $j$ for which

$$d(x_i, x_j) = \beta_u^{(i)} .$$

Thus for every $i$

$$\sum_{u=1}^{s_i} \alpha_u^{(i)} = n - 1 . \tag{3}$$

Now if (2) would be false, then $s_i \leqslant \varepsilon_k n$ for every $i$. Thus by an elementary inequality we obtain from (3)

$$\sum_{u=1}^{s_i} \binom{\alpha_u^{(i)}}{2} > \frac{n}{4\varepsilon_k} \tag{4}$$

$\left(\sum_{u=1}^{s_i}\binom{\alpha_u^{(i)}}{2}\right.$ is a minimum if the $\alpha_u^{(i)}$ are as nearly equal as possible$\left.\right)$. From (4) we have for sufficiently small $\varepsilon_k$

$$\sum_{i=1}^{n}\sum_{u=1}^{s_i}\binom{\alpha_u^{(i)}}{2} > \frac{n^2}{4\varepsilon_k} > (k-1)\binom{n}{2}\,. \tag{5}$$

The left side of (5) has the following geometric interpretation. Take all possible pairs $(x_u, x_v)$ which are equidistant from one of the $x_i$'s. In view of (5) at least one pair $(x_u, x_v)$ is equidistant from $k$ $x_i$'s. Thus the perpendicular bisector of $(x_u, x_v)$ goes through at least $k$ $x_i$'s. This contradiction proves our assertion.

SZEMERÉDI now conjectures the following generalization of Altman's result. Let $x_1, \ldots, x_n$ be $n$ points no three of them on a line. Then $D_2(x_1, \ldots, x_n) \geqslant [n/2]$ and in fact

$$\max_{1\leqslant i\leqslant n} d_2(x_i) \geqslant \left[\frac{n}{2}\right]. \tag{6}$$

Szemerédi's proof if carried out a little more carefully gives $\max\limits_{1\leqslant i\leqslant n} d_2(x_i) \geqslant [n/3]$.

These problems can be of course extended to $k$-dimensional space. The lattice points in $k$-dimensional space immediately give

$$f_k(n) < c_k n^{2/k} \tag{7}$$

and perhaps (7) is best possible. An easy induction process gives $f_k(n) > n^{\varepsilon_k}$ for some $\varepsilon_k > 0$.

For $k = 3$ Altman proved that if $x_1, \ldots, x_n$ are the vertices of a convex polyhedron, then $D_3(x_1, \ldots, x_n) > cn$. If no three of the points are on a line, perhaps the same holds, but Szemerédi's proof only gives $D_3(x_1, \ldots x_n) > cn^{\frac{1}{2}}$ which may hold for every set of points in $E_3$. Szemerédi's idea easily gives $D_3(x_1, \ldots, x_n) > cn$ if we assume that no four points are on a plane.

Before ending this chapter I would like to state a few more questions on $D_2(x_1, \ldots, x_n)$. Assume that no three $x_i$ are on a line and no four on a circle. What can be said about $D_2(x_1, \ldots, x_n)$. Is it true that

$$\lim_{n=\infty} D_2(x_1, \ldots, x_n)/n = \infty\,? \tag{9}$$

Assume next that no three $x$'s determine an isosceles triangle (i.e. assume that for every $1 \leqslant i < j < l$ $D_2(x_i, x_j, x_l) = 3$). What can be said about $\min D_2(x_1, \ldots, x_n)$. This question seems to be non trivial even for small values of $n$ e.g. $n = 6$. HAMBURGER and RUZSA showed that in this case $D_2(x_1, \ldots, x_6) \geqslant 6$. Similarly we can assume $D_2(x_i, x_j, x_k, x_l) \geqslant 4$ or $\geqslant 5$ and ask about $\min D_2(x_1, \ldots, x_n)$. I did not investigate any of these questions carefully and some of them may be trivial. Clearly many further related questions can be asked but I leave this to the reader. By the way if we assume $D_2(x_i, x_j, x_k, x_l) = 6$ (for every $1 \leqslant i < j < k < l \leqslant n$) then clearly $D_2(x_1, ..., x_n) = \binom{n}{2}$.

P. ERDÖS, *On sets of distances of n points*, Amer. Math. Monthly, **53** (1946), pp. 248-250.
L. MOSER, *On the different distances determined by n points*, Amer. Math. Monthly, **59** (1952), pp. 85-91.
E. ALTMAN, *On a problem of P. Erdös*, Amer. Math. Monthly, **70** (1963), pp. 148-157; see also *Some theorems on convex polygons*, Canad. Math. Bull., **15** (1972), pp. 329-340.

