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On Space-Like Hypersurfaces with Constant Mean Curvature of a Lorentz Space Form

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Dedicated to Professor Tadashi Nagano on his 60th birthday

Abstract. We study complete space-like hypersurfaces with constant mean curvature of a Lorentz space form.

Introduction.

Let R_1^m be an m-dimensional Minkowski space and $S_1^m(c)$ (resp. $H_1^m(c)$) be an m-dimensional de Sitter space (resp. an anti-de Sitter space) of constant curvature c and of index 1. The class of these indefinite Riemannian manifolds of index 1 is called a *Lorentz space form*, which is denoted by $M_1^m(c)$. A hypersurface M of a Lorentz space form is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite.

Now, let M be an entire space-like hypersurface with constant mean curvature of a Minkowski space R_1^{n+1}. Then Cheng and Yau [5] estimated the norm of the second fundamental form of M, by which the Bernstein-type problem in the Lorentz version is affirmatively solved.

On the other hand, it is pointed out by Marsden and Tipler [10] that space-like hypersurfaces with constant mean curvature of arbitrary spacetimes get interested in relativity theory. An entire space-like hypersurface with constant mean curvature of a Minkowski space is investigated by Goddard [8] and Treibergs [15]. As standard models of space-like hypersurfaces with constant mean curvature of a Lorentz space form $M_1^{n+1}(c)$, $n \geq 3$, it is seen that there are four classes of hypersurfaces $H^k(c_1) \times S^{n-k}(c_2)$ and R^n, $H^k(c_1) \times R^{n-k}$ and $H^k(c_1) \times H^{n-k}(c_2)$, where $k = 0, 1, \cdots, n$, according as $c > 0$,

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$=0$ or <0. Here an m-dimensional sphere (resp. a hyperbolic space) of constant curvature c is denoted by $S^m(c)$ (resp. $H^m(c)$) and an m-dimensional Euclidean space is denoted by R^m. In particular, $H^1(c_1)\times M^{n-1}(c_2)$ is called a *hyperbolic cylinder* and $H^{n-1}(c_1)\times S^1(c_2)$ is also called a *spherical cylinder*, and moreover if $k\neq 0$ or n, then it is called a *hyperboloid.* For Goddard's conjecture [8] concerning hyperboloids, Stumbles [14] and Treibergs [15] however constructed many entire such hypersurfaces of a Minkowski space R_1^{n+1} different from the hyperboloid.

For a complete space-like hypersurface with constant mean curvature h/n of $S_1^{n+1}(c)$ it is also seen by Akutagawa [3] and Ramanathan [13] that if $n\geq 3$ and if $h^2<4(n-1)c$, then M is totally umbilic. The total umbilicness of such hypersurfaces is investigated by Cheng and one of the authors [4] from the different point of view. Complete space-like hypersurfaces with constant mean curvature of a de Sitter space $S_1^{n+1}(c)$ are also studied by many authors [1], [11], [16] and so on. In particular, the recent study of Choi and Treibergs [6] about complete space-like hypersurfaces with constant mean curvature of a Minkowski space is very important.

Under this situation it seems to be very interesting to investigate the manifold structure about complete space-like hypersurfaces with constant mean curvature of a Lorentz space form. In this paper, we shall prove the following

THEOREM 1. *Let M be a complete space-like hypersurface with constant mean curvature of a Lorentz space form $M_1^{n+1}(c)$. If it satisfies one of the following properties*:

(1) $c\leq 0$,

(2) $c>0$, $n\geq 3$ *and* $h^2\geq 4(n-1)c$,

(3) $c>0$, $n=2$ *and* $h^2>4c$,

then $S\leq S_+(1)$, where S is the norm of the second fundamental form of M and $S_+(1)$ is a positive constant whose square is defined by

$$-nc+\frac{h\{nh+(n-2)D(1)^{1/2}\}}{2(n-1)}, \qquad D(1)=h^2-4(n-1)c\,.$$

If the equality holds at a point p, then $\nabla\alpha(p)=0$.

As an application to the main theorem, a hyperbolic cylinder of a Lorentz space form can be characterized.

THEOREM 2. *The hyperbolic cylinder is the only complete space-like hypersurface with non-zero constant mean curvature of $M_1^{n+1}(c)$, whose norm of the second fundamental form is equal to $S_+(1)$.*

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1. Preliminaries.

