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ON SOME PROPERTIES OF DEDDENS ALGEBRAS

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ABSTRACT. Deddens algebras and Shulman subspaces are introduced and their properties are studied. The descriptions of Deddens algebras associated with nilpotent and idempotent elements are given.

1. Introduction. Let H be a Hilbert space, $B(H)$ be an algebra of all bounded linear operators in H. In [**1**], Deddens determined for any invertible operator A from $B(H)$ the following algebra:

$$B_A \stackrel{\text{def}}{=} \Big\{X \in B(H) : \sup_{n\geq 0} \Big\|A^n X A^{-n}\Big\| \stackrel{\text{def}}{=} C_X < +\infty\Big\}.$$

It was proved in [**1**] that, for $A \geq 0$, B_A coincides with the nest algebra generated by the nest $\{E_A([0,\lambda]) : \lambda \geq 0\}$ (where E_A is the spectral measure of A) that gives a suitable characterization of nest algebras in all respects. Recently Todorov [**7**] has extended this result to weakly or strongly closed bimodules of a nest algebra. In [**2**] Deddens and Wong have proved that if $A = \lambda I + N$, where $\lambda \in \mathbf{C} \setminus \{0\}$ is a complex number, and N is a nilpotent operator, then the algebra B_A coincides with the commutant $\{A\}'$ of A. In their proof of the last statement the Hilbert property of the space H is essentially used.

The main aim of this paper is to show that the result of Deddens and Wong is valid in any unital Banach algebra.

2. Deddens algebras. Let B be a Banach algebra with the unit e. For any invertible element $a \in B$ put

$$B_a \stackrel{\text{def}}{=} \Big\{x \in B : \sup_{n\geq 0} \|a^n x a^{-n}\| \stackrel{\text{def}}{=} C_x < +\infty\Big\}.$$

We call the algebra B_a the Deddens algebra.

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Our main result is the following.

Theorem 1. *Let B be a Banach algebra with a unit e. If $a = e + b$, where b is a nilpotent element of the algebra B, then the Deddens algebra B_a coincides with the commutant $\{a\}'$, i.e., $B_a = \{a\}'$.*

Before passing to the proof of the theorem, we prove the following general lemma.

Lemma 2. *Let B, a, b be the same as in Theorem 1. Let $a_n \in B$, $n = 0, 1, 2, \dots$, be such that*

1) $\|a_n\| = O(n^\alpha)$, $n \to +\infty$, *for some* α, $0 \le \alpha < 1$;

2) *for some* $c \in B$

$$a_n a = a a_{n-1} + c, \quad n = 1, 2, \dots .$$

Then $a_0 = a_1 = a_2 = \dots$.

Proof. It is sufficient to prove the lemma in the case $c = 0$. Indeed, it follows from the equality

$$a_n a = a a_{n-1} + c, \quad n \ge 1,$$

that

$$d_n a = a d_{n-1}, \quad n \ge 1, \tag{1}$$

where $d_n \overset{\text{def}}{=} a_n - a_{n-1}$. It is clear that $\|d_n\| = O(n^a)$ for $n \to +\infty$. Assume that the lemma is valid for $c = 0$. Taking into account (1) and applying our hypothesis to the sequence (d_n), we obtain the equality

$$d_0 = d_1 = d_2 = \dots ,$$

that is,

$$a_1 - a_0 = a_2 - a_1 = \dots = x.$$

Hence

$$a_n = a_0 + nx, \quad n \ge 1,$$

whence it follows from condition 1) that

$$\|x\| \leq \frac{\|a_n - a_0\|}{n} \longrightarrow 0 \quad \text{for } n \to +\infty,$$

i.e., $x = 0$. Therefore, $a_0 = a_1 = a_2 = \cdots$.

So it is sufficient to prove the statement of the lemma for $c = 0$.

