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On Spatially-averaged Electrokinetics of Point Charges and Maxwell's Equations

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Abstract Maxwell-like field relations which describe spatially-averaged kinematic behaviour of electrons and atomic nuclei (modelled as point charges) are obtained at any prescribed scale using weighting function methodology. Upon appeal to the experimental laws of Coulomb and Biot-Savart, and to dimensional considerations, these relations yield the macroscopic Maxwell equations as they pertain to electrostatics and magnetostatics. Generalisation to classical macroscopic electrodynamics is effected by taking account of signal transmission delay and selection of appropriate retardation potentials. Unlike previous derivations, no appeal is made to the microscopic field relations of Lorentz.

Keywords Macroscopic electrokinetics · Maxwell's equations · Weighting functions

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1 Introduction

The foundations of electromagnetic theory were established in the nineteenth century to explain experimental evidence associated with a wide variety of phenomena concerning electric charge, its flow, and links with magnetic effects. In a two-volume treatise Maxwell in 1873 presented a meticulous and comprehensive survey [1] of phenomena and experimentation, and developed a theoretical framework within which results could be interpreted. In a radical departure from the Newtonian concept of action at a distance, Maxwell's account introduced relations between fields (that is, continuous functions of position and time) holding in the aether, on the basis of which he conjectured the electromagnetic nature of light. In particular Maxwell's equations included (using modern notation: see Hendry [2], Chap. 6)

$$\operatorname{div}\mathbf{D} = \rho, \tag{1.1}$$

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$$\partial \mathbf{D}/\partial t + \mathbf{j} = \operatorname{curl} \mathbf{H}, \tag{1.2}$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \tag{1.3}$$

and

$$\mathbf{f} = \mathbf{v} \times \mathbf{B} - \partial \mathbf{A}/\partial t - \nabla \psi. \tag{1.4}$$

Here $\mathbf{D}$ denotes electric displacement, ρ charge density, $\mathbf{j}$ electric current density, $\mathbf{H}$ magnetic displacement, $\mathbf{B}$ magnetic field, $\mathbf{A}$ magnetic potential, $\mathbf{f}$ electromotive force, $\mathbf{v}$ velocity (relative to the aether), and ψ electrostatic potential.

Immediately (1.3) yields

$$\operatorname{div} \mathbf{B} = 0. \tag{1.5}$$

Further, on taking the divergence of relation (1.2) and the time derivative of (1.1) it follows that

$$\partial \rho/\partial t + \operatorname{div} \mathbf{j} = 0. \tag{1.6}$$

If the electric field $\mathbf{E}$ is defined via

$$\mathbf{E} := -\partial \mathbf{A}/\partial t - \nabla \psi, \tag{1.7}$$

then the time derivative of (1.3) yields

$$\partial \mathbf{B}/\partial t = -\operatorname{curl} \mathbf{E}, \tag{1.8}$$

and relation (1.4) may be written as

$$\mathbf{f} = \mathbf{v} \times \mathbf{B} + \mathbf{E}. \tag{1.9}$$

The relations which are now considered to be the most fundamental, and which bear Maxwell's name, are (1.1), (1.2), (1.5) and (1.8).

Attempts to deduce Maxwell's equations from dynamical principles were restricted by contemporaneous understanding of the nature of matter. In 1902 Lorentz [3, 4] proposed a system of Maxwell-like equations which related to individual small isolated regions of charged matter he termed electrons. Each electron was characterised by a charge density field ρ, which vanished outside the region it occupied, together with fields of velocity $\mathbf{v}$, electric displacement $\mathbf{d}$, and magnetic force $\mathbf{h}$. In stationary aether these were to satisfy (cf. [4], equations I, II, IV, V, VI)

$$\operatorname{div} \mathbf{d} = \rho, \qquad \partial \rho/\partial t + \operatorname{div} \rho \mathbf{v} = 0, \tag{1.10}$$

$$\operatorname{div} \mathbf{h} = 0, \qquad \operatorname{curl} \mathbf{h} = c^{-1}(\partial \mathbf{d}/\partial t + \rho \mathbf{v}), \tag{1.11}$$

and

$$\partial \mathbf{h}/\partial t = -c \operatorname{curl} \mathbf{d}, \tag{1.12}$$

where c denotes the speed of light in the aether. Further, 'the force, reckoned per unit charge, which the aether exerts on a charged element of volume', was postulated (cf. [4], equation VII) to be

$$\mathbf{f} = \mathbf{d} + c^{-1} \mathbf{v} \times \mathbf{h}. \tag{1.13}$$

Here **d** was identified as 'the electric force that would act on an immovable charge'. The foregoing fields were considered to change rapidly and irregularly, and observable/measurable quantities were to be identified with local spatial averages computed over many charges (cf. [3]). Specifically, the average of any field F, evaluated at location $\mathbf{x}$, is

$$\langle F\rangle(\mathbf{x}) := V^{-1}\int_{C(\mathbf{x})} F\,dv. \tag{1.14}$$

Here the volume integral is taken over the averaging region $C(\mathbf{x})$ of volume V. Relations (1.10)–(1.13) were averaged using properties[1]

$$\partial/\partial t\big\{\langle F\rangle\big\} = \langle \partial F/\partial t\rangle, \qquad \nabla\big\{\langle F\rangle\big\} = \langle \nabla F\rangle. \tag{1.15}$$

Distinguishing between electrons associated with conduction, polarisation, and magnetisation, Lorentz obtained the macroscopic Maxwell Eqs. (1.1), (1.5), (1.8), and a version of (1.2) of the form

$$\operatorname{curl}\mathbf{H} = \mathbf{C} \quad \text{with } \operatorname{div}\mathbf{C} = 0. \tag{1.16}$$

Identification of composite current $\mathbf{C}$ with $\partial\mathbf{D}/\partial t + \mathbf{j}$ yields (1.2) and, via (1.1) and $(1.16)_1$, also (1.6). Further, Lorentz effected decompositions

$$\mathbf{D} = \mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{H} = \mathbf{B} - \mathbf{M}, \tag{1.17}$$

where $\mathbf{P}$ and $\mathbf{M}$ represented electric and magnetic polarisation densities.

Many studies have subsequently refined Lorentz' pioneering work. In particular, once electrons and atomic nuclei were established to be the fundamental and discrete carriers of charge, it became natural to model these entities as point charges, and to attempt to introduce such knowledge into the microscopic relations. Further, averaging procedures, together with modelling assumptions concerning subatomic behaviour, have been clarified. Specifically, averaging may be purely spatial (Van Vleck [6]), jointly in space and time (Rosenfeld [7]), or statistical (via designation of an appropriate ensemble: Mazur and Nijboer [8]). These approaches were reviewed by de Groot [9] who also discussed covariant derivations. A more rigorous approach to spatial averaging via the introduction of a weighting function was employed by Russakoff [10].

