ON SOME PROPERTIES OF A SUBMANIFOLD WITH CONSTANT MEAN CURVATURE IN A RIEMANN SPACE

By

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Introduction. It has been proved by H. Liebmann [1][1)] that the only ovaloids with constant mean curvature in an Euclidean space E^3 are the spheres. The same problem for closed hypersurfaces in an n-dimensional Euclidean space E^n has been investigated by W. Süss [2], T. Bonnesen and W. Fenchel [3], (cf. p. 118), H. Hopf [4], C. C. Hsiung [5] and A. D. Alexandrov [7]. The analogous problem for closed hypersurfaces in an n-dimensional Riemann space R^n has been discussed by C. C. Hsiung [6], A. D. Alexandrov [8], Y. Katsurada [9], [10], [11], K. Yano [12] and T. Ôtsuki [15].

It is the aim of the present authors to investigate the analogous problem for an m-dimensional closed submanifold V^m in the n-dimensional Riemann space R^n. The generalized Minkowski formulas for V^m in R^n are given in §**1**. In §**2** and §**3**, we derive the second and the third integral formulas which are valid for V^m in R^n under some conditions. Making use of those integral formulas, certain property of V^m in constant Riemann curvature space is proved in §**4**. Also, we prove a theorem for V^m in R^n which admits an one-parameter group of homothetic transformations.

§1. Generalized Minkowski formulas for a submanifold. We consider a Riemann space R^n ($n \geqq 3$) of class C^ν ($\nu \geqq 3$) which admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau \, , \tag{1.1}$$

where x^i are local coordinates in R^n and ξ^i are the components of a contravariant vector ξ. We suppose that the paths of these transformations cover R^n simply and that ξ is everywhere continuous and $\neq 0$. If the vector ξ is a Killing vector, a homothetic Killing, a conformal Killing etc. ([**13**], p. 32),

1) Numbers in brackets refer to the references at the end of the paper.

then the group is called isometric, homothetic, conformal etc. respectively.

We now consider a closed orientable submanifold V^m of class C^3 imbedded in R^n, locally given by

$$x^i = x^i(u^\alpha)^{2)} .$$

Let the contravariant unit vectors $\underset{\lambda}{i^i}$ ($\lambda = 1, 2, \cdots, m$) span the tangent vector space at each point of V^m and they be orthogonal to one another. We shall indicate by $\underset{P}{n^i}$ ($P = m+1, m+2, \cdots, n$) the contravariant unit vectors normal to V^m and suppose that they are mutually orthogonal.

Putting

$$B^i_j = \sum_{\lambda=1}^{m} \underset{\lambda}{i^i}\underset{\lambda}{i_j}, \qquad C^i_j = \sum_{P=m+1}^{n} \underset{P}{n^i}\underset{P}{n_j} \tag{1.2}$$

we have

$$B^i_j + C^i_j = \delta^i_j . \tag{1.3}$$

The first fundamental tensor $g_{\alpha\beta}$ of V^m is given by

$$g_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}$$

and $g^{\alpha\beta}$ are defined by $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$, where g_{ij} denotes the first fundamental tensor of R^n.

Denoting by " ; " the operation of D-symbol due to van der Waerden-Bortolotti ([**16**], p. 254), we have

$$\left(\frac{\partial x^i}{\partial u^\alpha}\right)_{;\beta} = H_{\alpha\beta}{}^i , \tag{1.4}$$

where $H_{\alpha\beta}{}^i$ means the Euler–Schouten curvature tensor ([**16**], p. 256). Then, putting $H_{\alpha\beta}{}^i \underset{P}{n_i} = \underset{P}{b_{\alpha\beta}}$, we have

$$H_{\alpha\beta}{}^i = \sum_{P=m+1}^{n} \underset{P}{b_{\alpha\beta}}\underset{P}{n^i} . \tag{1.5}$$

Multiplying (1.5) by $g^{\alpha\beta}$ and contracting, we get

$$g^{\alpha\beta}H_{\alpha\beta}{}^i = \sum_{P=m+1}^{n} m\underset{P}{H_1}\underset{P}{n^i} , \tag{1.6}$$

where we put $\underset{P}{H_1} = \dfrac{1}{m} g^{\alpha\beta}\underset{P}{b_{\alpha\beta}}$ and $\underset{P}{H_1}$ is the first mean curvature of V^m for the normal direction $\underset{P}{n^i}$.