Let $k \geq 2$ be the nilpotency degree of the element $b \in B$, that is, $b^k = 0$, but $b^{k-1} \neq 0$. Then for any $n \geq k$,

$$(e + b)^n = e + \alpha_1 b + \alpha_2 b^2 + \cdots + \alpha_{k-1} b^{k-1},$$

where $\alpha_m \overset{\text{def}}{=} C_n^m = \dfrac{n!}{m!(n-m)!}$, $m = 1, 2, \dots, k-1$. The inverse element of $(e + b)^n$ has the form

$$(e + b)^{-n} = e + \beta_1 b + \beta_2 b^2 + \cdots + \beta_{k-1} b^{k-1}$$

for some numbers $\beta_1, \beta_2, \dots, \beta_{k-1}$. Taking the equality

$$(e + \beta_1 b + \cdots + \beta_{k-1} b^{k-1})(e + \alpha_1 b + \cdots + \alpha_{k-1} b^{k-1}) = e,$$

then removing the parentheses and identifying the coefficients, we obtain the system that connects the numbers $\alpha_1, \dots, \alpha_{k-1}$ with the numbers $\beta_1, \dots, \beta_{k-1}$:

$$\text{(2)} \qquad \begin{cases} \beta_1 + \alpha_1 = 0 \\ \alpha_1 \beta_1 + \beta_2 + \alpha_2 = 0 \\ \alpha_2 \beta_1 + \alpha_1 \beta_2 + \beta_3 + \alpha_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{k-2} \beta_1 + \alpha_{k-3} \beta_2 + \cdots + \beta_{k-1} + \alpha_{k-1} = 0. \end{cases}$$

From the equality

$$a_n a = a a_{n-1}$$

we have

$$a_n (e + b)^n = (e + b)^n a_0,$$

that is,

$$a_n = (e + b)^n a_0 (e + b)^{-n}$$

or

$$a_n = (e + \alpha_1 b + \cdots + \alpha_{k-1} b^{k-1}) a_0 (e + \beta_1 b + \cdots + \beta_{k-1} b^{k-1}),$$

for all $n \geq k$. Hence we have

$$\begin{aligned}
a_n - a_0 = {} & (\beta_1 a_0 b + \beta_2 a_0 b^2 + \cdots + \beta_{k-1} a_0 b^{k-1}) \\
& + (\alpha_1 b a_0 + \alpha_1 \beta_1 b a_0 b + \alpha_1 \beta_2 b a_0 b^2 + \cdots \\
& + \alpha_1 \beta_{k-2} b a_0 b^{k-2} + \alpha_1 \beta_{k-1} b a_0 b^{k-1}) \\
& + (\alpha_2 b^2 a_0 + \alpha_2 \beta_1 b^2 a_0 b + \alpha_2 \beta_2 b^2 a_0 b + \cdots \\
& + \alpha_2 \beta_{k-1} b^2 a_0 b^{k-1}) + \cdots \\
& + (\alpha_{k-2} b^{k-2} a_0 + \alpha_{k-2} \beta_1 b^{k-2} a_0 b \\
& + \alpha_{k-2} \beta_2 b^{k-2} a_0 b^2 + \cdots \\
& + \alpha_{k-2} \beta_{k-2} b^{k-2} a_0 b^{k-2} + \alpha_{k-2} \beta_{k-1} b^{k-2} a_0 b^{k-1}) \\
& + (\alpha_{k-1} b^{k-1} a_0 + \alpha_{k-1} \beta_1 b^{k-1} a_0 b \\
& + \alpha_{k-1} \beta_2 b^{k-1} a_0 b^2 + \cdots + \alpha_{k-1} \beta_{k-2} b^{k-1} a_0 b^{k-2} \\
& + \alpha_{k-1} \beta_{k-1} b^{k-1} a_0 b^{k-1}).
\end{aligned} \tag{3}$$

Since

$$\begin{aligned}
\alpha_m = \alpha_m(n) &= \frac{1}{m!} [n(n-1)(n-2) \cdots (n-m+1)] \\
&= n\varphi_1(m) + n^2 \varphi_2(m) + \cdots + n^m \varphi_m(m)
\end{aligned}$$

($m = 1, 2, \dots, k-1$) where φ_i, $i = 1, 2, \dots, m$, do not depend on n, then as we see from system (2), β_m is also calculated by the formula

$$\beta_m = n\psi_1(m) + n^2 \psi_2(m) + \cdots + n^p \psi_p(m),$$

where $p \leq k-1$, and coefficients ψ_i, $i = 1, 2, \dots, p$, do not depend on n. Therefore, after some simple calculations we can write the equality (3) as follows:

$$\begin{aligned}
a_n - a_0 = {} & n f_1(k, a_0, b) + n^2 f_2(k, a_0, b) + \cdots \\
& + n^{2(k-1)} f_{2(k-1)}(k, a_0, b)
\end{aligned} \tag{4}$$

where $f_j(k, a_0, b) \in B$, $j = 1, 2, \dots, 2(k-1)$, do not depend on n. For convenience we determine

$$J_{2(k-1)} \overset{\text{def}}{=} (a_n - a_0) - n^{2(k-1)} f_{2(k-1)}(k, a_0, b).$$