**2.** – Let there be given $n$ distinct points in $k$-dimensional space whose diameter is 1 (i.e. $\max_{1 \leqslant i < j \leqslant n} d(x_i, x_j) = 1$). Denote by $M_k(n)$ the maximum number of pairs satisfying $d(x_i, x_j) = 1$ (the maximum is taken over all sets $x_1, ..., x_n$ of diameter 1). Trivially $M_1(n) = 1$ and ERIKA PANNWITZ proved $M_2(n) = n$. Thirty five years ago VÁZSONYI conjectured $M_3(2n) = 2n - 2$. This conjecture was proved independently by GRÜNBAUM, HEPPES and STRASZIEVICZ in 1956. LENZ made the surprising observation that $M_4(n) \geqslant [n^2/4]$ and I proved

(1) $$\lim_{n=\infty} M_k(n)/n^2 = \frac{1}{2} - \frac{1}{2[k/2]} .$$

Here I mention the following classical conjecture of Borsuk: let $s_k$ be a set in $k$-dimensional space of diameter 1. Is it true that $s_k$ can be decomposed into $k+1$ sets of diameter less than 1. This is trivial for $k=1$ and easy for $k=2$. For $k=3$ it was proved by EGGLESTON and later a simpler proof was found by GRÜNBAUM and HEPPES. For $k > 3$ the conjecture is still undecided.

Assume now that $\min_{1 \leqslant i < j \leqslant n} d(x_i, x_j) = 1$. Denote by $m_k(n)$ the maximum number of pairs satisfying $d(x_i, x_j) = 1$. It is easy to see that $m_1(n) = n - 1$ and $m_2(n) < 3n$. The later inequality follows from the fact that there can be at most six points at distance 1 from $x_i$ (otherwise 1 would clearly not be the minimum distance). $m_3(n) < 6n$ since there are at most 12 points on the unit sphere so that the distance between any two of them is 1. $m_k(n) < r_k(n)$ is easy to see, but the best value of $r_k$ is not known for $k > 3$.

It is easy to improve $m_2(n) < 3n$. We obtain with very little trouble that

$$3n - c_1 n^{1/2} < m_2(n) < 3n - c_2 n^{1/3} \; ; \quad 6n - c_3 n^{2/3} < m_3(n) < 6n - c_4 n^{2/3} .$$

Perhaps

(2) $$m_2(3n^2 + 3n + 1) = 9n^2 + 6n .$$

If true (2) is best possible; $m_2(3n^2 + 3n + 1) \geqslant 9n^2 + 3n$ follows if we consider the points of a triangular lattice inside and on a regular hexagon of sidelength $n$.

V. REUTHER, *Recently conjectured* $m_2(n) = 3n - (12n-3)^{\frac{1}{2}}$, *Elemente der Math.*, **27** (1972), p. 19, this conjecture was proved by HARBOTH.

Denote by $P_k(n)$ the maximum number of pairs $(x_i, x_j)$ for which $d(x_i, x_j)$ assumes the same value (i.e. $P_k(n)$ is the maximum number of pairs $(x_i, x_k)$ with say say $d(x_i, x_j) = r$.

Trivially $P_1(n) = n-1$. For $k=2$ and $k=3$ it is surprisingly difficult to give a good estimation for $P_k(n)$. I showed

(3) $$n^{1+c_1/\log\log n} < P_2(n) < 2n^{3/2}.$$

I expect that in (3) the lower bound gives the right order of magnitude for $P_2(n)$, but I was not even able to show $P_2(n) = o(n^{3/2})$. SZEMERÉDI and JÒZSA just proved this, but their ingenious proof is complicated and will appear in the proceedings of the Keszthely meeting held in 1973.

For $k=3$, I showed

$$c_2 n^{4/3} \log\log n < P_3(n) < c_3 n^{5/3} .$$

It is curious that for $k \geqslant 4$ $P_k(n)$ is easier to handle. I proved that if $k=2l$, $n \equiv 0 \pmod{2k}$, $n > n_0(k)$ then

$$P_k(n) = \frac{n^2(l-1)}{4} + n .$$

For odd $k$ the results are slightly less precise.

