ON SOME PROPERTIES OF π-STRUCTURES ON DIFFERENTIABLE MANIFOLD

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D. C. Spencer [1][1)] considered under the name "complex almost-product structure" the structure on the n-dimensional differentiable manifold V_n defined by giving two differentiable distributions T_1, T_2 which assign two complemented subspaces of dimension $\geqq 1$ in the complexified tangent space T_x^C at each point $x \in V_n$. G. Legrand [2] called such structure as a π-structure and studied it by generalizing most properties of the almost complex structure which can be regarded as a special case of it [3].

In the following, we assume that on the manifold a structure is defined by giving r $(2 \leqq r \leqq n)$ differentiable distributions $T_1, \ldots\ldots, T_r$ which assign r complemented subspaces of dimension $\geqq 1$ in the complexified tangent space $T_x^C(T_x^C = T_1 + \ldots\ldots + T_r$: direct sum) at each point $x \in V_n$. We call such structure as an r-π-structure if we want to express the number of the distributions explicitly. Whereas we call it simply as a π-structure if we need not (or can not) express the number r definitely. We generalize some properties of π-structure in the sense of Legrand to the r-π-structure.

In this note we assume that the differentiable manifold V_n as well as the distributions $T_1, \ldots\ldots, T_r$ are of class C^∞ unless we state it explicitly. It is also assumed that the manifold is arc-wise connected and the second countability axiom is satisfied.

1. Fundamental tensor of the π-structure. Suppose the differentiable manifold V_n has a π-structure defined by r differentiable distributions $T_1, \ldots\ldots, T_r$. Let the projection operations from T_x^C to T_α be denoted as $\mathfrak{P}_\alpha$, then we have

$$\mathfrak{P}_\alpha^2 = \mathfrak{P}_\alpha, \quad \mathfrak{P}_\alpha \mathfrak{P}_\beta = 0 \ (\alpha \neq \beta), \tag{1.1}$$

$$\mathfrak{P}_1 + \ldots\ldots + \mathfrak{P}_r = \mathfrak{J}, \tag{1.2}$$

where $\mathfrak{J}$ denotes the identity transformation and the Greek indices vary from 1 to r. Define a transformation $\mathfrak{F}$ on T_x^C by the following:

$$\mathfrak{F}v = \lambda \sum_\alpha w_\alpha \mathfrak{P}_\alpha v, \tag{1.3}$$

1) Numbers in bracket refer to the reference at the end of the paper.

where v is any vector in T_x^C, λ is a non zero complex constant and $w_\alpha(\alpha = 1, \ldots \ldots, r)$ are the r-th power roots of unity. It is obvious that

$$\mathfrak{F}^s v = \lambda^s \sum_\alpha w_\alpha{}^s \mathfrak{P}_\alpha v \quad (1 \leqq s \leqq r), \tag{1.4}$$

thus we have

$$\mathfrak{F}^r v = \lambda^r v, \quad \text{i. e.,} \quad \mathfrak{F}^r = \lambda^r \mathfrak{F}. \tag{1.5}$$

On the manifold there exists a complex tensor field which induces $\mathfrak{F}$ at each T_x^C. Let this tensor field be denoted as $F_j{}^i$, then we have from (1.5) the following:

$$\overset{r}{F}_j{}^i \equiv F_{h_1}{}^i F_{h_2}{}^{h_1} F_{h_3}{}^{h_2} \ldots\ldots F_j{}^{h_{r-1}} = \lambda^r \delta_j^i. \tag{1.6}$$

Conversely, if the manifold has a non trivial tensor $F_j{}^i$ satisfying (1.6), and $\mathfrak{F}$ be the transformation induced at T_x^C by $F_j{}^i$, then it is obvious that the proper values of $\mathfrak{F}$ are among λw_α $(\alpha = 1, \ldots\ldots, r)$. If $\mathfrak{F}$ has actually $s(s \geqq 2$, because $F_j{}^i$ is non trivial) of them as its proper values, then the number s and the proper values do not vary when the point x varies on the manifold, because of the differentiability of the considered tensor field $F_j{}^i$ and the connectedness of the manifold. Consequently the manifold has s differentiable distributions constituted of s proper subspaces in T_x^C at each point x. Thus we have

THEOREM 1. 1. *The manifold is endowed with a π-structure if and only if the manifold has a non trivial tensor field $F_j{}^i$ satisfying* (1.6) *for some r: $2 \leqq r \leqq n$.*

A tensor satisfying (1.6) is said to be *degenerate* if the number of its different proper values $s < r$. An example of degenerate tensor of the type (1.6) is given by:

$$\underset{1}{F}_j{}^i = \left(\cos\frac{\pi}{r}\right)\delta_j^i + \left(\sin\frac{\pi}{r}\right)\phi_j{}^i \text{ 2)}, \tag{1.7}$$

where $\phi_j{}^i$ is assumed to be a tensor defining an almost complex structure on the manifold, i. e., it is a real tensor such that $\overset{2}{\phi}_j{}^i \equiv \phi_h{}^i \phi_j{}^h = -\delta_j^i$. It is obvious that $\overset{r}{\underset{1}{F}}_j{}^i = -\delta_j^i$ and $\underset{1}{F}_j{}^i$ has only two different proper values.

Now, if the manifold has an r-π-structure, then the tensor $F_j{}^i$ defined by (1.3) is non degenerate. For, from (1.2) and (1.4) we have

2) I was informed by Mr. Hatakeyama of the construction of a tensor $F_j{}^i$ satisfying $\overset{r}{F}_j{}^i = -\delta_j^i$ starting from the tensor defining an almost complex structure.

$$\mathfrak{P}_\alpha v = \frac{1}{r}\sum_{s=0}^{r-1}\frac{1}{(\lambda w_\alpha)^s}\mathfrak{F}^s v \tag{1.8}$$

from which it follows that

$$\mathfrak{F}v_\alpha = (\lambda w_\alpha)v_\alpha, \quad \text{where} \quad v_\alpha \equiv \mathfrak{P}_\alpha v \in T_\alpha. \tag{1.9}$$

From (1.9) and dim $T_\alpha \equiv n_\alpha > 0$, it follows that λw_α $(\alpha = 1, \ldots\ldots, r)$ is actually a proper value of $\mathfrak{F}$.

Conversely if the manifold has a non degenerate tensor field $F_j^{\,i}$ satisfying (1.6), then the r proper subspaces corresponding to the r different proper values at each point induce r differentiable distributions which define an r-π-structure. Thus we have:

THEOREM 1.2. *For the manifold to have an r-π-structure, it is necessary and sufficient that the manifold has a non degenerate tensor $F_j^{\,i}$ satisfying* (1.6).

The tensor corresponding to the r-π-structure insisted in the above theorem is called the *fundamental tensor of the r-π-structure* as it plays an important role in the study of r-π-structure.

In the sequel, the following notations are used for the convenience sake:

$$F_{h_1}^{\,i}F_{h_2}^{\,h_1}\ldots\ldots F_j^{\,h_{s-1}} \equiv \overset{s}{F}{}_j^{\,i}, \quad F_j^{\,i} \equiv \overset{1}{F}{}_j^{\,i} \quad \text{and} \quad \overset{0}{F}{}_j^{\,i} \equiv \delta_j^{\,i}. \tag{1.10}$$

By use of these notations (1.5) is expressed as

$$\overset{r}{F}{}_j^{\,i} = \lambda^r \delta_j^{\,i}. \tag{1.5$'$}$$

Moreover, if we define the following for a tensor satisfying (1.6):

$$\overset{-s}{F}{}_j^{\,i} = \frac{1}{\lambda^{ar}}\overset{ar-s}{F}{}_j^{\,i} \qquad (a, s: \text{positive integers}, \ r > ar - s \geqq 0), \tag{1.11}$$

then we have

$$\overset{s}{F}{}_h^{\,i}\overset{t}{F}{}_j^{\,h} = \lambda^r \overset{s+t-r}{F}{}_j^{\,i}.$$

2. Adapted bases for an r-π-structure. In the sequel, we assume that the indices take the following ranges:

$$1 \leqq a_1, b_1, c_1, \ldots\ldots \leqq n_1,$$
$$n_1 + 1 \leqq a_2, b_2, c_2, \ldots\ldots \leqq n_1 + n_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n_1 + \ldots\ldots + n_{r-1} + 1 \leqq a_r, b_r, c_r, \ldots\ldots \leqq n_1 + n_2 + \ldots\ldots + n_r,$$

whereas

$$1 \leqq i, j, k, \ldots\ldots \leqq n,$$

$$1 \leqq \alpha, \beta, \gamma, \ldots\ldots \leqq r.$$

Moreover, we assume that $\bar{a}_\alpha, \bar{b}_\alpha, \bar{c}_\alpha, \ldots\ldots, (1 \leqq \alpha \leqq r)$ take all integers $[(n - n_\alpha)$ in number] between 1 and n except for n_α integers between $n_1 + \ldots\ldots + n_{\alpha-1} + 1$ and $n_1 + \ldots\ldots + n_{\alpha-1} + n_\alpha$.

A basis (e_i) in T_x^C is called an adapted basis at x if $e_{a_\alpha} \in T_\alpha$ for all $\alpha = 1, \ldots\ldots, r$. Since T_α is the proper subspace corresponding to the proper value λw_α of $\mathfrak{F}$, the tensor F_j^i satisfying $(1.5)'$ has the following components with respect to such an adapted basis:

$$F_{b_\alpha}^{a_\alpha} = \lambda w_\alpha \delta_{b_\alpha}^{a_\alpha}, \quad F_{b_\beta}^{a_\alpha} = 0 \quad \text{for} \quad \alpha \neq \beta. \tag{2.1}$$

More generally, we have

$$\overset{s}{F}{}_{b_\alpha}^{a_\alpha} = (\lambda w_\alpha)^s \delta_{b_\alpha}^{a_\alpha}, \quad \overset{s}{F}{}_{b_\beta}^{a_\alpha} = 0 \quad \text{for} \quad \alpha \neq \beta,\ 1 \leqq s \leqq r. \tag{2.2}$$

The transformation from an adapted basis to any other adapted basis is expressed as follows:

$$e_{b'_1} = A_{b'_1}{}^{a_1} e_{a_1}, \quad e_{b'_2} = A_{b'_2}{}^{a_2} e_{a_2}, \ldots\ldots, e_{b'_r} = A_{b'_r}^{a_r} e_{a_r}, \tag{2.3}$$

where

$$A_1 = (A_{b'_1}{}^{a_1}),\ A_2 = (A_{b'_2}{}^{a_2}),\ \ldots\ldots,\ A_r = (A_{b'_r}^{a_r}) \tag{2.4}$$

is respectively an $n_1 \times n_1$, $n_2 \times n_2, \ldots\ldots, n_r \times n_r$ non singular matrix.