We have from equality (4) by virtue of condition 1)

$$\|f_{2(k-1)}(k,a_0,b)\| \le \frac{\|a_n - a_0\|}{n^{2(k-1)}} + \frac{\|J_{2(k-1)}\|}{n^{2(k-1)}} \longrightarrow 0$$

for $n \to +\infty$. Hence we conclude that

$$f_{2(k-1)}(k,a_0,b) = 0.$$

The sequential repetition of this argument shows us that all summands in equality (4) equal zero, and thus we obtain

$$a_n - a_0 = 0$$

for any $n \ge k$, that is, $a_n = a_0$, $n \ge k$. It remains to show that

$$a_{k-1} = a_{k-2} = \cdots = a_1 = a_0.$$

Since

$$a_n = (e+b)^{n-m} a_m (e+b)^{-(n-m)}, \quad m = 1, 2 \dots, k-1,$$

then, by using similar arguments, we see that

$$a_n - a_m = 0,$$

and so $a_n = a_m$ for each $n \ge k$ and m, $1 \le m \le k-1$.

Thus

$$a_0 = a_1 = a_2 = \dots .$$

The lemma is proved. □

Now we prove the theorem.

Proof of Theorem 1. Let $x \in B_a$ be any element. Put

$$c_n \overset{\text{def}}{=} a^n x a^{-n}, \quad n \ge 0.$$

Then

$$c_n a = a^n x a^{-n} a = a(a^{n-1} x a^{-(n-1)}) = a c_{n-1},$$

that is,

$$c_n a = a c_{n-1}, \quad n \geq 1. \tag{5}$$

Since $x \in B_a$, there exists a constant $c_x > 0$ such that $\|c_n\| \leq c_x$, $n \geq 0$. Taking into account these inequalities and equality (5), we state by means of Lemma 2 that $c_n = c_{n-1}$, $n \geq 1$, and in particular, $c_1 = c_0$, and thus

$$axa^{-1} = x.$$

Hence, $ax = xa$, $x \in \{a\}'$. Consequently, $B_a \subset \{a\}'$. The inclusion $\{a\}' \subset B_a$ is obvious, and therefore $B_a = \{a\}'$. The theorem is proved. □

Let B be a Banach algebra with the idempotent p (i.e., $p^2 = p$) and with the unit e. We introduce the following notation

$$S_p \stackrel{\text{def}}{=} \{x \in B : px(e-p) = 0\}.$$

Our next theorem describes the Deddens algebra associated with the idempotent.

Theorem 3. *Let B be a Banach algebra with an idempotent p and with the unit e. Then*

$$B_{e+p} = S_p.$$

Proof. Think of the algebra B as having a (2×2)-matrix decomposition relative to the decomposition of the identity $e = p + (e-p)$; thus elements of B have the form $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Relative to this decomposition, p takes the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $e + p = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. An easy calculation then shows that $(e+p)^n \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} (e+p)^{-n}$ is a bounded sequence if and only if $b_{12} = 0$, which is equivalent to the desired result. □

Corollary 4. *Let $a \in B$ be regular by a von Neumann element* (*that is, there exists an element $b \in B$ satisfying the condition $a = aba$*). *Let*

$$B^a \stackrel{\text{def}}{=} \{x \in B : xa = ay \text{ for some } y \stackrel{\text{def}}{=} y_x\}.$$

Then B^a is an algebra and $B^a \subset B_{e+p_a}$, where $p_a \overset{\text{def}}{=} ab$ is an idempotent element of B.

Proof. It is clear that B^a is an algebra and p_a is an idempotent. Now we show that $B^a \subset B_{e+p_a}$. We will let $x \in B$. Then it is clear from the equality $xa = ay$ that

$$bxa = bay.$$

We have

$$abxa = abay = ay = xa,$$

whence

$$(e - ab)xa = 0,$$

i.e.,

$$(e - ab)xab = 0;$$

consequently,

$$(e - p_a)xp_a = 0.$$

This equality means by virtue of Theorem 3 that $x \in S_{p_a} = B_{e+p_a}$. The proof is completed. □

In the remainder of this section we are concerned with Deddens operator algebras.

