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On space-time, reference frames and the structure of relativity groups

by

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ABSTRACT. — A general formulation of the notions of space-time, reference frame and relativistic invariance is given in essentially topological terms. Reference frames are axiomatized as C^0 mutually equivalent real four-dimensional C^0-atlases of the set M denoting space-time, and M is given the C^0-manifold structure which is defined by these atlases. We attempt ti give an axiomatic characterization of the concept of equivalent frames by introducing the new structure of equiframe. In this way we can give a precise definition of space-time invariance group $\mathfrak{Q}$ of a physical theory formulated in terms of experiments of the yes-no type. It is shown that, under an obvious structural requirement and provided a suitable assumption is made on the space-time domains of experiments, the group $\mathfrak{Q}$ can be realized isomorphically in a unique way onto a group $\mathcal{G}$ of homeomorphisms of the topological space M. We call $\mathcal{G}$ the relativity group of the theory. A physically acceptable topology on $\mathcal{G}$ is discussed. Lastly, the general formalism is applied to introduce in an axiomatic way the notion of inertial frame. It is shown that in a theory for which the equivalent frames are the inertial frames, the space-time invariance group is isomorphic either to the Poincaré group or to the inhomogeneous Galilei group.

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1. INTRODUCTION

In this paper we give a mathematical formulation of the concept of space-time in terms of the notion of reference frame and we study in a general context the significance of the concept of equivalent reference frames for a given class of physical phenomena. As an application, the geometrical theory of special relativity is constructed, starting from a minimal set of assumptions.

The concept of reference frame is formalized in a natural way by postulating an injection of the set $\mathcal{R}$ of reference frames into the set of C^0 mutually equivalent four-dimensional real atlases of M of class C^0. In this way, space-time acquires naturally a structure of real four-dimensional manifold of class C^0 (the space-time manifold M) and a reference frame corresponds to an atlas of the manifold M. Since there is a one to one correspondence between n-dimensional C^0-manifolds and locally euclidean spaces of the same dimension, the preceding characterization is actually equivalent to the costumarily accepted topological definition of space-time as a four-dimensional locally euclidean space (with a choice of a class of C^0-atlases). The usual Hausdorff separation axiom is also assumed for the topology of M. In the preceding points lies the essence of the content of section 2.

In its substance, the notion of equivalent reference frames is formulated in a classical way : two reference frames are equivalent in regard to the description of a certain class $\mathcal{F}$ of physical phenomena, if the laws which govern the phenomena of $\mathcal{F}$ are the same in the two frames. From a technical point of view we adopt a scheme in which physical laws are expressed in terms of experiments of the yes-no type and of states, along the lines of the modern formulations of quantum theory [1]. Thus, to every reference frame R is associated a set Q_R whose elements represent the yes-no experiments which can be carried out in R, and a set Π_R whose elements represent, vaguely speaking, the possible states of physical systems as seen in R. Mathematically, an element of Π_R is a function defined on Q_R and taking its values in [0,1], whose value at a given experiment a is the probability for the positive outcome of a when a is carried out on the state which the function represents. We further assume that the experiments are localized, so that to every element of Q_R is associated an open relatively compact region of M. This framework is briefly discussed in section 3.

With the above formalism we can give the notion of equivalent reference frames in the following terms. Given two reference frames R and R′ we say that they are equivalent if the following conditions are realized. First, $Q_R = Q_{R'} = Q$ and $\Pi_R = \Pi_{R'} = \Pi$. Second, there is a permutation, $\gamma_{RR'}$ of Q such that, if $\pi \in \Pi$ and if we define a mapping $\pi^{RR'}$ of Q into [0,1] as $\pi^{RR'}(a) = \pi(\gamma_{RR'}^{-1})(a))$, $a \in Q$, then $\pi^{RR'} \in \Pi$. This is interpreted as follows. $\gamma_{RR'}$ (a) (respectively $\pi^{RR'}$) represents an experiment (respectively a state) which in the language of R′ is the same as a (respectively π) is in the language of R, so that the above written equation expresses exactly the identity of physical laws in the two reference frames. These concepts are mathematically formulated in section 4, which is centered on the definition of the structure of equiframe. Substantially, this structure is employed to select a class $\mathcal{E}$ of equivalent reference frames in such a way that to this class is associated a group $\mathcal{Q}$ of permutations of the experiments, which forms the *space-time invariance group* of the set of natural laws for which $\mathcal{E}$ is the class of equivalent frames.

In section 5 it is shown that under an assumption which is wholly natural in the context which we adopted, the group $\mathcal{Q}$ can be isomorphically realized onto a group $\mathcal{G}$ of homeomorphisms of M, in such a way that every $g \in \mathcal{G}$ transforms the domain of an experiment a into the domain of the transformed of a by the element of $\mathcal{Q}$ to which g corresponds. Further, this realization is unique. The group $\mathcal{G}$ is called the *relativity group* for the class of equivalent frames.

In section 6 we define on $\mathcal{G}$ a topology compatible with the group structure, by means of which it is possible to endow the set $\mathcal{E}$ of equivalent reference frames with a topology which translates in a physically acceptable way the notion of closeness for a pair of reference frames.

In section 7 the concepts of the preceding sections are applied in the context of the theory of special relativity. The class of equivalent reference frames is characterized by means of a certain number of axioms, in such a way that it can be assumed as the class of inertial reference frames. These axioms express in essence the property of translational homegeneity of space-time and of isotropy of space which are costumarily assumed in the theory of special relativity and they allow to establish the result that the relativity group can be indentified in this case with the Poincaré group or, as a limiting case, to the Galilei group.

2. THE SPACE-TIME MANIFOLD

We accept uncritically the picture of space-time as a real four-dimensional continuum. In this section we attempt to introduce a mathematical structure which should represent as better as possible the physical concepts connected with this picture and which should provide a minimal framework for the subsequent development. We refer to [2] for the mathematical terminology.

We shall assume as primitive the notion of *event*. *Space-time* is the set M of all events. In the sequel we shall normally refer to events simply as points. If $p \in M$, a *local coordinate system* on M at p is a real four-dimensional chart (U, φ) of M whose domain contains p, namely $p \in U \subseteq M$, φ a bijection of U onto an open subset of R^4. A local coordinate system will be termed *global* if its domain is M.

Next we need the notion of a reference frame to which events can be referred. Specifically, by a *reference frame* R we mean a collection of physical objects in terms of which, $\forall\, p \in M$, a local coordinate system can be defined on M at p. Hence, physically, a local coordinate system (for a frame R) is an operationally well defined rule which associates to each event of a certain regions of space-time a system of four real numbers uniquely determining the event in that region ([1]). The continuum picture can be formulated by assuming that if R is a reference frame, any two local coordinate systems for R are C^0-compatible.

For a given reference frame R we shall denote by $\mathcal{A}_R$ the set of local coordinate systems (charts) for R. The above requirements imply that $\mathcal{A}_R$ is a C^0-atlas of M. Thus the basic objects of our theory are the space-time set M, the set $\mathcal{R}$ of reference frames on M and a map $\mathcal{A} : R \to \mathcal{A}_R$ of $\mathcal{R}$ into the set of C^0-atlases of M.

The first axiom expresses the indistuinguishability of frames producing the same local coordinate systems.

AXIOM 2.1. — *The map* $\mathcal{A} : R \to \mathcal{A}_R$ *is injective.*

The second axiom gives a restriction on the allowed atlases.

