

## ON SOME SECOND ORDER CESÀRO DIFFERENCE SPACES OF NON-ABSOLUTE TYPE

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ABSTRACT. The second order Cesàro sequence spaces of non-absolute type  $X_p(\Delta^2)$  for  $1 \leq p \leq \infty$  were defined and studied in [1]. It seems, however, that the characterizations of their  $\beta$ -duals given there do not hold for  $1 < p \leq \infty$ . In this paper, we determine the  $\beta$ -duals  $(X_p(\Delta^2))^\beta$  for  $1 \leq p \leq \infty$ .

### 1. INTRODUCTION

Let  $\omega$  denote the set of all complex sequences  $x = (x_k)_{k=1}^\infty$ . We write  $\ell_\infty$ ,  $c$ ,  $c_0$  and  $cs$  and  $\ell_1$  for the sets of all bounded, convergent, null sequences and for the sets of all convergent and absolutely convergent series, respectively, and  $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$  for  $1 < p < \infty$ . As usual,  $e$  and  $e^{(n)}$  ( $n = 1, 2, \dots$ ) are the sequences with  $e_k = 1$  ( $k = 1, 2, \dots$ ), and  $e_n^{(n)} = 1$  and  $e_k^{(n)} = 0$  ( $k \neq n$ ). We write  $\mathbf{n}^\lambda = (n^\lambda)_{n=1}^\infty$ ,  $\mathbf{k}^\lambda = (k^\lambda)_{k=1}^\infty$  for  $\lambda \in \mathbb{R}$  and  $\mathbf{1/n} = (1/n)_{n=1}^\infty$ .

Let  $x, y \in \omega$  and  $X \subset \omega$ . We write  $xy = (x_k y_k)_{k=1}^\infty$ ,  $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ ,  $x^\beta = x^{-1} * cs$  and

$$X^\beta = \bigcap_{x \in X} x^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k \text{ converges for all } x \in X\}$$

for the  $\beta$ -dual of  $X$ .

Given any infinite matrix  $A = (a_{nk})_{n,k=1}^\infty$  of complex numbers and any sequence  $x$ , we write  $A_n = (a_{nk})_{k=1}^\infty$  for the sequence in the  $n$ -th row of  $A$ ,  $A_n(x) = \sum_{k=1}^\infty a_{nk} x_k$  ( $n = 1, 2, \dots$ ) and  $A(x) = (A_n(x))_{n=1}^\infty$ , provided  $A_n \in x^\beta$  for all  $n$ . Furthermore,  $X_A = \{x \in \omega : A(x) \in X\}$  denotes the *matrix domain of  $A$  in  $X$* . We define the matrices  $\Sigma$ ,  $\Delta$  and  $E$  by  $\Sigma_{nk} = 1$  ( $1 \leq k \leq n$ ),  $\Sigma_{nk} = 0$  ( $k > n$ ),  $\Delta_{nn} = 1$ ,  $\Delta_{n,n+1} = -1$ ,  $\Delta_{nk} = 0$  (otherwise),  $e_{nk} = 1$  ( $k \geq n + 1$ ) and  $e_{nk} = 0$  ( $1 \leq k \leq n$ ) for all  $n$ , and write  $\Delta^2 = \Delta\Delta$ . Then the sets  $X_p(\Delta^2) = ((\mathbf{1/n})^{-1} * \ell_p)_{\Sigma\Delta^2}$  are the *second order Cesàro*

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*difference sequence spaces of non-absolute type* (cf. [1]). Throughout, let  $q = \infty$  for  $p = 1$ ,  $q = p/(p-1)$  for  $1 < p < \infty$  and  $q = 1$  for  $p = \infty$ . We write  $R = E(a)$  for  $a \in cs$ , that is  $R_n = \sum_{k=n+1}^{\infty} a_k$  ( $n = 1, 2, \dots$ ). For any subset  $X$  of  $\omega$ , we write  $S(X) = \{x \in X : x_1 = x_2 = 0\}$ . Since  $(\Delta\Sigma)(x) = (-x_{n+1})_{n=1}^{\infty}$  for all  $x \in S(X)$ , we have  $S(X_p(\Delta^2)) = S((\mathbf{1}/\mathbf{n})^{-1} * \ell_p)_\Delta$ . Furthermore, obviously  $(X_p(\Delta^2))^\beta = (S(X_p(\Delta^2)))^\beta$ . In [1], it was stated that

$$(X_p(\Delta^2))^\beta = (\mathbf{n}^{-1} * \ell_q)_E \text{ for } 1 \leq p \leq \infty. \quad (1.1)$$

