An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 217–222

ON SOME SUFFICIENT CONDITIONS FOR UNIVALENCE

Dorina Răducanu, Halit Orhan and Erhan Deniz

Abstract

In this paper the method of subordination chains is used to establish some sufficient conditions for univalence for analytic functions defined in the open unit disk.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

In order to prove our main result we need a brief summary of the method of subordination chains.

A function $L(.,t) : \mathbb{U} \to \mathbb{C}, t \geq 0$ is said to be a *subordination chain* or a *Loewner chain* if:

(i) $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$.

(ii) $L(z,t) \prec L(z,s)$ for all $0 \leq t \leq s < \infty$, where the symbol "$\prec$" stands for subordination.

The following result is due to Ch. Pommerenke [6].

Key Words: Analytic functions, subordination chains, univalence conditions
2010 Mathematics Subject Classification: 30C45
Received: June, 2010
Accepted: September, 2010

Theorem 1.1. *Let $L(z,t)=a_1(t)z+\dots$ be an analytic function in $\mathbb{U}$ for all $t\geq 0$. Suppose that:*

(i) $L(z,t)$ is a locally absolutely continuous function of $t\in[0,\infty)$, locally uniform with respect to $z\in\mathbb{U}$;

(ii) $a_1(t)$ is a complex valued continuous function on $[0,\infty)$ such that $a_1(t)\neq 0$, $\lim_{t\to\infty}|a_1(t)|=\infty$ and

$$\left\{\frac{L(.,t)}{a_1(t)}\right\}_{t\geq 0}$$

is a normal family of functions in $\mathbb{U}$;

(iii) there exists an analytic function $p:\mathbb{U}\times[0,\infty)\to\mathbb{C}$ satisfying $\Re p(z,t)>0$ for all $(z,t)\in\mathbb{U}\times[0,\infty)$ and

$$\frac{\partial L}{\partial t}(z,t)=p(z,t)z\frac{\partial L}{\partial z}(z,t)\ ,\ z\in\mathbb{U}\ ,\ a.e\ \ t\geq 0.$$

Then, for all $t\geq 0$, the function $L(z,t)$ is a subordination chain.

2 Sufficient conditions for univalence

In this section, making use of Theorem 1.1, we obtain various conditions for univalence which generalize some known results.

Theorem 2.1. *Consider $f\in\mathcal{A}$. Let m be a positive real number and let α be a complex number such that $\alpha\neq 1$, $\left|\frac{\alpha}{1-\alpha}\right|<1$. If the inequalities*

$$\left|\frac{\alpha}{f'(z)-\alpha}-\frac{m-1}{2}\right|<\frac{m+1}{2} \tag{2}$$

and

$$\left|\frac{\alpha|z|^{m+1}+(1-|z|^{m+1})zf''(z)}{f'(z)-\alpha}-\frac{m-1}{2}\right|\leq\frac{m+1}{2} \tag{3}$$

are satisfied for all $z\in\mathbb{U}$, then the function f is univalent in $\mathbb{U}$.

Proof. Define the function $L(.,t):\mathbb{U}\to\mathbb{C}\ ,t\geq 0$

$$L(z,t)=f(e^{-t}z)+\left(e^{mt}z-e^{-t}z\right)\left(f'(e^{-t}z)-\alpha\right). \tag{4}$$

We will prove that the function $L(z,t)$ satisfies the conditions of Theorem 1.1.

Since the function $f(e^{-t}z)$ is analytic in $\mathbb{U}$, it is easy to see that the function $L(z,t)$ is also analytic in $\mathbb{U}$ for all $t \geq 0$. We have

$$\frac{\partial L}{\partial t}(z,t) = -e^{-t}z\left[\alpha + (e^{mt} - e^{-t})zf''(e^{-t}z)\right] + me^{mt}z\left[f'(e^{-t}z) - \alpha\right].$$

It follows that $\left|\frac{\partial L}{\partial t}(z,t)\right|$ is bounded on $[0,T]$, for any fixed $T > 0$ and $z \in \mathbb{U}$. Therefore, the function $L(z,t)$ is locally absolutely continuous on $[0,\infty)$, locally uniform with respect to $z \in \mathbb{U}$.