AXIOM 2.2. — $\forall$ R, $R' \in \mathcal{R}$, *the atlases* $\mathcal{A}_R$ *and* $\mathcal{A}_{R'}$ *are* C^0*-equivalent.*

Let $\mathfrak{C}$ be the set of C^0-atlases of M which are C^0-equivalent to some atlas $\mathcal{A}_R$. Then, M endowed with $\mathfrak{C}$ is a pure four-dimensional real manifold of class C_0 which we call the *space-time manifold.* We call *the topological space* M the set M endowed with the (locally euclidean) topology generated by the domains of the charts of the atlases of $\mathfrak{C}$. We assume

AXIOM 2.3. — *The topological space* M *is Hausdorff and connected.*

From this it follows that M is locally compact and hence uniformizable [4].

We conclude this section with the following remarks. Axiom 2.2 allows us to give M a natural, frame independent manifold structure, which in turn leads to a natural frame independent topological structure

(1) For a physical discussion concerning this point (see f. i. [3]).

on M (note that it is in general possible to endow a set X with distinct and even non isomorphic C^0-manifold structures).

At this point the objection can be raised that for a concrete constructive theory of the space-time continuum the very general scheme given by the axioms above is insufficient and it is of course true that in the theory of relativity a richer structure has to be assumed, notably a suitable C^r-differentiability order and a pseudo-riemannian metric [5]. Nevertheless, we mantain that this is not necessary here, in view of the purpose of this paper which lies mainly in the formulation of the notions of equivalent frames and of space-time invariance of a physical theory (compare the following sections).

3. EXPERIMENTS AND STATES

In the preceding section space-time M was considered from the point of view of its pure mathematical structure and no reference was made to its being the theatre of physical happenings. Accordingly, a reference frame was considered as a collection of physical objects which were apt to merely electing the manifold structure of M. In this section we shall go much further, by supposing that to every reference frame is associated a collection of laboratories equipped with measuring devices, by means of which physical phenomena can be observed and studied.

In this context, our aim is to formulate the notions of a class of equivalent reference frames and of a relativity group for this class. We shall define these concepts in an axiomatic way, but what we have in mind here is the classical idea that there are reference frames which are equivalent in that the laws of nature appear to be the same in all of them.

But what is a law of nature ? It lies beyond the scope of this paper to discuss thoroughly this concept. For our purposes the following few remarks will suffice. Generally speaking, laws of nature are regularities observed in a given field of phenomena. These regularities can be theoretically described in many different ways, but in physics, and especially in quantum physics, the following scheme seems to be the most convenient one. A physical law tells us that, once a physical system has been prepared with a definite laboratory procedure, we can predict the probabilities for the outcomes of all conceivable experiments which we can make on the system. It is precisely the very possibility of giving these probabilities which expresses the existence of regularities in the field of those phenomena which are interpreted as the manifestations of the physical system in the realm of our perceptions.

We shall now formalize these considerations. It is conceivable that, at least in principle, all physically meanigful information about a physical system S can be obtained by performing laboratory experiments of the yes-no type. By a *yes-no experiment* we mean an experiment which ascertains whether or not the system has a given property P (2). It has to be remarked here that, in general, there is a whole class of equivalent yes-no experiments which are capable of ascertaining whether the system has the property P. In other words one must allow for several experiments devised to answer one and the same experimental question, but possibly performed in different ways. In this way we obtain an equivalence relation T on the set χ (S) of all possible yes-no experiments which can be performed on the system S in some reference frame. Thus, given a reference frame R, we are led to introduce first a subset Q_R (S) of χ (S)/T which is defined in the following way : $a \in Q_R$ (S) iff there exists some yes-no experiment of the equivalence class a which can be performed in R. In the sequel, the elements of Q_R (S) will be referred simply as the *experiments* (on S) in R.

Next we introduce a set Π_R (S) of maps of Q_R (S) into [0,1]. We call the elements of Π_R (S) *the (mathematical) states of the system.* These objects are supposed to be in correspondence with all possible laboratory procedures of preparing the system in such a way that, for $a \in Q_R$ (S), $\pi \in \Pi_R$ (S), π (a) gives the probability for the positive outcome of the experiment a on the system prepared with the procedure to which π corresponds. Thus the functions π express in mathematical form correlations between the manifestations of S in the world of phenomena and hence constitute the formulation of the (possibility statistical) laws of nature concerning the behaviour of S. Our success in formulating a satisfactory physical theory depends on how correctly we have established the correspondence between mathematical states and laboratory procedures of preparing physical systems, and this can only be judged on an empirical basis.

In its general lines, the preceding discussion follows some expositions of the quantum theoretical formalism which have recently been given by several authors [1]. However, we have made no reference to the points concerning the structure properties (of partially ordered set, of lattice, etc.) of the set Q_R (S) and the conditions on the states related to these properties. For our purposes it is in fact not necessary to touch these important problems here. *En revanche*, it will be essential in the sequel an assumption concerning the local character of experiments ([1 (*d*)], [7]). Since actual laboratories and measuring apparatuses

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(2) For a thorough analysis of this type of experiments (filters) (see f. i. [6]).

have a finite extension in space and actual measuring processes have a finite duration, we assume that the actual space-time region of interaction of the measuring instruments with a physical system in the course of a yes-no experiment has a limited extension (this term will presently be given an exact topological meaning). It is further conceivable that the space-time interaction regions associated with two yes-no experiments which belong to the same equivalence class are the same, even if the measuring apparatuses extend in general to different regions. This can be translated into mathematical language by introducing a map

$$\Delta_{RS}: \quad a \rightarrow \Delta_{RS}(a)$$

of $Q_R(S)$ into the set of *open relatively compact* subsets of M. $\Delta_{RS}(a)$ represents the actual space-time interaction region for the experiment a performed on the system S in the reference frame R and will be called the *domain* of a. The choice of the domains of experiments to be open sets lies in the fact that the inclusion of boundary points seems to imply the presence therein of measuring components which are localized with an infinite precision, and we regard this as an necessary idealization. On the other hand, the condition of relative compactness is intended to give a precise form to the assumption that actual experiments involve space-time regions of finite extension. We can justify this as follows. Let Δ be the domain of an experiment. By our assumptions there exists a compact set C which contains Δ. Let p be a point of C and let (U_p, φ_p) be a chart for a reference frame R such that $p \in U_p$. Let V_p be open neighbourhood of p contaned in U_p and such that $\varphi_p(V_p)$ is a bounded subset of R^4. We can find a finite family $\Phi = \{V_{p_1}, V_{p_2}, \ldots, V_{p_n}\}$ which is a covering of C hence of Δ. The $\varphi_{p_i}(V_{p_i} \cap \Delta)'s$ $(i = 1, 2, \ldots, n)$ are bounded subsets of R^4. Hence Δ can be partitioned in a finite number of parts, each of which can be described by a set of coordinates whose values are bounded. Loosely speaking, the condition of relative compactness does not allow for experiments the domains of which reach the boundary (if there is any) of the universe.

The preceding discussion concerning the local character of experiments might provide a justification for the assumption in section 2, axiom 2.2., that the space-time topology is frame independent. Indeed, if it were not so, an open relatively compact subset in one frame might not appear as such in another, and this would give the locality assumption an ambiguous character.

In general, a given class $\mathcal{F}$ of physical phenomena will involve several physical systems forming a set $\mathcal{S}$. The systems of $\mathcal{S}$ shall be considered together to form a supersystem for which the set Q_R of experiments contains the set $\bigcup_{S \in \mathcal{S}} Q_R(S)$. Thus, in order to formulate the physical laws governing the phenomena of the class $\mathcal{F}$, we can avoid any reference to a particular physical system and consider only the following objects :

(*a*) the set Q_R of experiments in the reference frame R;

(*b*) a set Π_R of maps of Q_R into [0,1], *i. e.* the set of states of the supersystem in R;

(*c*) a map $\Delta_R : a \to \Delta_R(a)$ of Q_R into the set of open relatively compact subsets of the space-time manifold M. We remark that, by our previous assumptions, if $a \in Q_R \cap Q_{R'}$, then $\Delta_R(a) = \Delta_{R'}(a)$.