The aforementioned works are based upon the microscopic relations (1.10)–(1.13), even if atomicity is introduced via expressions for ρ and $\mathbf{j}\,(:= \rho\mathbf{v})$ in terms of sums involving δ-functions and instantaneous point charge locations and velocities. Such atomicity is somewhat at variance with the interpretations of **d** and **h**, no matter how locally these fields are defined. It is helpful to bear in mind relevant spatial scales. The respective sizes of nuclei and atoms are of orders 10^{-15} m and 10^{-10} m, and so the Lorentz equations, postulated to model electromagnetic behaviour within atoms, relate to behaviour at scale 10^{-10} m or less. On the other hand Maxwell's equations pertain to reproducible macroscopic behaviour at scales often described as 'physically infinitesimal' but 'microscopically large'. Practically speaking, field values must be related to measurements, and hence to spatial and temporal sensitivity of monitoring devices. In any specific physical context it is thus for experimentation to determine the scales of length and time at which Maxwell's relations provide a valid description.

[1] Derivation of these results can be made rigorous by implementing cellular averaging in terms of a weighting function: cf. Murdoch [5], p. 61 and pp. 197–199.

Here the approach is from the outset entirely atomistic and no appeal is made to postulated microscopic relations. Electrons and atomic nuclei are modelled as point charges. Spatial averaging of the kinetic behaviour of any set of charges is effected in terms of a scale-dependent weighting function w. The starting point in Sect. 2 is the definition of a charge density field ρ_w whose time derivative immediately introduces a current density $\mathbf{j}_w$ which satisfies (1.6). Any solution $\mathbf{a}$ to $\operatorname{div}\mathbf{a} = w$ leads directly to relations of forms (1.1) and (1.2), with specific definitions of the electrokinetic fields $\mathbf{D}_w$ and $\mathbf{H}_w$. A natural choice of w is introduced in Sect. 3 which accords equal weighting to charges within a prescribed distance ϵ from any point $\mathbf{x}$ at which an average is to be computed, and zero weighting to charges further than $\epsilon + \delta$ from $\mathbf{x}$, with $\delta \ll \epsilon$. For distances in the range $[\epsilon, \epsilon + \delta]$ weighting corresponds to a choice of mollifier which ensures that w is everywhere smooth. The assumption that the corresponding function $\mathbf{a}$ be isotropic leads to natural decompositions $\mathbf{D}_w = \mathbf{P}_w + \boldsymbol{\mathcal{E}}_w$ and $\mathbf{H}_w = -\mathbf{M}_w + \boldsymbol{\mathcal{B}}_w$, together with potentials ψ_w and $\boldsymbol{\mathcal{A}}_w$ for which $\boldsymbol{\mathcal{E}}_w = -\nabla\psi_w$, $\boldsymbol{\mathcal{B}}_w = \operatorname{curl}\boldsymbol{\mathcal{A}}_w$ and $\partial\psi_w/\partial t + \operatorname{div}\boldsymbol{\mathcal{A}}_w = 0$. The contributions to values $\boldsymbol{\mathcal{E}}_w(\mathbf{x})$ and $\boldsymbol{\mathcal{B}}_w(\mathbf{x})$ from any charge q_i distant further than $\epsilon + \delta$ from $\mathbf{x}$ are shown to be $-q_i\mathbf{u}_i/4\pi u_i^3$ and $q_i\mathbf{u}_i \times \mathbf{v}_i/4\pi u_i^3$, respectively. (Here $\mathbf{u}_i$ denotes the displacement of q_i from $\mathbf{x}$, u_i its magnitude, and $\mathbf{v}_i$ its velocity.) Additional time averaging is introduced in Sect. 4 to elucidate the physical interpretations of fields by considering bound and free/diffusive electrons in simple systems. In particular, $\mathbf{P}_w$ is seen to be a density of a measure of time-averaged electron charge distribution about parent nuclei: individual contributions are time-averaged dipole moments which measure orbital asymmetry. In Sect. 5 the electrokinetic fields $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w$ are linked with the force-related electrostatic and magnetostatic fields $\mathbf{E}_w^s$ and $\mathbf{B}_w^s$ via the experimental results of Coulomb and Biot-Savart, respectively. Specifically, $\mathbf{E}_w^s = \epsilon_0^{-1}\boldsymbol{\mathcal{E}}_w$ and $\mathbf{B}_w^s = \mu_0\boldsymbol{\mathcal{B}}_w$, where ϵ_0 and μ_0 are determined by experiment, satisfy $\epsilon_0\mu_0 = c^{-2}$, and serve to ensure dimensional consistency. A formal generalisation to dynamical contexts motivated by (1.7) leads to a complete set of Maxwell relations, consequent upon knowledge of the instantaneous location and velocity of every charge. However, such global instantaneous information can never be known, but requires time to be communicated. This issue is addressed in Sect. 6. It is assumed that information is transferred at the local speed of light and is visualised via the artifice of hypothetical radar signal reflection. The consequent fully-dynamical Maxwell relations correspond to knowledge of the apparent locations and velocities as monitored at any given location and time. Such information has required time to be transmitted and corresponds to an earlier (retarded) time. In Sect. 7 the consequences of force relation (1.9) holding in a general dynamical context are explored. The individual contribution $\mathbf{B}_i^d$ of a charge P_i to the dynamic magnetic field at location $\mathbf{x}$ is shown to be orthogonal both to the corresponding dynamic electric field contribution $\mathbf{E}_i^d$ and to the apparent displacement of P_i from $\mathbf{x}$. The force $\mathbf{F}_i$ on a charge at $\mathbf{x}$ due to P_i is expressed as a linear transformation acting on $\mathbf{E}_i^d$. A brief summary and concluding remarks are appended in Sect. 8.

The notation employed is direct (that is, free of co-ordinate considerations: cf. [5]) and standard identities involving scalar and vector fields are employed without reference.

2 Spatial Averaging and Weighting Functions

Electromagnetic phenomena derive from the behaviour of electrons and atomic nuclei which are here modelled as point charges. Relations which describe spatially-averaged kinetic behaviour of assemblies of such charges are derived: these are formally identical to Maxwell's Eqs. (1.6), (1.1) and (1.2).

Here m_i, q_i and $\mathbf{x}_i(t)$ denote the mass, charge, and location at time t, of a typical charge P_i. For any assembly of charges, the net charge within any region $\mathcal{R}$ at time t, divided by the volume V of R, yields a volumetric average (that is, a *density*) $\rho(\mathcal{R}, t)$ at time t. Symbolically,

$$\rho(\mathcal{R}, t) := \sum_i{}' q_i / V, \tag{2.1}$$

where the primed sum is taken only over those charges in $\mathcal{R}$ at time t. Equivalently,

$$\rho(\mathcal{R}, t) = \sum_i q_i w_i(t), \tag{2.2}$$

where the sum is over *all* assembly charges, and $w_i(t) = 1$ or 0 according to whether or not P_i lies in $\mathcal{R}$ at time t.