Let (θ^i) and $(\theta^{i'})$ be respectively the dual cobasis of (e_i) and $(e_{i'})$, then we have

$$\theta^{a_1} = A_{b'_1}{}^{a_1} \theta^{b'_1}, \quad \theta^{a_2} = A_{b'_2}{}^{a_2} \theta^{b'_2}, \quad \ldots\ldots, \quad \theta^{a_r} = A_{b'_r}^{a_r} \theta^{b'_r}. \tag{2.5}$$

Denote $E_\pi(V_n)$ as the set of all adapted bases relative to all points in V_n, and p as the mapping which assigns each adapted basis in T_x^C to x. Then $E_\pi(V_n)$ is a principal fibre space having p as projection and a subgroup $G(n_1, n_2, \ldots\ldots, n_r)$ of $GL(n, C)$ as structure group. Here $G(n_1, n_2, \ldots\ldots, n_r)$ is the group which consists of all matrices of the following form:

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_r \end{pmatrix}, \tag{2.6}$$

where $A_1 \in GL(n_1, C), \ldots\ldots, A_r \in GL(n_r, C)$, hence

$$G(n_1, n_2, \ldots\ldots, n_r) \cong GL(n_1, C) \times GL(n_2, C) \times \ldots\ldots \times GL(n_r, C).$$

3. Torsion of r-π-structure. Assume that V_n has an r-π-structure. Consider a local section of $E_\pi(V_n)$ of class C^∞ in each neighborhood of V_n, then at every point of the neighborhood U there is associated an adapted basis (e_i). Let (θ^i) be the dual cobasis of (e_i), then we have

$$d\theta^i = \frac{1}{2} C^i{}_{jk}\theta^j \wedge \theta^k, \tag{3.1}$$

where

$$C^i{}_{jk} + C^i{}_{kj} = 0. \tag{3.2}$$

Let U' be any other neighborhood and $(\theta^{i'})$, $C^{i'}{}_{j'k'}$ are defined by the same way, then for any $x \in U \cap U'$ we have (2.5). If we put

$$A_{b'_\beta}^{a_\alpha} = 0 \quad \text{for} \quad \alpha \neq \beta, \tag{3.3}$$

then (2.5) is expressed as follows:

$$\theta^i = A_{i'}{}^i\theta^{i'}, \tag{3.4}$$

from which we have

$$d\theta^i = dA_{i'}{}^i \wedge \theta^{i'} + A_{i'}{}^i d\theta^{i'}. \tag{3.5}$$

Substitute (3.1) and the corresponding formula for $(\theta^{i'})$, and then make use of (3.4), we have

$$\frac{1}{2} C_{jk}{}^i A_{j'}{}^j A_{k'}{}^k \theta^{j'} \wedge \theta^{k'} = dA_{j'}{}^i \wedge \theta^{j'} + \frac{1}{2} A_{i'}{}^i C^{i'}_{j'k'} \theta^{j'} \wedge \theta^{k'}. \tag{3.6}$$

Let i take the integers in the range of a_α, and then compare the term $\theta^{b'_\beta} \wedge \theta^{c'_\gamma}$ $(\beta \neq \alpha,\ \gamma \neq \alpha)$ in the both sides, we have

$$C^{a_\alpha}_{b_\beta c_\gamma} A^{b_\beta}_{b'_\beta} A^{c_\gamma}_{c'_\gamma} = A^{a_\alpha}_{a'_\alpha} C^{a'_\alpha}_{b'_\beta c'_\gamma} \quad (\beta \neq \alpha,\ \gamma \neq \alpha),$$

that is,

$$C^{a'_\alpha}_{b'_\beta c'_\gamma} = A^{a'_\alpha}_{a_\alpha} A^{b_\beta}_{b'_\beta} A^{c_\gamma}_{c'_\gamma} C^{a_\alpha}_{b_\beta c_\gamma}. \tag{3.7}$$

Hence, if we define t_{jk}^i as follows:

$$t^{a_\alpha}_{\bar{b}_\alpha \bar{c}_\alpha} = C^{a_\alpha}_{\bar{b}_\alpha \bar{c}_\alpha}\ ; \quad t_{jk}{}^i = 0 \quad \text{for other indices}, \tag{3.8}$$

then t_{jk}^i is a tensor. We call this tensor the *torsion tensor* of the r-π-structure and call the following form the *torsion form* of the r-π-structure:

(3.9) $$T^i = \frac{1}{2} t_{jk}^i \theta^j \wedge \theta^k.$$

4. Integrability of the r-π-structure. By definition an r-π-structure defined by r distributions $T_1, \ldots\ldots, T_r$ is said to be *integrable* if at each point of V_n there exist a neighborhood and n complex valued functions z^i of the local coordinates in the neighborhood such that each T_α is expressed by $dz^{\bar{a}_\alpha} = 0$ at every point in the neighborhood.

Suppose that the considered r-π-structure is integrable, then as T_α is expressed by $dz^{\bar{a}_\alpha} = 0$, $\theta^i = dz^i$ may be regarded as the dual cobasis of the adapted basis given by a local section of $E_\pi(V_n)$ on the neighborhood. Hence (3.1) and consequently the following relations hold for $\theta^i = dz^i$:

(4.1) $$d\theta^{a_\alpha} = \frac{1}{2} C_{b_\alpha c_\alpha}^{a_\alpha} \theta^{b_\alpha} \wedge \theta^{c_\alpha} + C_{b_\alpha \bar{c}_\alpha}^{a_\alpha} \theta^{b_\alpha} \wedge \theta^{\bar{c}_\alpha} + T^{a_\alpha} \quad (\alpha = 1, \ldots\ldots, r),$$

where

(4.2) $$T^{a_\alpha} = \frac{1}{2} C_{\bar{b}_\alpha \bar{c}_\alpha}^{a_\alpha} \theta^{\bar{b}_\alpha} \wedge \theta^{\bar{c}_\alpha}.$$

On the other hand, let $\overline{T}_s$ be the direct sum of all T_α's except for T_s, then $\overline{T}_s$ is expressed by $dz^{a_s} = 0$ in the considered neighborhood. Thus the distribution given by $\overline{T}_s$ is integrable, and $d\theta^{a_s}$ belong to the ideal defined by $\theta^{a_s} = dz^{a_s}$. Therefore from (4.1) we have $T^{a_s} = 0$, that is the torsion tensor of the r-π-structure vanishes.

Conversely, assume that the torsion tensor of the r-π-structure vanishes and moreover, that both the considered manifold and the r-π-structure are of class C^ω. Under this situation, both the real and imaginary part of the tensor F_j^i are real analytic functions of the local coordinates x^i. Since T_α is spanned by the proper vectors of $\mathfrak{F}$ corresponding to the proper value λw_α, it is expressed by the following equations in the local coordinates:

(4.3) $$(F_j^i - \lambda w_\alpha \delta_j^i) dx^j = 0.$$

As T_α is n_α-dimensional, this system is of rank $n - n_\alpha \equiv \bar{n}_\alpha$. Hence (4.3) is equivalent with a system which consists of $\bar{n}_\alpha$ independent equations, say:

(4.4) $$S_\alpha : \; B_i^{\bar{a}_\alpha} dx^i = 0.$$

As $T_1 \cap T_2 = \{0\}$, the system

(4.5) $$S_1 + S_2 : \; B_i^{\bar{a}_1} dx^i = 0, \; B_i^{\bar{a}_2} dx^i = 0$$

has zero vector as its only solution. Thus the system $S_1 + S_2$ is of rank n. Consequently we can select n_1 forms θ^{a_1} out of S_2 such that the system consisting of $B_i^{\bar{a}_1}dx^i$ and θ^{a_1} is independent. Let (e_i) be the dual basis of this system and $<e_{b_1}, \theta^{a_1}> = \delta_{b_1}^{a_1}$, then $<e_{b_1}, B_i^{\bar{a}_1}dx^i> = 0$, hence e_{b_1} form a basis of T_1. As $<e_{\bar{b}_1}, \theta^{a_1}> = 0$ and $T_s(s > 1)$ is spanned by some vectors in $(e_{\bar{b}_1})$, it follows that θ^{a_1} are linear combinations of forms in $S_s(s > 1)$. As θ^{a_1} are linear combinations of forms in S_2 and the system made up by S_2 and S_1 is of rank n, we can select n_2 forms θ^{a_2} out of S_1 such that the system (S_2, θ^{a_2}) is linearly independent. By labeling the indices of $e_{\bar{b}_1}$ adequately we have $<e_{b_2}, \theta^{a_2}> = \delta_{b_2}^{a_2}$, $<e_{b_2}, B_i^{\bar{a}_2} dx^i> = 0$ and $<e_{\bar{b}_2}, \theta^{a_2}> = 0$. Thus e_{b_2} form a basis of T_2 and θ^{a_2} are linear combinations of forms in S_t $(t \neq 2)$. As the rank of the system (S_3, S_1) is also n and θ^{a_1}, θ^{a_2} are linear combinations of the forms in S_3, we can select n_3 forms θ^{a_3} out of $S_1 - (\theta^{a_2})$ and then continue the same processes as above. At last we can split up the system S_1 in $(r-1)$ subsystems $(\theta^{a_2}), (\theta^{a_3}),\ldots\ldots, (\theta^{a_r})$ such that (θ^{a_s}) are linear combinations of the forms in S_t $(t \neq s)$ and each system forms the dual cobasis in $T_2,\ldots\ldots, T_r$. Thus $(\theta^{\bar{a}_s})$ are linear combinations of forms in S_s and the rank of the system is $n - n_s$, hence the system $(\theta^{\bar{a}_s})$ is equivalent with S_s. Now as we assume that the torsion tensor of the r-π-structure $t_{jk}^i = 0$, we have from (4. 1) and (4. 2) that

$$d\theta^{a_\alpha} = \frac{1}{2} C_{b_\alpha c_\alpha}^{a_\alpha} \theta^{b_\alpha} \wedge \theta^{c_\alpha} + C_{b_\alpha \bar{c}_\alpha}^{a_\alpha} \theta^{b_\alpha} \wedge \theta^{\bar{c}_\alpha}, \tag{4. 6}$$

from which it follows that $d\theta^{a_\alpha}$ are contained in the ideal defined by (θ^{a_α}). I. e., the system

$$\theta^{a_\alpha} = 0 \tag{4. 7}$$

is completely integrable (This means that the distribution $\bar{T}_\alpha$ is integrable). Therefore, there exist n_α complex valued functions z^{a_α} of class C^ω such that the system $\theta^{a_\alpha} = 0$ is equivalent with the system $dz^{a_\alpha} = 0$ $(\alpha = 1,\ldots\ldots, r)$. As the system $(F_j^i - \lambda w_\alpha \delta_j^i)dx^j = 0$ is equivalent with S_α which is in turn equivalent with $(\theta^{\bar{a}_\alpha} = 0)$, which is again equivalent with $dz^{\bar{a}_\alpha} = 0$, it follows that the T_α is expressed by $dz^{\bar{a}_\alpha} = 0$, i. e. the considered r-π-structure is integrable. Thus we have

THEOREM 4. 1. *If the r-π-structure is integrable, then the torsion tensor of the r-π-structure $t_{jk}^i = 0$. Conversely if $t_{jk}^i = 0$ and moreover both the manifold and the considered r-π-structure are of class C^ω, then the r-π-structure is integrable.*

From the course of the above proof, it is also evident that the definition of the integrability of an r-π-structure stated above is equivalent to the one made by Walker [4].