Let (M, g) be a space-like hypersurface of an $(n+1)$-dimensional Lorentz space form $M_1^{n+1}(c)$. We choose a local field of orthonormal frames $e_1, \cdots, e_n$ adapted to the Riemannian metric induced from the indefinite Riemannian metric on the ambient space and let $\omega_1, \cdots, \omega_n$ denote the dual coframe on M. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$(1.1)\qquad \begin{aligned} &d\omega_i+\sum\omega_{ij}\wedge\omega_j=0\,, \quad \omega_{ij}+\omega_{ji}=0\,,\\ &d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}\,,\\ &\Omega_{ij}=-\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,,\end{aligned}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M. The *second fundamental form* on M is given by the quadratic form

$$\alpha=-\sum h_{ij}\omega_i\omega_j e_0$$

with values in the normal bundle, where e_0 is a unit time-like normal vector and the scalar $H=\sum h_{jj}/n$ is called the *mean curvature* of the hypersurface.

The Gauss equation, the Codazzi equation and the Ricci formula for the second fundamental form are given by

$$(1.2)\qquad R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})-h_{il}h_{jk}+h_{ik}h_{jl}\,,$$

$$(1.3)\qquad h_{ijk}=h_{ikj}\,,$$

$$(1.4)\qquad h_{ijkl}-h_{ijlk}=-\sum h_{rj}R_{rikl}-\sum h_{ir}R_{rjkl}\,,$$

where h_{ijk} and h_{ijkl} denote the components of the covariant differential $\nabla\alpha$ of α and the second covariant differential $\nabla^2\alpha$, respectively.

Now, making use of the last relationship repeatedly and taking account of the Codazzi equation (1.3), the Gauss equation (1.2) and the first Bianchi identity, one can compute h_{ijkl} as follows:

$$h_{ijkl}=h_{klij}+c(h_{ij}\delta_{kl}-h_{il}\delta_{jk}+h_{jk}\delta_{il}-h_{kl}\delta_{ij})+(h_{ij})^2h_{kl}-(h_{il})^2h_{jk}+(h_{jk})^2h_{il}-(h_{kl})^2h_{ij}\,,$$

where $(h_{ij})^2=-\sum h_{ir}h_{rj}$. Accordingly a Laplacian of the second fundamental form is given by

$$(1.5)\qquad \Delta h_{ij}=\sum h_{ijkk}=\sum h_{kkij}+c(nh_{ij}-h\delta_{ij})+h(h_{ij})^2-h_2h_{ij}\,,$$

where $h=\sum h_{jj}$ and $h_2=\sum(h_{jj})^2$. By utilizing (1.5), the Laplacian of the function h_2 can be determined as follows:

$$(1.6)\qquad \frac{1}{2}\Delta h_2=-\sum h_{ijk}h_{ijk}-\sum h_{ij}h_{kkij}+c(nh_2+h^2)-hh_3-h_2^2\,.$$

From now on, we assume that M is a complete space-like hypersurface with constant mean curvature of $M_1^{n+1}(c)$. For the shape operator A we define a symmetric linear transformation $P=A-hI/n$, where I denotes the identity transformation. Then it should be clear that

$$(1.7)\qquad \operatorname{Tr}P=0\,,\quad \operatorname{Tr}P^2=\operatorname{Tr}A^2-\frac{h^2}{n}\,,\quad \operatorname{Tr}P^3=\operatorname{Tr}A^3-\frac{3h}{n}\operatorname{Tr}P^2-\frac{h^3}{n^2}\,.$$

Now, a non-negative function f is defined by $f^2=\operatorname{Tr}P^2$, i.e., $f^2=-h_2-h^2/n$. Since h is constant, we have

$$\frac{1}{2}\Delta f^2\geq ncf^2+hh_3+\left(f^2+\frac{h^2}{n}\right)^2,$$

by virtue of (1.6), where the equality holds at a point p if and only if $\sum h_{ijk}h_{ijk}(p)=0$, i.e., $\nabla\alpha(p)=0$. Substituting (1.7) into the above equation and using the results that $\operatorname{Tr}A^2=-h_2$ and $\operatorname{Tr}A^3=-h_3$, we get

$$(1.8)\qquad \frac{1}{2}\Delta f^2\geq f^2\left(nc-\frac{h^2}{n}+f^2\right)-h\operatorname{Tr}P^3\,.$$

