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ON SOME VECTOR VALUED GENERALIZED DIFFERENCE MODULAR SEQUENCE SPACES

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Abstract

In this paper we generalize the modular sequence space $\ell\{M_k\}$ by introducing the sequence space $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$. We give various properties relevant to algebraic and topological structures of this space and derived some other spaces .

1 Introduction

By $w(X)$, we shall denote the space of all X -valued sequences spaces, where (X,q) is a seminormed space, seminormed by q. For $X = C$, the space of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\overline{\theta} = (\theta, \theta, \theta, ...)$ where θ is the zero element of X.

The notion of difference sequence space was introduced by Kizmaz [5], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [3] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

Let r, s be non-negative integers and $v = v_k$ be a sequence of non-zero scalars Also let $Z = \{\ell_\infty, c, c_0\}$. Dutta [2] define the following sequence spaces

$$Z\left(\Delta^s_{(vr)}\right) = \left\{x = (x_k) \in w : \left(\Delta^s_{(vr)} x_k\right) \in Z\right\},$$

where $\left(\Delta^s_{(vr)} x_k\right) = \left(\Delta^{s-1}_{(vr)} x_k - \Delta^{s-1}_{(vr)} x_{k-r}\right)$ and $\Delta^0_{(vr)} x_k = v_k x_k$ for all $k \in N$, which is equivalent to the following binomial representation :

$$\Delta^s_{(vr)} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} v_{k-ri} x_{k-ri}.$$

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In this expansion we take $v_k = 0$ and $x_k = 0$ for non-positive values of $k \in N$. Dutta [2] shown that these spaces can be made BK-spaces under the norm

$$\|x\| = \sup_k \left|\Delta^s_{(vr)} x_k\right|.$$

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous , non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If convexity of Orlicz function M is replaced by

$$M(x+y) \le M(x) + M(y),$$

then this function is called a modulus function introduced by Nakano [9].

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M becomes a Banach space , with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

which is called an Orlicz space .The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$. Another generalization of Orlicz sequence spaces due to Woo [13]. Let $\{M_k\}$ be a sequence of Orlicz functions . Define the vector space $\ell\{M_k\}$ by

$$\ell\{M_k\} = \left\{ x \in w : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \le \infty, \text{ for some } \rho > 0 \right\}$$

and this space has a norm defined by

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

Then $\ell\{M_k\}$ becomes a Banach space and is called a modular sequence space. The space $\ell\{M_k\}$ also generalizes the concept of modulared sequence space introduced by Nakano [10], who considered the space $\ell\{M_k\}$ when $M_k(x) = x^{\alpha_k}$, where $1 \le \alpha_k < \infty$ for $k \ge 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \le kM(u)$ $(u \ge 0)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \le kluM(u)$ for all values of u and for $l > 1$ (see;[7]). The Δ_2-condition implies $M(lu) \le Kl^{\log_2 K} M(u)$ for all values $u > 0$, $l > 1$.

Karakaya [6], Bektaş and Altin [1], Parasar and Choudhary [11], Mursaleen , Khan and Qamaruddin [8], Tripathy and Dutta [12] and many others have studied sequence spaces using Orlicz functions.

In [14], it is shown that a BK-spaces is a Banach space of complex sequences $x = (x_k)$ in which the co-ordinate maps are continuous , that is, $|x_k^n - x_k| \to 0$, whenever $\|x^n - x\| \to 0$ as $n \to \infty$, where $x^n = (x_k^n)$ for all $n \in N$ and $x = (x_k)$.

Let A denotes the set of all complex sequences which have only a finite number of non-zero coordinates, λ denotes a $BK-$space of sequences $x = (x_k)$ which contains A. An element $x = (x_k)$ of λ will be called sectionally convergent if

$$x^n = \sum_{k=1}^{n} x_k e_k \to x \text{ as } n \to \infty \ ,$$

where $e_k = (\delta_{ki})$, where $\delta_{kk} = 1$, $\delta_{ki} = 0$, for $k \neq i$.

