ON SOME SINGULAR CONVOLUTION OPERATORS[1]

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In this note, we state some results on the boundedness of certain operators on $L^p(R^n)$. The operators which we study are too singular to be handled by the ordinary Calderón-Zygmund techniques of [1].

Our first theorem concerns a sublinear operator g_λ^* which arises in Littlewood-Paley theory. If f is a real-valued function on R^n, set $u(x, t)$ equal to the Poisson integral of f, defined on $R_+^{n+1} = R^n \times (0, \infty)$. Then for $\lambda > 1$, the g_λ^*-function on R^n is defined by the equation

$$g_\lambda^*(f)(x) = \left(\int_{R_+^{n+1}} \left(\frac{t}{|x-y| + t} \right)^{n\lambda} t^{1-n} |\nabla u(y, t)|^2 dy dt \right)^{1/2}.$$

(∇ denotes the gradient in R^{n+1}.)

It is known [4] that if $p > 2/\lambda$ then the operator $f \to g_\lambda^*(f)$ is bounded on $L^p(R^n)$. On the other hand, if $p < 2/\lambda$ then there are L^p functions f such that $g_\lambda^*(f)(x) = +\infty$ for every $x \in R^n$. The behavior of g_λ^* on L^p for $p = 2/\lambda$ is more subtle, and the methods of [1] and [4] are inadequate to deal with it.

THEOREM 1. *Let* $1 < p < 2$, $p = 2/\lambda$. *Then the operator* $f \to g_\lambda^*(f)$ *has weak-type* (p, p), *i.e.*

$$\text{measure}(\{x \in R^n \mid g_\lambda^*(f)(x) > \alpha\}) \leqq (A/\alpha^p)\|f\|_p^p$$

for any $\alpha > 0$ *and* $f \in L^p(R^n)$, *and the "constant"* A *is independent of* f *and* α.

This result implies the positive theorem about $p > 2/\lambda$, for the case $p \leqq 2$, by the Marcinkiewicz interpolation theorem.

An argument almost identical to the proof of Theorem 1 gives information on fractional integration. In particular, suppose that $f \in L^p(R^n)$ and $0 < \beta < 1$. Stein [5] has shown that the fractional integral $F = I^\beta(f)$ satisfies the smoothness condition

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$$\mu_\beta(F) = \left(\int_{R^n} \frac{|F(x) - F(x-y)|^2}{|y|^{n+2\beta}}\,dy\right)^{1/2} \in L^p(R^n),$$

provided that $2n/(n+2\beta)<p$; and that conversely, any function $F\in L^p(R^n)$ for which $\mu_\beta(F)$ belongs to L^p, has a fractional derivative $I^{-\beta}F$ in L^p. This result follows from the study of g_λ^*, since one can prove a pointwise inequality $\mu_\beta(f)(x) \leqq C_{g\lambda}^*(f)(x)$, for $n(\lambda-1)>2\beta$, $0<\beta<1$.

THEOREM 1′. *For* $1<p<2$, $2n/(n+2\beta)=p$, *and* $0<\beta<1$, *the operator* $f\to\mu_\beta(I^\beta f)$ *has weak-type* (p, p).

Theorem 1′ is the best possible positive result for μ_β.

The above theorems exhibit various nonlinear operators which are bounded on some L^p spaces, but not on all. There are also some known examples of linear operators which are bounded only on some of the L^p spaces. For example, consider the operator

$$T_{a\alpha}: f \to \left(\frac{\exp[i/|x|^a]}{|x|^{n+\alpha}}\right) * f,$$

defined for $f\in C_0^\infty(R^n)$*. The convolution makes sense if we interpret $\exp[i/|x|^a]/|x|^{n+\alpha}$ as a temperate distribution on R^n. Fix an $a>0$ and an $\alpha>0$. For which p does $T_{a\alpha}$ extend to a bounded linear operator on $L^p(R^n)$? If α were negative, then $k=\exp[i/|x|^a]/|x|^{n+\alpha}$ would be locally L^1; so if we ignore difficulties at infinity (say by cutting off k outside of $|x|<1$), we find that $T_{a\alpha}$ is bounded on L^p for every p $(1\leqq p\leqq+\infty)$, if $\alpha<0$. On the other hand, by computing the Fourier transform of $\exp[i/|x|^a]/|x|^{n+\alpha}$, we can deduce that $T_{a\alpha}$ is bounded on $L^2(R^n)$ exactly when $\alpha\leqq(n/2)a$. (Since $T_{a\alpha}$ is defined only on $C_0^\infty(R^n)$, the statement "$T_{a\alpha}$ is bounded on L^p" means that $T_{a\alpha}$ extends to a bounded operator on L^p, or equivalently, that the a priori inequality $\|T_{a\alpha}f\|_p\leqq A\|f\|_p$ holds, for $f\in C_0^\infty(R^n)$.)