Let $x_1, \ldots, x_n$ be $n$ points in the plane. $d_1, \ldots, d_l$ the distinct distances determined by the points. What are the possible values of $l$. Clearly $f_k(n) \leqslant l \leqslant \binom{n}{2}$ ($f_k(n)$ is defined in 1), but it is not clear what are the possible values of $l$. I can show that there is a $c$ so that $l$ can take every value between $cn^{3/2}$ and $\binom{n}{2}$ (I think this result fails for $\varepsilon n^{3/2}$ instead of $cn^{3/2}$). Denote by $u_i$ the number of pairs satisfying $d(x_r, x_s) = d_i$, $u_1 \geqslant u_2 \geqslant \ldots \geqslant u_l$, $\sum u_i = \binom{n}{2}$, and $u_l \leqslant n$ by the result of ERIKA PANNWITZ, but $u_l = n$ is possible e.g. $n$ odd and the $x_i$ form a regular polygon, here of course $u_1 = \ldots = u_{[n/2]} = n$. How many distinct values can the $u$'s take. At most $n-1$, but I do not think $n-1$ can be attained for $n > 4$. Also what in the largest possible $t_n$ for which $\sum_{u_i < t_n} u_i > \frac{1}{2}\binom{n}{2}$? The lattice points show that $t_n$ can be as large as $n(\log n)^{\gamma_1}$ but it is quite possible that for a certain $\gamma$

$$\sum_{u_i > n(\log n)^{\gamma}} u_i = o(n^2)$$

i.e. there are relatively few distances which occur more often than $n(\log n)^{\gamma}$ times.

PURDY and I considered the following questions. Let there be given $n$ points $x_1, \ldots, x_n$ in $k$-dimensional space. Denote by $g_k^{(r)}(n)$ the maximum number of $r$-dimesional simplices whose vertices are chosen amongst the $x_i$'s and which all have the same non zero $r$-dimensional volume. We proved

$$c_4 n^2 \operatorname{loglog} n < g_2^{(2)}(n) < 4n^{5/2} . \tag{5}$$

Probably the lower bound in (5) is not very far from the truth.

In our paper we state a few problems which as far as I know are still unsolved. Let $x_1, \dots, x_n$ be $n$ distinct points in the plane how many quadruplets can one form so that not all the six distances should be different. Let us call such quadruplets degenerate. We can show that one can give $n$ points with $c_5 n^3 \log n$ degenerate quadruplets, also that the number of degenerate quadruplets is always less than $c_6 n^{7/2}$. We conjectured that it is less than $n^{3+\varepsilon}$.

Let there be given $n$ points in the plane. How many triangles can one have which have the maximal (or minimal) non zero area. We only have trivial results: The maximum are can occur at most $c_7 n^2$ times and it can occur $c_8 n$ times.

Let there be given $n$ points in $k$-dimensional space. What is the largest set of pairwise congruent (similar) triangles? What is the largest set of equilateral or (isosceles) triangles? One specific question: By the method of Lenz one can give $3n$ points in 6-dimensional space the vertices of which determine $n^3$ equilateral triangles of size 1. One would suspect that we can not have $n^3 + 1$ such triangles.

P. ERDÖS, *On sets of distances of n points*, Amer. Math. Monthly, **53** (1946), pp. 248-250.

P. ERDÖS, *On some applications of graph theory to geometry*, Canad. J. Math., **19** (1967), pp. 968-971; see also *On sets of distances of n points in Euclidean space*, Publ. Math., Inst. Hungar. Acad. Sci., **5** (1960), pp. 165-169.

P. ERDÖS - G. PURDY, *Some extremal problems in geometry*, J. Combinatorial Theory, **10** (series A) (1971), pp. 246-252, see also a forthcoming paper of Purdy in Discrete Mathematics.

For further literature on results quoted in this chapter see Proc. Symp. in Pure Math., Vol. VII, *Convexity*, Amer. Math. Soc., (1963), in particular the paper of L. DANZER, B. GRÜNBAUM and V. KLEE, *Helly's theorem and its relatives*, pp. 101-180 and B. GRÜNBAUM, *Borsuk's problem and related questions*, pp. 271-284.