2) Throughout the present paper the Latin indices run from 1 to n and the Greek indices from 1 to m ($m \leqq n-1$).

Let $\underset{E}{n}^i$ be the unit vector of the same direction to the vector $g^{\alpha\beta}H_{\alpha\beta}{}^i$. Then, we may consider $\underset{E}{n}^i$ as one of the unit normal vectors of V^m, that is, $\underset{m+1}{n}^i = \underset{E}{n}^i$. In this case, we obtain from (1.6)

$$g^{\alpha\beta}H_{\alpha\beta}{}^i = mH_1\underset{E}{n}^i \,, \tag{1.7}$$

where H_1 is the first mean curvature of V^m.

To the vector ξ, there belongs a covariant vector $\bar{\xi}$ of V^m with the components

$$\bar{\xi}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\xi_i \,. \tag{1.8}$$

Covariantly differentiating[3] the vector $\bar{\xi}_\alpha$, by means of (1.4) we have

$$\bar{\xi}_{\alpha;\beta} = H_{\alpha\beta}{}^i\xi_i + \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\xi_{i;j} \,. \tag{1.9}$$

Multiplying (1.9) by $g^{\alpha\beta}$ and contracting, we get by (1.7)

$$g^{\alpha\beta}\bar{\xi}_{\alpha;\beta} = mH_1\underset{E}{n}^i\xi_i + \frac{1}{2}g^{\alpha\beta}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\underset{\xi}{\pounds}g_{ij} \,, \tag{1.10}$$

where $\underset{\xi}{\pounds}g_{ij}$ is the Lie derivative of g_{ij} with respect to the infinitesimal transformation (1.1) ([**13**], p. 5). If we put

$$\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\underset{\xi}{\pounds}g_{ij} = \underset{\xi}{\pounds}g_{\alpha\beta} \,,$$

then (1.10) is rewritten as follows:

$$\frac{1}{m}\bar{\xi}^\alpha{}_{;\alpha} = H_1\underset{E}{n}^i\xi_i + \frac{1}{2m}g^{\alpha\beta}\underset{\xi}{\pounds}g_{\alpha\beta} \,.$$

Since V^m is closed and orientable, we have

$$\int\cdots\int_{V^m}\bar{\xi}^\alpha{}_{;\alpha}\,dA = 0 \,,$$

where dA is the area element of V^m ([**14**], p. 31). Thus we obtain the following integral formula:

$$\int\cdots\int_{V^m}H_1\underset{E}{n}^i\xi_i\,dA + \frac{1}{2m}\int\cdots\int_{V^m}g^{\alpha\beta}\underset{\xi}{\pounds}g_{\alpha\beta}\,dA = 0. \tag{I'}$$

3) In this paper, covariant differentiation means always the operation of D-symbol.

Let the group G be conformal, that is, ξ^i satisfy the equation

$$\underset{\xi}{\mathfrak{L}}\, g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij} \tag{1.11}$$

(cf. [13], p. 32). Then (I′) becomes

$$\int_{V^m}\cdots\int H_1 \underset{E}{n^i} \xi_i dA + \int_{V^m}\cdots\int \Phi dA = 0\,. \tag{I′)$_c$ ($$

Let G be homothetic, that is, $\Phi \equiv c = \text{const}$. Then

$$\int_{V^m}\cdots\int H_1 \underset{E}{n^i} \xi_i dA + c\int_{V^m}\cdots\int dA = 0\,. \qquad (\mathrm{I}')_h$$