The weighted sum in (2.2) can be generalised to yield a candidate electric charge density *field* via

$$\rho_w(\mathbf{x}, t) := \sum_i q_i \, w\big(\mathbf{u}_i(\mathbf{x}, t)\big), \tag{2.3}$$

where the *displacement* of P_i from location $\mathbf{x}$ at time t

$$\mathbf{u}_i(\mathbf{x}, t) := \mathbf{x}_i(t) - \mathbf{x}. \tag{2.4}$$

Suppressing arguments, (2.3) may be written as

$$\rho_w = \sum_i q_i w(\mathbf{u}_i). \tag{2.5}$$

Here *weighting function* w is defined on the space $\mathcal{V}$ of all displacements in three-dimensional Euclidean space, takes real values with physical dimension L^{-3}, and assigns greater values to charges near $\mathbf{x}$ than those far therefrom. In order that the integral of ρ_w over all space should yield the total net assembly charge it is sufficient that

$$\int_{\mathcal{V}} w(\mathbf{u}) d\mathbf{u} = 1. \tag{2.6}$$

Consideration of an assembly consisting of a single charge indicates that normalisation condition (2.6) is also necessary: cf. [5], p. 45.

Remark 2.1 While the physical interpretation of ρ_w depends upon the specific form of w, it is instructive to proceed formally before making a natural, scale-dependent, choice in Sect. 3.

From (2.3), noting that $\mathbf{v}_i := d\mathbf{x}_i/dt$ does not depend upon $\mathbf{x}$, and using the chain rule,[2]

$$\begin{aligned} \partial\rho_w/\partial t &= \sum_i q_i \nabla_{\mathbf{u}} w \,.\, \mathbf{v}_i = \sum_i q_i (-\nabla_{\mathbf{x}} w \,.\, \mathbf{v}_i) \\ &= -\sum_i q_i \, \mathrm{div}\{\mathbf{v}_i w\} = -\mathrm{div}\left\{\sum_i q_i \mathbf{v}_i w\right\}. \end{aligned} \tag{2.7}$$

[2] $\nabla_{\mathbf{u}} w$ and $\nabla_{\mathbf{x}} w$ denote derivatives with respect to arguments $\mathbf{u}$ and $\mathbf{x}$, respectively. Direct (that is, co-ordinate-free) notation is employed: cf., e.g., Murdoch [5], in particular Lemma 4.2.1.

Thus (cf. (1.6))

$$\partial \rho_w / \partial t + \operatorname{div} \mathbf{j}_w = 0, \tag{2.8}$$

where a candidate *current density* field is

$$\mathbf{j}_w := \sum_i q_i \mathbf{v}_i \, w(\mathbf{u}_i). \tag{2.9}$$

Remark 2.2 If q_i is replaced by m_i, then the *mass density* field

$$\rho_w^m := \sum_i m_i \, w(\mathbf{u}_i) \tag{2.10}$$

is similarly seen to satisfy

$$\partial \rho_w^m / \partial t + \operatorname{div} \mathbf{p}_w^m = 0, \tag{2.11}$$

where

$$\mathbf{p}_w^m := \sum_i m_i \mathbf{v}_i \, w(\mathbf{u}_i). \tag{2.12}$$

Here $\mathbf{p}_w^m$ is a *momentum density* field. Writing

$$\mathbf{v}_w^m := \mathbf{p}_w^m / \rho_w^m \tag{2.13}$$

yields

$$\partial \rho_w^m / \partial t + \operatorname{div}\{\rho_w^m \, \mathbf{v}_w^m\} = 0, \tag{2.14}$$

and $\mathbf{v}_w^m$ is a (mass) *velocity* field. Definition (2.13) makes sense if $\rho_w^m \neq 0$; that is, wherever and whenever matter is found.

The electrokinetic analogue of (2.13), namely

$$\mathbf{v}_w = \mathbf{j}_w / \rho_w, \tag{2.15}$$

makes sense only where and when $\rho_w \neq 0$, and hence is not a sensible construct for electrically-neutral assemblies. However, for assemblies consisting only of electrons, or only of nuclei, (charge) velocity fields are well-defined in terms of the relevant densities of current and charge. Indeed, for an assembly of identical charges (so $q_i = q$ and $m_i = m$, say) the constancy of ratio q_i / m_i $(= q/m)$ mandates that the mass and charge velocities be the same:

$$\begin{aligned} \mathbf{v}_w = \mathbf{j}_w / \rho_w &= \sum_i q \mathbf{v}_i \, w(\mathbf{u}_i) / \sum_i q \, w(\mathbf{u}_i) = \sum_i \mathbf{v}_i \, w(\mathbf{u}_i) / \sum_i w(\mathbf{u}_i) \\ &= \sum_i m \, \mathbf{v}_i \, w(\mathbf{u}_i) / \sum_i m w(\mathbf{u}_i) = \mathbf{p}_w^m / \rho_w^m = \mathbf{v}_w^m. \end{aligned} \tag{2.16}$$

Now suppose that, for some function $\mathbf{a} : \mathcal{V} \to \mathcal{V}$,

$$\mathbf{a}_i(\mathbf{x}, t) := \mathbf{a}\big(\mathbf{u}_i(\mathbf{x}, t)\big) \tag{2.17}$$

is a solution to

$$\operatorname{div} \mathbf{a}_i = w(\mathbf{u}_i). \tag{2.18}$$

Accordingly,

$$\rho_w = \sum_i q_i\, w(\mathbf{u}_i) = \sum_i q_i \,\mathrm{div}\,\mathbf{a}_i = \mathrm{div}\left\{\sum_i q_i \mathbf{a}_i\right\}, \tag{2.19}$$

and so

$$\mathrm{div}\,\mathbf{D}_w = \rho_w, \tag{2.20}$$

where (cf. (2.17))

$$\mathbf{D}_w := \sum_i q_i\, \mathbf{a}_i = \sum_i q_i \mathbf{a}(\mathbf{u}_i) \tag{2.21}$$

is a candidate *electric displacement* field.

From (2.17), use of the chain rule yields

$$\nabla_{\mathbf{x}}\, \mathbf{a}_i = (\nabla_{\mathbf{u}}\, \mathbf{a})(\nabla_{\mathbf{x}} \mathbf{u}_i) = (\nabla_{\mathbf{u}}\, \mathbf{a})(-\mathbf{1}) = -\nabla_{\mathbf{u}}\, \mathbf{a}. \tag{2.22}$$

Since $\mathbf{v}_i$ is independent of $\mathbf{x}$, from (2.21), (2.17), the chain rule, and (2.22) it follows that[3]

$$\begin{aligned} \partial \mathbf{D} w/\partial t &= \partial/\partial t \left\{\sum_i q_i \mathbf{a}_i\right\} = \sum_i q_i\, \partial/\partial t \{\mathbf{a}(\mathbf{u}_i)\} \\ &= \sum_i q_i (\nabla_{\mathbf{u}}\, \mathbf{a}) \partial \mathbf{u}_i/\partial t = \sum_i q_i (-\nabla_{\mathbf{x}}\, \mathbf{a}_i) \mathbf{v}_i \\ &= -\sum_i q_i \,\mathrm{div}\{\mathbf{a}_i \otimes \mathbf{v}_i\} = -\mathrm{div}\left\{\sum_i q_i \mathbf{a}_i \otimes \mathbf{v}_i\right\}. \end{aligned} \tag{2.23}$$