Let $a_1,\cdots,a_n$ be eigenvalues of P. Then it satisfies $\sum a_j=0$. For a positive number k such that $\sum a_j^2=k^2$, solving the problem for the conditional extremum leads to

$$(1.9)\qquad |\sum a_j^3|\leq\frac{(n-2)k^3}{\{n(n-1)\}^{1/2}}\,,$$

where the equality holds if and only if

$$a_1=\pm\sqrt{\frac{n-1}{n}}k\,,\qquad a_2=\cdots=a_n=\mp\sqrt{\frac{1}{n(n-1)}}k\,,$$

except the order. From which together with (1.8) it follows that we have

$$(1.10)\qquad \frac{1}{2}\Delta f^2\geq f^2\left[f^2-\frac{n-2}{\{n(n-1)\}^{1/2}}|h|f+\left(nc-\frac{h^2}{n}\right)\right],$$

where the equality holds at a point p if and only if $\nabla\alpha(p)=0$.

First of all, a fundamental property for the generalized principle due to Omori [12] and Yau [17] is next introduced.

THEOREM 1.1. *Let M be an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M, then for any $\varepsilon>0$ there exists a point p in M such that*

(1.11) $$|\nabla F(p)|<\varepsilon, \quad \Delta F(p)>-\varepsilon, \quad \inf F+\varepsilon>F(p),$$

where $|\nabla F|$ *denotes the norm of the gradient* ∇F *of* F.

2. Standard models.

This section is concerned with some standard models of complete space-like hypersurfaces with constant mean curvature of a Lorentz space form $M_1^{n+1}(c)$. In particular, we only consider non-totally umbilic cases and the norm of the second fundamental form of some standard such hypersurfaces are calculated. Without loss of generality, we may assume that the mean curvature h/n is non-negative. An $(n+1)$-dimensional Minkowski space R_1^{n+1} can be first regarded as a product manifold of R_1^{k+1} and R^{n-k}. With respect to the standard orthonormal basis of R_1^{n+1} a class of space-like hypersurfaces $H^k(c_1)\times R^{n-k}$ of R_1^{n+1} is defined by

$$H^k(c_1)\times R^{n-k}=\{(x, y)\in R_1^{n+1}=R_1^{k+1}\times R^{n-k} : |x|^2=-1/c_1>0\},$$

where $k=1, \cdots, n-1$ and $|\ \ |$ denotes the norm defined by the scalar product on R_1^{k+1} which is given by $\langle x, x\rangle=-x_0^2+\sum_{j=1}^k x_j^2$. The number of distinct principal curvatures of each hypersurface in this family is exactly two, say $(-c_1)^{1/2}$ and 0, with multiplicities k, $n-k$, respectively. In particular, if $k=1$, then $H^1(c_1)$ is a part γ of the hyperbolic curve in R_1^2 and $\gamma\times R^{n-1}$ is a class of space-like hypersurfaces of R_1^{n+1}.

We next define a family of space-like hypersurfaces $H^k(c_1)\times S^{n-k}(c_2)$ of $S_1^{n+1}(c)$ by

$$H^k(c_1)\times S^{n-k}(c_2)=\{(x, y)\in S_1^{n+1}(c)\subset R_1^{n+2}=R_1^{k+1}\times R^{n-k+1} : \\ |x|^2=-1/c_1, |y|^2=1/c_2\},$$

where $c_1<0, c_2>0, 1/c_1+1/c_2=1/c$ and $k=1, \cdots, n-1$. The number of distinct principal curvatures of such hypersurfaces in this family is exactly two and they are constant. A principal curvature is equal to $(c-c_1)^{1/2}$ with multiplicity k and the other is equal to $(c-c_2)^{1/2}$ with multiplicity $n-k$. In particular, if $k=1$, then $H^1(c_1)$ is a part γ of the hyperbolic curve in R_1^2 and $\gamma\times S^{n-1}(c_2)$ is a class of space-like hypersurfaces of $S_1^{n+1}(c)$.