Let G be isometric, that is, $c=0$. Then

$$\int_{V^m}\cdots\int H_1 \underset{E}{n^i} \xi_i dA = 0\,. \qquad (\mathrm{I}')_i$$

§2. The second integral formula. By virtue of (1.2) and (1.3) it follows that

$$\begin{aligned} \underset{E}{n^i}{}_{;\alpha} &= C^i_{j;k} \underset{E}{n^j} \frac{\partial x^k}{\partial u^\alpha} \\ &= -\sum_{\lambda=1}^{m} \left(\underset{\lambda}{i}_{j;k} \underset{E}{n^j} \frac{\partial x^k}{\partial u^\alpha} \right) \underset{\lambda}{i^i}\,. \end{aligned} \tag{2.1}$$

Then we may put

$$\underset{E}{n^i}{}_{;\alpha} = \underset{E}{\gamma^\gamma_\alpha} \frac{\partial x^i}{\partial u^\gamma}\,. \tag{2.2}$$

Multiplying (2.2) by $g_{ij}\dfrac{\partial x^j}{\partial u^\delta}$ and summing for i and j, we have

$$g_{ij}\frac{\partial x^j}{\partial u^\delta} \underset{E}{n^i}{}_{;\alpha} = \underset{E}{\gamma^\gamma_\alpha} g_{\gamma\delta}\,. \tag{2.3}$$

Since we have

$$\underset{E}{b}_{\delta\alpha} = g_{ij}\left(\frac{\partial x^j}{\partial u^\delta}\right)_{;\alpha} \underset{E}{n^i} = -g_{ij}\frac{\partial x^j}{\partial u^\delta} \underset{E}{n^i}{}_{;\alpha}\,, \tag{2.4}$$

we obtain from (2.2), (2.3) and (2.4)

$$\underset{E}{n^i}{}_{;\alpha} = -\underset{E}{b^\gamma_\alpha} \frac{\partial x^i}{\partial u^\gamma}\,, \tag{2.5}$$

where $\underset{E}{b^\gamma_\alpha} = g^{\beta\gamma} \underset{E}{b}_{\alpha\beta}$.

To the vector ξ given in **§1**, we put

(2.6) $$\eta_\alpha = \underset{E}{n}{}^i{}_{;\alpha}\xi_i\,.$$

Covariantly differentiating (2.6), by means of (1.4) and (2.5) we get

$$\eta_{\alpha;\beta} = -\left(\underset{E}{b}{}^{\gamma}_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma}\xi_i + \underset{E}{b}{}^{\gamma}_{\alpha}H_{\gamma\beta}{}^{i}\xi_i + \underset{E}{b}{}^{\gamma}_{\alpha}\frac{\partial x^i}{\partial u^\gamma}\frac{\partial x^j}{\partial u^\beta}\xi_{i;j} - F_{\alpha\beta}\right),$$

where $F_{\alpha\beta} \overset{\text{def.}}{=} \Gamma''^{P}_{E\beta}\underset{P}{n}{}^i{}_{;\alpha}\xi_i$ and $\Gamma''^{P}_{Q\alpha}$ means $\nabla_j \underset{Q}{n}{}^k B^j_\alpha \overset{P}{n}_k$.

Multiplying the above equation by $g^{\alpha\beta}$ and contracting, we obtain

(2.7) $$g^{\alpha\beta}\eta_{\alpha;\beta} = -\left(g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma}\xi_i + g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}H_{\gamma\beta}{}^{i}\xi_i + g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}\frac{\partial x^i}{\partial u^\gamma}\frac{\partial x^j}{\partial u^\beta}\xi_{i;j} - g^{\alpha\beta}F_{\alpha\beta}\right).$$

We shall first calculate the first term of the right hand side of (2.7):