**3.** – Denote by $f(n; k)$ the smallest integer so that any set of $f(n; k)$ points in $k$-dimensional space contains a subset of $n$ points any two distances of which are distinct. It is not hard to see that $f(n; k) < n^{c_k}$ but I do not know the best exponent $c_k$. I conjectured

$$f(n; 1) = (1 + o(1))\, n^2.$$

TURÁN and I proved $f(n; 1) \geqslant (1 + o(1))\, n^2$ and recently KOMLÓS, SULYOK and SZEMERÉDI proved by a very ingenious and general number theoretic argument that $f(n; 1) < cn^2$, their proof is not yet published and will appear in Acta Math. Sci. Hungar.

I proved $f(3, 2) = 7$ and Croft proved $f(3, 3) = 9$ (i.e. 9 points in Euclidean 3-space always contain three points which do not form an isosceles triangle). STRAUS and I proved $f(n; k) < c_n^k$, our proof is not yet published. Probably $\lim_{k\to\infty} f(n; k)^{1/k} = 1$, but we have not been able to prove this even for $n = 3$.

L. M. KELLY raised the following question. Let $g(n; k)$ be the largest integer so that there are $g(n; k)$ points in $k$-dimensional space which determine at most $n$ distinct distances. STRAUS and I proved $g(n; k) < c^{k^{1-\beta_n}}$, our proof is not yet published. $g(n; k) > ck^n$ is easy and perhaps $\lim_{k=\infty} g(n; k)/k^n$ exists. $g(2; 1) = 3$ is trivial, $g(2; 2) = 5$ is easy and CROFT proved $g(2; 3) = 6$. The $2^k$ vertices of the $k$-dimensional cube determine $k$ distinct distances, thus $g(k+1;\ k) \geqslant 2^k$. It would be interesting to get a good upper bound for $g(k+1, k)$.

I proved that if $s$ is a set of power $m$ in $k$-dimensional space then $s$ has a subset $s_1$ of power $m$ so that any two distnces of $s_1$ are distinct. This completely fails in Hilbert space. KAKUTANI and I constructed in Hilbert space a set of power $c$ so that all the distances are rational. Also one can construct in Hilbert space a set of power $c$ all triangles of which are isosceles and acute angled. PÓSA disproving a conjecture of mine constructed in Hilbert space a set $s$ of power $c$ so that all subsets $s_1 \subset S$ of power $c$ have an infinite subset $s_2$ any two points of which are equidistant. PÓSA uses $2^{\aleph_0} = \aleph_1$.

H. T. CROFT, 9 *point and* 7 *point configurations in* 3-*space*, Proc. London Math. Soc., **12** (1962), pp. 400-424.

P. ERDÖS - P. TURÁN, *On the problem of Sidon in additive number theory and on some related problems*, Journal London Math. Soc., **16** (1941), pp. 212-215.

P. ERDÖS, *Some remarks on set theory*, II, Proc. Amer. Math. Soc., **1** (1950), pp. 127-141.

L. M. KELLY - E. A. NORDHAUS, *Distance sets in metric spaces*, Trans. Amer. Math. Soc., **71** (1951), pp. 440-456, see p. 451.

**4.** – Let there be given $n$ points in the plane not all on a line. Is it true that there always is a line which goes through precisely two of the points? Such a line is called an ordinary line. This beautiful question was posed in 1893 by SYLVESTER and nobody solved it at that time. I rediscovered the question in 1933 and communicated it to T. GALLAI who soon found a simple proof. Other proofs were found later the simplest in my opinion is due to L. M. KELLY. This question and its generalizations have a large literature a small part of which I try to give at the end of this paragraph.

DE BRUIJN and I conjectured that if $f(n)$ is the minimum number of ordinary lines determined by $n$ points then $f(n)$ tends to infinity. This conjecture was proved by MOTZKIN and later L. M. KELLY and W. MOSER proved $f(n) \geqslant [3n/7]$, equality for $n = 7$. MOTZKIN conjectured that for $n > n_0$ $f(n) \geqslant n/2$ and observed that for even $n$ there is equality.

Let there be given $n$ points in the plane no $n - k$ are on a line. I conjectured that these points then determine at least $ckn$ lines (where $c$ is an absolute constant independent of $k$ and $n$). Some very precise results in this direction were obtained by KELLY and MOSER.

GRAHAM conjectured that if there are given any $n$ points in the plane not all on a line. Then the lines determined by the points never have property $B$ (i.e. every subset of the $n$ points which meets all the lines contains all the points on at least one of the lines). This conjecture was recently proved by M. O. RABIN and independently by MOTZKIN.

