Nonlinear Analysis: Modelling and Control, 2014, Vol. 19, No. 3, 335–349 335
http://dx.doi.org/10.15388/NA.2014.3.2

On source identification problem for a delay parabolic equation*

Allaberen Ashyralyev[a,b], **Deniz Agirseven**[c]

[a]Department of Mathematics, Fatih University, Istanbul, Turkey
aashyr@fatih.edu.tr

[b]Department of Applied Mathematics, ITTU, Ashgabat, Turkmenistan

[c]Department of Mathematics, Trakya University, Edirne, Turkey
denizagirseven@trakya.edu.tr

Received: 5 January 2014 / **Revised:** 26 March 2014 / **Published online:** 30 June 2014

Abstract. In the present study, the inverse problem of a delay parabolic equation with nonlocal conditions is investigated. The stability estimates in Hölder norms for the solution of this problem are established.

Keywords: delay parabolic equation, Banach spaces, positive operators, stability estimates, nonlocal conditions.

1 Introduction

Delay parabolic equations (DPEs) have important applications in a wide range of applications such as physics, chemistry, biology and ecology and other fields. For example, diffusion problems where the current state depends upon an earlier one give rise to parabolic equations with delay. In mathematical modeling, DPEs are used together with boundary conditions specifying the solution on the boundary of the domain. Dirichlet and Neumann conditions are examples of classical boundary conditions (see [1] and the references given therein). In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions usually are identified as nonlocal boundary conditions and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. The well-posedness of various nonlocal boundary value problems for partial differential and difference equations has been studied extensively by many researchers (see [1–12] and the references given therein).

*This research was supported by Trakya University Scientific Research Projects Unit (project No. 2013-143).

Identification problems take an important place in applied sciences and engineering applications and have been studied by many authors (see [13–20] and the references given therein). Solving the direct problem permits the computation of various system outputs of physical interest. On the other hand, when some of the required inputs are not available we may instead be able to determine the missing inputs from outputs that are measured rather than computed by formulating and solving an appropriate inverse problem. In particular, when the missing input is unknown source term in the partial differential equation, the problem is called a source identification problem. The theory and applications of source identification problems for partial differential equations were given in various papers (see [21–23] and the references given therein). The well-posedness of the unknown source identification problem for a parabolic equation has been well-investigated when the unknown function p is dependent on space variable (see [24–29] and the references given therein). Nevertheless when the unknown function p is dependent on t the well-posedness of the source identification problem for a parabolic equation was investigated in [30–34].

The initial-boundary value problems for delay partial differential equations when the delay term is an operator of lower order with respect to other operator term were widely investigated (see [35–38] and the references given therein). In the case where the delay term is an operator of the same order with respect to other operator term is studied mainly if H is a Hilbert space (see, for example, [39] and the references given therein). In fact, there are very few papers which allow E to be a general Banach space (see [40–44]) and in these works, authors look only for partial differential equations under regular data. Moreover, approximate solutions of the delay parabolic equations in the case where the delay term is a simple operator of the same order with respect to other operator term were studied recently in papers [45–49]. However, the well-posedness of the source identification problem for a delay parabolic equation is not well-investigated (see [50]). In this paper, we investigate the source identification problem for a delay parabolic equation with nonlocal conditions

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) - b\left[a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2} - \sigma u(t-\omega,x)\right] \\
&\quad + p(t)q(x) + f(t,x), \quad 0 < x < l,\ 0 < t < \infty, \\
u(t,0) &= u(t,l), \quad u_x(t,0) = u_x(t,l), \quad 0 \leqslant t < \infty, \\
u(t,x) &= \varphi(t,x), \quad 0 \leqslant x \leqslant l,\ -\omega \leqslant t \leqslant 0, \\
u(t,x^*) &= \rho(t), \quad 0 \leqslant x^* \leqslant l,\ 0 \leqslant t < \infty,
\end{aligned} \tag{1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $\rho(t)$, $\varphi(t,x)$, $a(x)$, and $f(t,x)$ are sufficiently smooth functions, $a(x) \geqslant \delta > 0$, $b \in R^1$ and $\sigma > 0$ is a sufficiently large number with assuming that:

(a) $q(x)$ is a sufficiently smooth function,
(b) $q(x)$ and $q'(x)$ are periodic with length l,
(c) $q(x^*) \neq 0$.

In the present study, the source identification problem (1) for a delay parabolic equation with nonlocal conditions is investigated. The stability estimates in Hölder norms for the solution of this problem are established.