5. A formula on torsion form of the r-π-structure. Assume that the manifold is endowed with an r-π-structure $(2 \leqq r \leqq n)$. We generalize the operations C and M considered by Lichnerowicz [3] and Legrand [2] as follows:

Let $v_1, \ldots\ldots, v_t$ be any t vectors of T_x^C and φ be a t-form, then define

$$\overset{s}{C}\varphi(v_1, \ldots\ldots, v_t) = \varphi(\mathfrak{F}^s v_1, \ldots\ldots, \mathfrak{F}^s v_t), \tag{5.1}$$

$$\overset{s}{M}\varphi(v_1, \ldots\ldots, v_t) = \sum_{k=1}^{t} \varphi(v_1, \ldots\ldots, v_{k-1}, \mathfrak{F}^s v_k, v_{k+1}, \ldots\ldots, v_t), \tag{5.2}$$

$$(1 \leqq s < r).$$

If $\varphi_{i_1 \cdots\cdots i_t}$ are components of φ with respect to a basis at a point x, then the components of $\overset{s}{C}\varphi$ and $\overset{s}{M}\varphi$ are respectively as follows:

$$(\overset{s}{C}\varphi)_{i_1 \cdots\cdots i_t} = \overset{s}{F}_{i_1}{}^{j_1} \cdots\cdots \overset{s}{F}_{i_t}{}^{j_t} \varphi_{j_1 \cdots\cdots j_t}, \tag{5.3}$$

$$(\overset{s}{M}\varphi)_{i_1 \cdots\cdots i_t} = \sum_{k=1}^{t} \overset{s}{F}_{i_k}{}^{h} \varphi_{i_1 \cdots\cdots i_{k-1} h i_{k+1} \cdots\cdots i_t}. \tag{5.4}$$

Let (θ^i) be the dual cobasis of an adapted basis at x, then we say that the form

$$\varphi = \frac{1}{t!} \varphi_{i_1 \cdots\cdots i_t} \theta^{i_1} \wedge \cdots\cdots \wedge \theta^{i_t}$$

is pure of the type $(p_1, \ldots\ldots, p_r)$ if the only non zero term in the above expression is the term which is of degree p_α with respect to θ^{a_α} $(\alpha = 1, \ldots\ldots, r)$. It is evident that this definition is independent of the adapted basis used at x. Let $\varphi_{p_1, p_2, \ldots, p_r}$ be pure of the type $(p_1, p_2, \ldots\ldots, p_r)$, then from (2.2), (5.3) and (5.4) it follows that

$$\overset{s}{C}\varphi_{p_1, p_2, \ldots, p_r} = w_1{}^{sp_1} w_2{}^{sp_2} \cdots\cdots w_r{}^{sp_r} \lambda^{s(p_1+p_2+\cdots+p_r)} \varphi_{p_1, p_2, \ldots, p_r}, \tag{5.5}$$

$$\overset{s}{M}\varphi_{p_1, p_2, \ldots, p_r} = (p_1 w_1{}^s + p_2 w_2{}^s + \cdots\cdots + p_r w_r{}^s) \lambda^r \varphi_{p_1, p_2, \ldots, p_r}. \tag{5.6}$$

As we shall concern principally with 2-forms in the sequel, we list here some relations on the operations $\overset{s}{C}$ and $\overset{s}{M}$ which hold only when applied on 2-forms. Let φ be a 2-form with components φ_{pq}, then

$$(\overset{s}{M}\varphi)_{jk} = (\delta_j{}^p \overset{s}{F}_k{}^q + \delta_k{}^q \overset{s}{F}_j{}^p) \varphi_{pq}, \tag{5.7}$$

$$(\overset{s}{C}\varphi)_{jk} = \overset{s}{F}_j{}^p \overset{s}{F}_k{}^q \varphi_{pq}. \tag{5.8}$$

Hence

$$(\overset{s}{C}\overset{t-s}{M}\varphi)_{jk} = (\overset{s}{F}_j{}^p \overset{t}{F}_k{}^q + \overset{s}{F}_k{}^q \overset{t}{F}_j{}^p) \varphi_{pq}. \tag{5.9}$$

From these relations we have immediately

$$(5.10)\qquad \begin{cases} \overset{0}{M} = 2, & \overset{0}{C} = 1, \\ \overset{ar+b}{M} = \lambda^{ar}\overset{b}{M}, & \overset{ar+b}{C} = \lambda^{2ar}\overset{b}{C}, \\ \overset{ar}{M} = 2\,\lambda^{ar}, & \overset{ar}{C} = \lambda^{2ar}, \end{cases}$$

where a is any positive integer and $0 \leqq b < r$. Moreover, we have

$$(5.11)\qquad \overset{s}{M}\overset{t}{M} = \overset{s+t}{M} + \overset{s}{C}\overset{t-s}{M},$$

hence

$$(5.12)\qquad (\overset{s}{M})^2 = \overset{2s}{M} + 2\overset{s}{C}, \quad \overset{s}{C} = \frac{1}{2}\{(\overset{s}{M})^2 - \overset{2s}{M}\},$$

and

$$(5.13)\qquad \overset{s}{C}\overset{u}{M} = \overset{s}{M}\overset{u+s}{M} - \overset{2s+u}{M}.$$

From the last relation we have

$$(5.14)\qquad \overset{s}{C}\overset{u}{M} = \lambda^r \overset{u+s-r}{C}\overset{r-u}{M} \qquad \text{for} \quad u+s \geqq r,\ u > r,\ s\,;$$

and

$$(5.15)\qquad \overset{s}{C}\overset{r-s}{M} = \lambda^r \overset{s}{M}.$$

Now assume that the manifold is endowed with an r-π-structure, and let T^i be the torsion form of the structure. Denote

$$(5.16)\qquad \varphi\circ T \equiv \varphi_i T^i = \frac{1}{2}\varphi_i t^i_{jk}\theta^j \wedge \theta^k$$

for any 1-form φ, then we have the following:

$$(5.17)\qquad r^2\lambda^r d\varphi + \sum_{t=1}^{r}\Big(-r\overset{t}{M} + \frac{1}{2}\frac{1}{\lambda^{2r}}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t+2r-2s}{M}\Big)d\overset{r-t}{C}\varphi = r^2\lambda^r\varphi\circ T.$$

If we extend the definition of $\overset{s}{M}$ so as (5.7) to hold also for negative integer, then instead of (5.17) we have

$$(5.18)\qquad \boxed{r^2\lambda^r d\varphi + \sum_{t=1}^{r}\Big(-r\overset{t}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)d\overset{r-t}{C}\varphi = r^2\lambda^r\varphi\circ T.}$$

PROOF. Let φ be a pure form of the type $(1, 0, \ldots\ldots, 0)$, say $\varphi \equiv \varphi_{1,0,\ldots,0} = \varphi_{a_1}\theta^{a_1}$, then by (3.1), $d\varphi = d\varphi_{a_1}\theta^{a_1} + \varphi_{a_1}d\theta^{a_1}$ is the sum of the pure form of the type $\varphi_{2,0,\dots,0}$ [i. e., $p_1 = 2$, $p_2 = \dots\dots = p_r = 0$]; pure forms of the type $\varphi_{1,0,\dots,0,1,0,\dots,0}$ [i. e., $p_1 = 1$, $p_2 = \dots\dots = p_r = 0$ except for an integer $s\ (2 \leqq s \leqq r)$ and $p_s = 1$]; pure forms of the type $\psi_{0,\dots,0,2,0,\dots,0}$ [i. e., $p_1 = \dots\dots = p_r = 0$ except for an integer $s\ (2 \leqq s \leqq r)$ and $p_s = 2$] and pure forms of the type $\psi_{0,\dots,0,1,0,\dots,0,1,0,\dots,0}$ [i. e., $p_1 = \dots\dots = p_r = 0$ except for two integers $s, t, s \neq t\ (2 \leqq s, t \leqq r)$ and $p_s = p_t = 1$]. Whereas $\varphi \circ T$ is the sum of pure forms of the type $\psi_{0,\dots,0,2,0,\dots,0}$ and pure forms of the type $\psi_{0,\dots,0,1,0,\dots,0}$. As $\varphi = \varphi_{1,0,\dots,0}$ is a pure form, with respect to the adapted basis given by the local section, we have

$$d\overset{r-t}{C}\varphi = w_1{}^{(r-t)}\lambda^{(r-t)}d\varphi. \tag{5.19}$$

Therefore the coefficients of

$$\begin{cases} \varphi_{2,0,\cdots,0}\ ; & \varphi_{1,1,0,\cdots,0}\ ; \\ \psi_{0,2,0,\cdots,0}; & \psi_{0,1,1,0,\cdots,0}; \end{cases} \tag{5.20}$$

in $\frac{1}{2}\overset{s}{C}\overset{t-2s}{M}d\overset{r-t}{C}\varphi$ are respectively the following:

$$\begin{cases} \lambda^r; & \frac{1}{2}\left\{\left(\frac{w_2}{w_1}\right)^s + \left(\frac{w_2}{w_1}\right)^t\left(\frac{w_1}{w_2}\right)^s\right\}\lambda^r; \\ \left(\frac{w_2}{w_1}\right)^t\lambda^r; & \frac{1}{2}\left\{\left(\frac{w_2}{w_1}\right)^t\left(\frac{w_3}{w_2}\right)^s + \left(\frac{w_3}{w_1}\right)^t\left(\frac{w_2}{w_3}\right)^s\right\}\lambda^r. \end{cases} \tag{5.21}$$

Taking the summation $\sum_{t=1}^{r}\sum_{s=0}^{r-1}$ of each term in (5. 21), we get the following respective coefficients of (5. 20) in $\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}d\overset{r-t}{C}\varphi$:

$$\begin{cases} r^2\lambda^r; & 0; \\ 0\ ; & 0. \end{cases} \tag{5.22}$$

On the other hand the coefficients of the four forms of (5. 20) in $-r\overset{t}{M}d\overset{r-t}{C}\varphi$ are respectively

$$\begin{cases} -2r\lambda^r; & -r\left\{1 + \left(\frac{w_2}{w_1}\right)^t\right\}\lambda^r; \\ -2r\left(\frac{w_2}{w_1}\right)^t\lambda^r; & -r\left\{\left(\frac{w_2}{w_1}\right)^t + \left(\frac{w_3}{w_1}\right)^t\right\}\lambda^r. \end{cases} \tag{5.23}$$

Taking the summation $\sum_{t=1}^{r}$ of each term in (5. 23) we get the following respective coefficient of the forms of (5. 20) in $\sum_{t=1}^{r}(-r)\overset{t}{M}d\overset{r-t}{C}\varphi$:

$$(5.24)\qquad \begin{cases} -2r^2\lambda^r; & -r^2\lambda^r; \\ 0; & 0. \end{cases}$$

From (5. 22) and (5. 24) it follows immediately that the respective coefficients of the forms of (5. 20) in the left hand side of (5. 18) are respectively

$$(5.25)\qquad \begin{cases} 0; & 0; \\ r^2\lambda^r; & r^2\lambda^r. \end{cases}$$

Whereas the corresponding coefficients in $\varphi\circ T$ are respectively

$$(5.26)\qquad \begin{cases} 0; & 0; \\ 1; & 1. \end{cases}$$

Therefore, the relation (5. 18) holds for $\varphi\equiv\varphi_{1,0,\cdots,0}$ because we can get similar results for the other forms appearing in both sides of (5. 18). It is easily seen by the same way that (5. 18) holds also for any other pure forms $\varphi_{0,\cdots,0,1,0,\cdots,0}$ [i. e., $p_1=\cdots\cdots=p_r=0$ except for an integer $s\,(2\leqq s\leqq r)$ and $p_s=1$]. Thus we have the relation (5. 18) for any 1-form.