Applying a strong form of the Riesz-Thorin convexity theorem, we can interpolate between the L^1 inequality and the L^2 inequality, to obtain the following theorem. Let a, $\alpha>0$, and let $\beta=(a+1)(na/2-\alpha)$ be positive. (The significance of β is that it turns out that

$$\left|\left(\frac{\exp[i/|x|^a]}{|x|^\alpha}\right)^\wedge(y)\right| = O(|y|^{-\beta})$$

as $|y|\to\infty$.) Then $T_{a\alpha}$ is bounded on $L^p(R^n)$ if

$$\left|\frac{1}{2}-\frac{1}{p}\right|<\frac{\beta}{n}\left[\frac{n/2+\alpha}{\beta+\alpha}\right].$$

Easy examples show that $T_{a\alpha}$ cannot even have weak-type (p, p) if

$$\left|\frac{1}{2}-\frac{1}{p}\right|>\frac{\beta}{n}\left[\frac{n/2+\alpha}{\beta+\alpha}\right].$$

The question has been raised, whether $T_{a\alpha}$ is bounded on $L^{p_0}(R^n)$ where

$$\left|\frac{1}{2}-\frac{1}{p_0}\right|=\frac{\beta}{n}\left[\frac{n/2+\alpha}{\beta+\alpha}\right].$$

But no a priori L^{p_0} inequalities of any sort were known previously. We have proved the following partial result.

THEOREM 2. *Let α, a and p_0 be as above, and let q_0 be the exponent conjugate to p_0. Then $T_{a\alpha}$ extends to a bounded linear operator from $L^{p_0}(R^n)$ to the Lorentz space $L_{p_0q_0}(R^n)$. (For an exposition of Lorentz spaces, see* [**3**].)

Theorem 2 follows, using complex interpolation, from the two special cases $p=1$ and $p=2$. The case $p=2$ is immediate from the Plancherel theorem, and the case $p=1$ is just an example of the following generalization of the Calderón-Zygmund inequality.

THEOREM 2'. *Let K be a temperate distribution on R^n, with compact support; and let $0<\theta<1$ be given. Suppose that K is a locally integrable function, away from zero, and that*

(i) *The temperate distribution $\hat{K}$ is a function, and*

$$|\hat{K}(x)| \leqq A(1+|x|)^{-n\theta/2} \quad \textit{for } x\in R^n.$$

(ii) $\int_{|x|>2|y|^{1-\theta}}|K(x)-K(x-y)|\,dx\leqq A$ *for all* $y\in R(|y|<1)$.
Then the operator $f\to K*f$, defined for $f\in C_0^\infty(R^n)$ extends to an operator T of weak-type $(1, 1)$.

Obviously, then, T is a bounded operator on $L^p(R^n)$, for $1<p<+\infty$.

A concrete example of a K satisfying (i) and (ii) is the kernel $K(x)=\exp^{[i/|x|]}/x$ for $x\in R^1$, $|x|<1$, and $K(x)=0$ otherwise.

Theorem 2' can be strengthened in various ways. First of all, under reasonable assumptions on K, we can prove a weak-type inequality for the "maximal operator"

$$Mf(x) \equiv \sup_{\epsilon>0} \left| \int_{|y|<\epsilon} K(y)f(x-y)dy \right| .$$

Secondly, a proof almost identical to that of Theorem 2′ establishes a weak-type inequality for convolutions with kernels whose singularities lie at infinity, instead of at zero.

For a discussion of $T_{a\alpha}$ and similar operators, see Hirschmann [2] for the one-dimensional case, and Wainger [7] and Stein [6] for the n-dimensional case.

The operators we have discussed so far are only slightly more singular than the Calderón-Zygmund operators of [1], or operators which reduce to them by interpolation. We now discuss L^p inequalities for highly singular operators, for which the techniques of [1], [4], and [6] break down completely.

Let $T_\alpha: f \to f * (\sin|x|/|x|^\alpha)$, for $f \in C_0^\infty(R^n)$. T_α has an especially neat interpretation if $\alpha = (n+1)/2$. In fact, the operator S, given by $(Sf)\hat{}(x) = \chi(x)\cdot\hat{f}(x)$ (χ denotes the characteristic function of the unit ball in R^n), differs from $T_{(n+1)/2}$ by an error term which is relatively small, so that, roughly speaking, S and $T_{(n+1)/2}$ are the same.

It is easy to show that for $p \leqq 2n/(n+1)$ or $p \geqq 2n/(n-1)$, the operator S cannot be extended to a bounded operator on $L^p(R^n)$. The question of whether S(or $T_{(n+1)/2}$) extends to a bounded operator on $L^p(R^n)$ for $2n/(n+1) < p < 2n/(n-1)$, or for that matter, for any p other than 2, is a well-known unsolved problem.

By interpolation between $p=2$, $\alpha=(n+1)/2$, and $p=1$, $\alpha=n+\epsilon$, it is easy to prove that T_α is bounded on $L^p(R^n)$, for

$$\left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{n-1}{2}\right) < \alpha - \frac{n+1}{2}, \quad 1 < p < 2, \quad \frac{n+1}{2} < \alpha < n.$$

See [6]. But we have every right to expect a far stronger inequality. For if we assume the conjecture that $T_{(n+1)/2}$ is bounded on $L^{2n/(n+1)+\epsilon}(R^n)$, then it follows (at least heuristically) by interpolation, that T_α is bounded on $L^p(R^n)$ for p in the larger range $n/\alpha < p < 2$, $(n+1)/2 < \alpha < n$. This is the "right" range, since for $p \leqq n/\alpha$ it is easily seen that T_α does not extend to a bounded operator on $L^p(R^n)$.

THEOREM 3. *Let $n/\alpha < p < 2$, and $p < 4n/(3n+1)$. Then T_α extends to a bounded linear operator on $L^p(R^n)$.*

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