(2.8) $$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma}\xi_i = g^{\alpha\beta}g^{\gamma\delta}\underset{E}{b}_{\alpha\delta;\beta}\frac{\partial x^i}{\partial u^\gamma}\xi_i\,.$$

Since the Codazzi equations hold good for the submanifold V^m in a Riemann space R^n, we have

$$\underset{E}{b}_{\alpha\delta;\beta} - \underset{E}{b}_{\alpha\beta;\delta} = -R_{ikjl}\underset{E}{n}{}^i\frac{\partial x^k}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\delta}\frac{\partial x^l}{\partial u^\beta} \qquad \text{([16], p. 266)},$$

where R_{ikjl} is the curvature tensor of R^n.

Then, from (2.8) we get

(2.9) $$\begin{aligned} g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma}\xi_i &= g^{\alpha\beta}g^{\gamma\delta}\left(\underset{E}{b}_{\alpha\beta;\delta} - R_{ikjl}\underset{E}{n}{}^i\frac{\partial x^k}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\delta}\frac{\partial x^l}{\partial u^\beta}\right)\frac{\partial x^h}{\partial u^\gamma}\xi_h \\ &= \left(mH_{1;\delta} - R_{ikjl}\underset{E}{n}{}^i\frac{\partial x^k}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\delta}\frac{\partial x^l}{\partial u^\beta}g^{\alpha\beta}\right)g^{\gamma\delta}\frac{\partial x^h}{\partial u^\gamma}\xi_h\,. \end{aligned}$$

Next, we calculate the second term of the right hand side of (2.7). By means of (1.5), it follows that

(2.10) $$g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}H_{\gamma\beta}{}^{i}\xi_i = g^{\alpha\beta}\underset{E}{b}{}^{\gamma}_{\alpha}\left(\underset{E}{b}_{\gamma\beta}\underset{E}{n}{}^i + \sum_{P=m+2}^{n}\underset{E}{b}_{\gamma\beta}\underset{P}{n}{}^i\right)\xi_i\,.$$

Now, we assume that at each point on V^m the contravariant vector ξ^i is contained in the vector space spanned by $m+1$ independent vectors $\dfrac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1, 2, \cdots, m)$ and $\underset{E}{n}{}^i$. This assumption is always satisfied for the case $m=n-1$, that is, V^m is a hypersurface in R^n. Especially, if we consider a closed curve in 3-dimensional Euclidean space, the above condition for the vector ξ^i means that ξ^i is contained in the osculating plane at each point on the curve. Now, we consider a closed plane curve and take a point in the interior of the curve as the origin of an euclidean coordinates x^i. Attaching to each point x the vector $\xi(x)$ with components $\xi^i = x^i$, the vector ξ satisfies our assumption.

From our assumption for ξ^i and (2.10), it follows that

$$g^{\alpha\beta} \underset{E}{b_\alpha^\gamma} H_{\gamma\beta}{}^i \xi_i = g^{\alpha\beta} \underset{E}{b_\alpha^\gamma} \underset{E}{b_{\gamma\beta}} \underset{E}{n^i} \xi_i .$$

Then, we have

$$g^{\alpha\beta} \underset{E}{b_\alpha^\gamma} H_{\gamma\beta}{}^i \xi_i = m \left\{ m H_1^2 - (m-1) \underset{E}{H_2} \right\} \underset{E}{n^i} \xi_i . \tag{2.11}$$