4. EQUIFRAMES [3]

In this section we give a formal figure to the notion of equivalent frames for a given field of phenomena $\mathcal{F}$. We accomplish this by giving a suitable structure to the set $\mathcal{R}$ of reference frames of M according to the following definition.

DÉFINITION 4.1. — An *equiframe* is a triplet $\mathfrak{E} = (\mathcal{E}, \gamma, \omega)$ where,

(4.i) $\mathcal{E}$ is a subset of $\mathcal{R}$ such that

(4.i.1) $$Q_R = Q_{R'} = Q, \quad \forall\, R, R' \in \mathcal{E}$$

(hence, by the remark which concludes the preceding section,

$$\Delta_R = \Delta_{R'} = \Delta, \quad \forall\, R, R' \in \mathcal{E});$$

(4.i.2) $$\Pi_R = \Pi_{R'} = \Pi, \quad \forall\, R, R' \in \mathcal{E};$$

(4.i.3) $$\mathrm{Card}\, \mathcal{A}_R = \mathrm{Card}\, \mathcal{A}_{R'}, \quad \forall\, R, R' \in \mathcal{E}.$$

(4.ii) γ is a map $(R, R') \to \gamma_{RR'}$ of $\mathcal{E} \times \mathcal{E}$ into P (Q) satisfying

(4.ii.1) $$\gamma_{R'R''} \circ \gamma_{RR'} = \gamma_{RR''}, \quad \forall\, R, R', R'' \in \mathcal{E};$$

(4.ii.2) $\forall\, R, R', R'' \in \mathcal{E}, \exists\, R''' \in \mathcal{E}$ such that $\gamma_{R''R'''} = \gamma_{RR'}$;

(4.ii.3) $$\gamma_{RR'} = 1_Q \Rightarrow R = R'.$$

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[3] In the sequel, if X is a set, we shall denote by P (X) the group of permutations of X and by 1_X the corresponding neutral element. Further, we shall indicate by $\mathcal{C}$ (W) the group of homeomorphisms of the topological space W.

(4.iii) $\forall$ R, $R' \in \mathcal{E}$ and $\forall \pi \in \Pi$, the map $\pi^{RR'}$ of Q into [0,1] defined by

(4.iii.1) $$\pi^{RR'}(a) = \pi(\gamma_{RR'}^{-1}(a))$$

is again an element of Π, and

(4.iii.2) $$\pi = \pi^{RR'}, \quad \forall \pi \in \Pi \Rightarrow R = R'.$$

(4.iv) Let B (R, R′) be the set of bijections of $\mathcal{A}_R$ onto $\mathcal{A}_{R'}$. Then ω is a map $(R, R') \to \omega_{RR'}$ of $\mathcal{E} \times \mathcal{E}$ into $B = \bigcup_{(R, R') \in \mathcal{E} \times \mathcal{E}} B(R, R')$ satisfying

(4.iv.1) $$\omega_{RR'} \in B(R, R') \quad \text{and} \quad \omega_{R'R''} \circ \omega_{RR'} = \omega_{RR''}, \quad \forall R, R' R'' \in \mathcal{E};$$

further, whenever $\mathcal{A}_R \cap \mathcal{A}_{R'}$ is non void, $\forall R'' \in \mathcal{E}$, $\exists R''' \in \mathcal{E}$ such that, if $c \in \mathcal{A}_R \cap \mathcal{A}_{R'}$,

(4.iv.2) $$\omega_{RR''}(c) = \omega_{R'R'''}(c).$$

(4.v) Let R, $R' \in \mathcal{E}$, $a \in Q$, $(U, \varphi) \in \mathcal{A}_R$ and

$$\left(U_{\omega_{RR'}}, \varphi_{\omega_{RR'}}\right) = \omega_{RR'}(U, \varphi).$$

Then

(4.v.1) $$\varphi(\Delta(a) \cap U) = \varphi_{\omega_{RR'}}(\Delta(\gamma_{RR'}(a)) \cap U_{\omega_{RR'}}).$$

The elements of $\mathcal{E}$ are, in common language, reference frames which are equivalent in regard to the formulation of the laws of nature governing the field of phenomena $\mathcal{F}$. Indeed, the entities γ and ω and the " axioms " (4.i)-(4.v) are simply intended to give a mathematical form to the whole body of ideas which are usuelly implicitly attached at the concept of equivalent frames. (4.i.1) expresses the assumption that, given two equivalent frames R and R′, every yes-no experiment which can be performed in R can, at least in principle, be performed in R′ as well, if not in the same, at least in an equivalent way [1 (*e*)]. The motivation for this assumption lies in that, given two reference frames, it is a natural requirement that it should be possible in principle to carry out, in each of them separately, a fairly complete experimental study of an objective class of phenomena, if it has to be concluded with a sufficient degree of certainty that they are equivalent in regard to the formulation of the laws which govern the phenomena under consideration. As to (4.i.2), it states that in equivalent frames one can prepare the same set of states. Physically, since the preparation of a state corresponds to performing certain experiments, this is already implicit in (4.i.1). (4.i.3) and (4.iv) tell us essentially that we can use the same language to describe the way by which coordinates are introduced in equivalent frames. Given a chart (U, φ) for a reference frame R, the chart $\left(U_{\omega_{RR'}}, \varphi_{\omega_{RR'}}\right)$ is interpreted as the one which in the language of the reference frame R′ plays the same role as (U, φ) in the language of R. Similarly, as regards (4.ii), the experiments a and $\gamma_{RR'}(a)$ are regarded as experiments which, in the respective languages of R and R′, are described in the same way (in other words, subjectively identical experiments). In particular (4.ii.2) states that given any three frames R, R′, R″ there a fourth frame R‴ bearing the same relation to R′ as R″ to R. Note that (4.ii.3) implies that if subjectively identical experiments in two reference frames are objectively identical, the two frames actually identify. This is of course an obvious assumption, provided the class of phenomena under consideration is sufficiently vast.

(4.iii) plays fundamental rôle. It asserts that, given two reference frames R, $R' \in \mathcal{E}$ and a state π, there is a state π' such that $\pi'(\gamma_{RR'}(a)) = \pi(a)$, for every $a \in Q$. Then, if we interpret π' as the state which, in the language of R′, is described in the same way as the state π is described in the language of R, (4.iii.1) expresses just the identity of the physical laws in equivalent frames. In the same way as for the set of experiments, (4.iii.2) expresses the identity of two reference frames in which subjectively identical states are objectively identical. As to (4.v), it expresses the natural requirement that the domains of corresponding experiments should be described by the same set of coordinates in corresponding charts.

We observe that from (4.ii) it follows that $\mathfrak{Q} = \gamma(\mathcal{E} \times \mathcal{E})$ is a subgroup of the group P (Q) of permutations of experiments. The group $\mathfrak{Q}$ is determined by the invariance of the theory with respect to all frames of $\mathcal{E}$ and will be called the *invariance group* of the equiframe $\mathfrak{E}$.

We conclude this section by giving two propositions which are related to the above defined structure and which open the way to the main theorem to be proved in the next section.

PROPOSITION 4.1. — *Let* $\mathfrak{E} = (\mathcal{E}, \gamma, \omega)$ *be an equiframe and let* $a, b \in Q$ *such that* $\Delta(a) = \Delta(b)$. *Then,* $\forall\, \sigma \in \mathfrak{Q}$, $\Delta(\sigma(a)) = \Delta(\sigma(b))$.