2 Preliminaries. Main results

To formulate our results, we introduce the Banach space $\mathring{C}^{\alpha}[0,l]$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $[0,l]$ with $\phi(0)=\phi(l)$ satisfying a Hölder condition for which the following norm is finite:

$$\|\phi\|_{\mathring{C}^{\alpha}[0,l]} = \max_{0\leqslant x\leqslant l}\bigl|\phi(x)\bigr| + \sup_{0\leqslant x<x+h\leqslant l}\frac{|\phi(x+h)-\phi(x)|}{h^{\alpha}}.$$

With the help of the positive operator A we introduce the fractional spaces $E_\alpha = E_\alpha(E,A)$, $0<\alpha<1$, consisting of all $v\in E$ for which the following norm is finite [51]:

$$\|v\|_{E_\alpha} = \|v\|_E + \sup_{\lambda>0}\lambda^{1-\alpha}\bigl\|A\mathrm{e}^{-\lambda A}v\bigr\|_E. \tag{2}$$

In the present paper, $C([-\omega,0],E)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[-\omega,0]$ with values in E equipped with the norm

$$\|\varphi\|_{C([-\omega,0],E)} = \max_{-\omega\leqslant t\leqslant 0}\bigl\|\varphi(t)\bigr\|_E$$

and $L_1([0,\infty),E)$ stands for the Banach space of all strongly measurable E-valued functions $v(t)$ defined on $[0,\infty)$ for which the following norm is finite:

$$\|v\|_{L_1([0,\infty),E)} = \int_0^\infty \bigl\|v(t)\bigr\|_E\,\mathrm{d}t.$$

Finally, we introduce a differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \sigma u \tag{3}$$

with the domain $D(A^x) = \{u\in C^{(2)}[0,l]\colon\ u(0)=u(l),\ u'(0)=u'(l)\}$.

It is well known that $A = A^x$ is the strongly positive operator in $C[0,l]$ of all continuous functions $\phi(x)$ defined on $[0,l]$ with norm

$$\|\phi\|_{C[0,l]} = \max_{0\leqslant x\leqslant l}\bigl|\phi(x)\bigr|$$

and, for this operator, the following estimates hold:

$$\bigl\|\mathrm{e}^{-tA}\bigr\|_{C[0,l]\to C[0,l]} \leqslant M\mathrm{e}^{-\delta t}, \quad t\geqslant 0, \tag{4}$$

$$\bigl\|A^{\alpha}\mathrm{e}^{-tA}\bigr\|_{C[0,l]\to C[0,l]} \leqslant Mt^{-\alpha}, \quad t>0, \tag{5}$$

where $\delta, M>0$ [43].

Positive constants have different values in time and they will be indicated with M. On the other hand, $M(\alpha, \beta, \ldots)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \ldots$.

Moreover, we have the following theorem on the structure of the fractional space $E_\alpha = E_\alpha(C[0,l], A^x)$.

Theorem 1. *For $\alpha \in (0,1/2)$, the norms of the space $E_\alpha(C[0,l], A^x)$ and the Hölder space $\mathring{C}^\alpha[0,l]$ are equivalent* [43, 51].

The main result of present paper is the following theorem on stability of (1) in spaces $C([0,\infty), \mathring{C}^{2\alpha}[0,l])$, $\alpha \in (0,1/2)$.

Theorem 2. *Assume that*

$$|b| \leqslant \frac{1-\alpha}{M 2^{2-\alpha}}. \tag{6}$$

Let $\varphi(t,x), \varphi_{xx}(t,x) \in C([-\omega,0], \mathring{C}^{2\alpha}[0,l])$, $\varphi_t(t,x) \in C([-\omega,0], \mathring{C}^{2\alpha+2}[0,l])$, $f_t(t,x) \in L_1([0,\infty), \mathring{C}^{2\alpha}[0,l])$, $f(0,x) \in \mathring{C}^{2\alpha}[0,l]$ and $\rho'(t) \in L_1[0,\infty)$. Then, for the solution of problem (1)*, the following stability estimates hold for all $t \geqslant 0$*:

$$\begin{aligned}
&\big\|u_t(t)\big\|_{\mathring{C}^{2\alpha}[0,l]} + \big\|u(t)\big\|_{\mathring{C}^{2\alpha+2}[0,l]} + \big|p(t)\big| \\
&\quad \leqslant M(a,\delta,\sigma,\alpha,x^*,q,l) \mathrm{e}^{M(\alpha,x^*,q,l)t} \big[\|\varphi\|_{C([-\omega,0],\mathring{C}^{2\alpha+2}[0,l])} + \big\|f(0)\big\|_{\mathring{C}^{2\alpha}[0,l]} \\
&\qquad + \|\varphi'\|_{C([-\omega,0],\mathring{C}^{2\alpha}[0,l])} + \|f'\|_{L_1([0,\infty),\mathring{C}^{2\alpha}[0,l])} + \|\rho'\|_{L_1[0,\infty)}\big], \\
&M(\alpha,x^*,q,l) = \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}, \quad 0 < \alpha < \frac{1}{2}.
\end{aligned}$$

