

## ON SOME SUBMANIFOLDS OF A LOCALLY PRODUCT MANIFOLD

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An investigation of properties of submanifolds of the almost product or locally product Riemannian manifolds has been started in the last years, many interesting results being obtained. So, Okumura [8], Adati and Miyazawa [1], Miyazawa [7], studied the hypersurfaces of such manifolds, Adati [2], defined and studied the invariant, anti-invariant and non-invariant submanifolds, while Bejancu [4], analyzed the semi-invariant submanifolds which are corresponding to CR-submanifolds of a Kaehler manifold [3].

The purpose of this paper is to give some properties of the anti-invariant and semi-invariant submanifolds, by using cohomology groups.

In §1 we recall the definition of these submanifolds and some known results, already. An example of semi-invariant submanifold is given.

In §2 we associate to a semi-invariant submanifold a de Rham cohomology class (as in [5] for CR-submanifolds) and we obtain a connection between the properties of the invariant and anti-invariant distributions and the cohomology of the submanifold (theorem 2.2).

The stability of some anti-invariant submanifolds of a locally product Riemannian manifold is studied in §3 and we give algebraic conditions for stability.

**§1. Anti-invariant and semi-invariant submanifolds of a locally product Riemannian manifold.** Let  $(\tilde{M}, g, F)$  be a  $C^\infty$ -differentiable almost product Riemannian manifold, where  $g$  is a Riemannian metric and  $F$  is a non-trivial tensor field of type (1.1). Moreover  $g$  and  $F$  satisfy the following conditions

$$(1.1) \quad F^2 = I, \quad (F \neq \pm I); \quad g(FX, FY) = g(X, Y), \quad X, Y \in \mathfrak{X}(\tilde{M})$$

where  $I$  is the identity and  $\mathfrak{X}(\tilde{M})$  is the Lie algebra of vector fields on  $\tilde{M}$ .

We denote by  $\tilde{\nabla}$  the Levi-Civita connection on  $\tilde{M}$  with respect to  $g$  and furthermore we assume that  $\tilde{M}$  is locally product, that is

$$(1.2) \quad \tilde{\nabla}_X F = 0 \quad X \in \mathfrak{X}(\tilde{M}).$$

Let  $M$  be a Riemannian manifold isometrically immersed in  $\tilde{M}$  and denote

by the same symbol  $g$  the Riemannian metric induced on  $M$ .

If on  $M$  there exist two complementary and orthogonal distributions  $D$  and  $D^\perp$ , satisfying the following conditions :

$$F(D_x)=D_x; \quad F(D_x^\perp)\subset T_xM^\perp \quad \text{for each } x\in M$$

then  $M$  is called a *semi-invariant submanifold* of the locally product Riemannian manifold  $\tilde{M}$  [4].

Particularly if  $\dim D_x=\dim T_xM$  ( $\dim D_x=0$ ) for each  $x\in M$  then  $M$  is an *invariant submanifold (anti-invariant submanifold)* of  $\tilde{M}$  [2].

We have the following results

**THEOREM 1.1.** [2] *In a 2n-dimensional locally product Riemannian manifold every anti-invariant submanifold of dimension n is totally geodesic.*

**THEOREM 1.2.** [4] *For a semi-invariant submanifold of a locally product Riemannian manifold the following assertions hold:*

a) *the distribution  $D^\perp$  is involutive if and only if*

$$g(h(Y, Z), FX)=0 \quad X, Y\in D^\perp, \quad Z\in D$$

where  $h$  denote the second fundamental form of  $M$ ;

b)  *$M$  is  $D$ -geodesic (i. e.  $h(X, Y)=0$  for  $X, Y\in D$ ) if and only if  $D$  is involutive and each leaf of  $D$  is totally geodesic immersed in  $\tilde{M}$ ;*

c) *the distribution  $D$  is involutive if and only if*

$$h(X, FY)=h(FX, Y) \quad X, Y\in D.$$

*Example.* Let  $M$  be a normal metric almost paracontact manifold [9], [10], with the structure  $(\varphi, \xi, \eta, g)$  and denote by  $D^*$  the distribution  $\{X\in TM: \eta(X)=0\}$ . It is well-known that by putting

$$(1.3) \quad FX=\varphi(X), \quad X\in D^*; \quad F\xi=\frac{d}{dt}; \quad F\left(\frac{d}{dt}\right)=\xi$$