Further, from (2.9) and (2.18),

$$\mathbf{j}_w = \sum_i q_i \mathbf{v}_i \,\mathrm{div}\,\mathbf{a}_i = \sum_i q_i \,\mathrm{div}\{\mathbf{v}_i \otimes \mathbf{a}_i\} = \mathrm{div}\left\{\sum_i q_i \mathbf{v}_i \otimes \mathbf{a}_i\right\}. \tag{2.24}$$

Accordingly, (2.23) and (2.24) yield

$$\partial \mathbf{D}_w/\partial t + \mathbf{j}_w = \mathrm{div}\, \boldsymbol{\mathcal{H}}_w, \tag{2.25}$$

where

$$\boldsymbol{\mathcal{H}}_w := \sum_i q_i (\mathbf{v}_i \otimes \mathbf{a}_i - \mathbf{a}_i \otimes \mathbf{v}_i). \tag{2.26}$$

Relation (2.25) and definition (2.26) may be re-written by introducing relevant axial vectors.

Remark 2.3 To any skew-symmetric tensor $\mathbf{W}$ corresponds an axial vector $ax(\mathbf{W}) =: \mathbf{w}$ for which

$$\mathbf{W}\mathbf{k} = \mathbf{w} \times \mathbf{k} \tag{2.27}$$

[3]The tensor (or dyadic) product $\mathbf{a} \otimes \mathbf{b}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ is that linear transformation which maps any vector $\mathbf{k}$ into $(\mathbf{b}.\mathbf{k})\mathbf{a}$: cf. [5], p. 317. For any order 2 tensor field $\mathbf{L}$, $\mathrm{div}\,\mathbf{L}$ is the vector field for which $(\mathrm{div}\,\mathbf{L}).\mathbf{k} = \mathrm{div}(\mathbf{L}^T \mathbf{k})$ for any fixed vector $\mathbf{k}$: cf. [5], p. 390.

for any vector $\mathbf{k}$. (Cf., e.g., Murdoch [5], A.15.) In particular,

$$ax(\mathbf{a} \wedge \mathbf{b}) = -\mathbf{a} \times \mathbf{b}, \tag{2.28}$$

where

$$\mathbf{a} \wedge \mathbf{b} := \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}. \tag{2.29}$$

Further, if $\mathbf{W}$ denotes a skew-symmetric field and $\mathbf{w}$ the corresponding axial vector field, then for any fixed vector $\mathbf{k}$

$$\begin{aligned}(\operatorname{div} \mathbf{W}).\,\mathbf{k} &= \operatorname{div}\left\{\mathbf{W}^T\mathbf{k}\right\} = \operatorname{div}\{-\mathbf{W}\mathbf{k}\} = -\operatorname{div}\{\mathbf{W}\mathbf{k}\} \\ &= -\operatorname{div}\{\mathbf{w} \times \mathbf{k}\} = -\mathbf{k}\,.\operatorname{curl}\mathbf{w} + \mathbf{w}\,.\operatorname{curl}\mathbf{k} \\ &= -(\operatorname{curl}\mathbf{w}).\,\mathbf{k}. \end{aligned} \tag{2.30}$$

Accordingly, since $\mathbf{k}$ is arbitrary,

$$\operatorname{div}\mathbf{W} = -\operatorname{curl}\mathbf{w}. \tag{2.31}$$

It follows from (2.28) that with $\mathbf{W} = \mathbf{v}_i \wedge \mathbf{a}_i$,

$$\operatorname{div}\{\mathbf{v}_i \wedge \mathbf{a}_i\} = -\operatorname{curl}\{-\mathbf{v}_i \times \mathbf{a}_i\} = \operatorname{curl}\{\mathbf{v}_i \times \mathbf{a}_i\}. \tag{2.32}$$

Remark 2.3 enables (2.25) to be written as

$$\partial \mathbf{D}_w/\partial t + \mathbf{j}_w = \operatorname{curl}\mathbf{H}_w, \tag{2.33}$$

where (cf. (1.2)) candidate *magnetic displacement* field

$$\mathbf{H}_w := \sum_i q_i\, \mathbf{v}_i \times \mathbf{a}_i = \sum_i q_i \mathbf{v}_i \times \mathbf{a}(\mathbf{u}_i). \tag{2.34}$$

While relations (2.25) and (2.33) are equivalent, the former requires no appeal to orientation (that is, to 'right-' and 'left-handedness') which is necessary both for the definition of a vector product and of the curl operator.

Remark 2.4 Relations (2.8), (2.20) and (2.33) are formally identical with Maxwell relations (1.6), (1.1) and (1.2). However, the physical interpretation of all fields depends upon the choice of weighting function w. From an analytical viewpoint, relations (2.6) and (2.7) require that w be, respectively, integrable and differentiable. Indeed, the regularity of all fields depends precisely upon that associated with choice w. In the following section a specific natural choice (appropriate to any prescribed length scale) is made. Such choice provides physical insight, and mandates a natural binary decomposition both of $\mathbf{D}_w$ and $\mathbf{H}_w$, thereby introducing electric and magnetic fields which figure in Maxwell equations (1.5) and (1.8).

3 Electrokinetics via a Natural Choice of Weighting Function

The weighting function w introduced in (2.3) *et seq* does not depend explicitly upon location $\mathbf{x}$ and its values are invariant under translation, consistent with regarding space to be homogeneous. That space also be isotropic requires that

$$w(\mathbf{Q}\mathbf{u}) = w(\mathbf{u}) \tag{3.1}$$

for every displacement $\mathbf{u}$ and every proper orthogonal tensor $\mathbf{Q}$. That is, $w(\mathbf{u})$ is independent of the direction of $\mathbf{u}$, so

$$w(\mathbf{u}) = w(u\hat{\mathbf{u}}) =: \tilde{w}(u), \tag{3.2}$$

where

$$u := \|\mathbf{u}\| \quad \text{and} \quad \hat{\mathbf{u}} := \mathbf{u}/u. \tag{3.3}$$

Thus homogeneity and isotropy of space have mandated that w values be Galilean invariant and depend only upon separations.

The simplest form of w is given by

$$\left.\begin{array}{ll} w_\epsilon(u) = V_\epsilon^{-1} & \text{if } 0 \le u < \epsilon \\ w_\epsilon(u) = 0 & \text{if } u \ge \epsilon \end{array}\right\}, \tag{3.4}$$

where $V_\epsilon := 4\pi\epsilon^3/3$. This corresponds to spherical averaging regions and a specific choice ϵ of length scale.