By means of (2.9) and (2.11), we get from (2.7)

$$\begin{aligned} \eta^\alpha_{;\alpha} = & -\left(m H_{1;\delta} - R_{ikjl} \underset{E}{n^i} \frac{\partial x^k}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\delta} \frac{\partial x^l}{\partial u^\beta} g^{\alpha\beta} \right) g^{\gamma\delta} \frac{\partial x^h}{\partial u^\gamma} \xi_h \\ & - m \left\{ m H_1^2 - (m-1) \underset{E}{H_2} \right\} \underset{E}{n^i} \xi_i - \frac{1}{2} \underset{E}{H^{\beta\gamma}} \underset{\xi}{\pounds} g_{\beta\gamma} + g^{\alpha\beta} F_{\alpha\beta} , \end{aligned} \tag{2.12}$$

where $\underset{E}{H^{\beta\gamma}} = g^{\alpha\beta} g^{\gamma\delta} \underset{E}{b_{\alpha\delta}}$ and $\underset{E}{H_2}$ denotes the second mean curvature of V^m for the normal direction $\underset{E}{n^i}$. By means of (2.12) we have the following integral formula :

$$\begin{aligned} & \int \cdots \int_{V^m} \left(m H_{1;\delta} - R_{ikjl} \underset{E}{n^i} \frac{\partial x^k}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\delta} \frac{\partial x^l}{\partial u^\beta} g^{\alpha\beta} \right) g^{\gamma\delta} \frac{\partial x^h}{\partial u^\gamma} \xi_h dA \\ & = -\int \cdots \int_{V^m} \left[m \left\{ \left(m H_1^2 - (m-1) \underset{E}{H_2} \right) \underset{E}{n^i} \xi_i + \frac{1}{2m} \underset{E}{H^{\beta\gamma}} \underset{\xi}{\pounds} g_{\beta\gamma} \right\} - g^{\alpha\beta} F_{\alpha\beta} \right] dA . \end{aligned} \tag{II'}$$

If the group G of transformations is conformal, (II′) becomes

$$\begin{aligned} & \int \cdots \int_{V^m} \left(m H_{1;\delta} - R_{ikjl} \underset{E}{n^i} \frac{\partial x^k}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\delta} \frac{\partial x^l}{\partial u^\beta} g^{\alpha\beta} \right) g^{\gamma\delta} \frac{\partial x^h}{\partial u^\gamma} \xi_h dA \\ & = -\int \cdots \int_{V^m} \left[m \left\{ \left(m H_1^2 - (m-1) \underset{E}{H_2} \right) \underset{E}{n^i} \xi_i + \Phi H_1 \right\} - g^{\alpha\beta} F_{\alpha\beta} \right] dA . \end{aligned} \tag{II'$_c$}$$

§3. The third integral formula. Putting

$$\rho = \underset{E}{n^i} \xi_i \tag{3.1}$$

by means of (2.6) we have

$$\rho_{;\alpha} = \eta_\alpha + \underset{E}{n^i} \xi_{i;j} \frac{\partial x^j}{\partial u^\alpha} . \tag{3.2}$$

By covariant differentiation of (3.2), it follows that

$$\rho_{;\alpha;\beta}=\eta_{\alpha;\beta}+\underset{E}{n}^{i}{}_{;\beta}\xi_{i;j}\frac{\partial x^{j}}{\partial u^{\alpha}}-F_{\alpha\beta}+\underset{E}{n}^{i}\xi_{i;j;k}\frac{\partial x^{j}}{\partial u^{\alpha}}\frac{\partial x^{k}}{\partial u^{\beta}}+\underset{E}{n}^{i}\xi_{i;j}H_{\alpha\beta}{}^{j}\,. \tag{3.3}$$

Multiplying (3.3) by $g^{\alpha\beta}$ and contracting, by virtue of (1.7) and (2.5) we get

$$\begin{aligned} g^{\alpha\beta}\rho_{;\alpha;\beta}=\eta^{\alpha}{}_{;\alpha}-g^{\alpha\beta}\underset{E}{b}_{\beta}{}^{\gamma}\xi_{i;j}\frac{\partial x^{i}}{\partial u^{\gamma}}\frac{\partial x^{j}}{\partial u^{\alpha}}-g^{\alpha\beta}F_{\alpha\beta}\\ +\underset{E}{n}^{i}\xi_{i;j;k}\frac{\partial x^{j}}{\partial u^{\alpha}}\frac{\partial x^{k}}{\partial u^{\beta}}g^{\alpha\beta}+mH_{1}\underset{E}{n}^{i}\underset{E}{n}^{j}\xi_{i;j}\,. \end{aligned} \tag{3.4}$$