The space λ will be called AK-space if and only if each of its elements is sectionally convergent .Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions, X be a seminormed space with seminorm q, $p = (p_k)$ be a sequence of positive real numbers and $v = (v_k)$ be a fixed sequence of non-zero scalars. Then for non-negative real numbers s, m and n, we define

$$\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\} = \left\{x \in w(X) : \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)}}{\rho}\right)\right)\right]^{p_k}\right\} < \infty \text{ for some } \rho > 0.$$

Considering $X = C$, $q(x) = |x|$, $p_k = l$, $v_k = 1$ for all $k \in N$, $s = 0$ and $n = 0$, we get the modular space $\ell\{M_k\}$ introduced and studied by Woo [13].

2 Main Results

In this section, we give the theorems that chracterize the structure of the class of sequences $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ and some other spaces which can be derived from this space.

Theorem 1. *Let $p = (p_k)$ be bounded sequence of positive reals, then $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ is a linear space over the field C.*

Proof. Let x, $y \in \ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ and α, $\beta \in C$. Then there exist some $\rho_1 > 0$ and $\rho_2 > 0$ such *that*

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)} x_k}{\rho_1}\right)\right)\right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)} x_k}{\rho_2}\right)\right)\right]^{p_k} < \infty \ .$$

We consider $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since each M_k is non-decreasing and convex, and since q is a seminorm ,

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)}(\alpha x_k+\beta y_k)}{\rho_3}\right)\right)\right]^{p_k}\\
\leq\ &\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)}(\alpha x_k)}{\rho_3}\right)+q\left(\frac{\Delta^n_{(vm)}(\beta y_k)}{\rho_3}\right)\right)\right]^{p_k}\\
\leq\ &\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho_1}\right)+q\left(\frac{\Delta^n_{(vm)}y_k}{\rho_2}\right)\right)\right]^{p_k}\\
\leq\ &D\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho_1}\right)\right)\right]^{p_k}\\
&+D\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)}y_k}{\rho_2}\right)\right)\right]^{p_k}\\
<\ &\infty
\end{aligned}$$

where, $D=\max\left\{1,2^{H-1}\right\}$ and $H=\sup_k p_k$.Hence this completes the proof. ∎

Theorem 2. $\ell\left\{M_k,p,q,s,\Delta^n_{(vm)}\right\}$ *is a paranormed space (need not total paranorm) space with paranorm g, defined as follows .*

$$g(x)=\inf\left\{\rho^{\frac{p_n}{H}}:\sum_{k=1}^{\infty} k^{-s}M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right)\leq 1\ ,\ n=1,2,...\right\},$$

where $H=\sup_k p_k$.

Proof. Clearly $g(x)=g(-x)$. Since $M_k(0)=0$, for all $k\in N$, we get $\inf\left\{\rho^{\frac{p_n}{H}}\right\}=0$ for $x=\theta$. Now let $x,y\in\ell\left\{M_k,p,q,s,\Delta^n_{(vm)}\right\}$ and let us choose $\rho_1>0$ and $\rho_2>0$ such that

$$\sum_{k=1}^{\infty} k^{-s}M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho_1}\right)\right)\leq 1$$

and

$$\sum_{k=1}^{\infty} k^{-s}M_k\left(q\left(\frac{\Delta^n_{(vm)}y_k}{\rho_2}\right)\right)\leq 1$$

Let $\rho=\rho_1+\rho_2$. Then we have

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{-s}M_k\left(q\left(\frac{\Delta^n_{(vm)}(x_k+y_k)}{\rho}\right)\right) \leq\ &\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sum_{k=1}^{\infty} k^{-s}M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho_1}\right)\right)\\
&+\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sum_{k=1}^{\infty} k^{-s}M_k\left(q\left(\frac{\Delta^n_{(vm)}y_k}{\rho_2}\right)\right)\\
\leq\ &1
\end{aligned}$$

Hence $g(x+y) \leq g(x) + g(y)$.