Elementary calculations give

$$a_1(t) = e^{mt}[\alpha e^{-(m+1)t} + 1 - \alpha].$$

From $\alpha \neq 1$ and $\left|\frac{\alpha}{1-\alpha}\right| < 1$, it follows easily that $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.

Let $r \in (0,1)$ and let $K = \overline{\{z \in \mathbb{C} : |z| \leq r\}}$. Since the function $L(z,t)$ is analytic in $\mathbb{U}$, there exists $M > 0$ such that $|L(z,t)| \leq Me^{mt}$ for $z \in K$ and $t \geq 0$. Also, for $t \geq 0$, it is easy to see that there exists $N > 0$ such that $|a_1(t)| > Ne^{mt}$. It follows that $\left|\frac{L(z,t)}{a_1(t)}\right| \leq \frac{M}{N}$ for $z \in K$ and $t \geq 0$. Thus, $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family in $\mathbb{U}$.

Consider the function $p(z,t)$ defined by

$$p(z,t) = \frac{\partial L}{\partial t}(z,t) / z\frac{\partial L}{\partial z}(z,t).$$

In order to prove that the function $p(z,t)$ is analytic and has positive real part in $\mathbb{U}$, we will show that the function

$$w(z,t) = \frac{1 - p(z,t)}{1 + p(z,t)} \tag{5}$$

is analytic in $\mathbb{U}$ and

$$|w(z,t)| < 1 \text{ , for all } z \in \mathbb{U} \text{ , } t \geq 0. \tag{6}$$

Elementary calculations give

$$w(z,t) = \frac{2}{m+1}F(z,t) - \frac{m-1}{m+1},$$

where

$$F(z,t) = e^{-(m+1)t} \cdot \frac{\alpha + (e^{mt} - e^{-t})zf''(e^{-t}z)}{f'(e^{-t}z) - \alpha}.$$

The inequality (2.5) is therefore equivalent to

$$\left|F(z,t) - \frac{m-1}{2}\right| < \frac{m+1}{2}\ , z \in \mathbb{U}\ , t \geq 0. \tag{7}$$

If $t = 0$ the last inequality yields

$$\left|\frac{\alpha}{f'(z) - \alpha} - \frac{m-1}{2}\right| < \frac{m+1}{2}.$$

Define $G(z,t) = F(z,t) - \dfrac{m-1}{2}$.

Since $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $t > 0$ it follows that $G(z,t)$ is an analytic function in $\bar{\mathbb{U}}$. Making use of the maximum modulus principle we obtain that for each fixed $t > 0$, there exists $\theta \in \mathbb{R}$ such that :

$$|G(z,t)| < \max_{|z|=1} |G(z,t)| = |G(e^{i\theta},t)|\ , z \in \mathbb{U}.$$

Let $u = e^{-t}e^{i\theta}$. We have $|u| = e^{-t}$ and $e^{-(m+1)t} = (e^{-t})^{m+1} = |u|^{m+1}$. Therefore,

$$|G(e^{i\theta},t)| = \left|\frac{\alpha|u|^{m+1} + (1-|u|^{m+1})uf''(u)}{f'(u) - \alpha} - \frac{m-1}{2}\right|.$$

Inequality (2.2), from the hypothesis, yields

$$|G(e^{i\theta},t)| \leq \frac{m+1}{2}. \tag{8}$$

From (2.1) and (2.7) it follows that the inequality (2.6) is satisfied for all $z \in \mathbb{U}$ and $t \geq 0$.

Since all the conditions of Theorem 1.1 are satisfied we obtain that the function $L(z,t)$ is a subordination chain. If $t = 0$, we have $L(z,0) = f(z)$ and thus, the function f is univalent in $\mathbb{U}$. □

Remark 2.1. *Some particular cases of Theorem 2.1 are the following:*

(i) When $m = 1$ *and* $\alpha = 0$ *inequality (2.2) becomes*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1\ , z \in \mathbb{U}$$

which is Becker's condition of univalence [2].