Here $L_1[0,\infty)$ stands for the Banach space of all strongly measurable functions $v(t)$ defined on $[0,\infty)$ for which the following norm is finite:

$$\|v\|_{L_1[0,\infty)} = \int_0^\infty \big|v(t)\big| \,\mathrm{d}t.$$

Proof. Let us seek the substitution for the solution of the inverse problem in the following form:

$$u(t,x) = \eta(t) q(x) + w(t,x), \tag{7}$$

where

$$\eta(t) = \int_0^t p(s) \,\mathrm{d}s.$$

Taking derivatives from (7), we get

$$\frac{\partial u(t,x)}{\partial t} = p(t) q(x) + \frac{\partial w(t,x)}{\partial t}$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \eta(t)\frac{\mathrm{d}^2 q(x)}{\mathrm{d}x^2} + \frac{\partial^2 w(t,x)}{\partial x^2}.$$

Moreover, if we put $x = x^*$ in equation (1), we obtain

$$u(t,x^*) = \eta(t)q(x^*) + w(t,x^*) = \rho(t)$$

and

$$\eta(t) = \frac{\rho(t) - w(t,x^*)}{q(x^*)}. \tag{8}$$

Taking derivative of both sides of (8) with respect to t, we achieve

$$p(t) = \frac{\rho'(t) - w_t(t,x^*)}{q(x^*)}. \tag{9}$$

Using the triangle inequality and the identity (9), we obtain

$$\begin{aligned}|p(t)| &\leqslant \frac{1}{|q(x^*)|}\big[\big|\rho'(t)\big| + \big|w_t(t,x^*)\big|\big] \leqslant \frac{1}{|q(x^*)|}\Big(\big|\rho'(t)\big| + \max_{0\leqslant x\leqslant l}\big|w_t(t,x)\big|\Big)\\ &\leqslant \frac{1}{|q(x^*)|}\big(\big|\rho'(t)\big| + \big\|w_t(t)\big\|_{\mathring{C}^{2\alpha}[0,l]}\big)\end{aligned} \tag{10}$$

for any t, $t \in [0,\infty)$. Using equations (1), (7), (8) and under the same assumptions on $q(x)$, one can show that $w(t,x)$ is the solution of the following problem:

$$\begin{aligned}
\frac{\partial w(t,x)}{\partial t} &= a(x)\frac{\partial^2 w(t,x)}{\partial x^2} - \sigma w(t,x) - b\bigg[a(x)\frac{\partial^2 \varphi(t-\omega,x)}{\partial x^2} - \sigma\varphi(t-\omega,x)\bigg]\\
&\quad + \eta(t)\bigg[a(x)\frac{\partial^2 q(x)}{\partial x^2} - \sigma q(x)\bigg] + f(t,x), \quad 0<x<l,\ 0<t<\omega,\\
\frac{\partial w(t,x)}{\partial t} &= a(x)\frac{\partial^2 w(t,x)}{\partial x^2} - \sigma w(t,x) - b\bigg[a(x)\frac{\partial^2 w(t-\omega,x)}{\partial x^2} - \sigma w(t-\omega,x)\bigg]\\
&\quad + \big[\eta(t) - b\eta(t-\omega)\big]\bigg[a(x)\frac{\partial^2 q(x)}{\partial x^2} - \sigma q(x)\bigg] + f(t,x),\\
&\quad 0<x<l,\ \omega<t<\infty,\\
&w(t,0) = w(t,l), \quad w_x(t,0) = w_x(t,l), \quad 0 \leqslant t < \infty,\\
&w(0,x) = \varphi(0,x), \quad 0 \leqslant x \leqslant l.
\end{aligned} \tag{11}$$

So, the end of proof of Theorem is based on estimate (10) and the following theorem. □

Theorem 3. *For the solution of problem* (11)*, the following stability estimate holds for any* $t, t \geqslant 0$:

$$\begin{aligned}\|w_t\| &\leqslant M(a,\delta,\sigma,\alpha,x^*,q,l)\mathrm{e}^{M(\alpha,x^*,q,l)t}\big[\|\varphi\|_{C([-\omega,0],\mathring{C}^{2\alpha+2}[0,l])} + \big\|f(0)\big\|_{\mathring{C}^{2\alpha}[0,l]}\\ &\quad + \|\varphi'\|_{C([-\omega,0],\mathring{C}^{2\alpha}[0,l])} + \|f'\|_{L_1([0,\infty),\mathring{C}^{2\alpha}[0,l])} + \|\rho'\|_{L_1[0,\infty)}\big], \quad 0<\alpha<\frac{1}{2}.\end{aligned}$$