Proof. — If $\sigma \in \mathfrak{Q}$, $\sigma = \gamma_{RR'}$ for some R, $R' \in \mathcal{E}$. By (4.iv),

$$\forall\, (U', \varphi') \in \mathfrak{A}_{R'}, \quad \exists\, (U, \varphi) \in \mathfrak{A}_R$$

such that $(U', \varphi') = \left(U_{\omega_{RR'}}, \varphi_{\omega_{RR'}}\right)$. By (4.v),

$$\varphi'(\Delta(\sigma(a)) \cap U') = \varphi(\Delta(a) \cap U) = \varphi(\Delta(b) \cap U) = \varphi'(\Delta(\sigma(b)) \cap U')$$

and since φ' is an injection, $\Delta(\sigma(a)) \cap U' = \Delta(\sigma(b)) \cap U'$. Then the assertion follows since the domains of the charts for a given frame cover M ■

By virtue of this proposition, $\forall \sigma \in \mathcal{Q}$, $\exists$ a unique map d_σ of the set $\mathcal{D} = \{ \Delta(a) \mid a \in Q \}$ into itself such that

$$d_\sigma(\Delta(a)) = \Delta(\sigma(a)), \quad \forall a \in Q. \tag{4.1}$$

Further, it is easy to see that d_σ is a permutation of $\mathcal{D}$ whence we can define a map d of $\mathcal{Q}$ into $P(\mathcal{D})$ as

$$d : \sigma \to d_\sigma. \tag{4.2}$$

PROPOSITION 4.2. — *d is a group homomorphism.*

The proof is straightforward.

5. THE RELATIVITY GROUP OF AN EQUIFRAME

In this section we show that, provided we make a suitable assumption on the domains of experiments, the invariance group $\mathcal{Q}$ of a given equiframe can be realized isomorphically in a unique way onto a group $\mathcal{G}$ of homeomorphisms of M in such a manner that every $g \in \mathcal{G}$ transforms the domain of an experiment a into the domain of the transformed of a by the element of $\mathcal{Q}$ to which g corresponds. The assumption we make is expressed by the following

Separation Axiom. — $\forall p, q \in M$, $\exists a \in Q$ such that $p \in \Delta(a)$ and $q \notin \Delta(a)$.

This assumption is almost implicit in the axioms which define the structure of space-time in terms of reference frames, as have been introduced in section 2 : if p and q are two different space-time points, just to acknowledge that they are different we must admit that, at least in principle, we should have the possibility of distinguishing them be means of suitable (possibly idealized) physical operations, some of which extending over space-time regions containing one point and excluding the other.

We shall use the following equivalent statement of the axiom : $\forall p \in M$ denote by $\mathcal{D}_p$ the set $\{ \Delta \mid \Delta \in \mathcal{D}, p \in \Delta \}$; then

$$\bigcap_{\Delta \in \mathcal{D}_p} \Delta = p. \tag{5.1}$$

Before we establish the desired result we prove two lemmas.

LEMMA 5.1. — *Let $\mathcal{B}$ be a family of subsets of a given set* X *such that,* $\forall\, p \in X$,

$$\bigcap_{B \in \mathcal{B}_p} = p, \tag{5.2}$$

where $\mathcal{B}_p = \{B \mid B \in \mathcal{B}, p \in B\}$. *Let* K *be a subgroup of* P ($\mathcal{B}$) *such that,* $\forall\, p \in X$ *and* $\forall\, f \in K$, $f(\mathcal{B}_p) = \mathcal{B}_q$ *for some* $q \in X$. *Then* (*a*) $\forall\, f \in K$, $\exists$ *a unique* map $\tilde{f}$ *of* X *tnio itself such that* $\tilde{f}(A) = f(A)$, $\forall\, A \in \mathcal{B}$, *and* $\tilde{f}$ *is a permutation of* X; (*b*) *the* map $f \to \tilde{f}$ *is an isomorphism of* K *into* P (X).

Proof. — (*a*) By (5.2), if $f(\mathcal{B}_p) = \mathcal{B}_q$ for some q, q is unique. Hence, to every $f \in K$ we can associate a map

$$\tilde{f}: \quad p \to \tilde{f}(p) = q = \bigcap_{B \in \mathcal{B}_p} f(B) \tag{5.3}$$

of X into itself. Let $A \in \mathcal{B}$ and $r \in A$. Then $A \in \mathcal{B}_r$ whence $\tilde{f}(r) \in f(A)$ and $\tilde{f}(A) \subseteq f(A)$. Let $s \in f(A)$ and consider $t = \widetilde{f^{-1}}(s) \in f^{-1}(f(A)) = A$. We have

$$\tilde{f}(t) = \tilde{f}\left(\widetilde{f^{-1}}(s)\right) = \tilde{f}\left(\bigcap_{B \in \mathcal{B}_p} f^{-1}(B)\right) = \bigcap_{B \in \mathcal{B}_p} f(f^{-1}(B)) = \bigcap_{B \in \mathcal{B}_p} B = s.$$

Therefore $\tilde{f} \circ \widetilde{f^{-1}} = 1_X$ (and, similarly, $\widetilde{f^{-1}} \circ \tilde{f} = 1_X$) and $\tilde{f}(A) \supseteq f(A)$, so that $\tilde{f}$ satisfies the required property

$$\tilde{f}(A) = f(A) \tag{5.4}$$

and is a permutation of X. New suppose $\tilde{h}: X \to X$ such that $\tilde{h}(A) = f(A)$ $\forall\, A \in \mathcal{B}$. Then

$$\forall\, p \in X, \quad \tilde{h}(p) = \tilde{h}\left(\bigcap_{B \in \mathcal{B}_p} B\right) \subseteq \bigcap_{B \in \mathcal{B}_p} \tilde{h}(B) = \bigcap_{B \in \mathcal{B}_p} \tilde{f}(B) = \tilde{f}(p),$$

which proves the uniqueness.

(*b*) If $f, g \in K$, $p \in X$, by (5.3)

$$\widetilde{f \circ g}(p) = \bigcap_{B \in \mathcal{B}_p} f(g(B)) = \tilde{f}\left(\bigcap_{B \in \mathcal{B}_p} g(B)\right) = \tilde{f} \circ \tilde{g}(p).$$

Further, if $\tilde{f} = 1_X$ (5.4) gives $f(A) = A$, $\forall\, A \in \mathcal{B}$ namely $f = 1_{\mathcal{B}}$ ■

LEMMA 5.2. — *Let $\mathcal{Q}$ be the invariance group of an equiframe and let d be the homomorphism of $\mathcal{Q}$ into* P $(\mathcal{D})$ *defined by* (4.2). *Then,* $\forall \sigma \in \mathcal{Q}$ *and* $\forall p \in \mathrm{M}$, $d_\sigma(\mathcal{D}_p) = \mathcal{D}_q$ *for some* $q \in \mathrm{M}$.

Proof. — Let $p \in \mathrm{M}$ and $\sigma \in \mathcal{Q}$, $\sigma = \gamma_{\mathrm{RR}'}$ for some R, $\mathrm{R}' \in \mathcal{E}$. Fix $(\mathrm{U}, \varphi) \in \mathcal{A}_{\mathrm{R}}$ such that $p \in \mathrm{U}$ and let $\Delta \in \mathcal{D}_p$ ($\Delta = \Delta(a)$ for some $a \in \mathrm{Q}$). By (4.v.1) and (4.1) we have $\varphi(\Delta \cap \mathrm{U}) = \varphi_{\omega_{\mathrm{RR}'}}\left(d_\sigma(\Delta) \cap \mathrm{U}_{\omega_{\mathrm{RR}'}}\right)$. It follows that

$$q = \varphi^{-1}_{\omega_{\mathrm{RR}'}} \circ \varphi(p) \in d_\sigma(\Delta) \cap \mathrm{U}_{\omega_{\mathrm{RR}'}} \subseteq d_\sigma(\Delta) \Rightarrow d_\sigma(\mathcal{D}_p) \subseteq \mathcal{D}_q.$$