(ii) A result due to N. N. Pascu [4] is also obtained when $m = 1$.

The condition (2.2) of Theorem 2.1 can be replaced with a simpler one.

Corollary 2.1. *Consider $f \in \mathcal{A}$. Let m be a positive real number and let α be a complex number such that $\alpha \neq 1$ and $\left|\frac{\alpha}{1-\alpha}\right| < 1$. If*

$$\left|\frac{\alpha}{f'(z)-\alpha} - \frac{m-1}{2}\right| < \frac{m+1}{2}\ , z \in \mathbb{U}$$

and

$$\left|\frac{zf''(z)}{f'(z)-\alpha} - \frac{m-1}{2}\right| \leq \frac{m+1}{2}\ , z \in \mathbb{U} \tag{9}$$

then the function f is univalent in $\mathbb{U}$.

Proof. Making use of (2.1) and (2.8) we obtain

$$\left|\frac{\alpha|z|^{m+1} + (1-|z|^{m+1})zf''(z)}{f'(z)-\alpha} - \frac{m-1}{2}\right| =$$

$$= \left||z|^{m+1}\left(\frac{\alpha}{f'(z)-\alpha} - \frac{m-1}{2}\right) + (1-|z|^{m+1})\left(\frac{zf''(z)}{f'(z)-\alpha} - \frac{m-1}{2}\right)\right| <$$

$$< |z|^{m+1}\frac{m+1}{2} + (1-|z|^{m+1})\frac{m+1}{2} = \frac{m+1}{2}.$$

The conditions of Theorem 2.1 being satisfied it follows that the function f is univalent in $\mathbb{U}$. □

Remark 2.2. *Consider $\alpha < 0$. By elementary calculations we obtain that the inequality (2.1) is equivalent to*

$$\Re f'(z) > \frac{m}{\alpha(m+1)}|f'(z)|^2\ , z \in \mathbb{U}.$$

If in the last inequality we let $\alpha \to -\infty$ we obtain that

$$\Re f'(z) \geq 0.$$

Since (2.8) holds true for $\alpha \to -\infty$ it follows from Corollary 2.1 that the function f is univalent in $\mathbb{U}$.

Therefore, we can conclude that the univalence criterion due to Alexander-Noshiro-Warschawski [1], [3], [8] is a limit case of Corollary 2.1.

Acknowledgements

We wish to thank the referee for his/hers helpful comments.

The results contained in the present paper were obtained during the visit of the first author at Atatürk University, Erzurum. The visit was supported by Atatürk University.

References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17**(1915), 12-22.

[2] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math., **255**(1972), 23-43.

[3] K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Imp. Univ. Jap., (1), **2**(1934-1935), 129-155.

[4] N. N. Pascu, *Sufficient conditions for univalence*, Seminar on Geometric Functions Theory, (Preprint), **5**(1986), 119-122.

[5] N. N. Pascu, V. Pescar, *A generalization of Pfaltzgraff's Theorem*, Seminar on Geometric Function Theory, (Preprint), **2**(1991), 91-98.

[6] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math., **218**(1965), 159-173.

[7] D. Răducanu, I. Radomir, M. E. Gageonea, N. R. Pascu, *A generalization of Ozaki-Nunokawa's univalence criterion*, J. Inequal. Pure and Appl. Math., **5**(4), Art. 95(2004).

[8] S. E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc., **38**(1935), 310-340.

Faculty of Mathematics and Computer Science
Transilvania University of Braşov
50091, Iuliu Maniu, 50, Braşov, Romania,
Email: dorinaraducanu@yahoo.com

Faculty of Science and Arts
Department of Mathematics
Ataturk University, 25240, Erzurum Turkey
Email: horhan@atauni.edu.tr

Faculty of Science and Arts
Department of Mathematics
Ataturk University, 25240, Erzurum Turkey
Email: edeniz36@yahoo.com