Proof. We can rewrite the problem (11) in the following abstract form:

$$\begin{aligned}
&w_t(t) + Aw(t) = B\varphi(t-\omega) + (aq'' - \sigma q)\eta(t) + f(t), \quad 0 < t < \omega,\\
&w_t(t) + Aw(t) = Bw(t-\omega) + (aq'' - \sigma q)\big[\eta(t) - b\eta(t-\omega)\big] + f(t), \quad \omega < t < \infty,\\
&w(0) = \varphi(0)
\end{aligned}$$

in a Banach space $E = C[0,l]$ with the positive operator $A = A^x$ defined by formula (3) and the unbounded operator $B = bA^x$. Here $f(t) = f(t,x)$ and $w(t) = w(t,x)$ are, respectively, known and unknown abstract functions defined on $(0,\infty)$ with values in $E = C[0,l]$, $w(t,x^*)$ is unknown scalar function defined on $(0,\infty)$, $q = q(x)$, $q'' = q''(x)$, $\varphi = \varphi(x)$ and $a = a(x)$ are elements of $E = C[0,l]$ and $q(x^*)$ is a real number. Finally, we can rewrite condition (6) in the following form:

$$\big\|BA^{-1}\big\|_{C[0,l]\to C[0,l]} \leqslant \frac{1-\alpha}{M2^{2-\alpha}} \tag{12}$$

for any t, $t \in [0,\infty)$. Let us $0 \leqslant t \leqslant \omega$. Then, using the Cauchy formula, we establish

$$\begin{aligned}
w(t) = {}& \mathrm{e}^{-tA}\varphi(0) + \int_0^t \mathrm{e}^{-(t-s)A} B\varphi(s-\omega)\,\mathrm{d}s\\
&+ \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s.
\end{aligned}$$

Taking the derivative of both sides, we obtain that

$$\begin{aligned}
w_t(t) = {}& -A\mathrm{e}^{-tA}\varphi(0) - \int_0^t A\mathrm{e}^{-(t-s)A} B\varphi(s-\omega)\,\mathrm{d}s\\
&- \int_0^t A\mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,\mathrm{d}s - \int_0^t A\mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s\\
&+ B\varphi(t-\omega) + (aq'' - \sigma q)\eta(t) + f(t).
\end{aligned}$$

Applying formulas

$$\begin{aligned}
&- \int_0^t A\mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,\mathrm{d}s\\
&\quad = -(aq'' - \sigma q)\eta(t) + \mathrm{e}^{-tA}(aq'' - \sigma q)\eta(0) + \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta'(s)\,\mathrm{d}s\\
&\quad = -(aq'' - \sigma q)\eta(t) + \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)p(s)\,\mathrm{d}s - \int_0^t A\mathrm{e}^{-(t-s)A} B\varphi(s-\omega)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= -B\varphi(t-\omega) + \mathrm{e}^{-tA}B\varphi(-\omega) + \int_0^t \mathrm{e}^{-(t-s)A}B\varphi'(s-\omega)\,\mathrm{d}s \\
&\quad - \int_0^t A\mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s \\
&= -f(t) + \mathrm{e}^{-tA} f(0) + \int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s,
\end{aligned}
$$

we get

$$
\begin{aligned}
w_t(t) = \mathrm{e}^{-tA} w_t(0) &+ \int_0^t \mathrm{e}^{-(t-s)A}B\varphi'(s-\omega)\,\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)p(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} f'(s)\,\mathrm{d}s.
\end{aligned}
$$

Here

$$
w_t(0) = -A\varphi(0) + B\varphi(-\omega) + f(0).
$$

Applying this formula and the semigroup property, the condition (12) and the estimates (4), (5), we obtain

$$
\begin{aligned}
&\lambda^{1-\alpha}\big\|A\mathrm{e}^{-\lambda A} w_t(t)\big\|_E \\
&\quad \leqslant \lambda^{1-\alpha}\big\|A\mathrm{e}^{-(\lambda+t)A} w_t(0)\big\|_E + \lambda^{1-\alpha}\int_0^t \big\|A\mathrm{e}^{-((\lambda+t-s)/2)A}\big\|_{E\to E}\big\|BA^{-1}\big\|_{E\to E} \\
&\qquad \times \big\|A\mathrm{e}^{-((\lambda+t-s)/2)A}\varphi'(s-\omega)\big\|_E\,\mathrm{d}s \\
&\qquad + \lambda^{1-\alpha}\int_0^t \big\|A\mathrm{e}^{-(\lambda+t-s)A}Aq\big\|_E\big|p(s)\big|\,\mathrm{d}s + \lambda^{1-\alpha}\int_0^t \big\|A\mathrm{e}^{-(\lambda+t-s)A} f'(s)\big\|_E\,\mathrm{d}s \\
&\quad \leqslant \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\big\|w_t(0)\big\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}}\int_0^t \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}\,\mathrm{d}s \max_{0\leqslant t\leqslant\omega}\big\|\varphi'(t-\omega)\big\|_{E_\alpha} \\
&\qquad + \int_0^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\frac{1}{|q(x^*)|}\big(\big|\rho'(s)\big| + \big\|w_s(s)\big\|_{\mathring{C}^{2\alpha}[0,l]}\big)\,\mathrm{d}s\|Aq\|_{E_\alpha} \\
&\qquad + \int_0^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\big\|f'(s)\big\|_{E_\alpha}\,\mathrm{d}s
\end{aligned}
$$