Finally, let λ be a given non-zero scalar , then the continuity of the scalar multiplication follows from the following equality

$$\begin{aligned} g(\lambda x) &= \inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} k^{-s} M_k\left(q\left(\frac{\Delta^n_{(vm)}(\lambda x_k)}{\rho}\right)\right) \leq 1, n = 1, 2, \ldots\right\} \\ &= \inf\left\{(|\lambda| s)^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} k^{-s} M_k\left(q\left(\frac{\Delta^n_{(vm)}(\lambda x_k)}{s}\right)\right) \leq 1, n = 1, 2, \ldots\right\}, \end{aligned}$$

where $s = \frac{\rho}{|s|}$.This completes the proof. ∎

The proof of the following Theorem is easy, so it is omitted .

Theorem 3. *Let $\mathbf{M} = (M_k)$ and $\mathbf{T} = (T_k)$ be sequences of Orlicz functions .For any two sequences $p = (p_k)$ and $t = (t_k)$ of bounded positive real numbers and for any two seminorms q_1 and q_2 we have*

(i) If q_1 is stronger than q_2, then $\ell\left\{M_k, p, q_1, s, \Delta^n_{(vm)}\right\} \subset \ell\left\{M_k, p, q_2, s, \Delta^n_{(vm)}\right\}$,

(i i) $\ell\left\{M_k, p, q_1, s, \Delta^n_{(vm)}\right\} \cap \ell\left\{M_k, p, q_2, s, \Delta^n_{(vm)}\right\} \subset \ell\left\{M_k, p, q_1 + q_2, s, \Delta^n_{(vm)}\right\}$,

(i i i) $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\} \cap \ell\left\{T_k, p, q, s, \Delta^n_{(vm)}\right\} \subset \ell\left\{M_k + T_k, p, q, s, \Delta^n_{(vm)}\right\}$,

(i v) $\ell\left\{M_k, p, q_1, s, \Delta^n_{(vm)}\right\} \cap \ell\left\{M_k, t, q_2, s, \Delta^n_{(vm)}\right\} \neq \phi$,

(v) If $s_1 \leq s_2$, then $\ell\left\{M_k, p, q, s_1, \Delta^n_{(vm)}\right\} \subset \ell\left\{M_k, p, q, s_2, \Delta^n_{(vm)}\right\}$,

(v i) The inclusions $\ell\left\{M_k, p, q, s, \Delta^{n-1}_{(vm)}\right\} \subset \ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ are strict.

In general, $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\} \subset \ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ for $i = 1, 2, 3, ..., n-1$ and the inclusion is strict.

Theorem 4. *Let $\mathbf{M} = (M_k)$ and $\mathbf{T} = (T_k)$ be sequences of Orlicz functions which satisfy Δ_2-condition and $s > 1$, then*

$$\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\} \subseteq \ell\left\{T_k \circ M_k, p, q, s, \Delta^n_{(vm)}\right\}.$$

Proof. Let $x \in \ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $M(u) < \epsilon$ for $0 \leq u \leq \delta$. We write $y_k = M_k\left(q\left(\frac{\Delta^n_{(vm)} x_k}{\rho}\right)\right)$ and consider

$$\sum_{k=1}^{\infty} k^{-s}\left[T_k(y_k)\right]^{p_k} = \sum_{1} k^{-s}\left[T_k(y_k)\right]^{p_k} + \sum_{2} k^{-s}\left[T_k(y_k)\right]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and the second over $y_k > \delta$. Since $s > 1$, we have

$$\sum_{1} k^{-s}\left[T_k(y_k)\right]^{p_k} < \max\left(1, \epsilon^H\right) \sum_{k=1}^{\infty} k^{-s} < \infty.$$

For $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \left(\frac{y_k}{\delta}\right).$$

Since each T_k is non-decreasing and convex , it follows that, for each $k \in N$,

$$T_k\left(y_k\right) < T_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}T_k\left(2\right) + \frac{1}{2}T_k\left(2\frac{y_k}{\delta}\right).$$

Since each T_k is satsfy Δ_2-condition, we have

$$T_k\left(y_k\right) < \frac{1}{2}K\frac{y_k}{\delta}T_k\left(2\right) + \frac{1}{2}K\frac{y_k}{\delta}T_k\left(2\right) = Ky_k\delta^{-1}T_k\left(2\right).$$

