ON SOME TWO WAY CLASSIFICATIONS OF INTEGERS*

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In this note we use the method of generating functions to show that there is a unique way of splitting the non-negative integers into two classes in such a way that the sums of pairs of distinct integers will be the same (with same multiplicities) for both classes. We prove a similar theorem for products of positive integers and consider some related problems.

Suppose then that the non-negative integers are split into two classes A and B with

$$A: \; 0 = a_1 < a_2 < \dots$$

$$B: \; b_1 < b_2 < \dots \; .$$

Consider the corresponding generating functions

$$A \leftrightarrow f(x) = \sum_i x^{a_i}, \quad |x| < 1.$$

$$B \leftrightarrow g(x) = \sum_j x^{b_j}, \quad |x| < 1.$$

Since every non-negative integer belongs to exactly one class we have

(1) $$f(x) + g(x) = 1/(1-x).$$

The coefficient of x^n in $f^2(x) - f(x^2)$ will be twice the number of ways in which n can be represented as the sum of two a's.

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Thus if the sums of the a's two at a time are to be the same as the sums of the b's two at a time we will have

(2) $$f^2(x) - f(x^2) = g^2(x) - g(x^2) .$$

If we let

(3) $$h(x) = f(x) - g(x)$$

then (2) can be written in the form

(4) $$\{f(x) + g(x)\} h(x) = h(x^2) .$$

Now from (4) and (1) we obtain

(5) $$h(x)/h(x^2) = 1-x.$$

Throughout we have $|x| < 1$ so that $h(x^{2^k}) \to h(0)$ as $k \to \infty$. Further, since $a_1 = 0$ and $h(0) = 1$, iteration of (5) yields

(6) $$h(x) = (1-x)(1-x^2)(1-x^4)(1-x^8) \ldots .$$

Thus from (1), (3) and (6) we have

(7) $$f(x) = \frac{1}{2} \left[1/(1 - x) + \prod_{k=0}^{\infty} (1 - x^{2^k}) \right].$$

From (7) it is easy to determine which integers belong to A. Since every integer has a unique representation as a sum of distinct powers of 2 we have

(8) $$1/(1 - x) = \sum_{n=0}^{\infty} x^n = \prod_{k=0}^{\infty} (1 + x^{2^k}),$$

while

(9) $$\prod_{k=0}^{\infty} (1 - x^{2^k}) = \sum_{n=0}^{\infty} (-1)^{\varepsilon(n)} x^n,$$

where $\varepsilon(n)$ is the number of summands in the decomposition of n into distinct powers of 2, i.e. the sum of the digits in the binary representation of n.

Combining (7), (8) and (9) we find that A consists precisely of those integers for which $\varepsilon(n)$ is even. Thus the sequences A and B are

A : 0, 3, 5, 6, 9, 10, 12, 15, ... ,
B : 1, 2, 4, 7, 8, 11, 13, 14, ...

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We note that the classes A and B can also be constructed recursively as follows: 0 is in A and 1 in B and if

$$2^k \leq x < 2^{k+1}$$

then x and $x - 2^k$ are in opposite classes.

It is easy to see that our decomposition also leads to a unique decomposition of any arithmetic progression $\{a + nb\}$, $n = 0, 1, 2, \ldots$ into two classes having the same sums in pairs. Indeed we have only to put $a + nb$ into the same class as was previously assigned to n.

Let us next investigate the possibility of splitting the finite set of integers $0, 1, 2, \ldots, m-1$ into two classes in such a way that the sums in pairs in both classes are the same. Using the same definitions for f(x), g(x) and h(x) as before and following a similar procedure we obtain

(10) $$f(x) + g(x) = 1 + x + x^2 + \ldots + x^{m-1} = (1-x^m)/(1-x)$$

and

(11) $$h(x) = \frac{1-x}{1-x^m} \cdot \frac{1-x^2}{1-x^{2m}} \cdot \frac{1-x^4}{1-x^{4m}} \cdot \ldots .$$

Clearly h(x) is a polynomial if and only if $m = 2^k$, $k \geqslant 0$. In this case we have

(12) $$h(x) = (1-x)(1-x^2) \ldots (1-x^{2^{k-1}}),$$

with the understanding that this product is 1 when $k = 0$, and

(13) $$f(x) = \tfrac{1}{2}\left[(1-x^{2^k})/(1-x) + (1-x)(1-x^2)\ldots(1-x^{2^{k-1}})\right].$$

Thus the integers $0, 1, 2, \ldots, m-1$ can be split in the required manner if and only if $m = 2^k$ and in this case the split must be in accordance with the same rule as was used for the integers $0, 1, 2, \ldots$.