$$\leqslant \left\|w_t(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha} + \int_0^t \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left|\rho'(s)\right| \mathrm{d}s$$
$$+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]} \mathrm{d}s$$
$$\leqslant \left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^t \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left|\rho'(s)v\right| \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]} \mathrm{d}s$$

for every t, $0 \leqslant t \leqslant \omega$, and λ, $\lambda > 0$. This shows that

$$\left\|w_t(t)\right\|_{E_\alpha} \leqslant \left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^t \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left|\rho'(s)\right| \mathrm{d}s$$
$$+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]} \mathrm{d}s \tag{13}$$

for every t, $0 \leqslant t \leqslant \omega$. Applying the integral inequality, we obtain

$$\left\|w_t(t)\right\|_{E_\alpha} \leqslant \Bigg[\left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^\omega \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\Bigg]\int_0^\omega \left|\rho'(s)\right| \mathrm{d}s\, \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t} \tag{14}$$

for every t, $0 \leqslant t \leqslant \omega$. Now we consider the case $\omega \leqslant t < \infty$. Applying the mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality

$$\left\|w_t(t)\right\|_{E_\alpha} \leqslant \Bigg[\left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^{n\omega} \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^{n\omega} \left|\rho'(s)\right| \mathrm{d}s\Bigg]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t} \tag{15}$$

is true for t, $(n-1)\omega \leqslant t \leqslant n\omega$, $n = 1, 2, 3, \ldots$, for some n. Using the Cauchy formula, we establish

$$w(t) = e^{-(t-n\omega)A}w(n\omega) + \int_{n\omega}^{t} e^{-(t-s)A}Bw(s-\omega)\,ds + \int_{n\omega}^{t} e^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,ds + \int_{n\omega}^{t} e^{-(t-s)A}f(s)\,ds. \tag{16}$$

Taking the derivative of both sides with respect to t, we obtain

$$w_t(t) = -Ae^{-(t-n\omega)A}w(n\omega) - \int_{n\omega}^{t} Ae^{-(t-s)A}Bw(s-\omega)\,ds - \int_{n\omega}^{t} Ae^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,ds - A\int_{n\omega}^{t} e^{-(t-s)A}f(s)\,ds + Bw(t-\omega) + (aq'' - \sigma q)\eta(t) + f(t).$$

Applying formulas

$$\begin{aligned} -\int_{n\omega}^{t} Ae^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,ds &= -(aq'' - \sigma q)\eta(t) + e^{-(t-n\omega)A}(aq'' - \sigma q)\eta(n\omega) + \int_{n\omega}^{t} e^{-(t-s)A}(aq'' - \sigma q)\eta'(s)\,ds \\ &= -(aq'' - \sigma q)\eta(t) + e^{-(t-n\omega)A}(aq'' - \sigma q)\eta(n\omega) + \int_{n\omega}^{t} e^{-(t-s)A}(aq'' - \sigma q)p(s)\,ds, \end{aligned}$$

$$-\int_{n\omega}^{t} Ae^{-(t-s)A}Bw(s-\omega)\,ds = -Bw(t-n\omega) + e^{-(t-n\omega)A}Bw(n\omega-\omega) + \int_{n\omega}^{t} e^{-(t-s)A}Bw'(s-\omega)\,ds$$

$$-\int_{n\omega}^{t} Ae^{-(t-s)}f(s)\,ds = -f(t) + e^{-(t-n\omega)A}f(n\omega) + \int_{n\omega}^{t} e^{-(t-s)A}f'(s)\,ds,$$

we obtain

$$w_t(t) = \mathrm{e}^{-(t-n\omega)A} w_t(n\omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)} B w'(s-\omega)\,\mathrm{d}s + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)} (aq'' - \sigma q) p(s)\,\mathrm{d}s + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} f'(s)\,\mathrm{d}s.$$

Using this formula and the semigroup property, the condition (12) and the estimates (4), (5), we obtain