If $\Delta' \in \mathcal{D}_q$ we have

$$\begin{aligned}\varphi_{\omega_{\mathrm{RR}'}}\left(\Delta' \cap \mathrm{U}_{\omega_{\mathrm{RR}'}}\right) &= \varphi\left(d_\sigma^{-1}(\Delta') \cap \mathrm{U}\right)\\ &\Rightarrow \varphi^{-1} \circ \varphi_{\omega_{\mathrm{RR}'}}(q) = p \in d_\sigma^{-1}(\Delta') \cap \mathrm{U} \subseteq d_\sigma^{-1}(\Delta')\\ &\Rightarrow d_\sigma^{-1}(\Delta') \in \mathcal{D}_p \Rightarrow \Delta' \in d_\sigma(\mathcal{D}_p) \Rightarrow \mathcal{D}_q \subseteq d_\sigma(\mathcal{D}_p) \quad \blacksquare\end{aligned}$$

We are now in a position to establish the main

THEOREM 5.1. — *Let $\mathcal{Q}$ be the invariance group of an equiframe $\mathfrak{E} = (\mathcal{E}, \gamma, \omega)$. Then $\exists$ a unique homomorphism $\rho : \sigma \to \rho_\sigma$ of $\mathcal{Q}$ into* $\mathcal{O}(\mathrm{M})$ *such that*

$$(5.5) \qquad \rho_\sigma(\Delta(a)) = \Delta(\sigma(a)), \qquad \forall\, a \in \mathrm{Q} \quad \text{and} \quad \forall \sigma \in \mathcal{Q}.$$

Further,

(*a*) $\forall$ R, $\mathrm{R}' \in \mathcal{E}$ *and* $\forall$ $(\mathrm{U}, \varphi) \in \mathcal{A}_{\mathrm{R}}$,

$$(5.6) \qquad \left(\mathrm{U}_{\omega_{\mathrm{RR}'}}, \varphi_{\omega_{\mathrm{RR}'}}\right) = (\rho_\sigma(\mathrm{U}), \varphi \circ \hat{\rho}_\sigma^{-1}), \quad \sigma = \gamma_{\mathrm{RR}'},$$

where $\hat{\rho}_\sigma$ *denotes the restriction of* ρ_σ *to* U, *and*

(*b*) ρ *is injective.*

Proof. — By proposition 4.2 $d_{\mathcal{Q}}$ is a group of permutations of $\mathcal{D}$ and, by lemma 5.2 and the separation axiom the hypotheses of lemma 5.1 are satisfied with $\mathrm{X} = \mathrm{M}$, $\mathcal{B} = \mathcal{D}$ and $\mathrm{K} = d_{\mathcal{Q}}$. Then, $\forall \sigma \in \mathcal{Q}$, $\exists$ a unique map $\widetilde{d_\sigma} : \mathrm{M} \to \mathrm{M}$ such that

$$(5.7) \qquad \widetilde{d_\sigma}(\Delta(a)) = \Delta(\sigma(a)), \quad \forall\, a \in \mathrm{Q}$$

and $\widetilde{d_\sigma}$ is a permutation of **M**. Let $\sigma \in \mathcal{Q}$, $\sigma = \gamma_{\mathrm{RR}'}$ for some R, $\mathrm{R}' \in \mathcal{E}$ and let $(\mathrm{U}, \varphi) \in \mathcal{A}_{\mathrm{R}}$. Then, by lemmas 5.1 and 5.2,

$$(5.8) \qquad \widetilde{d_\sigma}(p) = \bigcap_{\Delta \in \mathcal{D}_p} d_\sigma(\Delta) = q = \varphi^{-1}_{\omega_{\mathrm{RR}'}} \circ \varphi(p) \in \mathrm{U}_{\omega_{\mathrm{RR}'}}, \quad \forall\, p \in \mathrm{U}$$

and therefore $\widetilde{d_\sigma}\ (\mathrm{U}) \subseteq \mathrm{U}_{\omega_{\mathrm{RR}'}}$. Conversely, if $\bar{q} \in \mathrm{U}_{\omega_{\mathrm{RR}'}}$ we have $p = \varphi^{-1} \circ \varphi_{\omega_{\mathrm{RR}'}}(\bar{q}) \in \mathrm{U}$ and obviously $\widetilde{d_\sigma}\,(\bar{p}) = \bar{q}$ so that $\widetilde{d_\sigma}\ (\mathrm{U}) \supseteq \mathrm{U}_{\omega_{\mathrm{RR}'}}$ and we get

$$\widetilde{d_\sigma}\,(\mathrm{U}) = \mathrm{U}_{\omega_{\mathrm{RR}'}}. \tag{5.9}$$

Then, since φ and $\varphi_{\omega_{\mathrm{RR}'}}$ are homeomorphisms we conclude from (5.8) and (5.9) that the restriction of $\widetilde{d_\sigma}$ to U is a homeomorphism of U onto $\mathrm{U}_{\omega_{\mathrm{RR}'}}$. Hence, by (4.iv) and since $\mathcal{U}_{\mathrm{R}} = \{\, \mathrm{U} \mid \mathrm{U} \subseteq \mathrm{M};\ (\mathrm{U}, \varphi) \in \mathcal{A}_{\mathrm{R}} \,\}$ and $\mathcal{U}_{\mathrm{R}'} = \{\, \mathrm{U}' \mid \mathrm{U}' \subseteq \mathrm{M};\ (\mathrm{U}', \varphi') \in \mathcal{A}_{\mathrm{R}'} \,\}$ are open coverings of M, $\widetilde{d_\sigma} \in \mathcal{O}\,(\mathrm{M})$.

Set $\rho = \widetilde{\ } \circ d$. Then ρ is a map of $\mathcal{Q}$ into $\mathcal{O}\,(\mathrm{M})$ which, by (5.7), satisfies (5.5) Further, ρ is a homomorphism of $\mathcal{Q}$ into $\mathcal{O}\,(\mathrm{M})$ because d is a homomorphism of $\mathcal{Q}$ into $\mathrm{P}\,(\mathcal{O})$ (by proposition 4.2) and $\widetilde{\ }$ is an isomorphism of $d_{\mathcal{Q}}$ into $\mathcal{O}\,(\mathrm{M})$ (by lemma 5.1 and by the above). The uniqueness of ρ is easily proved using (5.1).

(*a*) Since $\rho_\sigma \overset{\mathrm{Def}}{=} \widetilde{d_\sigma}$, (5.6) follows from (5.8) and (5.9).

(*b*) Let $\rho = 1_{\mathrm{M}}$. Then, by (5.6), $\mathcal{A}_{\mathrm{R}} = \mathcal{A}_{\mathrm{R}'}$, implying $\mathrm{R} = \mathrm{R}'$ by axiom 2.1. Hence $\sigma = \gamma_{\mathrm{RR}} = 1_{\mathrm{Q}}$ ■

The group $\rho_{\mathcal{Q}}$, which provides an isomorphic realization of the invariance group $\mathcal{Q}$ by space-time homeomorphisms will be called the *relativity group* of the given equiframe $\mathfrak{E}$ and henceforth denoted by $\mathcal{G}$.

Set $\eta = \rho \circ \gamma : \mathcal{E} \times \mathcal{E} \to \mathcal{G}$ and define

$$\mathcal{G}_{\mathrm{R}} = \{\, g \mid g \in \mathcal{G};\ g = \eta_{\mathrm{RR}'},\ \mathrm{R}' \in \mathcal{E} \,\}, \quad \mathrm{R} \in \mathcal{E}. \tag{5.10}$$

Then, by (4.ii) and the theorem,

$$\mathcal{G}_{\mathrm{R}} = \mathcal{G}, \quad \forall\ \mathrm{R} \in \mathcal{E} \tag{5.11}$$

and the map

$$\lambda_{\mathrm{R}} :\ \ \mathrm{R}' \to \eta_{\mathrm{RR}'} \tag{5.12}$$

is a bijection of $\mathcal{E}$ onto $\mathcal{G}$, $\forall\ \mathrm{R} \in \mathcal{E}$.