Remark 3.1 Choice $w = w_\epsilon$ in (2.3) yields $\rho_w(\mathbf{x}, t)$ as the sum of those charges which at time t lie within that sphere $S_\epsilon(\mathbf{x})$ of radius ϵ and centre $\mathbf{x}$, divided by the volume of the sphere. Similar interpretations of $\rho_w^m(\mathbf{x}, t)$ and $\mathbf{p}_w^m(\mathbf{x}, t)$ apply in respect of mass and momentum: see (2.10) and (2.12). Further, from (2.13) $\mathbf{v}_w^m(\mathbf{x}, t)$ is identifiable with the velocity of the mass centre of those charges within $S_\epsilon(\mathbf{x})$ at time t. Notice from (2.16) that, for any assembly of identical charges, $\mathbf{v}_w$ coincides with the corresponding mass centre velocity.

While choice (3.4) delivers precise definitions and physical interpretations, w_ϵ fails to be differentiable (indeed, suffers jump discontinuities) wherever $u = \epsilon$, and hence neither $\partial\rho_w/\partial t$ nor $\operatorname{div}\mathbf{j}_w$ are defined thereat. This wrinkle may be overcome by mollifying w_ϵ for $u \in [\epsilon, \epsilon + \delta]$ with $\delta(> 0)$ arbitrarily small. Specifically,

$$\begin{aligned} &w_{\epsilon,\delta}(u) := k \quad \text{if } 0 \le u < \epsilon, \\ &w_{\epsilon,\delta}(u) := k\varphi(\lambda) \quad \text{if } u = \epsilon + \lambda\delta \ \ (0 \le \lambda \le 1), \\ &w_{\epsilon,\delta}(u) := 0 \quad \text{if } u > \epsilon + \delta. \end{aligned} \tag{3.5}$$

Here k is a constant determined by normalisation (2.6), and φ is a monotonic decreasing function on $[0, 1]$ of class C^n for which $\varphi(0) = 1$, $\varphi(1) = 0$, and all derivatives up to order n vanish (one-sidedly) at $\lambda = 0$ and $\lambda = 1$.

Remark 3.2 It follows that any fields defined in terms of $w_{\epsilon,\delta}$ weighted sums inherit class C^n spatial regularity and also (via multiple use of the chain rule) class C^r temporal regularity if trajectories $\mathbf{x}_i(t)$ are of class C^s and $r = \min(n, s)$.

Normalisation condition (2.6) with (3.2) yield, on employing spherical polar coordinates,

$$4\pi \int_0^\infty u^2 \tilde{w}(u) du = 1. \tag{3.6}$$

It follows from (3.5) that (cf. [5], §4.3.5)

$$k = V_\epsilon^{-1}\left(1 + O(\delta/\epsilon)\right) \quad \text{as } \delta \to 0. \tag{3.7}$$

The simplest mollifier sufficient to render $w_{\epsilon,\delta}$ of class C^1 is

$$\varphi(\lambda) := 2\lambda^3 - 3\lambda^2 + 1. \tag{3.8}$$

From (3.5) and (3.6) such choice implies

$$\frac{1}{4\pi k} = \int_0^\epsilon u^2 du + \delta \int_0^1 (\epsilon + \lambda\delta)^2 \big(2\lambda^3 - 3\lambda^2 + 1\big) d\lambda. \tag{3.9}$$

Straightforward integration yields

$$\frac{1}{kV_\epsilon} = 1 + 3r/2 + 9r^2/10 + r^3/5, \tag{3.10}$$

where

$$r := \delta/\epsilon. \tag{3.11}$$

It follows (cf. (3.7)) that

$$k = V_\epsilon^{-1}\big(1 - 3r/2 + 27r^2/20 + O\big(r^3\big)\big) \quad \text{as } r \to 0. \tag{3.12}$$

Interpretation of $\mathbf{D}_w$ and $\mathbf{H}_w$ (cf. (2.21) and (2.34)) requires specification of $\mathbf{a}$ (cf. (2.17) and (2.18)), and hence depends upon both ϵ and mollifier φ. In view of spatial isotropy we assume that

$$\mathbf{a}(\mathbf{u}_i) = a(u_i)\mathbf{u}_i, \tag{3.13}$$

where a is scalar-valued.

Theorem 3.1

$$a(u) = -k/3 \quad \text{if } u < \epsilon, \tag{3.14}$$

$$a(u) = -1/4\pi u^3 \quad \text{if } u > \epsilon + \delta, \tag{3.15}$$

$$a(u) = -\big(k\epsilon^3/3u^3\big)F(\lambda) \quad \text{if } 0 \le \lambda \le 1, \tag{3.16}$$

where (*cf.* (3.11) *and* $(3.5)_2$) $\lambda = (u - \epsilon)/\delta$ *and*

$$F(\lambda) := 1 + 3\big(1 - \lambda^2 + \lambda^3/2\big)r\lambda + 3\big(1 - 3\lambda^2/2 + 4\lambda^3/5\big)r^2\lambda^2 + \big(1 - 9\lambda^2/5 + \lambda^3\big)r^3\lambda^3. \tag{3.17}$$

Proof The divergence theorem, (2.17) and (2.18) yield

$$\int_{S_R(\mathbf{x}_i)} w(\mathbf{u}_i) dV_\mathbf{x} = \int_{S_R(\mathbf{x}_i)} \operatorname{div} \mathbf{a}\, dV_\mathbf{x} = \int_{\partial S_R(\mathbf{x}_i)} \mathbf{a}\,.\,\mathbf{n}\, dS_\mathbf{x}. \tag{3.18}$$

Here $S_R(\mathbf{x}_i)$ denotes that spherical ball of radius R centred at $\mathbf{x}_i$ and $\partial S_R(\mathbf{x}_i)$ its boundary. The outward unit normal at point $\mathbf{x} \in \partial S_R(\mathbf{x}_i)$ is

$$\mathbf{n}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_i)/\|\mathbf{x} - \mathbf{x}_i\| = -\mathbf{u}_i/u_i =: -\hat{\mathbf{u}}_i. \tag{3.19}$$

Thus from (3.13) and (3.19), noting $u_i = R$ on $\partial S_R(\mathbf{x}_i)$,

$$\mathbf{a}\,.\,\mathbf{n} = a(u_i)\mathbf{u}_i\,.-\hat{\mathbf{u}}_i = -u_i\, a(u_i) = -Ra(R). \tag{3.20}$$

Hence (3.18) and (3.20) yield

$$\int_{S_R(\mathbf{x}_i)} w(\mathbf{u}_i) dV_{\mathbf{x}} = -Ra(R) \,.\, 4\pi R^2 = -4\pi R^3 a(R). \tag{3.21}$$

If $R < \epsilon$, then from $(3.5)_1$ and (3.21),

$$k \,.\, 4\pi R^3/3 = -4\pi R^3 a(R) \tag{3.22}$$

and hence if $u < \epsilon$, then $a(u) = -k/3$.