Therefore we have the following integral formula:

$$\int\cdots\int_{V^m}\left\{-\frac{1}{2}\underset{E}{H}^{\alpha\gamma}\underset{\xi}{\pounds}g_{\alpha\gamma}+\underset{E}{n}^{i}\xi_{i;j;k}\frac{\partial x^{j}}{\partial u^{\alpha}}\frac{\partial x^{k}}{\partial u^{\beta}}g^{\alpha\beta}+\frac{m}{2}H_{1}\underset{E}{n}^{i}\underset{E}{n}^{j}\underset{\xi}{\pounds}g_{ij}-g^{\alpha\beta}F_{\alpha\beta}\right\}dA=0\,. \tag{III'}$$

Let the group G be conformal. Then we have

$$\xi^{i}{}_{;j;k}=\underset{\xi}{\pounds}\left\{{i \atop jk}\right\}-R^{i}{}_{jhk}\xi^{h}\,, \tag{3.5}$$

$$\underset{\xi}{\pounds}\left\{{i \atop jk}\right\}=\delta^{i}_{j}\Phi_{k}+\delta^{i}_{k}\Phi_{j}-\Phi^{i}g_{jk}\,, \tag{3.6}$$

where $\left\{{i \atop jk}\right\}$ are the Christoffel symbols and $\Phi_{k}=\Phi_{;k}$, $\Phi^{i}=g^{ij}\Phi_{j}$ ([**13**], p. 160).

By means of (1.11), the first term of the left hand side of (III′) becomes

$$-\frac{1}{2}H^{\alpha\gamma}\underset{\xi}{\pounds}g_{\alpha\gamma}=-m\Phi H_{1}\,. \tag{3.7}$$

Now, we calculate the second term of the left hand side of (III′). By means of (3.5) and (3.6) it follows that

$$\underset{E}{n}^{i}\xi_{i;j;k}\frac{\partial x^{j}}{\partial u^{\alpha}}\frac{\partial x^{k}}{\partial u^{\beta}}g^{\alpha\beta}=-m\underset{E}{n}^{i}\Phi_{i}-R_{ijhk}\underset{E}{n}^{i}\xi^{h}\frac{\partial x^{j}}{\partial u^{\alpha}}\frac{\partial x^{k}}{\partial u^{\beta}}g^{\alpha\beta}\,. \tag{3.8}$$

We consider that the vector ξ^{i} satisfy the assumption stated in **§2**. Then, we may put

$$\xi^{h}=\varphi^{\gamma}\frac{\partial x^{h}}{\partial u^{\gamma}}+\rho\underset{E}{n}^{h}\,. \tag{3.9}$$

From (3.9), we get

$$\varphi^{\gamma}=g_{lm}\xi^{l}\frac{\partial x^{m}}{\partial u^{\delta}}g^{\gamma\delta}\,. \tag{3.10}$$

Making use of (3.9) and (3.10), it follows that

$$R_{ijhk}\underset{E}{n^i}\xi^h \frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}$$

$$(3.11) \qquad = R_{ijhk}\underset{E}{n^i}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^h}{\partial u^\gamma}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}g^{\gamma\delta}\xi_m \frac{\partial x^m}{\partial u^\delta}$$

$$+\rho R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}.$$

By means of (1.11), the third term of the left hand side of (III′) becomes

$$(3.12) \qquad \frac{m}{2} H_1 \underset{E}{n^i}\underset{E}{n^j}\underset{\xi}{\mathfrak{L}} g_{ij} = m\Phi H_1 .$$