6. Components of the torsion tensor of an r-π-structure. Let φ be any 1-form, then as $(\overset{s}{C}\varphi)_i=\overset{s}{F}{}_i^{\,m}\varphi_m$, we have

$$(6.1)\qquad d\overset{s}{C}\varphi=\overset{s}{\mathfrak{f}}\circ\varphi+\overset{s}{\mathfrak{G}},$$

where we have put

$$(6.2)\qquad \overset{s}{\mathfrak{f}}{}^m_{jk}=\frac{1}{2}(\partial_j\overset{s}{F}{}_k^{\,m}-\partial_k\overset{s}{F}{}_j^{\,m}),\quad (\overset{s}{\mathfrak{f}}\circ\varphi)_{jk}=\frac{1}{2}(\partial_j\overset{s}{F}{}_k^{\,m}-\partial_k\overset{s}{F}{}_j^{\,m})\varphi_m,$$

$$(6.3)\qquad \overset{s}{\mathfrak{G}}_{jk}=\frac{1}{2}(\overset{s}{F}{}_k^{\,m}\partial_j\varphi_m-\overset{s}{F}{}_j^{\,m}\partial_k\varphi_m).$$

Moreover, if we put

$$(6.4)\qquad \overset{(s,t)}{\mathfrak{H}}_{jk}=\frac{1}{2}(\overset{s}{F}{}_j^{\,p}\overset{t}{F}{}_k^{\,m}\partial_p\varphi_m-\overset{s}{F}{}_k^{\,q}\overset{t}{F}{}_j^{\,m}\partial_q\varphi_m),$$

then we have

$$(6.5)\qquad \begin{cases} \overset{0}{\mathfrak{f}}=0, \quad \overset{0}{\mathfrak{G}}=d\varphi, \quad \overset{ar+s'}{\mathfrak{G}}=\lambda^{ar}\overset{s'}{\mathfrak{G}}, \\ \overset{(0,0)}{\mathfrak{H}}=d\varphi, \quad \overset{(0,t)}{\mathfrak{H}}=\overset{t}{\mathfrak{G}}, \\ \overset{(ar+s',br+t')}{\mathfrak{H}}=\lambda^{(a+b)r}\overset{(s',t')}{\mathfrak{H}}, \quad \overset{(s,s)}{\mathfrak{H}}=\overset{s}{C}d\varphi, \\ \overset{(s,u)}{\mathfrak{H}}+\overset{(u,s)}{\mathfrak{H}}=\overset{s}{C}\overset{u-s}{M}d\varphi \quad \text{for} \quad u\geqq s, \end{cases}$$

$$\left\{\begin{aligned} &\overset{s}{M}\overset{t}{\mathfrak{G}} = \overset{s+t}{\mathfrak{G}} + \overset{(s,t)}{\mathfrak{H}}, \\ &\overset{s}{C}\overset{t}{\mathfrak{G}} = \overset{(s,s+t)}{\mathfrak{H}}, \qquad \overset{u}{C}\overset{(s,t)}{\mathfrak{H}} = \overset{(s+u,t+u)}{\mathfrak{H}}. \end{aligned}\right.$$

Putting (6. 1) in the left hand side of (5. 18) and then make use of (6. 5). we have

$$\begin{aligned} A &\equiv r^2\lambda^r d\varphi + \sum_{t=1}^{r}\Big(-r\overset{t}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)d\overset{r-t}{C}\varphi \\ &= r^2\lambda^r d\varphi + \Big(-r\overset{r}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{r-2s}{M}\Big)d\varphi \\ &+ \sum_{t=1}^{r-1}\Big(-r\overset{t}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)\overset{r-t}{\mathfrak{G}} \\ &+ \sum_{t=1}^{r-1}\Big(-r\overset{t}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)\overset{r-t}{\mathfrak{f}}\circ\varphi. \end{aligned} \tag{6.6}$$

As it is shown below, we have

$$B \equiv \sum_{t=1}^{r-1}\Big(-r\overset{t}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)\overset{r-t}{\mathfrak{G}} = -r^2\lambda^r d\varphi - \Big(-r\overset{r}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)d\varphi, \tag{6.7}$$

we get

$$A = \sum_{t=1}^{r-1}\Big(-r\overset{t}{M} + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\Big)\overset{r-t}{\mathfrak{f}}\circ\varphi. \tag{6.8}$$

From (6. 8), (5. 18), (5. 7) and (5. 9) we have

$$\boxed{\begin{aligned} t^i_{jk} &= \frac{1}{r^2\lambda^r}\sum_{t=1}^{r-1}\Big\{-r(\delta_j^p\overset{t}{F}_k{}^q + \delta_k^q\overset{t}{F}_j{}^p) \\ &+ \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{F}_j{}^{j_1}\overset{s}{F}_k{}^{k_1}(\delta_{j_1}^p\overset{t-2s}{F}_{k_1}{}^q + \delta_{k_1}^q\overset{t-2s}{F}_{j_1}{}^p)\Big\}(\partial_p\overset{r-t}{F}_q{}^i - \partial_q\overset{r-t}{F}_p{}^i). \end{aligned}} \tag{6.9}$$

The proof of (6. 7) is as follows: Making use of (6. 5) we have

$$B = -r(r-1)\lambda^r d\varphi - r\sum_{t=1}^{r-1}\overset{(t,r-t)}{\mathfrak{H}} + \frac{1}{2}(r-1)\sum_{s=0}^{r-1}\overset{(s,r-s)}{\mathfrak{H}} + \frac{1}{2}\sum_{t=1}^{r-1}\sum_{s=0}^{r-1}\overset{(t-s,r-t+s)}{\mathfrak{H}}.$$

It is shown below that

$$C \equiv \sum_{t=1}^{r-1}\sum_{s=0}^{r-1}\overset{(t-s,r-t+s)}{\mathfrak{H}} = (r-1)\sum_{u=1}^{r}\overset{(u,r-u)}{\mathfrak{H}}. \tag{6.10}$$

Hence we have

$$(6.11)\quad \begin{aligned} B = &- r(r-1)\lambda^r d\varphi - r\sum_{t=1}^{r-1}\overset{(t,r-t)}{\mathfrak{H}} + \frac{1}{2}(r-1)\overset{(0,r)}{\mathfrak{H}} + \frac{1}{2}(r-1)\sum_{s=1}^{r-1}\overset{(s,r-s)}{\mathfrak{H}} \\ &+ \frac{1}{2}(r-1)\sum_{u=1}^{r-1}\overset{(u,r-u)}{\mathfrak{H}} + \frac{1}{2}(r-1)\overset{(r,0)}{\mathfrak{H}}. \end{aligned}$$

As $\overset{(0,r)}{\mathfrak{H}} = \overset{(r,0)}{\mathfrak{H}} = \lambda^r \overset{(0,0)}{\mathfrak{H}} = \lambda^r d\varphi$, from (6.5) and (6.11) we have

$$\begin{aligned} B &= -(r-1)^2\lambda^r d\varphi - \sum_{t=1}^{r-1}\overset{(t,r-t)}{\mathfrak{H}} \\ &= -r^2\lambda^r d\varphi + r\overset{r}{M}d\varphi - \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{C}\overset{r-2s}{M}d\varphi. \end{aligned}$$

Finally, the proof of (6.10) is as follows:

$$\begin{aligned} C &= \sum_{t=1}^{r-1}\sum_{s'=t-r+1}^{t}\overset{(s',r-s')}{\mathfrak{H}} \qquad (\text{putting } s' = t - s \text{ in } C) \\ &= \sum_{t=1}^{r-1}\Big(\sum_{s'=1}^{t}\overset{(s',r-s')}{\mathfrak{H}} + \sum_{s'=t-r+1}^{0}\overset{(s',r-s')}{\mathfrak{H}}\Big). \end{aligned}$$

Since

$$\begin{aligned} \sum_{s'=t-r+1}^{0}\overset{(s',r-s')}{\mathfrak{H}} &= \sum_{s''=1}^{r-t}\overset{(t-r+s'',2r-t-s'')}{\mathfrak{H}} \qquad (\text{putting } s' = t-r+s'' \text{ in left hand side}) \\ &= \sum_{s''=1}^{r-1}\overset{(t+s'',r-t-s'')}{\mathfrak{H}} = \sum_{u=t+1}^{r}\overset{(u,r-u)}{\mathfrak{H}} \quad (u = t + s''), \end{aligned}$$

we have

$$\begin{aligned} C &= \sum_{t=1}^{r-1}\Big(\sum_{s'=1}^{t}\overset{(s',r-s')}{\mathfrak{H}} + \sum_{u=t+1}^{r}\overset{(u,r-u)}{\mathfrak{H}}\Big) \\ &= \sum_{t=1}^{r-1}\Big(\sum_{u=1}^{r}\overset{(u,r-u)}{\mathfrak{H}}\Big) = (r-1)\sum_{u=1}^{r}\overset{(u,r-u)}{\mathfrak{H}} \end{aligned}$$

7. An application of the relation (5.18). Let the infinitesimal transformation defined by the vector field X be denoted by $X\cdot f$ where f is a function. Let u, v be any two vector fields, φ be any 1-form, then it is known that the following relation holds:

$$(7.1)\qquad d\varphi(u, v) = u\cdot\varphi(v) - v\cdot\varphi(u) - \varphi([u, v]),$$

where $[u, v]$ is the Poisson's bracket of the two vector fields u, v. Making use of this formula we have

$$\overset{r-t}{dC}\varphi(u, v) = u\cdot\varphi(\mathfrak{F}^{r-t}v) - v\varphi(\mathfrak{F}^{r-t}u) - \varphi(\mathfrak{F}^{r-t}[u, v]), \tag{7.2}$$

from which we have moreover the following:

$$\begin{aligned}\overset{t}{M}\overset{r-t}{dC}\varphi(u, v) &= \overset{r-t}{dC}\varphi(\mathfrak{F}^{t}u, v) + \overset{r-t}{dC}\varphi(u, \mathfrak{F}^{t}v)\\ &= \mathfrak{F}^{t}u\cdot\varphi(\mathfrak{F}^{r-t}v) - \mathfrak{F}^{t}v\cdot\varphi(\mathfrak{F}^{r-t}u)\\ &+ \lambda^{r}\{d\varphi(u, v) + \varphi([u, v])\}\\ &- \varphi(\mathfrak{F}^{r-t}[\mathfrak{F}^{t}u, v]) - \varphi(\mathfrak{F}^{r-t}[u, \mathfrak{F}^{t}v]).\end{aligned} \tag{7.3}$$

Hence

$$\begin{aligned}\overset{s}{C}\overset{t-2s}{M}\overset{r-t}{dC}\varphi(u, v) &= \overset{t-2s}{M}\overset{r-t}{dC}\varphi(\mathfrak{F}^{s}u, \mathfrak{F}^{s}v)\\ &= \mathfrak{F}^{t-s}u\cdot\varphi(\mathfrak{F}^{r-(t-s)}v) - \mathfrak{F}^{t-s}v\cdot\varphi(\mathfrak{F}^{r-(t-s)}u)\\ &+ \mathfrak{F}^{s}u\cdot\varphi(\mathfrak{F}^{r-s}v) - \mathfrak{F}^{s}v\cdot\varphi(\mathfrak{F}^{r-s}u)\\ &- \varphi(\mathfrak{F}^{r-t}[\mathfrak{F}^{t-s}u, \mathfrak{F}^{s}v]) - \varphi(\mathfrak{F}^{r-t}[\mathfrak{F}^{s}u, \mathfrak{F}^{t-s}v]).\end{aligned} \tag{7.4}$$

Since the value $T(u, v)$ of the torsion form of an r-π-structure for the vector fields u, v defines a vector field, we have from (5.18) the following:

$$\begin{aligned}r^{2}\lambda^{r}\varphi(T(u, v)) = r^{2}\lambda^{r}d\varphi(u, v) &+ \sum_{t=1}^{r}(-r)\overset{t}{M}\overset{r-t}{dC}\varphi(u, v)\\ &+\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\overset{s}{C}\overset{t-2s}{M}\overset{r-t}{dC}\varphi(u, v).\end{aligned} \tag{7.5}$$

Putting (7.3) and (7.4) in the right hand side of the above formula, we have

$$\begin{aligned}r^{2}\lambda^{r}\varphi(T(u, v)) =& \sum_{t=1}^{r}(-r)\{\mathfrak{F}^{t}u\cdot\varphi(\mathfrak{F}^{r-t}v) - \mathfrak{F}^{t}v\cdot\varphi(\mathfrak{F}^{r-t}u)\} - r^{2}\lambda^{r}\varphi([u, v])\\ &+ r\sum_{t=1}^{r}\{\varphi(\mathfrak{F}^{r-t}[\mathfrak{F}^{t}u, v]) + \varphi(\mathfrak{F}^{r-t}[u, \mathfrak{F}^{t}v])\}\\ &+\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\{\mathfrak{F}^{t-s}u\cdot\varphi(\mathfrak{F}^{r-(t-s)}v) - \mathfrak{F}^{t-s}v\cdot\varphi(\mathfrak{F}^{r-(t-s)}u)\}\\ &+\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\{\mathfrak{F}^{s}u\cdot\varphi(\mathfrak{F}^{r-s}v) - \mathfrak{F}^{s}v\cdot\varphi(\mathfrak{F}^{r-s}u)\}\\ &-\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\{\varphi(\mathfrak{F}^{r-t}[\mathfrak{F}^{t-s}u, \mathfrak{F}^{s}v]) + \varphi(\mathfrak{F}^{r-t}[\mathfrak{F}^{s}u, \mathfrak{F}^{t-s}v])\}.\end{aligned} \tag{7.6}$$