If $R > \epsilon + \delta$, then normalisation (2.6), $(3.5)_3$ and (3.21) yield

$$1 = -4\pi R^3 a(R), \tag{3.23}$$

so

$$a(u) = -1/4\pi u^3 \quad \text{if } u > \epsilon + \delta.$$

If $\epsilon \le R \le \epsilon + \delta$, then $(3.5)_{1,2}$ and (3.21) yield

$$\begin{aligned} -4\pi R^3 a(R) &= 4\pi \int_0^R u^2 w_{\epsilon,\delta}(u) du \\ &= 4\pi k \int_0^\epsilon u^2 du + 4\pi k \int_\epsilon^R u^2 \varphi(x) du, \end{aligned} \tag{3.24}$$

where

$$x := (u - \epsilon)/\delta. \tag{3.25}$$

Thus

$$-R^3 a(R) = k\epsilon^3/3 + k\delta \int_0^\lambda (\epsilon + x\delta)^2 \varphi(x) dx, \tag{3.26}$$

where

$$\lambda := (R - \epsilon)/\delta. \tag{3.27}$$

Recalling (3.11) and (3.8),

$$\begin{aligned} &\int_0^\lambda (\epsilon + x\delta)^2 \varphi(x) dx \\ &= \epsilon^2 \int_0^\lambda (1 + rx)^2 \big(2x^3 - 3x^2 + 1\big) dx \\ &= \epsilon^2 \lambda \big\{ \big(1 - \lambda^2 + \lambda^3/2\big) + r\lambda \big(1 - 3\lambda^2/2 + 4\lambda^3/5\big) + r^2\lambda^2 \big(1/3 - 3\lambda^2/5 + \lambda^3/3\big) \big\} \\ &= \epsilon^2 \lambda \big(F(\lambda) - 1\big)/3r\lambda = \epsilon^3 \big(F(\lambda) - 1\big)/3\delta, \end{aligned} \tag{3.28}$$

with F given by (3.17). Hence (3.26) yields

$$-R^3 a(R) = k\epsilon^3/3 + k\epsilon^3 \big(F(\lambda) - 1\big)/3 = k\epsilon^3 F(\lambda)/3. \tag{3.29}$$

Writing $R = u$ yields (3.16). □

Remark 3.3 Theorem 3.1 highlights a marked difference in the nature of $a(u)$ for $u < \epsilon$ as compared with $u > \epsilon + \delta$. In particular, for $0 \leq u < \epsilon$,

$$a(u) = -k/3 = -w_{\epsilon,\delta}(u)/3. \tag{3.30}$$

If

$$a_1(u) := -w_{\epsilon,\delta}(u)/3 \quad \text{and} \quad a_2(u) := a(u) + w_{\epsilon,\delta}(u)/3, \tag{3.31}$$

then

$$a = a_1 + a_2. \tag{3.32}$$

Accordingly,

$$a_1(u_i) = -k/3 \quad \text{and} \quad a_2(u_i) = 0 \quad \text{if } 0 \leq u_i < \epsilon, \tag{3.33}$$

$$a_1(u_i) = 0 \quad \text{and} \quad a_2(u_i) = -1/4\pi u_i^3 \quad \text{if } u_i > \epsilon + \delta. \tag{3.34}$$

The binary decomposition (3.32) of a (and hence, via (3.13), of $\mathbf{a}$) motivates corresponding decompositions of fields $\mathbf{D}_w$ and $\mathbf{H}_w$. Specifically (cf. (1.17)),

$$\mathbf{D}_w = \mathbf{P}_w + \boldsymbol{\mathcal{E}}_w, \qquad \mathbf{H}_w = -\mathbf{M}_w + \boldsymbol{\mathcal{B}}_w. \tag{3.35}$$

Here, noting (3.31) and suppressing subscripts ϵ and δ,

$$\mathbf{P}_w := \sum_i q_i\, a_1(u_i)\mathbf{u}_i = -(1/3)\sum_i q_i\, w(u_i)\mathbf{u}_i, \tag{3.36}$$

$$\boldsymbol{\mathcal{E}}_w := \sum_i q_i\, a_2(u_i)\mathbf{u}_i = \sum_i q_i \big(a(u_i) + w(u_i)/3\big)\mathbf{u}_i, \tag{3.37}$$

$$-\mathbf{M}_w := \sum_i q_i\, a_1(u_i)\mathbf{v}_i \times \mathbf{u}_i = -(1/3)\sum_i q_i \mathbf{u}_i \times \mathbf{v}_i\, w(u_i), \tag{3.38}$$

$$\boldsymbol{\mathcal{B}}_w := \sum_i q_i\, a_2(u_i)\mathbf{v}_i \times \mathbf{u}_i = \sum_i q_i \big(a(u_i) + w(u_i)/3\big)\mathbf{v}_i \times \mathbf{u}_i. \tag{3.39}$$

Remark 3.4 Fields $\mathbf{P}_w$ and $\mathbf{M}_w$ are local spatial densities (cf. (2.5) and (2.9)) of electrokinetic variables $-q_i\mathbf{u}_i/3$ and $-q_i\mathbf{u}_i \times \mathbf{v}_i/3$. In particular, values $\mathbf{P}_w(\mathbf{x}, t)$ and $\mathbf{M}_w(\mathbf{x}, t)$ derive (cf. $(3.36)_1$, $(3.38)_1$ and $(3.34)_1$) only from charges within a distance $\epsilon + \delta$ from $\mathbf{x}$ at time t. In contrast, values $\boldsymbol{\mathcal{E}}_w(\mathbf{x}, t)$ and $\boldsymbol{\mathcal{B}}_w(\mathbf{x}, t)$ derive (cf. $(3.37)_1$, $(3.39)_1$ and $(3.33)_2$) only from charges further than ϵ from $\mathbf{x}$ at time t.

Remark 3.5 In view of (2.20) and $(3.35)_1$ it is natural to examine the separate contributions of $\operatorname{div} \boldsymbol{\mathcal{E}}_w$ and $\operatorname{div} \mathbf{P}_w$ to ρ_w. Now

$$\begin{aligned} \operatorname{div}\big\{a_2(u_i)\mathbf{u}_i\big\} &= \operatorname{div}\big\{a(u_i)\mathbf{u}_i\big\} + (1/3)\operatorname{div}\big\{w(u_i)\mathbf{u}_i\big\} \\ &= w(u_i) + (1/3)\big(w(u_i)\operatorname{div}\mathbf{u}_i + \nabla w(u_i)\,.\,\mathbf{u}_i\big). \end{aligned} \tag{3.40}$$

Since

$$\nabla \mathbf{u}_i = -\mathbf{1} \quad \text{and} \quad u_i^2 = \mathbf{u}_i\,.\,\mathbf{u}_i, \tag{3.41}$$

$$\operatorname{div} \mathbf{u}_i = -3 \tag{3.42}$$

and

$$2u_i \nabla u_i = 2(\nabla \mathbf{u}_i)^T \mathbf{u}_i = -2\mathbf{u}_i,$$

so

$$\nabla u_i = -\mathbf{u}_i / u_i =: -\hat{\mathbf{u}}_i. \tag{3.43}$$

Accordingly, (3.40) with (3.42) and (3.43) yield, via the chain rule,

$$\operatorname{div}\{a_2(u_i)\mathbf{u}_i\} = (1/3)w'(u_i)\nabla u_i \,.\, \mathbf{u}_i = -u_i w'(u_i)/3. \tag{3.44}$$