First we observe that (1.1) does not hold for  $1 < p \leq \infty$ . To see this, we put,

$$R_k = \begin{cases} \frac{1}{n^{2/q}} \frac{1}{2^n} & (k = 2^n) \\ 0 & (k \neq 2^n) \end{cases} \text{ and } x_k = \begin{cases} 0 & (k = 1, 2) \\ k^{1+\frac{1}{2q}} & (k = 3, 4, \dots) \end{cases}.$$

This yields

$$\sum_{k=1}^{\infty} |kR_k|^q = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \text{ that is } R \in \mathbf{n}^{-1} * \ell_q,$$

$$|\Delta_k(x)| = (k+1)^{1+\frac{1}{2q}} - k^{1+\frac{1}{2q}} \leq \left(1 + \frac{1}{2q}\right) (k+1)^{\frac{1}{2q}} \quad (k = 3, 4, \dots)$$

by the mean value theorem, hence, for  $1 < p < \infty$  with  $M = (1 + 1/(2q))^p$ ,

$$\sum_{k=3}^{\infty} \left(\frac{|\Delta_k(x)|}{k}\right)^p \leq 2^p M \cdot \sum_{k=1}^{\infty} \frac{1}{(k+1)^{(1-1/q)p + \frac{p}{2q}}} = 2^p M \cdot \sum_{k=1}^{\infty} \frac{1}{(k+1)^{1+\frac{p}{2q}}} < \infty,$$

that is  $x \in S(X_p(\Delta^2))$ , and we have for  $p = \infty$ , that is  $q = 1$

$$\frac{|\Delta_k(x)|}{k} \leq \frac{3\sqrt{k+1}}{2k} \quad (k = 3, 4, \dots), \text{ that is } x \in S(X_\infty(\Delta^2)),$$

but

$$|a_{2^n} x_{2^n}| = |(R_{2^{n-1}} - R_{2^n}) x_{2^n}| = \frac{1}{n^{2/q}} \frac{1}{2^n} 2^{n(1+\frac{1}{2q})} = \frac{2^{\frac{n}{2q}}}{n^{2/q}} \rightarrow \infty \quad (n \rightarrow \infty),$$

that is  $ax \notin cs$ .

In this paper, we determine the  $\beta$ -duals of  $X_p(\Delta^2)$  for  $1 \leq p \leq \infty$ .

## 2. THE $\beta$ -DUALS OF THE SETS $X_p(\Delta^2)$

Now we determine  $(S(X_p(\Delta^2)))^\beta$  for  $1 \leq p \leq \infty$ . We start with

*Remark 2.1.* We put

$$D_p^{(1)} = (\mathbf{n}^{-1} * \ell_q)_E \quad \text{and} \quad D_p^{(2)} = \begin{cases} \left( (\mathbf{n}^{2-1/p})^{-1} * \ell_\infty \right)_E & (1 \leq p < \infty) \\ ((\mathbf{n}^2)^{-1} * c_0)_E & (p = \infty). \end{cases}$$

Then  $D_1^{(1)} = D_1^{(2)}$ , but if  $1 < p \leq \infty$  then neither  $D_p^{(1)} \subset D_p^{(2)}$  nor  $D_p^{(2)} \subset D_p^{(1)}$ .

*Proof.* The identity  $D_1^{(1)} = D_1^{(2)}$  is obvious from the definition of the sets  $D_p^{(1)}$  and  $D_p^{(2)}$ . Let  $1 < p \leq \infty$ . We define the sequence  $a$  by  $a_1 = a_2 = 0$  and

$$a_n = \frac{1}{(\log(n-1))^{1/q}} \frac{1}{(n-1)^{1+1/q}} - \frac{1}{(\log n)^{1/q}} \frac{1}{n^{1+1/q}} \quad \text{for } n \geq 3.$$

Since  $1/p + 1/q = 1$  for  $1 < p < \infty$  and  $1/p = 0$  for  $p = \infty$ , we obtain

$$\begin{aligned} R_n &= \frac{1}{(\log n)^{1/q}} \frac{1}{n^{1+1/q}} \quad \text{for } n \geq 2, \\ |n^{2-1/p} R_n| &= |n^{1+1/q} R_n| = \frac{1}{(\log n)^{1/q}} \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{and} \\ \sum_{n=2}^{\infty} |n R_n|^q &= \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty, \quad \text{hence } a \in D_p^{(2)} \setminus D_p^{(1)}. \end{aligned}$$