For two arbitrarily chosen operators $L, M \in B(H)$, we introduce for consideration the following subspace of the algebra $B(H)$:

$$\mathcal{U}(L, M) \overset{\text{def}}{=} \{L\}' + \{L\}'M.$$

Such subspaces have been studied in detail by Shulman (see [**5**], [**6**]) for the integration operator V, $(Vf)(x) = \int_0^x f(t)\,dt$ and multiplication operator T, $(Tf)(x) = xf(x)$ in the space $L^2[0, 1]$ in relation with nontransitivity of root algebras. We call the subspace $\mathcal{U}(L, M)$ the Shulman subspace.

The relation between Deddens algebras and Shulman subspaces is established in the next theorem. Below, the number $\lambda \in C$ is assumed to be such that $L_\lambda \overset{\text{def}}{=} \lambda I + L$ is an invertible operator.

Theorem 5. *Let the operators $L, M \in B(H)$ satisfy the Kleinecke-Shirokov condition, i.e., $X \overset{\text{def}}{=} [M, L] \in \{L\}'$. Then the intersection of Deddens algebra $B_{\lambda I+L}$ and the weak closure of Shulman subspace $\mathcal{U}(L, M)$ coincide with a commutant of the operator L, that is,*

$$B_{\lambda I+L} \cap \overline{\mathcal{U}(L, M)}^w = \{L\}'.$$

Proof. Let $A \in B_{L_\lambda} \cap \overline{\mathcal{U}(L, M)}^w$ be any operator. Then there exist the sequences of operators X_n and Y_n from $\{L\}'$ such that

$$\lim_{n\to\infty} \langle (X_n + Y_n M)x, y\rangle = \langle Ax, y\rangle$$

for all $x, y \in H$. Then it is clear that

$$\lim_{n\to\infty} \langle (X_n + Y_n M)Lx, y\rangle = \langle ALx, y\rangle$$

and

$$\lim_{n\to\infty} \langle L(X_n + Y_n M)x, y\rangle = \langle LAx, y\rangle.$$

Since, by the condition of the theorem,

$$ML - LM = X \in \{L\}',$$

it follows that

$$\langle (AL - LA)x, y\rangle = \lim_{n\to\infty} \langle Y_n Xx, y\rangle$$

for all $x, y \in H$. Therefore

$$AL - LA \in \overline{\{L\}'X}^w \subset \{L\}',$$

or,

$$AL - LA = Y, \tag{6}$$

where $Y \in \{L\}'$. Therefore,

$$AL_\lambda - L_\lambda A = Y.$$

Hence
$$A - L_\lambda A L_\lambda^{-1} = Y L_\lambda^{-1},$$
that is,
$$L_\lambda A L_\lambda^{-1} = A - Y L_\lambda^{-1}. \tag{7}$$
By multiplying both sides of equality (7) from the left by L_λ, and from the right by L_λ^{-1}, and again considering (7), we get that
$$\begin{aligned} L_\lambda^2 A L_\lambda^{-2} &= L_\lambda A L_\lambda^{-1} - L_\lambda Y L_\lambda^{-2} \\ &= A - Y L_\lambda^{-1} - Y L_\lambda^{-1} = A - 2Y L_\lambda^{-1}, \end{aligned}$$
or simply,
$$L_\lambda^2 A L_\lambda^{-2} = A - 2Y L_\lambda^{-1}.$$
Thus we prove by induction that
$$L_\lambda^n A L_\lambda^{-n} = A - nY L_\lambda^{-1}, \quad n \geq 0.$$
Since $A \in B_{L_\lambda}$, we have
$$\|Y L^{-1}\| \leq \frac{\|A\| + C_A}{n} \longrightarrow 0, \quad n \to +\infty,$$
i.e.,
$$Y L_\lambda^{-1} = 0,$$
and therefore $Y = 0$. This means by virtue of (6) that $A \in \{L\}'$. Consequently,
$$B_{L_\lambda} \cap \overline{\mathcal{U}(L,M)}^w \subset \{L\}'.$$
The inverse inclusion is obvious and so
$$B_{L_\lambda} \cap \overline{\mathcal{U}(L,M)}^w = \{L\}'.$$
The theorem is proved. □