It is seen immediatley that

$$\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\{\mathfrak{F}^{s}u\cdot\varphi(\mathfrak{F}^{r-s}v)-\mathfrak{F}^{s}v\cdot\varphi(\mathfrak{F}^{r-s}u)\}$$
$$=\frac{1}{2}r\sum_{s=1}^{r}\{\mathfrak{F}^{s}u\cdot\varphi(\mathfrak{F}^{r-s}v)-\mathfrak{F}^{s}v\cdot\varphi(\mathfrak{F}^{r-s}u)\}. \tag{7.7}$$

As it is shown below that

$$D\equiv\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\{\mathfrak{F}^{t-s}u\cdot\varphi(\mathfrak{F}^{r-(t-s)}v)-\mathfrak{F}^{t-}v\cdot\varphi(\mathfrak{F}^{r-(t-s)}u)\}$$
$$=\frac{1}{2}r\sum_{t=1}^{r}\{\mathfrak{F}^{t}u\cdot\varphi(\mathfrak{F}^{r-t}v)-\mathfrak{F}^{t}v\cdot\varphi(\mathfrak{F}^{r-t}u)\}, \tag{7.8}$$

from (7.6), (7.7) and (7.8) we have the following

$$r^{2}\lambda^{r}T(u,v)=-r^{2}\lambda^{r}[u,v]+r\sum_{t=1}^{r}\{\mathfrak{F}^{r-t}[\mathfrak{F}^{t}u,v]+\mathfrak{F}^{r-t}[u,\mathfrak{F}^{t}v]\}$$
$$-\frac{1}{2}\sum_{t=1}^{r}\sum_{s=0}^{r-1}\{\mathfrak{F}^{r-t}[\mathfrak{F}^{t-s}u,\mathfrak{F}^{s}v]+\mathfrak{F}^{r-t}[\mathfrak{F}^{s}u,\mathfrak{F}^{t-s}v]\}. \tag{7.9}$$

To prove (7.8), put $t'=t-s$, then we have

$$D=\frac{1}{2}\sum_{t'=1}^{r}\sum_{s=0}^{r-t'}\{\mathfrak{F}^{t'}u\cdot\varphi(\mathfrak{F}^{r-t'}v)-\mathfrak{F}^{t'}v\cdot\varphi(\mathfrak{F}^{r-t'}u)\}$$
$$+\frac{1}{2}\sum_{t'=-(r-2)}^{0}\sum_{s=1-t'}^{r-1}\{\mathfrak{F}^{t'}u\cdot\varphi(\mathfrak{F}^{r-t'}v)-\mathfrak{F}^{t'}v\cdot\varphi(\mathfrak{F}^{r-t'}u)\}.$$

If we put $t''=t'+r$, then the last term of the above formula turns out to be

$$\frac{1}{2}\sum_{t''=2}^{r}\sum_{s=r+1-t''}^{r-1}\{\mathfrak{F}^{t''-r}u\cdot\varphi(\mathfrak{F}^{2r-t''}v)-\mathfrak{F}^{t''-r}v\cdot\varphi(\mathfrak{F}^{2r-t''}u)\}$$
$$=\frac{1}{2}\sum_{t''=2}^{r}(t''-1)\{\mathfrak{F}^{t''}u\cdot\varphi(\mathfrak{F}^{r-t''}v)-\mathfrak{F}^{t''}v\cdot\varphi(\mathfrak{F}^{r-t''}u)\}$$
$$=\frac{1}{2}\sum_{t''=1}^{r}(t''-1)\{\mathfrak{F}^{t''}u\cdot\varphi(\mathfrak{F}^{r-t''}v)-\mathfrak{F}^{t''}v\cdot\varphi(\mathfrak{F}^{r-t''}u)\}.$$

Hence we have (7.8).

For giving an application of (7.9), we first calculate some formulas to be used:

From (1.4) we have

$$\mathfrak{F}^{r-t}[\mathfrak{F}^t u, v] = \lambda^r \sum_{\beta=1}^{r} \sum_{\alpha=1}^{r} \left(\frac{w_\alpha}{w_\beta}\right)^t \mathfrak{P}_\beta[\mathfrak{P}_\alpha u, v]$$

and

$$\mathfrak{F}^{r-t}[\mathfrak{F}^{t-s} u, \mathfrak{F}^s v] = \lambda^r \sum_{\gamma=1}^{r} \sum_{\beta=1}^{r} \sum_{\alpha=1}^{r} \left(\frac{w_\beta}{w_\gamma}\right)^t \left(\frac{w_\alpha}{w_\beta}\right)^s \mathfrak{P}_\gamma[\mathfrak{P}_\beta u, \mathfrak{P}_\alpha v].$$

From these two formulas we have respectively

$$(7.10) \qquad \sum_{t=1}^{r} \mathfrak{F}^{r-t}[\mathfrak{F}^t u, v] = r\lambda^r \sum_{\alpha=1}^{r} \mathfrak{P}_\alpha[\mathfrak{P}_\alpha u, v],$$

$$(7.11) \qquad \sum_{t=1}^{r} \sum_{s=0}^{r-1} \mathfrak{F}^{r-t}[\mathfrak{F}^{t-s} u, \mathfrak{F}^s v] = r^2\lambda^r \sum_{\alpha=1}^{r} \mathfrak{P}_\alpha[\mathfrak{P}_\alpha u, \mathfrak{P}_\alpha v].$$

It is obvious that

$$(7.12) \qquad [u, v] = \sum_{\alpha=1}^{r} \mathfrak{P}_\alpha[u, v].$$

Substitute (7.10), (7.11) and (7.12) in (7.9), we have

$$(7.13) \qquad \begin{aligned} r^2\lambda^r T(u, v) &= r^2\lambda^r \sum_{a=1}^{r} \{ -\mathfrak{P}_\alpha[u, v] + \mathfrak{P}_\alpha[\mathfrak{P}_\alpha u, v] + \mathfrak{P}_\alpha[u, \mathfrak{P}_\alpha v] - \mathfrak{P}_\alpha[\mathfrak{P}_\alpha u, \mathfrak{P}_\alpha v] \} \\ &= r^2\lambda^r \sum_{\alpha=1}^{r} \mathfrak{P}_\alpha \{ -\mathfrak{P}_\alpha[u, v] + \mathfrak{P}_\alpha[\mathfrak{P}_\alpha u, v] + \mathfrak{P}_\alpha[u, \mathfrak{P}_\alpha v] - [\mathfrak{P}_\alpha u, \mathfrak{P}_\alpha v] \}. \end{aligned}$$

Let $N(P_\alpha)$ be the Nijenhuis tensor of the projection tensor $\underset{\alpha}{P}{}_j^i$ induced by $\mathfrak{P}_\alpha$. As it is known that

$$(7.14) \qquad N(P_\alpha)(u, v) = -\mathfrak{P}_\alpha[\mathfrak{P}_\alpha u, v] - \mathfrak{P}_\alpha[u, \mathfrak{P}_\alpha v] + \mathfrak{P}_\alpha[u, v] + [\mathfrak{P}_\alpha u, \mathfrak{P}_\alpha v],$$

we have from (7.13) the following

$$(7.15) \qquad T(u, v) = -\sum_{\alpha=1}^{r} \mathfrak{P}_\alpha N(P_\alpha)(u, v).$$

8. π-connections on the differentiable manifold endowed with an r-π-structure. Let V_n be a differentiable manifold having an r-π-structure. By definition a π-connection on V_n is an infinitesimal connection defined on the principal fibre space $E_\pi(V_n)$. Let $E_C(V_n)$ be the principal fibre space consisting of all complex bases at all points of V_n, and having $GL(n, C)$ as its structure group. It is evident that $E_\pi(V_n)$ can be seen as a subspace of $E_C(V_n)$, so a local section in $E_\pi(V_n)$ can also be regarded as a local section in $E_C(V_n)$. Thus a π-connection can also be regarded as a complex linear connection, that is an infinitesimal connection on $E_C(V_n)$. If a complex linear connection is determined by complex valued Pfaff forms (ω_j^i) with respect to the local section in $E_\pi(V_n)$, we say that (ω_j^i) defines a connection relative to the adapted basis of the r-π-structure. A complex linear connection (ω_j^i) defined relative to the adapted bases of the r-π-structure can be regarded as a π-connection if and only if the values of the forms (ω_j^i) belong to the Lie algebra of the structure group $G(n_1, n_2,\dots\dots, n_r)$ of $E_\pi(V_n)$, that is to say, the following condition are satisfied:

$$\omega_{\bar{a}_\alpha}^{a_\alpha} = 0 \quad (\alpha = 1, 2,\dots\dots, r). \tag{8.1}$$

Let ∇F_j^i be the absolute differential of the tensor F_j^i with respect to the connection (ω_j^i), then we have

$$\nabla F_j^i = dF_j^i + \omega_k^i F_j^k - \omega_j^k F_k^i. \tag{8.2}$$

Referring to an adapted basis of the r-π-structure and then make use of (2. 1) we have

$$\begin{aligned} \nabla F_{b_\alpha}^{a_\alpha} &= \lambda w_\alpha \omega_{b_\alpha}^{a_\alpha} - \lambda w_\alpha \omega_{b_\alpha}^{a_\alpha} = 0, \\ \nabla F_{b_\beta}^{a_\alpha} &= \lambda w_\beta \omega_{b_\beta}^{a_\alpha} - \lambda w_\alpha \omega_{b_\beta}^{a_\alpha} = \lambda(w_\beta - w_\alpha)\omega_{b_\beta}^{a_\alpha}, \\ &(\alpha, \beta = 1, 2,\dots\dots, r;\ \alpha \neq \beta). \end{aligned} \tag{8.3}$$

Therefore (8. 1) is equivalent to

$$\nabla F_j^i = 0. \tag{8.4}$$

Thus we have the following:

THEOREM 8. 1. *A complex linear connection can be regarded as a π-connection if and only if the absolute differential of the tensor F_j^i (the fundamental tensor of the r-π-structure) with respect to the considered connection vanishes.*

From (1. 3) and (1. 8) we have the following for the tensor fields $P_{\alpha j}{}^i$ induced by $\mathfrak{P}_\alpha$:

$$F_j^i = \lambda \sum_{\alpha=1}^{r} w_\alpha P_{\alpha j}{}^i,$$

$$P_{\alpha j}{}^i = \frac{1}{r}\sum_{s=0}^{r-1} \frac{1}{(\lambda w_\alpha)^s} \overset{s}{F}{}_j^i.$$

Hence (8. 4) is equivalent with the following:

$$\nabla P_{\alpha j}{}^i = 0 \quad (\alpha = 1,\dots\dots, r). \tag{8.5}$$

From (8. 6) we can see easily that a π-connection is the connection with respect to which each of the considered r-distributions is parallel (See Fukami [5]).

LEMMA. *Let (ω_j^i) be any complex linear connection defined relative to the adapted basis of the r-π-structure, then the following forms (π_j^i) determine a π-connection*:

$$\pi_{b_\alpha}^{a_\alpha} = \omega_{b_\alpha}^{a_\alpha}, \quad \pi_{\bar{a}_\alpha}^{a_\alpha} = 0\,; \quad (\alpha = 1,\ldots\ldots, r). \tag{8. 6}$$

For the proof, the only thing must be shown is that (π_j^i) defines a complex linear connection. But this is easily seen from its transformation rule with respect to the adapted basis.

The connection (π_j^i) stated in the above lemma is called the π-connection induced by the complex linear connection (ω_j^i). Let

$$\omega_j^i = \gamma_{jk}^i\theta^k, \quad \pi_j^i = l_{jk}^i\theta^k, \tag{8.7}$$

where (θ^i) is the dual cobasis of the adapted basis at each point defined by the considered local section.