If $\mathcal{A}_{\mathrm{R}}$ contains a single chart (denoted $(\mathrm{M}, \varphi_{\mathrm{R}})$), (5.6) writes [from (4.ii)] we have $\gamma_{\mathrm{RR}'}^{-1} = \gamma_{\mathrm{R}'\mathrm{R}}$, hence also $\eta_{\mathrm{RR}'}^{-1} = \eta_{\mathrm{R}'\mathrm{R}}$ $(\mathrm{M}, \varphi_{\mathrm{R}'}) = (\mathrm{M}, \varphi \circ \eta_{\mathrm{R}'\mathrm{R}})$, $\forall\ \mathrm{R}, \mathrm{R}' \in \mathcal{E}$, whence $\eta_{\mathrm{RR}'} = \varphi_{\mathrm{R}}^{-1} \circ \varphi_{\mathrm{R}'}$ so that, in this case,

$$\mathcal{G} = \{\, g \mid g \in \mathcal{O}\,(\mathrm{M});\ g = \varphi_{\mathrm{R}}^{-1} \circ \varphi_{\mathrm{R}'};\ \mathrm{R}, \mathrm{R}' \in \mathcal{E} \,\} \tag{5.13}$$

and also, by virtue of (5.11),

$$\mathcal{G} = \mathcal{G}_{\mathrm{R}} = \{\, g \mid g \in \mathcal{G};\ g = \varphi_{\mathrm{R}}^{-1} \circ \varphi_{\mathrm{R}'};\ \mathrm{R}' \in \mathcal{E} \,\},\ \mathrm{R} \in \mathcal{E}. \tag{5.14}$$

We conclude this section with two remarks. First we stress that, as one easily see, the separation axiom is essential for the proof of the theorem, even in the special case when the atlases of the frames of $\mathcal{E}$ contain each a single chart. Second, we note that if we give up axiom 2.1, ρ fails to be injective under the natural assumption $\mathcal{C}\!l_{R} = \mathcal{C}\!l_{R'} \Rightarrow \omega_{RR'} = 1_{\mathcal{C}\!l_R}$. The non unit elements σ of the kernel of ρ satisfy $\Delta(\sigma(a)) = \Delta(a)$, $\forall\, a \in Q$, thus appearing clearly as internal symmetry transformations. This shows axiom 2.1 to be essentially equivalent, as regards the invariances of a theory, to the statement that internal symmetry transformations do not allow for a « passive » interpretation.

6. A TOPOLOGY ON THE RELATIVITY GROUP

In this section we discuss the possibility of endowing the relativity group $\mathcal{G}$ of an equiframe $\mathfrak{E}$ with a topology $\mathcal{T}$ satisfying some acceptable requirements. First we would obviously ask $\mathcal{T}$ to make $\mathcal{G}$ into a (Hausdorff) topological group operating continuously in M by the map $(g, p) \to g(p)$, $g \in \mathcal{G}$, $p \in M$. Secondly, we would like $\mathcal{T}$ to have some reasonable features from the physical and intuitive point of view. To discuss this point suppose $\mathcal{G}$ has already been endowed with a certain topology $\mathcal{T}$. It is then natural to think at a notion of closeness among the frames of $\mathcal{E}$ by saying that two frames R and R′ are " sufficiently close to each other " if $\eta_{RR'}$ falls in a " sufficiently small " neighbourhood of the identity. This " closeness relation " also reveals clearly a uniform character over $\mathcal{E}$. To formalize these statements, let $\mathcal{N}(e)$ denote the neighbourhood filter of the identity of $\mathcal{G}$ and $\forall\, N \in \mathcal{N}(e)$ define $W(N) = \{ (R, R') \mid R, R' \in \mathcal{E}, \eta_{RR'} \in N \}$. It is then an easy matter to show that the family $\{ W(N) \}_{N \in \mathcal{N}(e)}$ is a fundamental system of surroundings for a uniform structure $\mathfrak{U}$ on $\mathcal{E}$ which is the image of the right uniform structure $\mathfrak{T}$ of $\mathcal{G}$ under the inverse map μ_R of (5.12) for any fixed $R \in \mathcal{E}$. In this way $\mathcal{E}$ receives canonically a uniform structure $\mathfrak{U}$ isomorphic to $\mathfrak{T}$ and a corresponding topology $\mathcal{U}$ homeomorphic to $\mathcal{T}$, which formalizes naturally the above intuitive idea of closeness of frames.

Now suppose M to be endowed with the finest uniformity $\mathfrak{V}$ which is compatible with its topology and denote by $\mathcal{V}$ the set of surroundings which define $\mathfrak{V}$. Let $R \in \mathcal{E}$ and consider a varying frame $R' \in \mathcal{E}$ which we ideally make to approach R (in the topology $\mathcal{U}$) (4). Let a be an

(4) This and the following statements will presently be correctly formalized. For the time being, we rely on their intuitive meaning.

experiment which we suppose to be performed in R and consider the equivalent experiment $\gamma_{RR'}(a)$ as performed in R'. We expect that in this idealized situation, as R' approaches R, $\gamma_{RR'}(a)$ " approaches " a, in the sense that the physical devices and operations by which $\gamma_{RR'}(a)$ is carried out become less and less different from those which realize a. In particular, the corresponding space-time domains $\Delta(a)$ and $\Delta(\gamma_{RR'}(a))$ will approach one another and overlap in the limit.

Since $\Delta(a)$ is a space-time region of finite extension, we would ask this process to take place uniformly in $\Delta(a)$, in the sense that $\forall\, V \in \mathcal{V}$ we can make $\gamma_{RR'}(p)$ close to p of the same order V, $\forall\, p \in \Delta(a)$, provided R' is taken close enough to R. Conversely, we would ask uniform convergence between domains of corresponding experiments to ensure convergence between frames.

These requirements can be fulfilled if we endow $\mathcal{G}$ with the topology of uniform convergence in the domains of experiments, as shown by the following.

PROPOSITION 6.1. — *Define $\mathcal{T}$ on $\mathcal{G}$ to be the topology of uniform convergence in the domains of experiments. Then (a) $\mathcal{T}$ makes $\mathcal{G}$ into a Hausdorff topological group operating continuously in M by the map $(g, p) \to g(p)$, $g \in \mathcal{G}$, $p \in M$ and (b) a filter Φ on $\mathcal{E}$ converges to $R \in \mathcal{E}$ in the topology $\mathcal{U} = \mu_R(\mathcal{T})$ on $\mathcal{E}$ iff it has the following property (α): $\forall\, V \in \mathcal{V}$ and $\forall\, a \in Q$, $\exists\, F \in \Phi$ such that if $R' \in F$, then $(p, \gamma_{RR'}(p)) \in V$, $\forall\, p \in \Delta(a)$.*

Proof. — (a) Since M is Hausdorff and the set $\mathcal{D}$ of domains of experiments covers M, $\mathcal{T}$ is Hausdorff [8]. Since every $\Delta \in \mathcal{D}$ is contained in a compact subset of M and every compact subset of M is contained in a finite union of Δ' s, $\mathcal{T}$ is identical to the topology of compact convergence [9]. Then, since M is locally connected [10], $\mathcal{T}$ is also identical to the topology of uniform convergence in M, when M is endowed with the uniformity $\mathfrak{B}'$ induced by the unique uniformity of its Alexandroff compactification [11]. Hence $\mathcal{T}$ is compatible with the group structure of $\mathcal{G}$ [12]. Further, it is the coarsest topology for which the map $(g, p) \to g(p)$ of $\mathcal{G} \times M$ onto M is continuous [13].