Example. Let V be the Volterra integration operator $f \to \int_0^x f(t)\,dt$ and T be the multiplication operator $f \to xf(x)$ in $L^2[0,1]$. It is easy to verify that
$$TV - VT = V^2.$$

Hence, the operators V and T satisfy the condition of Theorem 5 and therefore

$$B_{1+V} \cap \overline{\mathcal{U}(V,T)}^w = \{V\}'.$$

Before passing to the next result, we note the following:

The radical R of any complex normed algebra $\mathcal{D}$ with identity is defined as

$$R(\mathcal{D}) \overset{\text{def}}{=} \{x \in \mathcal{D} : xy \text{ is quasinilpotent for all } y \in \mathcal{D}\}.$$

For an invertible operator A, let

$$R_A \overset{\text{def}}{=} \{X \in B(H) : \lim_{k\to\infty} \|A^k X A^{-k}\| = 0\}.$$

It is known [**1**] that R_A is a bilateral ideal in algebra B_A contained in the radical $R(B_A)$.

Proposition 6. *Let $L, M \in B(H)$, and let $[M, L] \in \{L\}'$. Then*

$$R_{L_\lambda} \cap \overline{\mathcal{U}(L,M)}^w = \{0\}.$$

Proof. As we already proved in Theorem 5, for each $A \in \overline{\mathcal{U}(L,M)}^w$, there exists $Y \in \{L\}'$ such that

$$AL - LA = Y. \tag{8}$$

Then for each $n \geq 0$, we have

$$\begin{aligned}\|Y\| &= \|L_\lambda^n Y L_\lambda^{-n}\| = \|L_\lambda^n (AL - LA) L_\lambda^{-n}\| \\ &= \|L_\lambda^n A L L_\lambda^{-n} - L_\lambda^n L A L_\lambda^{-n}\| \\ &= \|L_\lambda^n A L_\lambda^{-n} L - L L_\lambda^n A L_\lambda^{-n}\| \leq 2\|L\| \|L_\lambda^n A L_\lambda^{-n}\|.\end{aligned}$$

Consequently,

$$\|L_\lambda^n A L_\lambda^{-n}\| \geq \frac{1}{2} \frac{\|Y\|}{\|L\|}, \tag{9}$$

$n \geq 0$. The statement of the proposition directly follows from (9). The proof is completed. □

Corollary 7. *Let $\Delta = \Delta_V$ be the inner derivation $X \to [V, X]$ of $B(L^2[0,1])$ and p_n be the polynomial of the form $p_n(z) = (1+z)^n$. Then*

$$\inf_{n\geq 0} \|p_n(V)\| \geq \frac{\pi}{4} \|\Delta_V | \ker \Delta_V^2\|,$$

where V is the Volterra integration operator in $L^2[0,1]$.

Proof. As Sarason [**4**] proved, $\{V\}' = \operatorname{alg}(V)$, the weak closed algebra generated by the operators V and I. Therefore, it follows from the results of Shulman (see [**6**, Theorem 1.1]) that

$$\ker \Delta_V^2 = \overline{\mathcal{U}(V,T)}^w, \tag{10}$$

where T is an operator of multiplication by independent variable in $L^2[0,1]$. Now by setting in Proposition 6 $L = V$, $M = T$ and taking into account the equality $\|V\| = \frac{2}{\pi}$ (see [**3**, Problem 188]), (8) and (10), we get from (9) that for any $A \in \ker \Delta_V^2$,

$$\frac{\pi}{4}\|AV - VA\| \leq \|(I+V)^n A (I+V)^{-n}\|, \quad n \geq 0.$$

Hence, taking into account the known equality $\|(I+V)^{-1}\| = 1$ (see [**3**, Problem 190]), we have

$$\frac{\pi}{4}\|AV - VA\| \leq \|(I+V)^n\| \|A\|,$$

that is,

$$\frac{\pi}{4}\|\Delta_V(A)\| \leq \|(I+V)^n\|\|A\|.$$

We have from this

$$\inf_{n\geq 0}\|p_n(V)\| \geq \frac{\pi}{4}\|\Delta_V | \ker \Delta_V^2\|.$$

The proof is completed. □

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