Put

$$\tau_{jk}^i = l_{jk}^i - \gamma_{jk}^i, \tag{8. 8}$$

then τ_{jk}^i is a tensor, and we have the following with respect to the adapted basis:

$$\tau_{b_\beta k}^{a_\alpha} = -\gamma_{b_\beta k}^{a_\alpha}, \quad \tau_{b_\alpha k}^{a_\alpha} = 0 \quad (\alpha \neq \beta). \tag{8. 9}$$

As the covariant derivatives $\nabla_k F_j^i$ of F_j^i with respect to the connection (ω_j^i) are defined by the following:

$$\nabla F_j^i = \nabla_k F_j^i \theta^k, \tag{8. 10}$$

from (8. 3) we have

$$\nabla_k F_{b_\alpha}^{a_\alpha} = 0, \quad \nabla_k F_{b_\beta}^{a_\alpha} = \lambda(w_\beta - w_\alpha)\gamma_{b_\beta k}^{a_\alpha} \quad (\alpha \neq \beta). \tag{8. 11}$$

Making use of (2. 2) we have generally

$$\nabla_k \overset{s}{F}{}_{b_\alpha}^{a_\alpha} = 0, \quad \nabla_k \overset{s}{F}{}_{b_\beta}^{a_\alpha} = \lambda^s\{(w_\beta)^s - (w_\alpha)^s\}\gamma_{b_\beta k}^{a_\alpha} \quad (\alpha \neq \beta,\ s = 1,\ldots\ldots, r). \tag{8. 12}$$

From the above formulas we have

$$(\nabla_k \overset{s}{F}{}_{b_\beta}^{c_\alpha})\overset{r-s}{F}{}_{c_\alpha}^{a_\alpha} = \lambda^r\left\{\left(\frac{w_\beta}{w_\alpha}\right)^s - 1\right\}\gamma_{b_\beta k}^{a_\alpha},$$

hence

$$\frac{1}{r\lambda^r}\sum_{s=1}^{r-1}(\nabla_k \overset{s}{F}{}_{b_\beta}^{c_\alpha})\overset{r-s}{F}{}_{c_\alpha}^{a_\alpha} = -\gamma_{b_\beta k}^{a_\alpha} = \tau_{b_\beta k}^{a_\alpha} \quad (\alpha \neq \beta\,;\ \alpha, \beta = 1,\ldots\ldots, r). \tag{8. 13}$$

From (8.12) we have moreover,

$$\frac{1}{r\lambda^r}\sum_{s=1}^{r-1}(\nabla_k \overset{s}{F}{}_{b_\beta}^{c_\alpha})\overset{r-s}{F}{}_{c_\alpha}^{a_\alpha} = 0 = \tau_{b}{}^{a}{}_{\alpha}{}^{\alpha}{}_{k} \quad (\alpha = 1,\ldots\ldots, r). \tag{8.14}$$

Therefore, it is evident that the tensor τ^i_{jk} has the following components in the local coordinate system:

$$\tau^i_{jk} = \frac{1}{r}\frac{1}{\lambda^r}\sum_{s=1}^{r-1}(\nabla_k \overset{s}{F}{}_j{}^l)\overset{r-s}{F}{}_l{}^i. \tag{8.15}$$

Thus we have the following:

THEOREM 8.2. *Let γ^i_{jk} be the parameters of a linear connection in the local coordinate system, then the following are parameters of a π-connection:*

$$l^i_{jk} = \gamma^i_{jk} + \frac{1}{r}\frac{1}{\lambda^r}\sum_{s=1}^{r-1}(\nabla_k \overset{s}{F}{}_j{}^l)\overset{r-s}{F}{}_l{}^i. \tag{8.16}$$

This connection is the π-connection induced by the linear connection γ^i_{jk} (Tachibana [6]).

Next, let $\bar{\pi}_j{}^i$ be any π-connection and let $\bar{\pi}_j{}^i = \bar{l}^i_{jk}\theta^k$. Put

$$\sigma^i_{jk} = \bar{l}^i_{jk} - l^i_{jk}, \tag{8.17}$$

then σ^i_{jk} is a tensor. As $\bar{\pi}_{b_\beta}^{a_\alpha} = \pi_{b_\beta}^{a_\alpha} = 0$, we have following with respect to the adapted basis:

$$\sigma_{b_\beta k}^{a_\alpha} = 0. \tag{8.18}$$

This is also the sufficient condition for $\bar{\pi}_j{}^i$ and $\pi_j{}^i$ be both the π-connection.

Since

$$\left\{\begin{aligned} &\overset{r-s}{F}{}_{b_\alpha}^{d_\alpha}\sigma_{d_\alpha k}^{c_\alpha}\overset{s}{F}{}_{c_\alpha}^{a_\alpha} = \lambda^r\sigma_{b_\alpha k}^{a_\alpha},\\ &\overset{r-s}{F}{}_{b_\beta}^{d_\beta}\sigma_{d_\beta k}^{c_\alpha}\overset{s}{F}{}_{c_\alpha}^{a_\alpha} = \lambda^r\left(\frac{w_\alpha}{w_\beta}\right)^s\sigma_{b_\beta k}^{a_\alpha} \end{aligned}\right. \tag{8.19}$$

with respect to adapted bases, we have

$$\left\{\begin{aligned} &\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\overset{r-s}{F}{}_{b_\alpha}^{d_\alpha}\sigma_{d_\alpha k}^{c_\alpha}\overset{s}{F}{}_{c_\alpha}^{a_\alpha} = (r-1)\sigma_{b_\alpha k}^{a_\alpha},\\ &\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\overset{r-s}{F}{}_{b_\beta}^{d_\beta}\sigma_{d_\beta k}^{c_\alpha}\overset{s}{F}{}_{c_\alpha}^{a_\alpha} = -\sigma_{b_\beta k}^{a_\alpha} \quad (\text{even if } \sigma_{b_\beta k}^{a_\alpha} \neq 0). \end{aligned}\right. \tag{8.20}$$

Hence it follows that

$$(8.21)\qquad \begin{cases} \dfrac{1}{r}\left(\sigma_{b_\alpha k}^{a_\alpha} + \dfrac{1}{\lambda^r}\sum_{s=1}^{r-1} \overset{r-s}{F}{}_{b_\alpha}^{d_\alpha} \sigma_{d_\alpha k}^{c_\alpha} \overset{s}{F}{}_{c_\alpha}^{a_\alpha}\right) = \sigma_{b_\alpha k}^{a_\alpha}, \\ \dfrac{1}{r}\left(\sigma_{b_\beta k}^{a_\alpha} + \dfrac{1}{\lambda^r}\sum_{s=1}^{r-1} \overset{r-s}{F}{}_{b_\beta}^{d_\beta} \sigma_{d_\beta k}^{c_\alpha} \overset{s}{F}{}_{c_\alpha}^{a_\alpha}\right) = 0. \end{cases}$$

Thus if (8. 18) is satisfied we have

$$(8.22)\qquad \frac{1}{r}\left(\sigma_{b_\beta k}^{a_\alpha} + \frac{1}{\lambda^r}\sum_{s=1}^{r-1} \overset{r-s}{F}{}_{b_\beta}^{d_\beta} \sigma_{d_\beta k}^{c_\alpha} \overset{s}{F}{}_{c_\alpha}^{a_\alpha}\right) = \sigma_{b_\beta k}^{a_\alpha}.$$

From (8. 21) and (8. 22) it is evident that if σ^i_{jk} satisfies (8. 18), then its components with respect to a local coordinate system are as follows:

$$(8.23)\qquad \frac{1}{r}\left(\sigma^i_{jk} + \frac{1}{\lambda^r}\sum_{s=1}^{r-1} \overset{r-s}{F}{}_j^d \sigma^c_{dk} \overset{s}{F}{}_c^i\right).$$

Conversely, for any tensor σ^i_{jk} the tensor having (8. 23) as its components satisfies (8. 18). Thus we have

THEOREM 8. 3. *Let γ^i_{jk} be the parameters of a linear connection in the local coordinate system, then any π-connection can be expressed as follows*:

$$(8.24)\qquad \gamma^i_{jk} + \frac{1}{r}\frac{1}{\lambda^r}\sum_{s=1}^{r-1}(\nabla_k \overset{s}{F}{}_j^l)\overset{r-s}{F}{}_l^i + \frac{1}{r}\left(\sigma^i_{jk} + \frac{1}{\lambda^r}\sum_{s=1}^{r-1} \overset{r-s}{F}{}_j^d \sigma^c_{dk} \overset{s}{F}{}_c^i\right),$$

where σ^i_{jk} is a tensor [6].

9. Distinguished π-connections.

LEMMA. *Let $\omega_j{}^i = \gamma^i_{jk}\theta^k$ be any complex linear connection defined relative to the adapted basis of the π-structure, then the following forms $(\hat{\pi}_j{}^i)$ determine a π-connection*:

$$(9.1)\qquad \hat{\pi}_{b_\alpha}^{a_\alpha} = \omega_{b_\alpha}^{a_\alpha} - \gamma_{\bar{b}_\alpha b_\alpha}^{a_\alpha}\theta^{\bar{b}_\alpha}, \quad \hat{\pi}_{\bar{d}_\alpha}^{a_\alpha} = 0.$$

For the proof, the only thing must be shown is that $\gamma_{\bar{b}_\alpha b_\alpha}^{a_\alpha}$ define a tensor. But this is easily seen.

Now assume that $\omega_j{}^i = \gamma^i_{jk}\theta^k$ is a symmetric linear connection (complex or real) defined relative to the adapted basis. Then as it is without torsion, we have

$$(9.2)\qquad \begin{aligned} d\theta^{a_\alpha} &= \theta^j \wedge \omega_j{}^{a_\alpha} \\ &= \theta^{b_\alpha} \wedge \omega_{b_\alpha}^{a_\alpha} + \gamma_{\bar{b}_\alpha c_\alpha}^{a_\alpha}\theta^{\bar{b}_\alpha} \wedge \theta^{c_\alpha} + \frac{1}{2}(\gamma_{\bar{b}_\alpha \bar{c}_\alpha}^{a_\alpha} - \gamma_{\bar{c}_\alpha \bar{b}_\alpha}^{a_\alpha})\,\theta^{\bar{b}_\alpha} \wedge \theta^{\bar{c}_\alpha}. \end{aligned}$$

Let T^i be the torsion form of the considered r-π-structure, we have

$$T^{a_\alpha} = \frac{1}{2}(\gamma^{a_\alpha}_{\bar{b}_\alpha \bar{c}_\alpha} - \gamma^{a_\alpha}_{\bar{c}_\alpha \bar{b}_\alpha})\, \theta^{\bar{b}_\alpha} \wedge \theta^{\bar{c}_\alpha} \qquad (\alpha = 1, \ldots\ldots, r). \tag{9.3}$$

Let $\hat{\mathfrak{T}}^i$ be the torsion form of the π-connection $(\hat{\pi}_j{}^i)$, then we have

$$\hat{\mathfrak{T}}^{a_\alpha} = d\theta^{a_\alpha} - \theta^{b_\alpha} \wedge \hat{\pi}_{b_\alpha}^{a_\alpha}. \tag{9.4}$$

Substituting (9. 1) in the above formula, we have

$$\hat{\mathfrak{T}}^{a_\alpha} = d\theta^{a_\alpha} - \theta^{b_\alpha} \wedge \omega_{b_\alpha}^{a_\alpha} - \gamma^{a_\alpha}_{\bar{b}_\alpha b_\alpha} \theta^{\bar{b}_\alpha} \wedge \theta^{b_\alpha}. \tag{9.5}$$

Then from (9. 2), (9. 3) and (9. 5) we get

$$\hat{\mathfrak{T}}^{a_\alpha} = T^{a_\alpha}, \quad \alpha = 1, \ldots\ldots, r. \tag{9.6}$$

Thus we have

THEOREM 9. 1. *There exists a π-connection having the torsion tensor of the considered r-π-structure as its torsion tensor. Hence the r-π-structure is without torsion if and only if there exists a symmetric π-connection.*

The connection insisted in the above theorem is called the distinguished π-connection for the simplicity of statements.