Another family of space-like hypersurfaces $H^k(c_1)\times H^{n-k}(c_2)$ of $H_1^{n+1}(c)$ are defined by

$$H^k(c_1)\times H^{n-k}(c_2)=\{(x, y)\in H_1^{n+1}(c)\subset R_2^{n+2}=R_1^{k+1}\times R_1^{n-k+1} : \\ |x|^2=-1/c_1, |y|^2=-1/c_2\},$$

where $c_1<0$, $c_2<0$, $1/c_1+1/c_2=1/c$ and $k=1, \cdots, n-1$. The number of distinct principal curvatures of such hypersurfaces in this family is exactly two and they are constant. A principal curvature is equal to $\pm(c-c_1)^{1/2}$ with multiplicity k and the other is equal to $\mp(c-c_2)^{1/2}$ with multiplicity $n-k$. In particular, if $k=1$, then $H^1(c_1)$ is a part γ of the hyperbolic curve in R_1^2 and $\gamma \times H^{n-1}(c_2)$ is a class of space-like hypersurfaces of $H_1^{n+1}(c)$. In the case where $k=n-1$, we have the same situation as that of $k=1$.

Now, we denote by $S(k)$ the norm of the second fundamental form of each hypersurface $H^k(c_1) \times R^{n-k}$, $H^k(c_1) \times S^{n-k}(c_2)$ or $H^k(c_1) \times H^{n-k}(c_2)$. Furthermore, two positive numbers $S_+(k)$ and $S_-(k)$ are given by

$$S_\pm(k)^2 = -nc + \frac{h\{nh \pm (n-2k)D(k)^{1/2}\}}{2k(n-k)}, \tag{2.1}$$

where $D(k)=h^2-4k(n-k)c \geq 0$. Then $S_+(k)$ is monotonously decreasing with respect to k for $k \leq n/2$ and moreover it satisfies

$$S_-(k)=S_+(n-k) \tag{2.2}$$

for all k. Assume that the mean curvature is positive. In the case where $c \neq 0$, by the relation between c, c_1 and c_2 we have

$$k^2c_1^2+(h^2-2nkc)c_1-c(h^2-n^2c)=0, \tag{2.3}$$

$$(n-k)^2c_2^2+\{h^2-2n(n-k)c\}c_2-c(h^2-n^2c)=0. \tag{2.4}$$

Since $h^2 \geq n^2c$, we get $S(k)=S_+(k)$ by the straightforward calculation. It is also seen that the norm $S(k)$ of $H^k(c_1) \times R^{n-k}$ is given by the similar discussion to the above matter. We find that the maximum of $S(k)$ is $S_+(1)$ from (2.2) and the above property.

REMARK 2.1. For the space-like hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, $n \geq 3$, the norm $S(1)$ of the second fundamental form is given by

$$S(1)^2 = -nc + \frac{h\{nh+(n-2)D(1)^{1/2}\}}{2(n-1)},$$

where $D(1)=h^2-4(n-1)c \geq 0$. When $c_1=-(n-2)c$ and $c_2=(n-2)c/(n-1)$, we have $D(1)=0$, which means that the estimate of h by Akutagawa [3] and Ramanathan [13] for the space-like hypersurface M to be totally umbilic is best possible.

3. Proof of theorems.

In this section we prove the main theorem stated in the introduction and as an application to this theorem we are able to characterize hyperbolic cylinders of a Lorentz space form.

PROOF OF THEOREM 1. Given any positive number a, a function F is defined by $(f^2+a)^{-1/2}$, which is bounded from above by the positive constant $a^{-1/2}$ and is bounded from below by 0. Since M is space-like, the Ricci curvature is bounded from below and so we can apply Theorem 1.1 due to Omori [12] and Yau [17] to the function F. Given any positive number ε there exists a point p at which it satisfies (1.11). Consequently the following relationship

$$\frac{1}{2}F(p)^4\Delta f^2(p) < 3\varepsilon^2 + F(p)\varepsilon \tag{3.1}$$

can be derived by the simple calculation. Thus, for any convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m > 0$ and $\varepsilon_m \to 0$ $(m \to \infty)$, there is a point sequence $\{p_m\}$ so that the sequence $\{F(p_m)\}$ satisfies (1.11) and converges to F_0 by taking a subsequence, if necessary, because the sequence is bounded. From the definition of the infimum and (1.11) we have $F_0 = \inf F$ and hence $f(p_m) \to f_0 = \sup f$, according to the definition of F. On the other hand, it follows from (3.1) that we have