$$\begin{aligned}
&\lambda^{1-\alpha} \|A\mathrm{e}^{-\lambda A} w_t(t)\|_E \\
&\quad\leqslant \lambda^{1-\alpha} \|A\mathrm{e}^{-(\lambda+t-n\omega)A} w_t(n\omega)\|_E + \lambda^{1-\alpha} \int_{n\omega}^{t} \|A\mathrm{e}^{-((\lambda+t-s)/2)A}\|_{E\to E} \|BA^{-1}\|_{E\to E} \\
&\qquad\times \|A\mathrm{e}^{-((\lambda+t-s)/2)A} w'(s-\omega)\|_E \,\mathrm{d}s \\
&\qquad+ \lambda^{1-\alpha} \int_{n\omega}^{t} \|A\mathrm{e}^{-(\lambda+t-s)A} Aq\|_E |p(s)|\,\mathrm{d}s + \lambda^{1-\alpha} \int_{n\omega}^{t} \|A\mathrm{e}^{-(\lambda+t-s)A} f(s)\|_E \,\mathrm{d}s \\
&\quad\leqslant \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \|w_t(n\omega)\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}} \int_{n\omega}^{t} \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}\,\mathrm{d}s \max_{n\omega\leqslant s\leqslant t} \|w'(s-\omega)\|_{E_\alpha} \\
&\qquad+ \int_{n\omega}^{t} \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}} \frac{1}{|q(x^*)|} \bigl(|\rho'(s)| + \|w_s(s)\|_{\mathring{C}^{2\alpha}[0,l]}\bigr)\,\mathrm{d}s \|Aq\|_{E_\alpha} \\
&\qquad+ \int_{n\omega}^{t} \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s \\
&\quad\leqslant \max_{n\omega-\omega\leqslant t\leqslant n\omega} \|w_t(t)\|_{E_\alpha} + \int_{n\omega}^{t} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} |\rho'(s)|\,\mathrm{d}s \\
&\qquad+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} \|w_s(s)\|_{\mathring{C}^{2\alpha}[0,l]}\,\mathrm{d}s
\end{aligned}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$, and λ, $\lambda > 0$. This shows that

$$\begin{aligned}
\|w_t(t)\|_{E_\alpha} &\leqslant \max_{n\omega-\omega\leqslant t\leqslant n\omega} \|w_t(t)\|_{E_\alpha} + \int_{n\omega}^{t} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s \\
&\quad+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} |\rho'(s)|\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} \|w_s(s)\|_{\mathring{C}^{2\alpha}[0,l]}\,\mathrm{d}s \qquad (17)
\end{aligned}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$. Applying the integral inequality, we obtain

$$\begin{aligned}\|w_t(t)\|_{E_\alpha} \leqslant \Biggl[&\max_{n\omega-\omega\leqslant t\leqslant n\omega}\|w'(t)\|_{E_\alpha} + \int\limits_{n\omega}^{n\omega+\omega}\|f'(s)\|_{E_\alpha}\,\mathrm{d}s \\ &+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{n\omega}^{n\omega+\omega}|\rho'(s)|\,\mathrm{d}s\Biggr]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)}\end{aligned} \tag{18}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$. Applying estimates (15) and (18), we get

$$\begin{aligned}&\|w_t(t)\|_{E_\alpha} \\ &\quad\leqslant \max_{n\omega-\omega\leqslant t\leqslant n\omega}\|w'(t)\|_{E_\alpha}\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \\ &\qquad+\Biggl[\int\limits_{n\omega}^{n\omega+\omega}\|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{n\omega}^{n\omega+\omega}|\rho'(s)|\,\mathrm{d}s\Biggr]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \\ &\quad\leqslant \Biggl[(1+|b|)\max_{-\omega\leqslant t\leqslant 0}\|A\varphi(t)\|_{E_\alpha} + \|f(0)\|_{E_\alpha}\max_{-\omega\leqslant t\leqslant 0}\|\varphi'(t)\|_{E_\alpha} \\ &\qquad+\int\limits_{0}^{n\omega}\|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{0}^{n\omega}|\rho'(s)|\,\mathrm{d}s\Biggr]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t} \\ &\qquad+\Biggl[\int\limits_{n\omega}^{n\omega+\omega}\|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{n\omega}^{n\omega+\omega}|\rho'(s)|\,\mathrm{d}s\Biggr]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \\ &\quad\leqslant \Biggl[(1+|b|)\max_{-\omega\leqslant t\leqslant 0}\|A\varphi(t)\|_{E_\alpha} + \|f(0)\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\|\varphi'(t)\|_{E_\alpha} \\ &\qquad+\int\limits_{0}^{(n+1)\omega}\|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{0}^{(n+1)\omega}|\rho'(s)|\,\mathrm{d}s\Biggr]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t}\end{aligned}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$. This result and Theorem 1 completes the proof of Theorem 2. □

3 Conclusion

In the present study, the source identification problem (1) for a delay parabolic equation with nonlocal conditions is investigated. The stability estimates in Hölder norms for the solution of this problem are established. Moreover, applying the result of the monograph [43], the high order of accuracy single-step difference schemes for the numerical solution of the source identification problem (1) for a delay parabolic equation with nonlocal conditions can be presented. Of course, the stability estimates for the solution of these difference schemes have been established without any assumptions about the grid steps.

Acknowledgment. The authors would like to thank Prof. Pavel Sobolevskii (Jerusalem, Israel) and referees for their helpful suggestions to the improvement of our paper.