Since $w'(u_i)$ vanishes for $u_i < \epsilon$ and $u_i > \epsilon + \delta$ (cf. (3.5)), the only contributions to $\operatorname{div} \boldsymbol{\mathcal{E}}_w$ evaluated at $\mathbf{x}$ come from charges for which $\epsilon \le u_i \le \epsilon + \delta$. It follows that if the contribution to $\boldsymbol{\mathcal{E}}_w$ from charge q_i is labelled $\boldsymbol{\mathcal{E}}_i$, and if $\mathcal{R}$ is a region for which $S_{\epsilon+\delta}(\mathbf{x}_i) \subset \mathcal{R}$ (cf. (3.18)), then (cf. $(3.5)_2$ and (3.11))

$$\begin{aligned}\int_{\mathcal{R}} \operatorname{div} \boldsymbol{\mathcal{E}}_i \, dV &= \int_{\mathcal{R}} -(1/3) q_i w'(u_i) u_i \, dV \\ &= -(4\pi q_i/3) \int_0^1 w'(\epsilon + \lambda\delta)(\epsilon + \lambda\delta)^3 \delta \, d\lambda \\ &= -\left(4\pi k \epsilon^3 q_i/3\right) \int_0^1 \varphi'(\lambda)(1 + \lambda r)^3 d\lambda. \end{aligned} \tag{3.45}$$

However, notice normalisation condition (3.9) may be written as

$$\begin{aligned} 1/4\pi k - \epsilon^3/3 &= \epsilon^2 \delta \int_0^1 \varphi(\lambda)(1 + \lambda r)^2 d\lambda \\ &= \epsilon^2 \delta \left\{ \left[\varphi(\lambda)(1 + \lambda r)^3/3r\right]_0^1 - (1/3r) \int_0^1 \varphi'(\lambda)(1 + \lambda r)^3 d\lambda \right\} \\ &= \epsilon^2 \delta \,.\, -1/3r - \left(\epsilon^3/3\right) \int_0^1 \varphi'(\lambda)(1 + \lambda r)^3 d\lambda, \end{aligned}$$

so[4]

$$\int_0^1 \varphi'(\lambda)(1 + \lambda r)^3 d\lambda = -3/4\pi k \epsilon^3. \tag{3.46}$$

Accordingly (3.45) becomes

$$\int_{\mathcal{R}} \operatorname{div} \boldsymbol{\mathcal{E}}_i \, dV = q_i \left(= \int_{\mathcal{R}} q_i w(u_i) dV \right). \tag{3.47}$$

Writing

$$\rho_i := q_i \, w(u_i), \qquad \mathbf{P}_i := q_i a_1(u_i) \mathbf{u}_i, \tag{3.48}$$

[4]This result could have been obtained by straightforward integration in (3.45). However, relating normalisation condition (3.9) to (3.45), via integration by parts, yields result (3.47) for *any* choice of mollifier.

it follows from (3.47), (2.20) and (3.36), that

$$\int_{\mathcal{R}} (\operatorname{div} \boldsymbol{\mathcal{E}}_i - \rho_i \, dV) = 0 = -\int_{\mathcal{R}} \operatorname{div} \mathbf{P}_i \, dV. \tag{3.49}$$

Accordingly, for any assembly of charges P_i each of which satisfies $S_{\epsilon+\delta}(\mathbf{x}_i) \subset \mathcal{R}$,

$$\int_{\mathcal{R}} (\operatorname{div} \boldsymbol{\mathcal{E}}_w - \rho_w) dV = 0 = -\int_{\mathcal{R}} \operatorname{div} \mathbf{P}_w \, dV. \tag{3.50}$$

It should be noted that this does *not* imply that

$$\operatorname{div} \boldsymbol{\mathcal{E}}_w - \rho_w = 0 = -\operatorname{div} \mathbf{P}_w,$$

since this would require (3.50) to hold for arbitrarily small regions (and hence condition $S_{\epsilon+\delta}(\mathbf{x}_i) \subset \mathcal{R}$ would fail to be satisfied). Indeed, in general (3.50) is approximate if there is any charge for which $S_{\epsilon+\delta}(\mathbf{x}_i) \cap \mathcal{R} \neq S_{\epsilon+\delta}(\mathbf{x}_i)$ (in such case part of the mollifying region would lie outside $\mathcal{R}$).

Fields $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w$ are expressible in terms of potential functions:

Theorem 3.2 *If f satisfies*

$$f'(u) = u a_2(u), \tag{3.51}$$

then

$$\boldsymbol{\mathcal{E}}_w = -\nabla \psi_w, \qquad \boldsymbol{\mathcal{B}}_w = \operatorname{curl} \boldsymbol{\mathcal{A}}_w, \tag{3.52}$$

where

$$\psi_w := \sum_i q_i f(u_i), \qquad \boldsymbol{\mathcal{A}}_w := \sum_i q_i \, f(u_i) \mathbf{v}_i. \tag{3.53}$$

Proof From the chain rule, (3.43) and (3.51),

$$\nabla f(u_i) = f'(u_i) \nabla u_i = -f'(u_i) \hat{\mathbf{u}}_i = -\mathbf{u}_i a_2(u_i), \tag{3.54}$$

and $(3.52)_1$ follows from $(3.37)_1$. Further, noting that $\mathbf{x}_i$ does not depend upon $\mathbf{x}$ so $\operatorname{curl} \mathbf{v}_i = \mathbf{0}$,

$$\operatorname{curl}\{ f(u_i) \mathbf{v}_i \} = \nabla f(u_i) \times \mathbf{v}_i = -a_2(u_i) \mathbf{u}_i \times \mathbf{v}_i = a_2(u_i) \mathbf{v}_i \times \mathbf{u}_i. \tag{3.55}$$

Thus $(3.52)_2$ follows via $(3.39)_1$. □

Corollary 3.2.1

$$\operatorname{curl} \boldsymbol{\mathcal{E}}_w = \mathbf{0}, \qquad \operatorname{div} \boldsymbol{\mathcal{B}}_w = 0. \tag{3.56}$$

Corollary 3.2.2

$$\partial \psi_w / \partial t + \operatorname{div} \boldsymbol{\mathcal{A}}_w = 0. \tag{3.57}$$