We define the sequence  $a$  by  $a_{2^n} = -a_{2^{n+1}} = -n^{-2} 2^{-n}$  ( $n = 1, 2, \dots$ ) and  $a_k = 0$  otherwise. This yields  $R_{2^n} = n^{-2} 2^{-n}$  ( $n = 1, 2, \dots$ ) and  $R_k = 0$  otherwise,  $\sum_{k=1}^{\infty} |k R_k|^q = \sum_{n=1}^{\infty} n^{-2q} < \infty$  and

$$\left| (2^n)^{2-1/p} R_{2^n} \right| = (2^n)^{1+1/q} \frac{1}{n^2 2^n} = \frac{2^{n/q}}{n^2} \rightarrow \infty \quad (n \rightarrow \infty),$$

hence  $a \in D_p^{(1)} \setminus D_p^{(2)}$ . □

**Theorem 2.1.** We put  $M_1 = D_1^{(1)}$  and  $M_p = D_p^{(1)} \cap D_p^{(2)}$  for  $1 < p \leq \infty$ . Then we have  $(S(X_p(\Delta^2)))^\beta = M_p$ . Moreover, if  $a \in (S(X_p(\Delta^2)))^\beta$  then

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} R_k \Delta(x_k) \quad \text{for all } x \in S(X_p(\Delta^2)). \quad (2.1)$$

*Proof.* We write  $Y = S(X_p(\Delta^2))$ .

First we assume  $a \in M_p$ . Let  $y \in Y$ . Then  $x = (\mathbf{n})^{-1} \Delta(y) \in \ell_p$ , and  $a \in D_p^{(1)}$ , that is  $\mathbf{n}R \in \ell_q$ , implies  $\|R\Delta(y)\|_1 \leq \|\mathbf{n}R\|_q \|x\|_p < \infty$ , hence

$$R\Delta(y) \in cs. \quad (2.2)$$

We define the matrix  $A$  and the sequence  $z$  by

$$a_{mk} = \begin{cases} -kR_m & (1 \leq k \leq m) \\ 0 & (k > m) \end{cases} \quad \text{and } z_m = A_m(x) \quad (m = 1, 2, \dots). \quad (2.3)$$

If  $p = 1$ , then  $\mathbf{n}R \in \ell_\infty$  implies that there is a constant  $C$  such that

$$|a_{mk}| = k|R_m| \leq \frac{k}{m}C \quad \text{for all } m, \quad (2.4)$$

and since obviously  $D_p^{(1)} \subset D_1^{(1)}$  for all  $p \geq 1$ , this inequality holds for all  $p$ . Thus

$$\lim_{m \rightarrow \infty} a_{mk} = 0 \quad \text{for each fixed } k. \quad (2.5)$$

Now (2.4) and (2.5) together imply  $A \in (l_1, c_0)$  by [2, Example 8.4.1A, p. 126].

If  $1 < p < \infty$ , then  $a \in D_p^{(2)}$  yields

$$\sup_m \sum_{k=1}^{\infty} |a_{mk}|^q = \sup_m \left( |R_m|^q \sum_{k=1}^m k^q \right) \leq \sup_m |m^{1+1/q} R_m|^q < \infty.$$

This and (2.5) together imply  $A \in (\ell_p, c_0)$  by [2, Example 8.4.5D, p. 129].

If  $p = \infty$ ,  $a \in D_\infty^{(2)}$  yields

$$\lim_{m \rightarrow \infty} \sum_{k=1}^{\infty} |a_{mk}| = \lim_{m \rightarrow \infty} R_m \frac{m(m+1)}{2} = 0.$$

This and (2.5) together imply  $A \in (\ell_\infty, c_0)$  by [2, Theorem 1.7.19, p. 17].

Finally, (2.2),  $A \in (\ell_p, c_0)$  and Abel's summation by parts

$$\sum_{k=1}^m a_k y_k = - \sum_{k=1}^m R_k \Delta_k(y) - R_m y_{m+1} = - \sum_{k=1}^m R_k \Delta_k(y) - z_m \quad \text{for all } m \quad (2.6)$$

together imply  $ay \in cs$ , that is  $a \in Y^\beta$ . Thus we have shown  $M_p \subset Y^\beta$ .