$F$  defines an almost product metric structure on  $M\times R$ . The product metric on  $M\times R$  satisfies condition (1.1) and as  $M$  is normal, it follows that  $M\times R$  with this structure is local product.  $M$  is a closed submanifold of  $M\times R$  and from (1.3) follows that  $F\xi\in TM^\perp$ ,  $FD^*=D^*$  and then  $M$  is semi-invariant in  $M\times R$ .

**§ 2. Cohomology of semi-invariant submanifolds.** Furthermore we assume  $M$  as a compact without boundary manifold.

If  $\dim D^\perp=q$  and  $\dim D=p$  then we denote by  $\mathcal{B}_{D^\perp}=\{X_1, \dots, X_q\}$ ,  $\mathcal{B}_D=\{X_{q+1}, \dots, X_{q+p}\}$  two orthonormal local bases in  $D^\perp$ , resp. in  $D$ .

**PROPOSITION 2.1.** *If the distribution  $D^\perp$  is involutive, then each leaf of  $D^\perp$  is minimal in  $M$ .*

*Proof.* The mean-curvature vector of  $D^\perp$  is

$$H_{D^\perp} = \frac{1}{q} \sum_{i=1}^q (\nabla_{X_i} X_i)^\perp$$

where  $(\nabla_X X)^\perp$  is the component of  $\nabla_X X$  in  $D$ .

By applying the Gauss formula

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad X, Y \in \mathfrak{X}(M)$$

where  $\nabla$  is the Riemannian connection on  $M$ , induced by  $\tilde{\nabla}$ , from the conditions (1.1) and (1.2) we have

$$(2.2) \quad g(Y, \nabla_X X) = g(FY, \tilde{\nabla}_X(FX)), \quad X \in \mathcal{B}_{D^\perp}, \quad Y \in D$$

Now, by using the Weingarten formula

$$(2.3) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad N \in TM^\perp$$

from (2.2) we can deduct

$$(2.4) \quad g(Y, \nabla_X X) = -g(FY, A_{FX} X)$$

and taking into account the known equality

$$(2.5) \quad g(h(X, Y), N) = g(A_N X, Y)$$

we can write

$$(2.6) \quad g(Y, \nabla_X X) = -g(h(X, FY), FX)$$

From (2.6) and the theorem 1.2, a) follows  $g(Y, \nabla_X X) = 0$  and then  $H_{D^\perp} = 0$ .

Q. E. D.

**PROPOSITION 2.2.** *For a  $D$ -geodesic semi-invariant submanifold the distribution  $D$  is minimal.*

*Proof.* If  $X \in \mathcal{B}_D$  then there exists  $\bar{X} \in D$  such that  $F\bar{X} = X$  and from (1.1), (1.2), (2.1), (2.3), (2.5) we obtain

$$(2.7) \quad g(Y, \nabla_X X) = g(FY, h(F\bar{X}, \bar{X})), \quad Y \in D^\perp.$$

But  $M$  is  $D$ -geodesic and then from (2.7) follows  $g(Y, \nabla_X X) = 0$ , hence  $D$  is minimal.

Q. E. D.

We define on  $M$  the 1-forms  $\omega^1, \dots, \omega^q$ , satisfying the following conditions

$$(2.8) \quad \omega^i(Z) = 0, \quad Z \in D; \quad \omega^i(X_j) = \delta_j^i, \quad X_j \in \mathcal{B}_{D^\perp}; \quad i, j \in \overline{1, q}.$$

Then we give the  $q$ -form  $\omega = \omega^1 \wedge \dots \wedge \omega^q$ , globally defined on  $M$  and we have

the following

**PROPOSITION 2.3.** *If the distribution  $D$  is involutive and  $D^\perp$  is minimal then the  $q$ -form  $\omega$  is closed.*