References

1. M. Sapagovas, K. Jakubelienė, Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition, *Nonlinear Anal. Model. Control*, **17**(1):91–98, 2012.
2. A. Ashyralyev, I. Karatay, P.E. Sobolevskii, On well-posedness of the nonlocal boundary value problem for parabolic difference equations, *Discrete Dyn. Nat. Soc.*, **2**:273–286, 2004, doi:10.1155/S1026022604403033.
3. F. Zouyed, F. Rebbani, N. Boussetila, On a class of multitime evolution equations with nonlocal initial conditions, *Abstr. Appl. Anal.*, **2007**, Article ID 16938, 26 pp., 2007, doi:10.1155/2007/16938.
4. A. Boucherif, R. Precup, Semilinear evolution equations with nonlocal initial conditions, *Dyn. Syst. Appl.*, **16**(3):507–516, 2007.
5. J. Jachimavičienė, M. Sapagovas, A. Štikonas, O. Štikonienė, On the stability of explicit finite difference schemes for a pseudoparabolic equation with nonlocal conditions, *Nonlinear Anal. Model. Control*, **19**(2):225–240, 2014.
6. R. Čiupaila, M. Sapagovas, O. Štikonienė, Numerical solution of nonlinear elliptic equation with nonlocal condition, *Nonlinear Anal. Model. Control*, **18**(4):412–426, 2013.
7. Y. Wang, S. Zheng, The existence and behavior of solutions for nonlocal boundary problems, *Bound. Value Probl.*, **2009**, Article ID 484879, 17 pp., 2009, doi:10.1155/2009/484879.
8. A. Ashyralyev, Nonlocal boundary-value problems for abstract parabolic equations: well-posedness in Bochner spaces, *J. Evol. Equ.*, **6**(1):1–28, 2006, doi:10.1007/s00028-005-0194-y.
9. F. Ivanauskas, T. Meskauskas, M. Sapagovas, Stability of difference schemes for two-dimensional parabolic equations with non-local boundary conditions, *Appl. Math. Comput.*, **215**(7):2716–2732, 2009.
10. F.F. Ivanauskas, Yu.A. Novitski, M.P. Sapagovas, On the stability of an explicit difference scheme for hyperbolic equations with nonlocal boundary conditions, *Differ. Equ.*, **49**(7):849–856, 2013.
11. S. Sajavičius, Stability of the weighted splitting finite-difference scheme for a two-dimensional parabolic equation with two nonlocal integral conditions, *Comput. Math. Appl.*, **64**(11):3485–3499, 2012.
12. A. Ashyralyev, A note on the Bitsadze–Samarskii type nonlocal boundary value problem in a Banach space, *J. Math. Anal. Appl.*, **344**(1):557–573, 2008, doi:10.1016/j.jmaa.2008.03.008.

13. A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York, 1987.

14. V. Isakov, *Inverse Problems for Partial Differential Equations*, Appl. Math. Sci., Vol. 127, Springer-Verlag, New York, 1998.

15. Yu.Ya. Belov, *Inverse Problems for Partial Differential Equations*, Inverse Ill-Posed Probl. Ser., Vol. 32, VSP, 2002.

16. T. Kimura, T. Suzuki, A parabolic inverse problem arising in a mathematical model for chromatography, *SIAM J. Appl. Math.*, **53**(6):1747–1761, 1993.

17. J.R. Cannon, Y.L. Lin, S. Xu, Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic differential equations, *Inverse Probl.*, **10**:227–243, 1994.

18. A. Hasanov, Identification of unknown diffusion and convection coefficients in ion transport problems from flux data: an analytical approach, *J. Math. Chem.*, **48**(2):413–423, 2010.

19. V. Serov, L. Päivärinta, Inverse scattering problem for two-dimensional Schrödinger operator, *J. Inverse Ill-Posed Probl.*, **14**(3):295–305, 2006.

20. Y.A. Gryazin, M.V. Klibanov, T.R. Lucas, Imaging the diffusion coefficient in a parabolic inverse problem in optical tomography, *Inverse Probl.*, **25**:373–397, 1999.

21. D. Orlovsky, S. Piskarev, The approximation of Bitzadze–Samarsky type inverse problem for elliptic equations with Neumann conditions, *Contemp. Anal. Appl. Math.*, **1**(2):91–97, 2013.

22. Ch. Ashyralyyev, M. Dedeturk, A finite difference method for the inverse elliptic problem with the Dirichlet condition, *Contemp. Anal. Appl. Math.*, **1**(2):132–155, 2013.

23. A. Ashyralyev, M. Urun, Determination of a control parameter for the Schrödinger equation, *Contemp. Anal. Appl. Math.*, **1**(2):156–166, 2013.

24. A.B. Kostin, The inverse problem of recovering the source in a parabolic equation under a condition of nonlocal observation, *Mat. Sb.*, **204**(10):1391–1434, 2013 (in Russian).

25. Y.S. Eidelman, *Boundary Value Problems for Differential Equations with Parameters*, PhD thesis, Voronezh State University, Voronezh, 1984 (in Russian).