(b) $\forall\, V \in \mathcal{V}$ and $\forall\, a \in Q$ define

$$W(a, V) = \{\, g \mid g \in \mathcal{G};\ (p, g(p)) \in V \text{ if } p \in \Delta(a) \,\}. \tag{6.1}$$

As V runs over $\mathcal{V}$ and a over Q, the set $\mathcal{X}$ of finite intersections of the $W(a, V)'$ s forms a neighbourhood base of the identity of $\mathcal{G}$ endowed with $\mathcal{T}$ [14]. Endow $\mathcal{E}$ with the topology $\mathcal{U}$ and let $\mathcal{N}_R$ denote the neighbourhood filter of $R \in \mathcal{E}$. Since $\mathcal{U} = \mu_R(\mathcal{T})$ a base for $\mathcal{N}_R$ is $\{\mu_R(X)\}_{X \in \mathcal{X}}$ and if $X = \bigcap_{i=1}^{n} W(a_i, V_i)$, $V_i \in \mathcal{V}$, $a_i \in Q$, we have, by (6.1),

$$\mu_R(X) = \{\, R' \mid R' \in \mathcal{E};\ (p, \gamma_{RR'}(p)) \in V_i \text{ if } p \in \Delta(a_i),\ i = 1, 2, \ldots, n \,\}. \tag{6.2}$$

Let Φ be a filter on $\mathcal{E}$ converging to R. Then $\forall\, V \in \mathcal{V}$ and $\forall\, a \in Q$, $\exists\, F \in \Phi$ such that $F \subseteq \mu_R(W(a, V))$, namely Φ has the property (α). Conversely, suppose Φ has the property (α) and let $N \in \mathcal{N}_R$. Then N contains a set $\mu_R(X)$ for a suitable sequence pair $V_i \in \mathcal{V}$, $a_i \in Q$, $i = 1, 2, \ldots, n$. $\exists\, F_i \in \Phi$ satisfying (α) with $V = V_i$, $a = a_i$ and by (6.2), $F = \bigcap_{i=1}^{n} F_i \subseteq \mu_R(X) \subseteq N$, which proves that Φ converges to R ■

We conclude this section with two remarks.

In the definition of $\mathcal{T}$ the choice of the uniformity on M to be the finest compatible with the space-time topology is immaterial. Indeed, on the set of continuous maps of M into itself the topology of compact convergence depends only on the topology of M [15].

In the particular case of special relativity, in which $\mathcal{E}$ is taken to be the set of inertial reference frames and $\mathcal{G}$ is the Poincaré group $\mathcal{P}$ or, as a limiting case, the Galilei group $\mathcal{M}$ (*see* next section) it can be shown by a slight extension of an argument by Bourbaki [16] that the topology of compact convergence is identical on $\mathcal{P}$ (respectively on $\mathcal{M}$) to the usual Lie group topology.

7. SPECIAL RELATIVITY

We devote this section to the construction of the geometrical theory of special relativity. This can be done by endowing the space-time manifold with a specific equiframe which we call an *inertial equiframe* and denote by $\mathfrak{I}$, and by giving a set of axioms for both space-time and $\mathfrak{I}$. The justification for the choice of the axioms will be that they imply that the corresponding relativity group $\mathcal{G}$ can be identified to the proper orthochronous Poincaré group (or, as a limiting case, to the Galilei group).

AXIOM 7.1. — *Let* M_0 *denote the space-time manifold of special relativity subject to the axioms of section* 2. *We assume that* M_0 *is simply connected and that there is a map* $(a, p) \to a + p$ *of* $R^4 \times M$ *into* M_0 *by which the additive group* R^4 *operates continuously and transitively in* M_0 *and such that,* $\forall\, a \in R^4$, *the map*

$$t(a):\quad p \to a + p \tag{7.1}$$

is an element of $\mathcal{G}$.

Axiom 7.1 can be regarded as expressing the homogeneity of space-time in special relativity. It implies :

PROPOSITION 7.1. — M_0 *is homeomorphic onto* R^4 *(hence it admits of global charts) and* R^4 *operats freely in* M_0 (5).

Proof. — Since R^4 operates transitively in M_0, $\forall\, p \in M_0$ the map

$$\alpha_p : \quad a \to a + p \tag{7.2}$$

of R^4 into M_0 is surjective. Denote by H_p the stabilizer of p and endow the coset space R^4/H_p with the usual quotient topology, namely, the finest topology for which the canonical map $\beta_p : a \to a + H_p$ is continous. α_p factorizes as $\alpha_p = \gamma_p \circ \beta_p : R^4 \overset{\beta_p}{\to} R^4/H_p \overset{\gamma_p}{\to} M_0$, where $\gamma_p : a + H_p \to a + p$ is a continuous bijection of R^4/H_p onto M_0. Then, since M_0 is locally compact and R^4 is locally compact and has a countable base, γ_p is a homeomorphism of R^4/H_p onto M_0 and H_p is closed [17] hence isomorphic to a direct product of the form $R^l \otimes Z^m$ $(0 \leq l + m \leq 4)$, where Z denotes the additive group of the integers [18].

Therefore, R^4/H_p is isomorphic to $R^{4-l-m} \otimes T^m$, where $T = R/Z$. By the simple connectedness of M_0, $m = 0$ and then, by the topological invariance of dimension, $l = 0$. Therefore $H_p = 0$, namely R^4 operates freely in M_0, and γ_p identifies to γ_p hence it is a homeomorphism of R^4 onto M_0 ■

We call the elements of R^4 *space-time translations* and R^4, which we identify to $t\,(R^4)$ by proposition 7.1, the *group of space-time translations.*

We call a global chart (M_0, φ) *affine* if there is a map $l_\varphi : R^4 \to R^4$ such that

$$\varphi\,(a + p) = \varphi\,(p) + l_\varphi\,(a), \qquad p \in M_0, \qquad a \in R^4. \tag{7.3}$$

PROPOSITION 7.2. — *Let* $\mathfrak{W} = \{\,(M_0, \varphi_\lambda)\,\}_{\lambda \in \Lambda}$ *be the set of affine charts of* M_0. *Then,*

(a) $\forall\, \lambda \in \Lambda$ *the map* $l\,\varphi_\lambda$ *is an automorphism of the vector space* R^4, *which we call the* automorphism *of the chart* (M_0, φ_λ);

(b) $\forall\, p \in M_0$ *and* $\forall$ *automorphism* l *of the vector space* R^4 $\exists$ *one and only one* $\lambda \in \Lambda$ *such that* (M_0, φ_λ) *has automorphism* l *and center* p (6);

(5) Given a group G operating in a set X we say that G operates freely in X if the stabilizer of every point of X is reduced to the neutral element of G.

(6) Given a chart (U, φ) if $p \in U$ and $\varphi\,(p) = 0$ we call p the center of the chart.

(c) *$\forall$ fixed $\mu \in \Lambda$, the set $\{\varphi_\lambda \circ \varphi_\mu^{-1}\}_{\lambda \in \Lambda}$ of homeomorphisms of R^4 is the affine group $AF_4(R)$ of R, namely the group of all transformations of R^4 of the type $a \to m(a) + b$, $b \in R^4$, m an automorphism of the vector space R^4.*

The proof is straightforward but tedious and will be omitted.

We call an atlas of M_0 *affine* if it contains only one chart and this is affine.