Therefore, by virtue of (3.7), (3.8), (3.11) and (3.12), (III′) is rewritten as follows:

$$(3.13) \qquad \int\cdots\int_{V^m}\Big\{m\underset{E}{n^i}\Phi_i + R_{ijhk}\underset{E}{n^i}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^h}{\partial u^\gamma}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}g^{\gamma\delta}\xi_m\frac{\partial x^m}{\partial u^\delta}$$

$$+\rho R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta} + g^{\alpha\beta}F_{\alpha\beta}\Big\} dA = 0 .$$

By means of $(\mathrm{II}')_c$ in **§2** and (3.13), finally we obtain the following integral formula:

$$\int\cdots\int_{V^m}\Big\{mH_{1;\delta} + \rho R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}\Big\} dA$$

$$= -\int\cdots\int_{V^m} m\Big\{\Big(mH_1^2-(m-1)\underset{E}{H_2}\Big)\underset{E}{n^i}\xi_i + \underset{E}{n^i}\Phi_i + \Phi H_1\Big\} dA . \qquad (\mathrm{III}')_c$$

If the group G is homothetic, we have

$$\int\cdots\int_{V^m}\Big\{mH_{1;\delta} + \rho R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} g^{\alpha\beta}\Big\} dA$$

$$= -\int\cdots\int_{V^m} m\Big\{\Big(mH_1^2-(m-1)\underset{E}{H_2}\Big)\underset{E}{n^i}\xi_i + cH_1\Big\} dA . \qquad (\mathrm{III}')_h$$

§4. Some properties of a closed orientable submanifold. If R^n is the constant Riemann curvature space and if V^m has the property $H_1=$const., then the left hand side of (II′) in **§2** vanishes and we have

$$\int\cdots\int_{V^m}\Big[\Big\{mH_1^2-(m-1)\underset{E}{H_2}\Big\}\underset{E}{n^i}\xi_i + \frac{1}{2m}\underset{E}{H^{\beta\gamma}}\underset{\xi}{\mathfrak{L}} g_{\beta\gamma} - g^{\alpha\beta}F_{\alpha\beta}\Big] dA = 0 . \qquad (\mathrm{II}'')$$

If the group G of transformations is concircular (II$''$) becomes

$$\int_{V^m}\cdots\int\left\{mH_1^2-(m-1)\underset{E}{H_2}\right\}\underset{E}{n^i}\xi_i dA+\int_{V^m}\cdots\int\Phi H_1 dA=0\,. \qquad (\mathrm{II}'')_{\mathrm{c}}$$

If the group G is homothetic,

$$\int_{V^m}\cdots\int\left\{mH_1^2-(m-1)\underset{E}{H_2}\right\}\underset{E}{n^i}\xi_i dA+c\int_{V^m}\cdots\int H_1 dA=0\,, \qquad (\mathrm{II}'')_{\mathrm{h}}$$

and if G is isometric

$$\int_{V^m}\cdots\int\left\{mH_1^2-(m-1)\underset{E}{H_2}\right\}\underset{E}{n^i}\xi_i dA=0\,. \qquad (\mathrm{II}'')_{\mathrm{i}}$$

Now, we shall prove the following theorem:

Theorem 4.1. *Let R^n be a constant Riemann curvature space and V^m a closed orientable submanifold with $\underset{E}{H_1}=const.$ We suppose that there exists a continuous one-parameter group G of concircular transformations generated by a vector ξ^i of R^n such that the scalar product $\rho=\underset{E}{n^i}\xi_i$ does not change the sign (and is not $\equiv 0$) on V^m, where the vector ξ is contained in the vector space spanned by $m+1$ vectors $\frac{\partial x^i}{\partial u^\alpha}$ $(\alpha=1,2,\cdots,m)$ and $\underset{E}{n^i}$. Then every point of V^m is umbilic with respect to Euler–Schouten vector $\underset{E}{n}$.*

