

ON SOME SEMI-GROUPS

BY REIKICHI YOSHIDA

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THEOREM 1. *Let Φ be an infinite field, $f(x, y)$ a polynomial in x, y over Φ . Let α, β be any elements in Φ . In order that Φ may be a semi-group under the operation $\alpha \circ \beta = f(\alpha, \beta)$, the possible types of $f(x, y)$ are the following:*

$$axy + b(x + y) + \frac{b(b - 1)}{a}, \quad x + y + d, \quad d, \quad x, \quad y,$$

where a, b, d denote any constants in Φ , and $a \neq 0$.

Moreover under these operations, Φ is respectively a commutative group with zero, a commutative group, a trivial semi-group (all composites are zero), an "anti-semi-groupe à droite" and an "anti-semi-groupe à gauche."

In the proof of this theorem, after we arrange lexicographically the terms of $f(x, y)$, and compare the leading terms of $(x \circ y) \circ z$ and $x \circ (y \circ z)$, we find that $f(x, y)$ does not include terms of degree ≥ 3 . Next, from the relations between the coefficients which express that $(x \circ y) \circ z = x \circ (y \circ z)$, we derive five types as a conclusion. Further we obtain Theorem 1 by investigating their structures. Moreover using the result of Theorem 1, we obtain

THEOREM 2. *Let Φ be an infinite field, $f(x, y), g(x, y)$ polynomials over Φ . Then, in order that Φ be an associative semi-ring under the operations $x \oplus y = f(x, y), x \otimes y = g(x, y)$, the possible types are limited to six. Of them, Φ is a field under*

$$x \oplus y = x + y + \frac{b'}{a'}, \quad x \otimes y = a'xy + b'(x + y) + \frac{b'(b' - 1)}{a'} \quad (a' \neq 0);$$

and Φ is a trivial ring under

$$x \oplus y = x + y + d, \quad x \otimes y = -d.$$

Further, as an application of this theorem, we can see that there exists no lattice with the defining polynomial operations:

$$x \cup y = f(x, y), \quad x \cap y = g(x, y).$$

On the other hand, we lay down the following

DEFINITION. Let $\bar{S} = \{s_1, s_2, \dots\}$ be a semi-group, and consider the set $S_i = \{s_i, s'_i, s''_i, \dots, s_i^{(m)}, \dots\}$, obtained by joining a finite or an infinite number of elements

$$s'_i, s''_i, \dots, s_i^{(m)}, \dots$$

to every element s_i . If we define products in

$$S = \{s_1, s'_1, \dots, s_2, s'_2, \dots, \dots\}$$

as follows:

$$s_i^{(m)} \circ s_j^{(n)} = s_i \circ s_j,$$

S becomes a semi-group, which we shall call the “prolongation” of \bar{S} . Then we have

THEOREM 3. Let Φ be an infinite field and let $a, b, c, d, k; a', b', c', d', k'$ be arbitrary constants in Φ . When we define a new operation by

$$(x, y) \circ (z, u) = (ax + by + cz + du + k, a'x + b'y + c'z + d'u + k'),$$

there exist only 22 types such that $\Phi \times \Phi$ forms a semi-group. They are:

- (1°) an anti-semi-groupe à droite: $(x, y) \circ (z, u) = (x, y)$.
- (2°) an anti-semi-groupe à gauche: $(x, y) \circ (z, u) = (z, u)$.
- (3°) a trivial semi-group: $(x, y) \circ (z, u) = (k, k')$.
- (4°) a commutative group: $(x, y) \circ (z, u) = (x + z + k, y + u + k')$.
- (5°) prolongations of the commutative groups:

$$(x, y) \circ (z, u) = (x + z + k, a'(x + z) + k'),$$

$$(x, y) \circ (z, u) = (k, a'(x + z) + (y + u) + k'),$$

$$(x, y) \circ (z, u) = \left(a(x + z) + b(y + u) + k,$$

$$\left(\frac{1 - a}{b} \right) [a(x + z) + b(y + u)] + k' \right).$$

- (6°) idempotent semi-groups:

$$(x, y) \circ (z, u) = \left(a(x - z) + b(y - u) + z,$$

$$\left[\frac{a(1 - a)}{b} \right] (x - z) - a(y - u) + y \right),$$

$$(x, y) \circ (z, u) = (x, a'(x - z) + u),$$

$$(x, y) \circ (z, u) = (z, a'(x - z) + y).$$

(7°) *prolongations of the anti-semi-groups à gauche:*

$$(x, y) \circ (z, u) = (z, c'z + k'),$$

$$(x, y) \circ (z, u) = (k, c'(z - k) + u),$$

$$(x, y) \circ (z, u) = \left(cz + du + k, \left(\frac{1 - c}{d} \right) (cz + du) - \frac{ck}{d} \right).$$

(8°) *prolongations of the anti-semi-groups à droite:*

$$(x, y) \circ (z, u) = (x, a'x + k'),$$

$$(x, y) \circ (z, u) = (k, a'(x - k) + y),$$

$$(x, y) \circ (z, u) = \left(ax + by + k, \left(\frac{1 - a}{b} \right) (ax + by) - \frac{ak}{b} \right).$$

(9°) *unions of infinite left ideals:*

$$(x, y) \circ (z, u) = ((x + k) + z, a'(x + k) + u),$$

$$(x, y) \circ (z, u) = (z, a'x + (y + u) + k'),$$

$$(x, y) \circ (z, u) = \left((ax + by + k) + z, \left(\frac{1 - a}{b} \right) (ax + by + k) + u \right).$$

(10°) *unions of infinite right ideals:*

$$(x, y) \circ (z, u) = ((z + k) + x, c'(z + k) + y),$$

$$(x, y) \circ (z, u) = (x, c'z + (y + u) + k'),$$

$$(x, y) \circ (z, u) = \left((cz + du + k) + x, \left(\frac{1 - c}{d} \right) (cz + du + k) + y \right).$$

b, d denote nonzero constants.

REFERENCES

1. R. H. Bruck, *A survey of binary systems*, Springer, Berlin, 1958.
2. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc., Providence, R. I., 1957.
3. N. Jacobson, *Lectures in abstract algebra*, Van Nostrand, New York, 1951.
4. A. Chatelet et P. Dubreil, *Séminaire, demi-groupes*, Faculté des Sciences de Paris, Paris, 1956.

RITSUMEIKAN UNIVERSITY, KYOTO, JAPAN