Hence

$$\sum_2 k^{-s}\left[T_k\left(y_k\right)\right]^{p_k} \leq \max\left(1, \left(K\delta^{-1}M\left(2\right)\right)^H\right)\sum_{k=1}^{\infty} k^{-s}\left(y_k\right)^{p_k} < \infty.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty} k^{-s}\left[T_k\left(y_k\right)\right]^{p_k} &= \sum_1 k^{-s}\left[T_k\left(y_k\right)\right]^{p_k} + \sum_2 k^{-s}\left[T_k\left(y_k\right)\right]^{p_k} \\ &\leq \max\left(1, \epsilon^H\right)\sum_{k=1}^{\infty} k^{-s} + \max\left(1, \left(K\delta^{-1}M\left(2\right)\right)^H\right)\sum_{k=1}^{\infty} k^{-s}\left(y_k\right)^{p_k} \\ &< \infty .\end{aligned}$$

Hence $x \in \ell\left\{T_k \circ M_k, p, q, s, \Delta^n_{(vm)}\right\}$ .This completes the proof . ∎

Taking $M_k\left(x\right) = x$, for all k in N, in the Theorem 4, we get the next Corollary.

Corollary 5. *Let* $\mathbf{M} = (M_k)$ *be any sequence of Orlicz functions which satisfy* Δ_2*-condition and* $s > 1$*, then*

$$\ell\left\{p, q, s, \Delta^n_{(vm)}\right\} \subseteq \ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}.$$

We will write $f \approx g$ *for non-negative functions* f *and* g *whenever* $C_1 f \leq g \leq C_2 f$ *for some* $C_j > 0, \ \ j = 1, 2.$

Theorem 6. *Let* $\mathbf{M} = (M_k)$ *and* $\mathbf{T} = (T_k)$ *be a sequence of Orlicz functions. If* $M_k \approx T_k$ *for each* $k \in N$*, then* $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\} = \ell\left\{T_k, p, q, s, \Delta^n_{(vm)}\right\}$.

Proof. Proof is obvious. ∎

Theorem 7. *Let* $\mathbf{M} = (M_k)$ *be a sequence of Orlicz functions. If* $\lim_{t\to 0}\frac{M_k(t)}{t} > 0$ *and* $\lim_{t\to 0}\frac{M_k(t)}{t} < \infty$ *for each* $k \in N$*, then*

$$\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\} = \ell\left\{p, q, s, \Delta^n_{(vm)}\right\}$$

Proof. If the given conditions are satisfied, we have $M_k(t) \approx t$ for each k and the proof follows from Theorem 5. ∎

If we take $s = 0$, the sequence space $\ell\left\{M_k, p, q, s, \Delta^n_{(vm)}\right\}$ reduce to the following sequence space:

$$\ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\} = \left\{x \in w(X) : \sum_{k=1}^{\infty}\left[M_k\left(q\left(\frac{\Delta^n_{(vm)} x_k}{\rho}\right)\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\}.$$

Theorem 8. *Let $p = (p_k)$ be bounded sequence of positive reals and (X, q) be a complete seminormed space , then $\ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\}$ is a complete paranormed space paranormed by h , defined by*

$$h(x) = \inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x_k}{\rho}\right)\right) \leq 1, n = 1, 2, \ldots\right\},$$

where $H = \sup_k p_k$.

Proof. Let $\left(x^i\right)$ be a Cauchy sequence in $\ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\}$. Let $\delta > 0$ be fixed and $r > 0$ be such that for a given $0 < \epsilon < 1$, $\frac{\epsilon}{r\delta} > 0$, and $r\delta \geq 1$. Then there exists a positive integer n_0 such that

$$h\left(x^i - x^j\right) < \frac{\epsilon}{r\delta}$$

for all i, $j \geq n_0$

$$h\left(x^i - x^j\right) = \inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k}{\rho}\right)\right) \leq 1\right\} < \frac{\epsilon}{r\delta}$$

for all $i, , j \geq n_0$. Hence we have

$$\sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k}{h(x^i - x^j)}\right)\right) \leq 1$$

for all $i, j \geq n_0$. It follows that

$$M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k}{h\left(x^i - x^j\right)}\right)\right) \leq 1$$

for all $i, j \geq n_0$ and $k \in N$. For $r > 0$ with $M_k\left(\frac{r\delta}{2}\right) \geq 1$, we have

$$M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k}{h\left(x^i - x^j\right)}\right)\right) \leq M_k\left(\frac{r\delta}{2}\right),$$

for all $i, j \geq n_0$ and $k \in N$. Since M_k is non-decreasing for each $k \in N$, we have

$$q\left(\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k\right) \leq \frac{r\delta}{2}.\frac{\epsilon}{r\delta} = \frac{\epsilon}{2}.$$