We next consider a corresponding problem for products instead of sums. We will show that there is a unique way of splitting the positive integers into two classes in such a way that products of pairs of distinct integers from either class occur with the same multiplicities. Suppose then that the positive integers are split into C and D with

$$C: 1 = c_1 < c_2 < \dots ,$$
$$D: \quad d_1 < d_2 < \dots .$$

This time we define the corresponding Dirichlet series

$$C \leftrightarrow F(s) = \sum_{i=1}^{\infty} c_i^{-s} \quad (\text{Rl}(s) > 1)$$

and $$D \leftrightarrow G(s) = \sum_{j=1}^{\infty} d_j^{-s} \quad (\text{Rl}(s) > 1) .$$

Here we have

(14) $$F(s) + G(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s) .$$

The condition that the products of pairs of distinct c's are the same as products of pairs of distinct d's becomes

(15) $$F^2(s) - F(2s) = G^2(s) - G(2s) .$$

Hence if we let

(16) $$H(s) = F(s) - G(s)$$

then

(17) $$H(s)/H(2s) = (\zeta(s))^{-1}.$$

Iterating (17) and using $H(\infty) = 0$ we obtain

(18) $$H(s) = [\zeta(s)\,\zeta(2s)\,\zeta(4s)\,\zeta(8s)\dots]^{-1}.$$

If we now use the well known Euler identity

(19) $$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (\text{Rl}\, s > 1),$$

where the product runs over all primes, we obtain

(20) $$H(s) = \prod_p (1 - p^{-s})(1 - p^{-2s})(1 - p^{-4s})(1 - p^{-8s}) \dots$$
$$= \prod_p (1 + \alpha(p)/p^s + \alpha(p^2)/p^{2s} + \alpha(p^3)/p^{3s} + \dots) = \sum_n \alpha(n)/n^s ,$$

where $\alpha(n)$ is the multiplicative function determined by

(21) $$\alpha(p^r) = (-1)^{\varepsilon(r)}$$

and

$$\alpha(p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}) = \alpha(p_1^{r_1})\,\alpha(p_2^{r_2}) \ldots \alpha(p_k^{r_k}) .$$

We are thus led to the following rule for the required classification:

$$n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$$

lies in the first class if and only if the total number of binary digits required to represent the set of numbers $r_1, r_2, \ldots, r_k$ is even. The classification begins

C : 1, 6, 8, 10, 12, 14, 15, 18, 20, 21, 22, 26, 27, 28, ...

D : 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23, 24, 25, 29, 30, ...

It is easy to see that no finite set 1, 2, ... ,n can be split into two classes so that the products two at a time are the same for both classes, if $n \geq 3$. Indeed, if $n > 3$, a consideration of the position of the largest four numbers n-3, n-2, n-1 and n already leads to this conclusion. Thus, if n is in the same class as n-1 or n-2 then the product of the largest numbers in this class is larger than any product in the second class. On the other hand, if n-1 and n-2 are in the same class, then their product is larger than that of the largest numbers which could be in the other class, namely n and n-3. The case n=3 is easily verified independently.

In conclusion we note that the determination of all sets of numbers which have a given set of numbers as sums of pairs appears to be a difficult problem. In particular, in a paper* of J.L. Selfridge and E. Straus which treats a number of related problems, it is conjectured that no three distinct sets of numbers can have the same sum of distinct pairs.

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* On the determination of numbers by their sums of a fixed order, Pacific J. of Math. 8 (1958), pp. 845-856.

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