Proof. Multiplying the formula (I$'$)$_{\mathrm{c}}$ in **§1** by H_1 (=const.), we obtain

$$\int_{V^m}\cdots\int H_1^2\rho dA+\int_{V^m}\cdots\int\Phi H_1 dA=0\,.$$

Therefore, subtracting this formula from the formula (II$''$)$_{\mathrm{c}}$ in **§4**, we find

$$(4.1)\qquad \int_{V^n}\cdots\int(m-1)(H_1^2-\underset{E}{H_2})\rho dA=0\,.$$

By means of

$$mH_1=\sum_\alpha \underset{E}{k_\alpha}\,,\qquad \binom{m}{2}\underset{E}{H_2}=\sum_{\alpha<\beta}\underset{E}{k_\alpha}\underset{E}{k_\beta}$$

we get

$$(4.2)\qquad H_1^2-\underset{E}{H_2}=\frac{1}{m^2(m-1)}\sum_{\alpha<\beta}(\underset{E}{k_\alpha}-\underset{E}{k_\beta})^2\,,$$

where $\underset{E}{k_1},\underset{E}{k_2},\cdots,\underset{E}{k_m}$ denote the principal curvatures for $\underset{E}{n}$.

Then we see that

(4.3) $$H_1^2 - \underset{E}{H_2} \geqq 0\,.$$

Therefore, because of (4.2), it follows that

$$\underset{E}{k_1} = \underset{E}{k_2} = \cdots = \underset{E}{k_m}$$

hold good at each point of V^m.

Theorem 4.2. *Let R^n be an n-dimensional Riemann space and V^m an m-dimensional closed orientable submanifold with $H_1 = const.$ We suppose that there exists a continuous one-parameter group G of homothetic transformations generated by a vector ξ^i of R^n such that the scalar product $\rho = \underset{E}{n^i}\xi_i$ does not change the sign (and is not $\equiv 0$) on V^m, where the vector ξ is contained in the vector space spanned by $m+1$ vectors $\frac{\partial x^i}{\partial u^\alpha}$ $(\alpha = 1, 2, \cdots, m)$ and $\underset{E}{n^i}$. If the relation*

$$R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\sum_{\lambda=1}^{m}\underset{\lambda}{i^j}\underset{\lambda}{i^k} \geqq 0$$

holds good on V^m, then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$.

Proof. Multiplying the formula $(\mathrm{I}')_\mathrm{h}$ in **§1** by H_1 (=const.), we have

(4.4) $$\int \cdots \int_{V^m} H_1^2 \underset{E}{n^i}\xi_i dA + \int \cdots \int cH_1 dA = 0\,.$$

From our assumption $H_1 =$ const., $(\mathrm{III}')_\mathrm{h}$ becomes

(4.5) $$\begin{aligned}&\frac{1}{m}\int \cdots \int_{V^m} \rho R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}g^{\alpha\beta}dA \\ &= -\int \cdots \int_{V^m} \left\{mH_1^2 - (m-1)\underset{E}{H_2}\right\}\underset{E}{n^i}\xi_i dA - \int \cdots \int_{V^m} cH_1 dA\,.\end{aligned}$$

By means of (3.1), (4.4) and (4.5), we have

(4.6) $$\int \cdots \int_{V^m} \rho\left\{(m-1)(H_1^2 - \underset{E}{H_2}) + \frac{1}{m}R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}g^{\alpha\beta}\right\}dA = 0\,.$$

Since we have

$$g^{\alpha\beta}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} = \sum_{\lambda=1}^{m}\underset{\lambda}{i^j}\underset{\lambda}{i^k}\,,$$

if $R_{ijhk}\underset{E}{n^i}\underset{E}{n^h}\sum_{\lambda=1}^{m}\underset{\lambda}{i^j}\underset{\lambda}{i^k} \geqq 0$ hold good on V^m, we get from (4.6) the relation

$$H_1^2 - \underset{E}{H_2} = 0 .$$

This means $\underset{E}{k_1} = \underset{E}{k_2} = \cdots = \underset{E}{k_m}$.

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