Hence $\left(\Delta^n_{(vm)} x^i_k\right)$ is a Cauchy sequence in (X, q) for each $k \in N$. But (X, q) is complete and so $\left(\Delta^n_{(vm)} x^i_k\right)$ is convergent in (X, q) for each $k \in N$.

Let $\lim\limits_{i\to\infty} \Delta^n_{(vm)} x^i_k = y_k$ for all $k \geq 1$. Let $k = 1$, then we have

$$\lim_{i\to\infty} \Delta^n_{(vm)} x^i_1 = \lim_{i\to\infty} \sum_{v=0}^{n} (-1)^v \binom{n}{v} v_{1-mv} x^i_{1-mv} = \lim_{i\to\infty} v_1 x^i_1 = y_1 \tag{1}$$

Similary we have ,

$$\lim_{i\to\infty} \Delta^n_{(vm)} x^i_k = \lim_{i\to\infty} v_k x^i_k = y_k \quad \text{, for } k = 1, \ldots, nm \tag{2}$$

Thus from (2.1) and (2.2),we have $\lim\limits_{i\to\infty} x^i_{1+nm}$ exists. Let $\lim\limits_{i\to\infty} x^i_{1+nm} = x_{1+nm}$ .Proceeding in this way inductively, we have $\lim\limits_{i\to\infty} x^i_k = x_k$ for each $k \in N$. Now we have for all $i, j \geq n_0$,

$$\inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k}{\rho}\right)\right) \leq 1\right\} < \epsilon.$$

Then we have

$$\lim_{j\to\infty}\left\{\inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x^j_k}{\rho}\right)\right) \leq 1\right\}\right\} < \epsilon$$

for all $i \geq n_0$. Using the continuity of Orlicz functions , we have

$$\inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} \lim\limits_{j\to\infty} x^j_k}{\rho}\right)\right) \leq 1\right\} < \epsilon$$

for all $i \geq n_0$. This implies

$$\inf\left\{\rho^{\frac{p_n}{H}} : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)} x^i_k - \Delta^n_{(vm)} x_k}{\rho}\right)\right) \leq 1\right\} < \epsilon$$

for all $i \geq n_0$. It follows that $\left(x^i - x\right) \in \ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\}$. Since $\left(x^i\right) \in \ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\}$ and $\ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\}$ is a linear space , so we have $x = x^i - \left(x^i - x\right) \in \ell\left\{M_k, p, q, \Delta^n_{(vm)}\right\}$ .This completes the proof ∎

If we take $s = 0$ and $p_k = l$, the sequence space $\ell \left\{ M_k, p, q, s, \Delta^n_{(vm)} \right\}$ reduce to the following sequence space :

$$\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\} = \left\{ x \in w(X) : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < \infty \text{ , for some } \rho > 0 \right\}.$$

Theorem 9. *Let (X, q) be a complete normed space , then $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is a Banach space normed by $\|.\|$, defined by*

$$\|x\| = \inf \left\{ \rho : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) \leq 1 \right\}.$$

Proof. We prove that $\|.\|$ is a norm on $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. The completeness part can be proved using similar arguments as applied to prove above Theorem .