$$\frac{1}{2}F(p_m)^4\Delta f^2(p_m) < 3\varepsilon_m^2 + F(p_m)\varepsilon_m\,, \tag{3.2}$$

and the right hand side of the above equation converges to 0, because the function F is bounded. Accordingly, for any positive number ε (<2) there is a sufficiently large integer m such that $F(p_m)^4\Delta f(p_m)^2 < \varepsilon$. This inequality and (1.10) yield

$$(2-\varepsilon)f(p_m)^4 - \frac{2(n-2)}{\{n(n-1)\}^{1/2}}|h|f(p_m)^3 + 2\left(nc - \frac{h^2}{n} - \varepsilon a\right)f(p_m)^2 - \varepsilon a^2 < 0\,,$$

which implies that the sequence $\{f(p_m)\}$ is bounded. Thus the infimum of F satisfies $F_0 \neq 0$ by the definition of F and hence the inequality (3.2) implies that $\limsup \Delta f^2(p_m) \leq 0$. This means that the supremum f_0 of the function f satisfies

$$f_0^2\left[f_0^2 - \frac{n-2}{\{n(n-1)\}^{1/2}}|h|f_0 + \left(nc - \frac{h^2}{n}\right)\right] \leq 0 \tag{3.3}$$

by (1.10). In the first case, we shall assume that $c \geq 0$ and M is maximal. Then, by (3.3) it turns out that we get $f_0 = 0$, which means that M is totally geodesic. We next consider the proof under the condition (1), (2) or (3) given in Theorem 1. Without loss of generality, we may assume that h is positive. Then the supremum f_0 is bounded from above by the constant $\{(n-2)h + nD(1)^{1/2}\}/2\{n(n-1)\}^{1/2}$, because the second factor of the left hand side can be regarded as the quadratic equation for f_0 and the discriminant D is equal to $nD(1)/(n-1)$. Since the norm S of the second fundamental form α is given by $S^2 = -h_2 = f^2 + h^2/n$, we get

$$S \leq S_+(1)\,,$$

where the equality holds at a point p in M if and only if $\nabla\alpha(p) = 0$.

Therefore the main theorem is proved. ■

REMARK 3.1. (1) The main theorem is first proved by Cheng and one of the authors [4] in the case where $c \leq 0$. The proof is essentially similar to that in [4].

(2) The main theorem is a generalization of that of Cheng and Yau [5]. In fact, in the case where $c=0$ we find $S_{+}(1)=h$, which implies that the norm S of the second fundamental form of a space-like hypersurface of R_1^{n+1} satisfies $S \leq h$. This is the estimation which is obtained by Cheng and Yau [5] in the case where M is entire.

(3) The main theorem is also a generalization of Choquet-Bruhat, Fisher and Marsden [7], which asserts that if $c>0$ and if M is compact and maximal, then it is totally geodesic.

REMARK 3.2. By (3.2) the inequality $S \leq S_{+}(1)$ is equivalent to saying that the scalar curvature of M is non-positive if $c \geq 0$. In particular, in the case where $n=2$, it is equivalent to the fact that the Gauss curvature of M is non-positive. This is another proof of Aiyama's theorem [2].

Theorem 2 can be derived from Theorem 1.

PROOF OF THEOREM 2. Let M be a complete space-like hypersurface with constant mean curvature of a Lorentz space form $M_1^{n+1}(c)$. Without loss of generality we may assume that h is positive. Suppose that $S=S_{+}(1)$. For any point p it is seen that we have $\nabla\alpha(p)=0$, which implies that $\nabla\alpha$ vanishes identically on M. It follows from this property together with (1.4) that any principal curvature is constant and moreover it satisfies

$$\sum h_{rj} R_{rikl} + \sum h_{ir} R_{rjkl} = 0\,,$$

which means that the number of distinct principal curvatures is at most two. If M is totally umbilic, then $f=0$ and $S^2=h^2/n<S_{+}(1)^2$, which contradicts to our assumption. Accordingly M has just two distinct constant principal curvatures. By the congruence theorem of Abe, Koike and Yamaguchi [1] M is isometric to $H^k(c_1)\times M^{n-k}(c_2)$, where $M^{n-k}(c_2)$ denotes $S^{n-k}(c_2)$, R^{n-k} or $H^{n-k}(c_2)$, according as $c>0$, $c=0$ or $c<0$. For the norm $S(k)$ of the second fundamental form, it is seen by (2.2) that $S(k)<S_{+}(1)$ if $k\neq 1$. This shows that M is only a hyperbolic cylinder. ■