Since π-connection is a connection with respect to which each of the r distributions of the π-structure is parallel, we have from Theorem 4. 1 and Theorem 9.1 the following:

COROLLARY. *For an r-π-structure, there exists a connection making each of the distributions parallel and moreover which is symmetric if the π-structure is integrable* (See Walker [4]).

We are now in the stage of obtaining the parameters of the distinguished π-connection $\hat{\pi}_j{}^i = \hat{l}^i_{jk}\theta^k$ defined in (9. 1). From (9. 1) we have

$$\begin{cases} \hat{l}^{a_\alpha}_{b_\alpha c_\beta} = \gamma^{a_\alpha}_{b_\alpha c_\beta} - \gamma^{a_\alpha}_{c_\beta b_\alpha} = l^{a_\alpha}_{b_\alpha c_\beta} - \gamma^{a_\alpha}_{c_\beta b_\alpha}, \quad \hat{l}^{a_\alpha}_{b_\alpha c_\alpha} = \gamma^{a_\alpha}_{b_\alpha c_\alpha} = l^{a_\alpha}_{b_\alpha c_\alpha}, \\ \hat{l}^{a_\alpha}_{b_\beta c_\gamma} = 0 = l^{a_\alpha}_{b_\beta c_\gamma}, \quad \hat{l}^{a_\alpha}_{b_\beta c_\alpha} = 0 = l^{a_\alpha}_{b_\beta c_\alpha}, \\ \qquad (\alpha \neq \beta,\ \alpha \neq \gamma) \end{cases} \tag{9.7}$$

where $\pi_j{}^i = l^i_{jk}\theta^k$ is the π-connection induced by the symmetric connection $\omega_j{}^i$. Let $\mathfrak{T}^i$ be the torsion form of the π-connection $\pi_j{}^i$, then we have

$$\mathfrak{T}^i = (\pi_j{}^i - \omega_j{}^i) \wedge \theta^j = -\tau^i_{jk}\theta^j \wedge \theta^k.$$

From (8. 9)

$$(9.9)\qquad \mathfrak{T}^{a_\alpha}=\frac{1}{2}(\gamma^{a_\alpha}_{b_\beta c_\gamma}-\gamma^{a_\alpha}_{c_\gamma b_\beta})\,\theta^{b_\beta}\wedge\theta^{c_\gamma}+\sum_{\beta}\gamma^{a_\alpha}_{b_\beta c_\alpha}\theta^{b_\beta}\wedge\theta^{c_\alpha},\quad(\alpha\neq\beta,\ \alpha\neq\gamma).$$

Let S^i_{jk} be the torsion tensor of the π-connection $(\pi_j{}^i)$, that is,

$$(9.10)\qquad \mathfrak{T}^i=-S^i_{jk}\theta^j\wedge\theta^k,\quad(S^i_{jk}=-S^i_{kj}),$$

then we have

$$(9.11)\qquad \begin{cases} S^{a_\alpha}_{b_\beta c_\gamma}=-\dfrac{1}{2}(\gamma^{a_\alpha}_{b_\beta c_\gamma}-\gamma^{a_\alpha}_{c_\gamma b_\beta}), & S^{a_\alpha}_{b_\beta c_\alpha}=-\dfrac{1}{2}\gamma^{a_\alpha}_{b_\beta c_\alpha},\\[2mm] S^{a_\alpha}_{b_\alpha c_\alpha}=0,\quad(\alpha\neq\beta,\ \alpha\neq\gamma). \end{cases}$$

On the other hand, since (8.21) holds for any tensor σ^i_{jk}, it follows that

$$(9.12)\qquad \begin{cases} -\dfrac{2}{r}\Big(S^{a_\alpha}_{b_\alpha c_\beta}+\dfrac{1}{\lambda^r}\displaystyle\sum_{s=1}^{r-1}\overset{r-s}{F}{}^{d_\alpha}_{b_\alpha}S^{e_\alpha}_{d_\alpha c_\beta}\overset{s}{F}{}^{a_\alpha}_{e_\alpha}\Big)=-2S^{a_\alpha}_{b_\alpha c_\beta}=-\gamma^{a_\alpha}_{c_\beta b_\alpha}=\hat{l}^{a_\alpha}_{b_\alpha c_\beta}-l^{a_\alpha}_{b_\alpha c_\beta},\\[2mm] -\dfrac{2}{r}\Big(S^{a_\alpha}_{b_\alpha c_\alpha}+\dfrac{1}{\lambda^r}\displaystyle\sum_{s=1}^{r-1}\overset{r-s}{F}{}^{d_\alpha}_{b_\alpha}S^{e_\alpha}_{d_\alpha c_\alpha}\overset{s}{F}{}^{a_\alpha}_{e_\alpha}\Big)=-2S^{a_\alpha}_{b_\alpha c_\alpha}=0=\hat{l}^{a_\alpha}_{b_\alpha c_\alpha}-l^{a_\alpha}_{b_\alpha c_\alpha},\\[2mm] -\dfrac{2}{r}\Big(S^{a_\alpha}_{b_\beta k}+\dfrac{1}{\lambda^r}\displaystyle\sum_{s=1}^{r-1}\overset{r-s}{F}{}^{d_\beta}_{b_\beta}S^{e_\alpha}_{d_\beta k}\overset{s}{F}{}^{a_\alpha}_{e_\alpha}\Big)=0=\hat{l}^{a_\alpha}_{b_\beta k}-l^{a_\alpha}_{b_\beta k}. \end{cases}$$

From these formulas, it is evident that the parameters of the distinguished connection in the local coordinates are as follows (because $\hat{l}^i_{jk}-l^i_{jk}$ is a tensor):

$$(9.13)\qquad \hat{l}^i_{jk}=l^i_{jk}-\frac{2}{r}\Big(S^i_{jk}+\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\overset{r-s}{F}{}_j{}^d S^c_{dk}\overset{s}{F}{}_c{}^i\Big).$$

Thus we have the following:

THEOREM 9.2. *Let l^i_{jk} be the π-connection induced by a symmetric connection and let S^i_{jk} be its torsion tensor, then the connection defined in* (9.13) *is a distinguished π-connection (in the local coordinates).*

From (9.3) we have the following expression for the torsion tensor t^i_{jk} $\Big(T^i=\dfrac{1}{2}t^i_{jk}\theta^j\wedge\theta^k\Big)$ of the r-π-structure:

$$(9.14)\qquad t^{a_\alpha}_{b_\beta c_\gamma}=(\gamma^{a_\alpha}_{b_\beta c_\gamma}-\gamma^{a_\alpha}_{c_\gamma b_\beta}),\quad t^{a_\alpha}_{b_\alpha c_\beta}=0\quad(\alpha\neq\beta,\ \alpha\neq\gamma).$$

Again from (8.21) we have

$$\begin{cases}\dfrac{1}{\lambda^r}\displaystyle\sum_{t=1}^{r-1}(\overset{t}{F}{}^{p_\beta}_{b_\beta}\delta^{q_\gamma}_{c_\gamma}+\delta^{p_\beta}_{b_\beta}\overset{t}{F}{}^{q_\gamma}_{c_\gamma})S^{h_\alpha}_{p_\beta q_\gamma}\overset{r-t}{F}{}^{a_\alpha}_{h_\alpha}=-2S^{a_\alpha}_{b_\beta c_\gamma}\quad(\alpha\neq\beta,\ \alpha\neq\gamma),\end{cases}$$

$$(9.15)\quad \begin{cases} \dfrac{1}{\lambda^r}\sum_{t=1}^{r-1}(\overset{t}{F}{}_{b_\beta}^{p_\beta}\delta_{c_\alpha}^{q_\alpha} + \delta_{b_\beta}^{p_\beta}\overset{t}{F}{}_{c_\alpha}^{q_\alpha})S_{p_\beta q_\alpha}^{h_\alpha}\overset{r-t}{F}{}_{h_\alpha}^{a_\alpha} = (r-2)S_{b_\beta c_\alpha}^{a_\alpha} \quad (\alpha \neq \beta), \\ \dfrac{1}{\lambda^r}\sum_{t=1}^{r-1}(\overset{t}{F}{}_{b_\alpha}^{p_\alpha}\delta_{c_\alpha}^{q_\alpha} + \delta_{b_\alpha}^{p_\alpha}\overset{t}{F}{}_{c_\alpha}^{q_\alpha})S_{p_\alpha q_\alpha}^{h_\alpha}\overset{r-t}{F}{}_{h_\alpha}^{a_\alpha} = 2(r-1)S_{b_\alpha c_\alpha}^{a_\alpha}. \end{cases}$$

Therefore, from these formulas and (9.11), (9.14) we get

$$(9.16)\quad \begin{cases} \dfrac{1}{r}\Big\{(r-2)S_{b_\beta c_\gamma}^{a_\alpha} - \dfrac{1}{\lambda^r}\sum_{t=1}^{r-1}(\overset{t}{F}{}_{b_\beta}^{p_\beta}\delta_{c_\gamma}^{q_\gamma} + \delta_{b_\beta}^{p_\beta}\overset{t}{F}{}_{c_\gamma}^{q_\gamma})S_{p_\beta q_\gamma}^{h_\alpha}\overset{r-t}{F}{}_{h_\alpha}^{a_\alpha} = S_{b_\beta c_\gamma}^{a_\alpha} = -\dfrac{1}{2}t_{b_\beta c_\gamma}^{a_\alpha}, \\ \dfrac{1}{r}\Big\{(r-2)S_{b_\beta c_\alpha}^{a_\alpha} - \dfrac{1}{\lambda^r}\sum_{t=1}^{r-1}(\overset{t}{F}{}_{b_\beta}^{p_\beta}\delta_{c_\alpha}^{q_\alpha} + \delta_{b_\beta}^{p_\beta}\overset{t}{F}{}_{c_\alpha}^{q_\alpha})S_{p_\beta q_\alpha}^{h_\alpha}\overset{r-t}{F}{}_{h_\alpha}^{a_\alpha} = 0 = -\dfrac{1}{2}t_{b_\beta c_\alpha}^{a_\alpha}, \\ \dfrac{1}{r}\Big\{(r-2)S_{b_\alpha c_\alpha}^{a_\alpha} - \dfrac{1}{\lambda^r}\sum_{t=1}^{r-1}(\overset{t}{F}{}_{b_\alpha}^{p_\alpha}\delta_{c_\alpha}^{q_\alpha} + \delta_{b_\alpha}^{p_\alpha}\overset{t}{F}{}_{c_\alpha}^{q_\alpha})S_{p_\alpha q_\alpha}^{h_\alpha}\overset{r-t}{F}{}_{h_\alpha}^{a_\alpha} = -S_{b_\alpha c_\alpha}^{a_\alpha} = 0 = -\dfrac{1}{2}t_{b_\alpha c_\alpha}^{a_\alpha}. \\ \qquad\qquad (\alpha \neq \beta,\ \alpha \neq \gamma) \end{cases}$$

Hence it is evident that the torsion tensor of the r-π-structure has the following components in the local coordinates:

$$(9.17)\qquad -\frac{r}{2}t_{jk}^{i} = (r-2)S_{jk}^{i} - \frac{1}{\lambda^r}\sum_{t=1}^{r-1}(\overset{t}{F}{}_j{}^p\delta_k{}^q + \delta_j{}^p\overset{t}{F}{}_k{}^q)S_{pq}^{h}\overset{r-t}{F}{}_h{}^i,$$

where S_{jk}^{i} is the torsion tensor of the π-connection induced by a symmetric connection.