If $x = \theta$, then it is obvious that $\|x\| = 0$. Conversely assume $\|x\| = 0$. Then using the definition of norm, we have

$$\inf \left\{ \rho : \sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) \leq 1 \right\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some ρ_ϵ $(0 < \rho_\epsilon < \epsilon)$ such that

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\rho_\epsilon} \right) \right) \leq 1.$$

Thus

$$M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\epsilon} \right) \right) \leq M_k \left(q \left(\frac{\Delta^n_{(vm)}}{\rho_\epsilon} \right) \right) \leq 1, \forall k \in N.$$

Suppose that $\Delta^n_{(vm)} x_{n_i} \neq 0$ for some i. Let $\epsilon \to 0$, then $\frac{\left| \Delta^n_{(vm)} x_{n_i} \right|}{\epsilon} \to \infty$. It follows that $M_k \left(\frac{\left| \Delta^n_{(vm)} x_{n_i} \right|}{\epsilon} \right) \to \infty$ as $\epsilon \to 0$ for some $n_i \in N$. This is a contradiction Therefore $\Delta^n_{(vm)} x_k = 0$ for all $k \in N$. Let $k = 1$, then $\Delta^n_{(vm)} x_1 = \sum_{i=0}^{n} (-1)^i \binom{n}{v} v_{1-mi} x_{1-mi} = 0$ and so $v_1 x_1 = 0$, by putting $v_{1-mi} = 0$ and $x_{1-mi} = 0$ for $i = 1, 2, ..., n$.

Hence $x_1 = 0$, since (λ_k) is a sequence of non-zero scalars. Similarly taking $k = 2, ..., mn$, we have $x_2 = \cdots = x_{mn} = 0$. Next let $k = mn + 1$, then $\Delta^n_{(vm)} x_{mn+1} = \sum_{i=0}^{n} (-1)^i \binom{n}{v} v_{1+mn-mi} x_{1+mn-mi} = 0$. Since $x_1 = x_2 = \cdots = x_{mn} = 0$,we must have $v_{mn+1} x_{mn+1} = 0$ and thus $x_{mn+1} = 0$. Proceeding in this way we can conclude that $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$. Again proof of the properties $\|x + y\| \leq \|x\| + \|y\|$ and for any scalar α, $\|\alpha x\| = |\alpha| \|x\|$ are similar to that Theorem 2 .It is easy to see that $\left\| x^i \right\| \to 0$ implies that $x^i_k \to 0$ for each $i \geq 1$. ∎

Proposition 10. $\ell\left\{M_k, q, \Delta^n_{(vm)}\right\}$ *is a* $BK-$*space.*

Now we study the $AK-$characteristic of the space $\ell\left\{M_k, q, s, \Delta^n_{(vm)}\right\}$. Before that we give a new definition and prove some results those will be required.

Definition 1. *For any sequence of Orlicz functions* $\mathbf{M} = (M_k)$,

$$h\left\{M_k, q, \Delta^n_{(vm)}\right\} = \left\{x \in w(X) : \sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right) < \infty \text{ , for every } \rho > 0\right\} .$$

Clearly $h\left\{M_k, q, \Delta^n_{(vm)}\right\}$ is a subspace of $\ell\left\{M_k, q, \Delta^n_{(vm)}\right\}$. The topology of $h\left\{M_k, q, \Delta^n_{(vm)}\right\}$ is the one it inherits from $\|.\|$.

Proposition 11. *Let* $\mathbf{M} = (M_k)$ *be a sequence of Orlicz functions which satisfy* Δ_2-*condition . Then*

$$\ell\left\{M_k, q, \Delta^n_{(vm)}\right\} = h\left\{M_k, q, \Delta^n_{(vm)}\right\}.$$

Proof. It is enough to prove that $\ell\left\{M_k, q, \Delta^n_{(vm)}\right\} \subseteq h\left\{M_k, q, \Delta^n_{(vm)}\right\}$. Let $x \in h\left\{M_k, q, \Delta^n_{(vm)}\right\}$, then for some $\rho > 0$,

$$\sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right) < \infty$$

Therefore

$$M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right) < \infty \text{ for every } k \geq 1 .$$

Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$, then $M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\eta}\right)\right) < \infty$ for every $k \geq 1$ and so

$$\sum_{k=1}^{\infty} M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\eta}\right)\right) < \infty$$

Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$. Since M satisfied the Δ_2-condition , there exists a constant K such that

$$M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\eta}\right)\right) \leq K\left(\frac{\rho}{\eta}\right)^{\log_2 K} M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right)$$

for every $k \geq 1$. Now we can find $U > 0$ with $s > 1$ such that

$$M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right) < Uk^{-s}$$

for the fixed $\rho > 0$ and for every $k \geq 1$ .Then it follows that for every $\eta > 0$, we have

$$\sum_{k=1}^{\infty} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\eta} \right) \right) < K \left(\frac{\rho}{\eta} \right)^{\log_2 K} M \sum_{k=1}^{\infty} k^{-s} < \infty \ .$$