REMARK 3.3. The estimate of the norm of the second fundamental form of maximal space-like n-dimensional submanifolds of $M_p^{n+p}(c)$ is already given by Ishihara [9].

4. Non-negative curvature.

In this section another interpretation for the estimate of the norm of the second fundamental form is considered. Let M be a complete space-like hypersurface with constant mean curvature of $M_1^{n+1}(c)$. Assume that the sectional curvature of M is of non-negative. For any point p in M we can choose a local frame field $e_1, \cdots, e_n$ so that the matrix (h_{ij}) is diagonalized at that point, say

$$h_{ij}=\lambda_i\delta_{ij}\,. \tag{4.1}$$

Under such frame field at p we have

$$h=\sum\lambda_i\,,\qquad h_2=-\sum\lambda_i^2\,,\qquad h_3=-\sum\lambda_i^3\,,$$

because M is space-like and hence the normal vector is time-like. Accordingly we get

$$\begin{aligned}c(nh_2+h^2)-hh_3-h_2^2&=c\{n(-\sum\lambda_i^2)+(\sum\lambda_i)^2\}+\sum\lambda_i\sum\lambda_i^3-(\sum\lambda_i^2)^2\\&=\frac{1}{2}\sum(-c+\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2\,,\end{aligned}$$

from which together with (1.6) it turns out that

$$\Delta h_2=-\sum(c-\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2-2|\nabla\alpha|^2\,, \tag{4.2}$$

where $|\nabla\alpha|$ denotes the norm of the covariant differential $\nabla\alpha$ of the second fundamental form α. We denote by K_{ij} the sectional curvature of the plane section spanned by e_i and e_j.

THEOREM 4.1. *Let M be a complete space-like hypersurface with non-zero constant mean curvature of $S_1^{n+1}(c)$, $n\geq 3$. Assume that the sectional curvature is non-negative. If* $\sup S$ *is less than $S_+(1)$, then M is totally umbilic and it is congruent to R^n or $S^n(c)$.*

PROOF. Since the sectional curvature K_{ij} is given by $K_{ij}=c-\lambda_i\lambda_j$ and since it is assumed to be non-negative, we have $c-\lambda_i\lambda_j\geq 0$ for any distinct indices i and j. Accordingly (4.2) means that

$$\Delta h_2(p_m)\leq -\sum(c-\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2\leq 0\,. \tag{4.3}$$

On the other hand, since M is space-like, the Ricci curvature is bounded from below by a constant $(n-1)c-h^2/4$ and h_2 is bounded, we can apply Theorem 1.1 to the function h_2. Thus we get

$$h_2(p_m)\to\inf h_2\,,\quad \Delta h_2(p_m)\to 0\qquad (m\to\infty)\,. \tag{4.4}$$

This implies that (4.3) and (4.4) give rise to

$$(c-\lambda_i\lambda_j)(\lambda_i-\lambda_j)^2(p_m)\to 0\qquad (m\to\infty) \tag{4.5}$$

for any distinct indices i and j.

Now, since $h_2=-\sum\lambda_j^2$ is bounded, any principal curvature λ_j is bounded and hence so is any sequence $\{\lambda_j(p_m)\}$. Then there exists a subsequence $\{p_{m'}\}$ of $\{p_m\}$ such that

$$\lambda_j(p_{m'})\to\lambda_{j0}\qquad (m'\to\infty) \tag{4.6}$$

for any j. In fact, since a sequence $\{\lambda_1(p_m)\}$ is bounded, it converges to λ_{10} by taking a subsequence $\{p_{m1}\}$ if necessary. For the point sequence $\{p_{m1}\}$ a sequence $\{\lambda_2(p_{m1})\}$ is also bounded and hence there is a subsequence $\{p_{m2}\}$ of $\{p_{m1}\}$ such that $\{\lambda_2(p_{m2})\}$