Axiom 7.2. — *M_0 is endowed with an eqviframe $\mathfrak{I} = (\mathcal{J}, \gamma, \omega)$ suc that $\mathcal{C}_S$ is an affine atlas, $\forall$ $S \in \mathcal{J}$.*

We call $\mathfrak{I}$ an *inertial equiframe* and if $S \in \mathcal{J}$, we call it an *inertial reference frame.* If $S \in \mathcal{J}$, we denote by (M_0, φ_S) the (unique) chart of $\mathcal{C}_S$. Let $p \in M_0$, $S \in \mathcal{J}$. Then the coordinates $\{\varphi_S^\mu(p)\}_{\mu=1,2,3,4}$ of the vector $\varphi_S(p)$ are the *space-time coordinates* of the event p relative to the inertial frame S. (The triplet $\{\varphi_S^i(p)\}_{i=1,2,3}$ is the set of space coordinates and $\varphi_S^4(p)$ is the time coordinate.) The center of (M_0, φ_S) is the *space-time origin* of the frame S. We denote by $\mathcal{J}_S$ the equivalence class of S with respect to the relation " S and S′ have the same space-time origin ".

Now consider $\mathcal{G}$, the relativity group of $\mathfrak{I}$. If S is an arbitrary fixed element of $\mathcal{J}$ we have, by (5.14),

$$(7.4) \qquad \mathcal{G} = \{ g \mid g \in \mathcal{O}(M_0);\ g = \varphi_S^{-1} \circ \varphi_{S'},\ S' \in \mathcal{J} \}.$$

By proposition 7.2 and by axiom 7.2 the map

$$(7.5) \qquad \nu_S : \ \eta_{S'S} = \varphi_S^{-1} \circ \varphi_{S'} \to \varphi_S \circ (\varphi_S^{-1} \circ \varphi_{S'}) \circ \varphi_S^{-1} = \varphi_{S'} \circ \varphi_S^{-1}$$

is an isomorphism of $\mathcal{G}$ onto a subgroup $\mathcal{S}$ of $AF_4(R)$ which can easily be seen to be independent of S. An element of $\mathcal{S}$ is of the form $\varphi_{S'} \circ \varphi_S^{-1}$, with S, $S' \in \mathcal{J}$, and it represents the affine transformation expressing the space-time coordinates of an event p, as measured in S′, as functions of the coordinates of the same event p, as measured in S. We call $\varphi_{S'} \circ \varphi_S^{-1}$ the transformation connecting S to S′ and write

$$(7.6) \quad \varphi_{S'} \circ \varphi_S^{-1} : \quad x_\mu \to x'_\mu = \sum_{\nu=1}^{4} L_{\mu\nu}^{(S,S')} x_\nu + a_\mu^{(S,S')}, \qquad (\mu = 1, 2, 3, 4),$$

where $x_\mu = \varphi_S^\mu(p)$, $x'_\mu = \varphi_{S'}^\mu(p)$, $p \in M_0$. We identify $\mathcal{G}$ to $\mathcal{S}$ and determine the structure of $\mathcal{S}$.

Proposition 7.3. — *$\mathcal{S}$ contains as a subgroup the group of translations of R^4, which is identical to $\nu_S(t(R^4))$, $S \in \mathcal{J}$.*

Proof. — Let $b \in R^4$, $S \in \mathcal{J}$, and let l_S denote the automorphism of the affine chart (M_0, φ_S). By axiom 7.1 $t(l_S^{-1}(b))$ is an element of $\mathcal{G}$ so

$\nu_S(t(l_S^{-1}(b))) \in \mathcal{S}$ and we have

$$\begin{aligned}\nu_S(t(l_S^{-1}(b))(a) &= \varphi_S \circ t(l_S^{-1}(b)) \circ \varphi_S^{-1}(a) \\ &= \varphi_S(l_S^{-1}(b) + \varphi_S^{-1}(a)) \\ &= \varphi_S(\varphi_S^{-1}(a)) + l_S(l_S^{-1})(b)) = a + b \quad \blacksquare\end{aligned}$$

By virtue of the above proposition we can make the identification

$$\mathcal{S} = R^4 \otimes \mathcal{L} \quad (7.7)$$ (⁷),

where $\mathcal{L}$ is the group of automorphisms of R^4 of the form

$$\varphi_{S'} \circ \varphi_S^{-1} : \quad x_\mu \to x'_\mu = \sum_{\nu=1}^{4} L_{\mu\nu}^{(S, S')} x_\nu \qquad (\mu, \nu = 1, 2, 3, 4) \quad (7.8)$$

where S can be fixed arbitrarily and $S' \in \mathcal{J}_S$. We identify cononically the group of automorphisms of R^4 to the group GL (4, R) of 4×4 non singular matrices. With this identification, $\mathcal{L}$ is the set of matrices $\{ L^{(S,S')} = (L_{\mu\nu}^{(S,S')}) \}_{S' \in \mathcal{J}_S}$, with S an arbitrary fixed element of $\mathcal{J}$.

Define the following subgroup of GL (4, R) :

$$\mathcal{H} = \{ H \mid H \in GL(4, R);\ H_{i4} = 0,\ i = 1, 2, 3 \}, \quad (7.9)$$

$$\mathcal{C} = \{ C \mid C \in GL(4, R);\ C_{i4} = C_{4i} = 0,\ i = 1, 2, 3;\ C_{44} = 1;\ (C_{ik})_{i,k=1,2,3} \in SO(3) \}, \quad (7.10)$$

$$\begin{cases} \mathcal{L}(\lambda) = \{ L \mid L \in GL(4, R);\ L^T g(\lambda) L = g(\lambda);\ g(\lambda) \\ \qquad = \mathrm{diag}(-\lambda, -\lambda, -\lambda, 1);\ \det L = 1,\ L_{44} \geq 1 \}, \\ \qquad 0 < \lambda < +\infty, \end{cases} \quad (7.11)$$

$$\mathcal{M} = \{ M \mid M \in GL(4, R);\ M_{4i} = 0,\ i = 1, 2, 3;\ M_{44} = 1;\ (M_{ik})_{i,k=1,2,3} \in SO(3) \}. \quad (7.12)$$

$\mathcal{L}(\lambda)$ is the proper orthochronous Lorentz group corresponding to an invariant velocity equal to $\lambda^{-\frac{1}{2}}$, while $\mathcal{M}$ is the proper orthochronous (homogeneous) Galilei group.

The subgroup $\mathcal{L}_R = \mathcal{L} \cap \mathcal{H}$ of $\mathcal{L}$ is physically interpreted to be the group of transformations connecting frames of $\mathcal{J}_S$ which are at rest relative to each other.

AXIOM 7.3. — (α) *$\mathcal{L}_R$ is a proper subgroup of $\mathcal{L}$ and* (β) $\mathcal{L}_R = \mathcal{C}$. Physically, this axiom states : (*a*) that relative motion between frames of $\mathcal{J}$

(⁷) The symbol $\otimes$ denotes semidirect product.

is allowed, (*b*) that in each frame clocks have been syncronized in a standard way, (*c*) that time has a unidirectional flow and (*d*) that with respect to each frame of $\mathcal{I}$ space is euclidean and isotropic. It is further implicit that the standards of length and time are the same for all frames.

The structure of $\mathcal{L}$ is determined throught axiom 7.3 by the following theorem, whose proof can be found in [19] (compare also [20] for a simpler proof under an additional continuity assumption) :

THEOREM 7.1. — *Let* Φ *denote the family of subgroups of* GL (4, R) *defined by* $\mathcal{L} \in \Phi$ *iff* $\mathcal{L} \cap \mathcal{K} = \mathcal{C}$ *and* $\mathcal{C}$ *is a proper subgroup of* $\mathcal{L}$. *Then* $\Phi = \{ \mathcal{G}_\lambda \}_{\lambda \in [0, +\infty)}$ *where* (*a*) *if* $0 < \lambda < +\infty$, $\mathcal{G}_\lambda = \mathcal{L}(\lambda)$ *and* (*b*) $\mathcal{G}_0 = \mathcal{M}$.

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