I then asked the following questions. Does there exist for every $k$ a set of points in the plane so that if one colors the points by two colors in an arbitrary way, there always should be at least one line which contains at least $k$ points and all whose points have the same color. GRAHAM and SELFRIDGE gave an affirmative answer for $k=3$, but the cases $k>3$ seem to be open.

TH. MOTZKIN, *The lines and planes connecting the points of a finite set*, Trans. Amer. Math. Soc., **70** (1951), pp. 451-464. For further literature see e.g. B. GRÜNBAUM, *Convex polytopes*, p. 404, *Pure and Applied Math.*, Vol. XVI, Interscience John Wiley and sons and Hadwiger Debrunner and Klee, Combinatorial geometry in the plane, Holt, Rinehart and Winston.

See also B. GRÜNBAUMS, *Arrangements and spreads*, Amer. Math. Soc. Providence, 1972, and a forthcoming paper of S. BURR, B. GRUNBAUM and N. J. A. SLOANE. These papers contain many very interesting unsolved problems and very extensive references. In fact the shortness of this chapter is due to the fact that I can refer to these beautiful papers.

**5.** – In 1931 Miss E. KLEIN asked the following question: Is it true that for every $k$ there is an $n_k$ so that if there are given $n$ points in the plane no three on a line one can always find $k$ of them which determine the vertices of a convex $k$-gon[2]. She proved $n_4=5$, MAKAI and TURÁN showed $n_5=9$. SZEKERES conjectured $n_k=2^{k-2}+1$, this is open for $k \geqslant 6$.

SZEKERES and I proved

$$2^{k-2}+1 \leqslant n_k \leqslant \binom{2k-4}{k-2}. \tag{1}$$

The proof of the lower bound contains some minor inaccuracies, which were all corrected by KALBFLEISCH.

SZEKERES and I proved that if there are given $2^n$ points in the plane then there are always three of them which determine an angle $>\pi(1-1/n)$. A previous result of SZEKERES shows that this result is best possible since to every $\varepsilon>0$ he constructs $2^n$ points so that all the angles are less than $\pi(1-1/n)+\varepsilon$. For $m$ points $2^n<m<2^{n+1}$ we do not have such sharp results, also there are few precise results in higher dimensions. I conjectured that $2^n+1$ points in $n$-dimensional space always determines an angle greater than $\pi/2$. This conjecture was proved by DANZER and GRÜNBAUM. CROFT proved that 6 points in 3-space always determine an angle $\geqslant\pi/2$. It is easy to see that this result is best possible.

L. DANZER - B. GRÜNBAUM, *Über zwei Probleme bezöglich konvexen Körper von P. Erdös and V. L. Klee*, Math. Zeitschrift, **9** (1962), pp. 90-99. P. ERDÖS - G. SZEKERES, *On some extremum problems in elementary geometry*, Ammales Univ. Sci. Budapest, **3-4** (1960-61), pp. 53-62.

**6.** – Before ending this paper I would like to state a few miscellaneous problems and conjectures. HEILBRONN posed more than 20 years ago the following problem. Let there be given $n$ points in the unit square. Put

$$A_k(n) = \max_{x_1,\dots,x_n} \min_{1\leqslant i_1<\dots i_k\leqslant n} A(x_{i_1}, \dots, x_{i_k})$$

where $A(x_1, \dots, x_k)$ is the area of the convex hull of $x_1, \dots, x_k$: It is easy to see that $A_3(n) > c_1/n^2 \cdot A_3(n) < c_2/n$ is obvious. The first non-trivial result was due to K. F. ROTH who proved

$$A_3(n) < \frac{c_3}{n(\log\log n)^{\frac{1}{2}}}.$$

Recently W. SCHMIDT proved (*Journal London Math. Soc.*, 1972) that $A_3(n) < c_4/n(\log n)^{\frac{1}{2}}$ and very recently Roth proved $A_3(n) < c_5/n^{1+c_6}$.

It would be very interesting to decide whether $A_3(n) < c_7/n^2$ is true. In his paper W. SCHMIDT constructs $n$ points in the unit square so that

$$A_4(n) > c_8/n^{\frac{3}{2}}.$$

Perhaps for every $k$

$$A_k(n) > c_k/n^{1+1/(k-2)}.$$

SCHMIDT points out that the proof of $A_4(n) = o(1/n)$ presents difficulties. It seems of course that $A_k(n) = o(1/n)$ for every $k$.