26. Y.S. Eidelman, The boundary value problem for differential equations with a parameter, *Differ. Uravn.*, **14**:1335–1337, 1978 (in Russian).

27. A. Ashyralyev, On a problem of determining the parameter of a parabolic equation, *Ukr. Math. J.*, **62**(9):1200–1210, 2010.

28. M. Choulli, M. Yamamoto, Generic well-posedness of a linear inverse parabolic problem with diffusion parameter, *J. Inverse Ill-Posed Probl.*, **7**(3):241–254, 1999.

29. A. Ashyralyev, A.S. Erdogan, O. Demirdag, On the determination of the right-hand side in a parabolic equation, *Appl. Numer. Math.*, **62**(11):1672–1683, 2012.

30. S. Saitoh, V.K. Tuan, M. Yamamoto, Reverse convolution inequalities and applications to inverse heat source problems, *JIPAM, J. Inequal. Pure Appl. Math.*, **3**(5), Article ID 80, 2002.

31. N.I. Ivanchov, On the determination of unknown source in the heat equation with nonlocal boundary conditions, *Ukr. Math. J.*, **47**(10):1647–1652, 1995.

32. V.T. Borukhov, P.N. Vabishchevich, Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation, *Comput. Phys. Commun.*, **126**:32–36, 2000.

33. M. Dehghan, An inverse problem of finding a source parameter in a semilinear parabolic equation, *Appl. Math. Modelling*, **25**:743-754, 2001.

34. A.A. Samarskii, P.N. Vabishchevich, *Numerical Methods for Solving Inverse Problems of Mathematical Physics*, Inverse Ill-Posed Probl. Ser., Vol. 52, De Gruyter, Berlin, New York, 2007.

35. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

36. D.J. Higham, T.K. Sardar, Existence and stability of fixed points for a discretised nonlinear reaction-diffusion equation with delay, *Appl. Numer. Math.*, **18**:155–173, 1995.

37. T.K. Sardar, D.J. Higham, Dynamics of constant and variable stepsize methods for a nonlinear population model with delay, *Appl. Numer. Math.*, **24**:425–438, 1997.

38. H. Poorkarimi, J. Wiener, Bounded solutions of nonlinear parabolic equations with time delay, *15th Annual Conference of Applied Mathematics, Univ. of Central Oklalhoma*, *Electron. J. Differ. Equ.*, Conference 02:87–91, 1999.

39. H. Tanabe, *Functional Analytic Methods for Partial Differential Equations*, Marcel Dekker, New York, 1997.

40. H. Akca, V.B. Shakhmurov, G. Arslan, Differential-operator equations with bounded delay, *Nonlinear Times Dig.*, **2**:179–190, 1989.

41. A. Sahmurova, V.B. Shakhmurov, Parabolic problems with parameter occuring in environmental engineering, *Numerical Analysis and Applied Mathematics, ICNAAM 2012, AIP Conf. Proc.*, **1470**:39–41, 2012.

42. A. Ashyralyev, P.E. Sobolevskii, On the stability of the delay differential and difference equations, *Abstr. Appl. Anal.*, **6**(5):267–297, 2001.

43. A. Ashyralyev, P.E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Operator Theory Advances and Applications, Vol. 148, Birkhäuser, Basel, Boston, Berlin, 2004.

44. G. Di Blasio, Delay differential equations with unbounded operators acting on delay terms, *Nonlinear Anal., Theory Methods Appl.*, **52**(1):1–18, 2003.

45. A. Ashyralyev, D. Agirseven, Approximate solutions of delay parabolic equations with the Neumann condition, in: *Numerical Analysis and Applied Mathematics, ICNAAM 2012, AIP Conf. Proc.*, **1479**:555–558, 2012.

46. A. Ashyralyev, D. Agirseven, On convergence of difference schemes for delay parabolic equations, *Comput. Math. Appl.*, **66**(7):1232–1244, 2013.

47. A. Ashyralyev, D. Agirseven, Finite difference method for delay parabolic equations, *Numerical Analysis and Applied Mathematics, ICNAAM 2011, AIP Conf. Proc.*, **1389**:573–576, 2011.

48. D. Agirseven, Approximate solutions of delay parabolic equations with the Dirichlet condition, *Abstr. Appl. Anal.*, **2012**, Article ID 682752, 31 pp., 2012, doi:10.1155/2012/682752.

49. A. Ashyralyev, D. Agirseven, Well-posedness of delay parabolic difference equations, *Adv. Difference Equ.*, **2014**(18), 20 pp., 2014, doi:10.1186/1687-1847-2014-18.

50. G. Di. Blasio, A. Lorenzi, Identification problems for parabolic delay differential equations with measurement on the boundary, *J. Inverse Ill-Posed Probl.*, **15**(7):709–734, 2007.

51. A. Ashyralyev, P.E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations*, Operator Theory Advances and Applications, Vol. 69, Birkhäuser, Basel, Boston, Berlin, 1994.