Conversely we assume  $a \in Y^\beta$ . Then  $ax \in cs$  for all  $x \in Y$ . First  $b = (0, 0, 1, \dots) \in Y$  implies  $ab \in cs$ , and so the sequence  $R$  is defined. Let  $y \in Y$  be given. Then  $x = \Delta(y) \in (\mathbf{1}/\mathbf{n})^{-1} * \ell_p$ ,  $y_k = -\sum_{j=1}^{k-1} x_j$  for  $k = 1, 2, \dots$ , and we have

$$\sum_{k=1}^n a_k y_k = - \sum_{k=1}^{n-1} \left( \sum_{j=k+1}^n a_j \right) x_k \quad \text{for } n = 1, 2, \dots \quad (2.7)$$

Defining the matrix  $B = (b_{nk})_{n,k=1}^\infty$  by

$$b_{nk} = \begin{cases} -\sum_{j=k+1}^n a_j & (1 \leq k \leq n-1) \\ 0 & (k > n) \end{cases} \quad (n = 1, 2, \dots),$$

we conclude  $B \in ((\mathbf{1}/\mathbf{n})^{-1} * \ell_p, c) \subset ((\mathbf{1}/\mathbf{n})^{-1} * \ell_p, \ell_\infty)$ . Now  $B \in ((\mathbf{1}/\mathbf{n})^{-1} * \ell_p, \ell_\infty)$  if and only if  $\tilde{B} \in (\ell_p, \ell_\infty)$  where  $\tilde{b}_{nk} = kb_{nk}$  for all  $n$  and  $k$ , and so

$$S_q = \sup_n \|\tilde{B}_n\|_q < \infty \tag{2.8}$$

by [2, Example 8.4.1A, p. 126 ( $p = 1$ ), 8.4.5D, p. 129 ( $1 < p < \infty$ ) and 8.4.5A, p. 129 ( $p = \infty$ )].

If  $p = 1$ , then (2.8) yields  $|kR_k| = \lim_{n \rightarrow \infty} |k \sum_{j=k+1}^n a_j| \leq S_1$  for all  $k$ , that is  $R \in \mathbf{k}^{-1} * \ell_\infty$ , hence  $a \in M_1$ . Thus we have shown  $Y^\beta \subset D_1^{(1)} = M_1$ .

Now let  $p > 1$  and  $m \in \mathbb{N}$  be given. Then (2.8) yields

$$\sum_{k=1}^{m-1} k^q \left| \sum_{j=k+1}^n a_j \right|^q \leq \|\tilde{B}_n\|_q^q \leq S_q^q \text{ for all } n \geq m,$$

and so  $\sum_{k=1}^{m-1} |kR_k|^q = \lim_{n \rightarrow \infty} \sum_{k=1}^{m-1} k^q |\sum_{j=k+1}^n a_j|^q \leq S_q^q$ . Since  $m \in \mathbb{N}$  was arbitrary, we have  $\mathbf{k}R \in \ell_q$ . that is  $a \in D_p^{(1)}$ . We define the matrix  $A$  and the sequence  $z$  as in (2.3). Then we have  $z \in c$  by (2.3), that is  $A \in (\ell_p, c)$ .

For  $1 < p < \infty$ , there is a constant  $C$  such that

$$C \left( |R_m| m^{1+1/q} \right)^q \leq |R_m|^q \sum_{k=1}^m k^q = \sup_n \|A_n\|_q^q \text{ for all } m,$$

and  $\sup_n \|A_n\|_q^q < \infty$  by [2, Example 8.4.5D, p. 129], that is  $R \in (\mathbf{n}^{1+1/q})^{-1} * \ell_\infty$ , hence  $a \in D_p^{(2)}$ . Thus we have shown  $Y^\beta \subset D_p^{(1)} \cap D_p^{(2)} = M_p$ .

If  $p = \infty$ , then  $\sum_{k=1}^\infty |a_{mk}| = 0$  by [2, Theorem 1.7.18 (ii), p. 15], since  $\lim_{m \rightarrow \infty} a_{mk} = k \lim_{m \rightarrow \infty} R_m = 0$  for each fixed  $k$ . But we have  $\sum_{k=1}^\infty |a_{mk}| = R_m \sum_{k=1}^m k = R_m m(m+1)/2$  and so  $R \in (\mathbf{n}^2)^{-1} * c_0$ , that is  $a \in D_\infty^{(2)}$ . Thus we have shown  $Y^\beta \subset D_\infty^{(1)} \cap D_\infty^{(2)} = M_\infty$ .

If  $a \in (S(X_p(\Delta^2)))^\beta$ , then (2.1) is obvious from (2.6). □

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