converges to λ_{20} as m_2 tends to infinity. Thus we can inductively show that there exists a point sequence $\{p_{m'}\}$ of $\{p_m\}$ such that the property (4.6) holds. By (4.5) and (4.6) we get

$$(c-\lambda_{i0}\lambda_{j0})(\lambda_{i0}-\lambda_{j0})^2=0 \tag{4.7}$$

for any distinct indices. By the simple algebraic calculation it is easily seen that the number of distinct limits in $\{\lambda_{i0}\}$ is at most two.

Next, we shall show that all limits λ_{i0} coincide with each other. Suppose that the number of distinct limits is equal to two, say λ and μ ($\lambda \neq \mu$). By (4.5) they satisfy $\lambda\mu=c$. By r and s the numbers of indices j such that $\lambda_j(p_{m1})\to\lambda$ and $\lambda_j(p_m)\to\mu$ are denoted. Then $r=1$ or $s=1$. In fact, suppose that $r, s\geq 2$. It follows from $r\geq 2$ that there are distinct indices i and j such that $\lambda_i(p_m)\to\lambda$ and $\lambda_j(p_m)\to\lambda$ $(m\to\infty)$ and hence we get

$$c-\lambda_i(p_m)\lambda_j(p_m)\to c-\lambda^2 \qquad (m\to\infty)\,.$$

Since the sectional curvature is of non-negative, we obtain $\lambda^2\leq c$. Similarly, the other satisfies $\mu^2\leq c$ and therefore they satisfy $\lambda^2\mu^2\leq c^2$, which implies $\lambda^2=\mu^2=c>0$. However, it turns out that $\lambda=-\mu$, because they are distinct and it yields $c=\lambda\mu=-\lambda^2>0$, a contradiction. So, suppose that $r=1$. Then $s\geq 2$ and by the above discussion we get $0<\mu^2<c$, from which the fact $\lambda^2>c$ is derived. Without loss of generality we may assume that h is positive. Then two limits λ and μ are positive provided that $c>0$. In this case we define a negative number c_1 and a positive number c_2 by $\lambda^2=c-c_1$ and $\mu^2=c-c_2$, respectively. Then these numbers satisfy $c_1<0$, $0<c_2<c$ and

$$(c-c_1)(c-c_2)=c^2\,, \quad \text{i.e.,} \quad \frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}\,. \tag{4.8}$$

By $h=\sum\lambda_i(p_m)$, we have

$$h=\lambda+(n-1)\mu=(c-c_1)^{1/2}+(n-1)(c-c_2)^{1/2}\,. \tag{4.9}$$

By the same argument as that in §2 and by the assumption of the mean curvature, the relationships (4.8) and (4.9) yield

$$\lambda^2+(n-1)\mu^2=S_+(1)^2$$

which contradicts to the assumption $\sup S<S_+(1)$. Thus it is seen that for the limits λ_{j0} we get

$$\lambda_{j0}=\lambda$$

for any index j.

Because of $h_2+h^2/n=-(1/2)\sum(\lambda_i-\lambda_j)^2$, it follows from the above property that the sequence $\{h_2(p_m)+h^2/n\}$ converges to zero as m tends to infinity and therefore we get

$$h_2(p_m)\to\sup h_2 \qquad (m\to\infty)\,,$$

which yields together with (4.4) that the function h_2 becomes a constant $-h^2/n$. Accordingly, the hypersurface M is totally umbilic.

In the case where $c>0$, the sphere $S^n(c_2)$ and the Euclidean space R^n which is defined by $x_1=x_{n+2}+t$ are the only totally umbilic space-like hypersurfaces of non-negative curvature of $S_1^{n+1}(c)$. The norm S of the second fundamental form α is given by $\{n(c-c_2)\}^{1/2}$ or $(nc)^{1/2}$ according as $S^n(c_2)$ or R^n. It completes the proof. ■

REMARK 4.1. The assumption of the sectional curvature is essential. In fact, space-like hypersurfaces $H^k(c_1)\times S^{n-k}(c_2)$ of $S_1^{n+1}(c)$ are not of non-negative curvature if k is greater than 1.

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