*Proof.* It is enough to prove that

$$(2.9) \quad d\omega(Y, X_1, \dots, X_q) = 0 \quad \text{for } Y \in D, \quad X_1, \dots, X_q \in \mathcal{B}_{D^\perp}$$

$$(2.10) \quad d\omega(Y_1, Y_2, X_1, \dots, X_{q-1}) = 0 \quad \text{for } Y_1, Y_2 \in D, \quad X_1, \dots, X_{q-1} \in \mathcal{B}_{D^\perp}.$$

From the definition of the forms  $\omega^i$  it follows

$$(2.11) \quad d\omega(Y, X_1, \dots, X_q) = \sum_{i=1}^q g([Y, X_i], X_i)$$

Now, the connection  $\tilde{\nabla}$  is Riemannian and then

$$(2.12) \quad g(X_i, \tilde{\nabla}_Y X_i) = 0 \quad X_i \in \mathcal{B}_{D^\perp}, \quad Y \in D.$$

Taking into account (2.12), (2.1) follows

$$(2.13) \quad g([Y, X_i], X_i) = g(\nabla_{X_i} X_i, Y)$$

But  $D^\perp$  is minimal and then from (2.11) and (2.13) we obtain (2.9).

The distribution  $D$  being involutive, (2.10) follows from the equality

$$(2.14) \quad d\omega(Y_1, Y_2, X_1, \dots, X_{q-1}) = -\omega([Y_1, Y_2], X_1, \dots, X_{q-1}).$$

In the same manner as above we can define on  $M$  the 1-forms  $\theta^{q+1}, \dots, \theta^{q+p}$  by

$$(2.15) \quad \theta^{q+i}(Z) = 0, \quad Z \in D^\perp; \quad \theta^{q+i}(X_{q+j}) = \delta_j^i, \quad X_{q+j} \in \mathcal{B}_D, \quad i, j \in \overline{1, p}.$$

By using a similar computation as in the proof of the proposition 2.3 we can state the following

**PROPOSITION 2.4.** *If  $D$  is minimal and if  $D^\perp$  is involutive then the  $p$ -form  $\theta = \theta^{q+1} \wedge \dots \wedge \theta^{q+p}$  is closed on  $M$ .*

Now from Propositions 2.3, 2.4 we have  $\theta = *\omega$  and by applying the Hodge-Rham theorem we obtain the

**THEOREM 2.1.** *For any compact semi-invariant submanifold  $M$ , of a Riemannian locally product manifold, having the distribution  $D$  involutive and  $D^\perp$  minimal, a cohomology de Rham class  $[\omega] \in H^q(M, \mathbb{R})$  is well-defined. This class is non trivial if  $D$  is minimal and  $D^\perp$  is involutive.*

From Proposition 2.1 and Theorem 2.1 follows

**THEOREM 2.2.** *Let  $M$  be a compact semi-invariant submanifold of a locally product Riemannian manifold. If the distributions  $D, D^\perp$  are involutives and  $D$  is minimal then*

$$H^q(M, R) \neq 0 \quad q = \dim D^\perp.$$

From Theorems 1.2, 2.2 and Proposition 2.2 we deduce the following

**PROPOSITION 2.5.** *For every compact and totally geodesic semi-invariant submanifold of a locally product Riemannian manifold the Betti number  $b_q, q = \dim D^\perp$ , not vanish.*

Now taking into account the Theorem 2.2 and the above example we obtain the following

**PROPOSITION 2.6.** *Let  $M$  be a compact normal metric almost paracontact manifold. If the distribution  $D^*$  is involutive and minimal then*

$$H^1(M, R) \neq 0$$

*Remark.* It is well-known that if  $M$  is a SP-Sasakian manifold then the distribution  $D^*$  is involutive [9], [10].

Next we can make some comments on the obtained results.

A. Let  $S^{2n+1}$  be the unit sphere in  $R^{2n+2}, n \geq 2$ , endowed with the standard Sasakian structure  $(f, \xi, \eta)$ . It is known that the tensor field  $F$  given by

$$g(FX, Y) = (\nabla_X \eta)Y$$

defines a SP-Sasakian structure on  $S^{2n+1}$ . Moreover  $H^1(S^{2n+1}, R) = 0$  for  $n \geq 1$  and then the sphere  $S^{2n+1}, n \geq 2$ , is a semi-invariant submanifold of  $S^{2n+1} \times R$ , so that the distribution  $D^*$  is not minimal.