ANNING and I proved the following theorem. Let there be given an infinite set of points in the plane. Assume that the distance between every two of them is integral. Then the points are on a line. ULAM asked the following question: Is there an infinite set in the plane which is everywhere dense so that the distance between every two of its points is rational? The answer is probably no but the proof seems to be nowhere in shight. It is known if one can find 6 points in the plane no three on a line no four on a circle so that all the distances are integral. Recently HARBOTH found such a set of five points. Let $G$ be a denumerable graph with the vertices $x_1, x_2, \dots$. What is the necessary and sufficient condition on $G$ that there should exist a set of points $x_1, x_2, \dots$ in the plane no three on a line so that the distance between $x_i$ and $x_j$ is an integer if and only if $x_i$ and $x_j$ are joined in $G$ by an edge. I proved that if $G$ contains a $K(3; \aleph_0)$ (i.e. a complete bipartite graph with 3 white and $\aleph_0$ black vertices) then this is impossible. It is possible (but I doubt it) that if $G$ does not contain a $K(3; \aleph_0)$ then such a set $x_1, \dots,$ exists. If we further assume that the set $x_1, \dots,$ in the plane does not contain four points on a circle we may get a completely new situation.

Denote by $F(n)$ the smallest integer for which one can color the points of $n$-dimensional space by $F(n)$ colors so that two points of the same color never have distance 1. NELSON conjectured $F(2) = 4$. W. and L. MOSER proved $F(2) \geqslant 4$ and it is known that $F(2) \leqslant 6$. In this connection L. MOSER asked the following question: let $s$ be a measurable set situated in a circle of radius $r$ ($r$ large) and no two points of $r$ are at distance 1. Is it true that the measure of $s$ is less than $\pi r/4$. Equality for $r = 1$. For large $n$, $F(n)$ and related problems are studied in a recent paper of LARMAN and ROGERS, $F(n) > cn^2$ is the best lower bound known. $F(n) > (1 + c)^n$ would follow fro the following combinatorial conjecture: Let $|S| = n$ $A_i \subset S$, $1 \leqslant i \leqslant k$. Assume that $A_i \cap A_j$ never has size $[n/4]$. Then $k < (2 - c_1)^n$. More generally I conjecture that for every $\eta > 0$ there is an $\varepsilon > 0$ so that if $|S| = n$, $A_i \subset S$, $1 \leqslant i \leqslant k$, $k > (2 - \varepsilon)^n$ then for every $r$, $\eta n < r < n(\frac{1}{2} - \eta)$ there are two integers $1 \leqslant i < j \leqslant k$ so that $|A_i \cap A_j| = r$.

V. T. Sós and I proved that if there are $n + 1$ triples in a set $S$ of $n$ elements, then there are always two of them whose intersection is a singleton, for $n \equiv 0 \pmod 4$ this is best possible. The simple proof can be left to the reader. We conjectured that if $l > 3$, $A_i \subset S$, $1 \leqslant i \leqslant k$, $|A_i| = l$, $n > n_0(l)$, $k > \binom{n-2}{l-2}$ then for some $1 \leqslant i < j \leqslant k$, $|A_i \cap A_j| = 1$. This conjecture if true is certainly best possible. To see this consider the $\binom{n-2}{l-2}$ $l$-tuples containing two fixed elements of $S$. KATONA porved our conjecture for $l = 4$ the unpublished proof is not very simple. The cases $l > 4$ are open.

The following problem is due to FEJES-TÓTH: Let there be given $n$ points $x_1, \ldots, x_n$ in the plane. Assume their minimum distance is 1. Minimize

$$\sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j)$$

FEJES-TÓTH conjectures that the minimum is assumed if the $x_i$'s are the vertices of a triangular lattice.

In a recent paper several collaborators and I studied the following problem: A finite set $S$ in $n$-dimensional space is called Ramsey if for every $k$ there is a finite set $S$ in $m$-dimensional space $m = m_0(S, n, k)$ so that if we color the points of $S'$ by $k$ colors, there always is a monochromatic set congruent to $S$. We prove in our first paper that if $S$ is a rectangular parallelepiped then it is RAMSEY. On the other hand not every set is RAMSEY; we show that a RAMSEY set is spherical (i.e. lies on a sphere). The simplest unsolved problem is whether every non-degenerate triangle is RAMSEY. Another problem is the following: Color the points of the plane by two colors. Is it true that all triangles can be monochromatically imbedded with the possible exception of at most one equilateral triangle. Many further problems will be stated in our papers on this subject.

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