10. Some other expressions of the torsion tensor of r-π-structure. Let $\nabla_p\overset{r-t}{F}{}_q{}^m$ be the covariant derivatives of the tensor $\overset{r-t}{F}{}_q{}^m$ with respect to the linear connection Γ_{jk}^{i}, then we have

$$(10.1)\quad \partial_p\overset{r-t}{F}{}_q{}^m - \partial_q\overset{r-t}{F}{}_p{}^m = (\nabla_p\overset{r-t}{F}{}_q{}^m - \nabla_q\overset{r-t}{F}{}_p{}^m) - (\overset{r-t}{F}{}_q{}^h\Gamma_{ph}^{m} - \overset{r-t}{F}{}_p{}^h\Gamma_{qh}^{m}) + 2S_{pq}^{h}\overset{r-t}{F}{}_h{}^m,$$

where

$$(10.2)\qquad S_{pq}^{h} = \frac{1}{2}(\Gamma_{pq}^{h} - \Gamma_{qp}^{h})$$

is the torsion tensor of the connection Γ_{pq}^{h}.

Put (10.1) in (6.9), then by some straightforward calculation we have

$$t_{jk}^{m} = \frac{1}{r^2\lambda^r}\sum_{t=1}^{r-1}\Big\{-r(\delta_j{}^p\overset{t}{F}{}_k{}^q + \delta_k{}^q\overset{t}{F}{}_j{}^p) + \frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{F}{}_j^{j_1}\overset{s}{F}{}_k^{k_1}(\delta_{j_1}^{p}\overset{t-2s}{F}{}_{k_1}^{q}$$

$$(10.3)\qquad + \delta_{k_1}^{q}\overset{t-2s}{F}{}_{j_1}^{p})\Big\}(\nabla_p\overset{r-t}{F}{}_q{}^m - \nabla_q\overset{r-t}{F}{}_p{}^m) + \frac{2}{r^2\lambda^r}\sum_{t=1}^{r-1}\Big\{-r(\delta_j{}^p\overset{t}{F}{}_k{}^q + \delta_k{}^q\overset{t}{F}{}_j{}^p)$$

$$+\frac{1}{2}\sum_{s=0}^{r=1}\overset{s}{F}{}_j^{j_1}\overset{s}{F}{}_k^{k_1}(\delta_{j_1}^{p}\overset{t-2s}{F}{}_{k_1}^{q}+\delta_{k_1}^{q}\overset{t-2s}{F}{}_{j_1}^{p})\Big\}S_{pq}^{h}\overset{r-t}{F}{}_h{}^m+\frac{1}{r^2\lambda^r}\Big\{(r-1)(2r-1)\lambda^r S_{jk}^{m}$$

$$+r\sum_{t=1}^{r-1}\overset{t}{F}{}_j{}^p\overset{r-t}{F}{}_k{}^q S_{pq}^{m}-\sum_{t=1}^{r-t}\sum_{s=1}^{r-1}\overset{t-s}{F}{}_j{}^p\overset{r-t+s}{F}{}_k{}^q S_{pq}^{m}\Big\}.$$

Thus, if Γ_{jk}^{i} is symmetric we have

$$t_{jk}^{m}=\frac{1}{r^2\lambda^r}\sum_{t=1}^{r-1}\Big\{-r(\delta_j{}^p\overset{t}{F}{}_k{}^q+\delta_k{}^q\overset{t}{F}{}_j{}^p)$$

(10.4)

$$+\frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{F}{}_j^{j_1}\overset{s}{F}{}_k^{k_1}(\delta_{j_1}^{p}\overset{t-2s}{F}{}_{k_1}^{q}+\delta_{k_1}^{q}\overset{t-2s}{F}{}_{j_1}^{p})\Big\}(\nabla_p\overset{r-t}{F}{}_q{}^m-\nabla_q\overset{r-t}{F}{}_p{}^m).$$

By some straightforward calculation, the right hand side of the above formula can also be written as follows:

$$\frac{1}{r^2\lambda^r}\Big\{-\sum_{t=1}^{r-1}(r-1)(\delta_j{}^p\overset{t}{F}{}_k{}^q+\delta_k{}^q\overset{t}{F}{}_j{}^p)+\frac{1}{2}\sum_{t=2}^{r-1}\sum_{s=1}^{t-1}(\overset{t}{F}{}_j{}^p\overset{t-s}{F}{}_k{}^q+\overset{s}{F}{}_k{}^q\overset{t-s}{F}{}_j{}^p)$$

(10.5)

$$+\frac{1}{2}\sum_{t=1}^{r-2}\sum_{s=t+1}^{r-1}(\overset{s}{F}{}_j{}^p\overset{t-s}{F}{}_k{}^q+\overset{s}{F}{}_k{}^q\overset{t-s}{F}{}_j{}^p)\Big\}(\nabla_p\overset{r-t}{F}{}_q{}^m-\nabla_q\overset{r-t}{F}{}_p{}^m).$$

If Γ_{jk}^{i} is a π-connection, we have

$$t_{jk}^{m}=\frac{2}{r^2\lambda^r}\sum_{t=1}^{r-1}\Big\{-r(\delta_j{}^p\overset{t}{F}{}_k{}^q+\delta_k{}^q\overset{t}{F}{}_j{}^p)+\frac{1}{2}\sum_{s=0}^{r-1}\overset{s}{F}{}_j^{j_1}\overset{s}{F}{}_k^{k_1}(\delta_{j_1}^{p}\overset{t-2s}{F}{}_{k_1}^{q}+\delta_{k_1}^{q}\overset{t-2s}{F}{}_{j_1}^{p})\Big\}S_{pq}^{h}\overset{r-t}{F}{}_h{}^m$$

(10.6)

$$+\frac{1}{r^2\lambda^r}\Big\{(r-1)(2r-1)\lambda^r S_{jk}^{m}+r\sum_{t=1}^{r-1}\overset{t}{F}{}_j{}^p\overset{r-t}{F}{}_k{}^q S_{pq}^{m}-\sum_{t=1}^{r-1}\sum_{s=1}^{r-1}\overset{t-s}{F}{}_j{}^p\overset{r-t+s}{F}{}_k{}^q S_{pq}^{m}\Big\}.$$

As it can be seen by some simple calculations, the right hand side of the above formula can also be written as follows:

$$\frac{1}{r^2\lambda^r}\Big\{2(r-1)^2\lambda^r S_{jk}^{m}+2\sum_{t=1}^{r-1}\overset{t}{F}{}_j{}^p\overset{r-t}{F}{}_k{}^q S_{pq}^{m}-2(r-1)\sum_{t=1}^{r-1}(\delta_j{}^p\overset{t}{F}{}_k{}^q+\delta_k{}^q\overset{t}{F}{}_j{}^p)S_{pq}^{h}\overset{r-t}{F}{}_h{}^m$$

(10.7)

$$+\sum_{u=1}^{r-2}\sum_{t=u+1}^{r-1}(\overset{t-u}{F}{}_j{}^p\overset{u}{F}{}_k{}^q+\overset{t-u}{F}{}_k{}^q\overset{u}{F}{}_j{}^p)S_{pq}^{h}\overset{r-t}{F}{}_h{}^m+\sum_{u=2}^{r-1}\sum_{t=1}^{u-1}(\overset{t-u}{F}{}_j{}^p\overset{u}{F}{}_k{}^q+\overset{t-u}{F}{}_k{}^q\overset{u}{F}{}_j{}^p)S_{pq}^{h}\overset{r-t}{F}{}_h{}^m\Big\}.$$

Or more simply,

$$t_{jk}^{m}=\frac{2(r-1)^2}{r^2}\Phi\Phi' S_{jk}^{m}, \tag{10.8}$$

where the operations Φ and Φ' are defined as follows:

$$\Phi\, S_{jk}^{m} = S_{jk}^{m} - \frac{1}{r-1}\frac{1}{\lambda^{r}}\sum_{s=1}^{r-1} \overset{s}{F}{}_{k}^{k_1} S_{jk_1}^{h} \overset{r-s}{F}{}_{h}^{m}, \tag{10.9}$$

$$\Phi' S_{jk}^{m} = S_{jk}^{m} - \frac{1}{r-1}\frac{1}{\lambda^{r}}\sum_{s=1}^{r-1} \overset{s}{F}{}_{j}^{j_1} S_{j_1 k}^{h} \overset{r-s}{F}{}_{h}^{m}. \tag{10.10}$$

Finally we add a remark: Denote

$$\varphi\circ\overset{s}{N} \equiv \overset{s}{C}d\varphi + d\overset{s}{C}\overset{s}{M}\varphi - \overset{s}{M}d\overset{s}{C}\varphi\,; \quad s = 1,\dots\dots, r-1, \tag{10.11}$$

where φ is any 1-form. Then it can be easily seen that for the cases $r = 3, 4, 5, 6$, $\varphi\circ T$ can be expressed by $\varphi\circ\overset{s}{N}$ $(s = 1,\dots\dots, r-1)$. For example, we have

$$\begin{cases} r=3: & \varphi\circ T = \dfrac{1}{9\lambda^3}\left\{\left(2\dfrac{1}{\lambda^3}\overset{2}{C} + \dfrac{1}{2}\overset{1}{M}\right)(\varphi\circ\overset{1}{N}) + \left(2\dfrac{1}{\lambda^3}\overset{1}{C} + \dfrac{1}{2}\dfrac{1}{\lambda^3}\overset{2}{M}\right)(\varphi\circ\overset{2}{N})\right\}, \\ r=4: & \varphi\circ T = \dfrac{1}{16\lambda^4}\left\{\left(3\dfrac{1}{\lambda^4}\overset{3}{C} - \overset{1}{C}\right)(\varphi\circ\overset{1}{N}) + \left(\dfrac{1}{2}\dfrac{1}{\lambda^4}\overset{1}{C}\overset{2}{M} + 3\right)(\varphi\circ\overset{2}{N}) \right. \\ & \qquad \left. + \left(3\dfrac{1}{\lambda^4}\overset{1}{C} - \dfrac{1}{\lambda^8}\overset{3}{C}\right)(\varphi\circ\overset{3}{N})\right\}. \end{cases} \tag{10.12}$$

But the same does not hold good generally. For example, it is easily shown that for the case $r = 7$, $\varphi\circ T$ can not be expressed by the same way.

11. Characteristic forms of r-π-structure. Groups of holonomy. Following Legrand [2] we can define the characteristic forms of r-π-structure and obtain some analogous results. Let (π_j^i) be a π-connection defined relative to the adapted bases of the considered local section. Its curvature forms are as follows:

$$\Omega_j^i = d\pi_j^i + \pi_k^i \wedge \pi_j^k, \tag{11.1}$$

where $\pi_{a_\alpha}^{a_\alpha} = 0$ $(\alpha = 1,\dots\dots, r)$. Put

$$\psi_\alpha = \lambda w_\alpha \Omega_{a_\alpha}^{a_\alpha} \qquad (\alpha = 1,\dots\dots, r), \tag{11.2}$$

then it is easily seen that each of the 2-forms ψ_α(called the characteristic forms of the π-connection) is closed and that the cohomology class of the characteristic form ψ_α is independent of the π-connection used. Moreover, it is also easily seen that $\dfrac{1}{w_1}\psi_1 + \cdots\cdots + \dfrac{1}{w_r}\psi_r$ is homologous to zero.

It is trivial that for the manifold having an r-π-structure the group of holonomy with respect to an adapted basis is the subgroup of $G(n_1, n_2,\dots\dots, n_r)$. In relation to the characteristic forms of the π-connection, the following theorem is easily proved:

THEOREM 11. 1. *For the restricted homogeneous group of holonomy to be a subgroup of* $S_\alpha G(n_1, \ldots\ldots, n_r)$, *it is necessary and sufficient that the* α*-th characteristic form* ψ_α *vanishes on* V_n.

In the statement of the above theorem, $S_\alpha G(n_1, \ldots\ldots, n_r)$ means a group consisting of the elements of the form:

$$\begin{pmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & A_r \end{pmatrix}$$

in which the determinant of A_α (α: fixed) is equal to 1.

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