B. Suppose  $M$  is a compact SP-Sasakian manifold totally geodesic immersed in the Riemannian locally product manifold  $M \times R$ . From Theorem 1.2 and Proposition 2.2 it follows that the distribution  $D^*$  is involutive and minimal. Hence the first Betti number of  $M$  not vanish.

**§ 3. Stability of anti-invariant submanifolds.** Let  $M$  be a compact  $n$ -dimensional anti-invariant submanifold of the  $2n$ -dimensional locally product Riemannian manifold  $\tilde{M}$ .

By applying (2.1), (2.3) and Theorem 1.1 we have the

**LEMMA 3.1.** *For every  $X, Y \in TM$  the next equalities holds*

$$\nabla_Y^\perp(FX) = F(\nabla_Y X) \quad A_{FX}Y = 0.$$

Let  $\{X_1, \dots, X_n\}$  be a orthonormal local basis in  $TM$  and lets denote by  $S$  and  $\tilde{S}$  the Ricci tensors associated to the manifolds  $M$  and  $\tilde{M}$ .

LEMMA 3.2. For every  $X \in TM$  we have

$$\sum_{i=1}^n \tilde{R}(X_i, FX, FX, X_i) = \tilde{S}(X, X) - S(X, X).$$

*Proof.* From (1.1), (1.2) it follows that

$$(3.1) \quad \tilde{R}(X_i, FX, FX, X_i) = \tilde{R}(FX_i, X, X, FX_i).$$

Now the required equality is a consequence of (3.1) and of the Gauss equation

$$(3.2) \quad R(U, V, W, T) = \tilde{R}(U, V, W, T) + g(h(U, T), h(V, W)) - g(h(U, W), h(V, T)).$$

Let  $N$  be a normal vector field and denote by  $\mathcal{CV}''(N)$  the second normal variation of  $M$  induced by  $N$ . Then we have ([6], chap. I)

$$(3.3) \quad \mathcal{CV}''(N) = \int_M \left\{ \|\nabla^\perp N\|^2 - \sum_{i=1}^n \tilde{R}(X_i, N, N, X_i) - \|A_N\|^2 \right\} dV$$

where  $dV$  is the volume form of  $M$ .

On the other hand if  $\eta$  is the 1-form associated to the vector field  $X \in TM$  then we have the well-known formula ([6], chap. V)

$$(3.4) \quad \int_M \left\{ S(X, X) + \|\nabla X\|^2 - \frac{1}{2} \|d\eta\|^2 - (\delta\eta)^2 \right\} dV = 0.$$

Taking into account the Lemma 3.1 and the Theorem 1.1, from (3.3), (3.4) follow the

PROPOSITION 3.1. The normal variation induced by the normal vector field  $N = FX$  of the compact anti-invariant submanifold  $M$  in a locally product Riemannian manifold is given by

$$\mathcal{CV}''(N) = \int_M \left\{ \frac{1}{2} \|d\eta\|^2 + (\delta\eta)^2 - \tilde{S}(N, N) \right\} dV.$$

Now we can state the following

THEOREM 3.1. Let  $M$  be a compact anti-invariant submanifold of the locally product Riemannian manifold  $\tilde{M}$ .

- a) If  $\tilde{S}$  is negative definite then  $M$  is stable.
- b) If  $H^1(M, R) \neq 0$  and  $\tilde{S}$  is positive definite then  $M$  is unstable.

*Proof.* a) is an immediate consequence of the Proposition 3.1 because we obtain  $\mathcal{CV}''(N) > 0$  for every  $N \in TM^\perp$ .

b) Since  $H^1(M, R) \neq 0$  there exists an harmonic 1-form  $\eta$  on  $M$  and if  $X$  is it associated vector field, we have  $d\eta = \delta\eta = 0$  and then  $\mathcal{CV}''(FX) < 0$ . Consequently  $M$  is unstable in  $\tilde{M}$ .

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