This completes the proof . ∎

Proposition 12. *Let* (X, q) *be a complete normed space , then* $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ *is an* $AK-$*space.*

Proof. Let $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. Then for each ϵ, $0 < \epsilon < 1$, we can find an s_o such that

$$\sum_{k \geq s_0} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\epsilon} \right) \right) \leq 1.$$

Hence for $s \geq s_0$,

$$\begin{aligned} \left\| x - x^{[s]} \right\| &= \inf \left\{ \rho > 0 : \sum_{k \geq s+1} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\epsilon} \right) \right) \leq 1 \right\} \\ &\leq \inf \left\{ \rho > 0 : \sum_{k \geq s} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) \leq 1 \right\} < \epsilon. \end{aligned}$$

Thus we can conclude that $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ is an AK space. ∎

Combining Proposition 1 and Proposition 2, we have the following Theorem .

Theorem 13. *Let* $\mathbf{M} = (M_k)$ *be a sequence of Orlicz functions which satisfy* Δ_2-*condition, then* $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ *is an* $AK-$*space.*

Proposition 14. $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ *is a closed subspace of* $\ell \left\{ M_k, q, \Delta^n_{(vm)} \right\}$.

Proof. Let $\{x^s\}$ *be a sequence in* $h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$ *such that* $\|x^s - x\| \to 0$, *where* $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$. *To complete the proof we need to show that* $x \in h \left\{ M_k, q, \Delta^n_{(vm)} \right\}$, *i.e. ,*

$$\sum_{k \geq 1} M_k \left(q \left(\frac{\Delta^n_{(vm)} x_k}{\rho} \right) \right) < \infty \quad \textit{for every } \rho > 0.$$

For $\rho > 0$, *there corresponds an* l *such that* $\left\| x^l - x \right\| \leq \frac{\rho}{2}$. *Then using convexity of* *each* M_k ,

$$\begin{aligned}
\sum_{k\geq 1} M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right) &= \sum_{k\geq 1} M_k\left(q\left(\frac{2\left|\Delta^n_{(vm)}x^l_k\right| - 2\left(\left|\Delta^n_{(vm)}x^l_k\right| - \left|\Delta^n_{(vm)}x_k\right|\right)}{2\rho}\right)\right)\\
&\leq \frac{1}{2}\sum_{k\geq 1} M_k\left(q\left(\frac{2\left|\Delta^n_{(vm)}x^l_k\right|}{\rho}\right)\right)\\
&\quad +\frac{1}{2}\sum_{k\geq 1} M_k\left(q\left(\frac{2\left|\Delta^n_{(vm)}\left(x^l_k - x_k\right)\right|}{\rho}\right)\right)\\
&\leq \frac{1}{2}\sum_{k\geq 1} M_k\left(q\left(\frac{2\left|\Delta^n_{(vm)}x^l_k\right|}{\rho}\right)\right)\\
&\quad +\frac{1}{2}\sum_{k\geq 1} M_k\left(q\left(\frac{2\left|\Delta^n_{(vm)}\left(x^l_i - x_i\right)\right|}{\|x^l - x\|}\right)\right)
\end{aligned}$$

Now from Theorem 8, using the definition of norm $\|.\|$, *we have*

$$\sum_{k\geq 1} M_k\left(q\left(\frac{2\left|\Delta^n_{(vm)}\left(x^l_i - x_i\right)\right|}{\|x^l - x\|}\right)\right) \leq 1$$

It follows that

$$\sum_{k\geq 1} M_k\left(q\left(\frac{\Delta^n_{(vm)}x_k}{\rho}\right)\right) < \infty \ \ \textit{for every } \rho > 0$$

Thus $x \in h\left\{M_k, q, \Delta^n_{(vm)}\right\}$ ∎

Hence we have the following Corollary

Corollary 15. $h\left\{M_k, q, \Delta^n_{(vm)}\right\}$ *is a BK- space .*

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