Proof From $(3.53)_1$,

$$\partial\psi_w/\partial t = \sum_i q_i\, f'(u_i)\partial u_i/\partial t, \tag{3.58}$$

where, from $(3.41)_2$,

$$2u_i\partial u_i/\partial t = 2\mathbf{u}_i\,.\,\mathbf{v}_i,$$

so (cf. $(3.43)_2$)

$$\partial u_i/\partial t = \hat{\mathbf{u}}_i\,.\,\mathbf{v}_i. \tag{3.59}$$

Accordingly,

$$\partial\psi_w/\partial t = \sum_i q_i\, f'(u_i)\hat{\mathbf{u}}_i\,.\,\mathbf{v}_i. \tag{3.60}$$

From $(3.53)_2$, (3.54) and (3.43), noting $\operatorname{div}\mathbf{v}_i = 0$,

$$\begin{aligned}\operatorname{div}\boldsymbol{\mathcal{A}}_i &= \sum_i q_i \operatorname{div}\big\{f(u_i)\mathbf{v}_i\big\} = \sum_i q_i\,\nabla f(u_i)\,.\,\mathbf{v}_i\\ &= -\sum_i q_i\, f'(u_i)\hat{\mathbf{u}}_i\,.\,\mathbf{v}_i.\end{aligned} \tag{3.61}$$

Adding (3.60) and (3.61) yields (3.57). □

Remark 3.6 From $(3.33)_2$ and $(3.34)_2$, $a_2(u_i) = 0$ if $u_i < \epsilon$ and $a_2(u_i) = -1/4\pi u_i^3$ if $u > \epsilon + \delta$. Thus, from (3.51), $f'(u_i) = 0$ if $u_i < \epsilon$ and $f'(u_i) = -1/4\pi u_i^2$ if $u_i > \epsilon + \delta$. Noting that the constant of integration plays no part in determining $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w$, without loss of generality

$$f(u_i) = 0 \quad (u_i < \epsilon), \qquad f(u_i) = 1/4\pi u_i \quad (u_i > \epsilon + \delta). \tag{3.62}$$

Remark 3.7 Selection of natural choice (3.4) of scale-dependent weighting function w (suitably mollified in (3.5)), and associated displacement function $\mathbf{a}$ (delineated via (2.17), (2.18) and (3.13)) have resulted in relations formally identical to equations in classical electromagnetic theory. In particular, (2.20) and (2.33) are explicitly scale-dependent versions of Maxwell relations (1.1) and (1.2). Additionally, decompositions (3.35), together with the existence of potential functions given in (3.53) and which satisfy (3.57), take standard forms. However, all relations derived here are of purely electrokinetic character. While this is indeed the actual physical nature of spatial densities ρ_w, $\mathbf{j}_w$, $\mathbf{P}_w$ and $\mathbf{M}_w$, and of $\mathbf{D}_w$ and $\mathbf{H}_w$, the fields $\mathbf{E}$ and $\mathbf{B}$ which appear in Maxwell relations (1.5) and (1.8) are intimately related to the transmission of force, via (1.9). Evidently the physical dimensions of $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w$ are not those of $\mathbf{E}$ and $\mathbf{B}$, respectively. Indeed, from (1.9) $\mathbf{E}$ and $\mathbf{v}\times\mathbf{B}$ have the dimensions of force per unit charge. Accordingly,

$$\dim\mathbf{E} = MLT^{-2}Q^{-1}, \qquad \dim\mathbf{B} = MLT^{-2}Q^{-1}\big(LT^{-1}\big)^{-1} = MT^{-1}Q^{-1}. \tag{3.63}$$

On the other hand, noting w, a, a_1 and a_2 have dimension L^{-3} (cf. (3.30) and (3.31)), from (3.37) and (3.39)

$$\dim\boldsymbol{\mathcal{E}}_w = L^{-2}Q, \qquad \dim\boldsymbol{\mathcal{B}}_w = L^{-1}T^{-1}Q. \tag{3.64}$$

This issue is resolved in Sect. 5 by appeal to the experimentally-based laws of Coulomb and Biot-Savart, after examining the interpretation of fields for simple atomic systems via time averaging.

Remark 3.8 Mollifier φ in (3.8) was the simplest choice. Further regularity may be ensured via polynomials of higher order: cf. [5], §4.3.6. Different choices change field definitions only over separations u_i in the range $[\epsilon, \epsilon + \delta]$ and k satisfies (3.7).

4 Time Averaging and Interpretation of Atomic Kinetics

Any theoretical description of material behaviour must be related to its observation and measurement. Since no measurement can reflect behaviour at a geometrical point nor instant in time, but is limited by both the spatial and temporal sensitivity of the monitoring procedure and apparatus (cf. [5], Sect. 8.2), measurement values represent local averages jointly in space and time. This motivates a further temporal averaging of the spatially-averaged relations obtained so far.

The Δ-time average f_Δ of any integrable function of time is

$$f_\Delta(t) := \frac{1}{\Delta}\int_{t-\Delta}^{t} f(\tau)d\tau. \tag{4.1}$$

It follows (cf. [5], p. 78) that if f is a class C^1 field, then

$$\partial/\partial t\{f_\Delta\} = (\partial f/\partial t)_\Delta. \tag{4.2}$$

Further, if $\mathbf{f}$ is a C^1 vector field, then (cf. [5], Remark 8.3.1)

$$(\operatorname{div}\mathbf{f})_\Delta = \operatorname{div}\{\mathbf{f}_\Delta\},\ (\operatorname{curl}\mathbf{f})_\Delta = \operatorname{curl}\{\mathbf{f}_\Delta\}. \tag{4.3}$$

Result $(4.3)_1$ also holds for tensor fields of any order.

Accordingly, time averaging relations (2.8), (2.20), (2.33), (3.35), (3.52), (3.56) and (3.57) yields versions formally identical to their originals but wherein all fields are replaced by their time-averaged counterparts.

Remark 4.1 Consider a simple model of a conductor as a system of atomic nuclei P_j (charge $Z_j\, e$) each of which has a fixed set of N_j bound electrons P_{j_k}, together with a system of free/diffusive electrons P_ℓ. Both systems give rise to a relation of form (2.8), namely

$$\partial\rho_w^b/\partial t + \operatorname{div}\mathbf{j}_w^b = 0, \qquad \partial\rho_w^f/\partial t + \operatorname{div}\mathbf{j}_w^f = 0. \tag{4.4}$$

Here 'b' and 'f' designate the *b*ound and *f*ree systems. Of course, for the system as a whole, (2.8) is satisfied with

$$\rho_w = \rho_w^b + \rho_w^f, \qquad \mathbf{j}_w = \mathbf{j}_w^b + \mathbf{j}_w^f. \tag{4.5}$$

Suppressing time dependence (cf. (2.3)),

$$\rho_w^b(\mathbf{x}) := \sum_j Z_j\, e w(\mathbf{x}_j - \mathbf{x}) + \sum_j \sum_{j_k=1}^{N_j} (-e) w(\mathbf{x}_{j_k} - \mathbf{x})$$

$$= \sum_j (Z_j - N_j) e w(\mathbf{x}_j - \mathbf{x}) + \sum_j \sum_{j_k=1}^{N_j} (-e) \big\{ w(\mathbf{x}_{j_k} - \mathbf{x}) - w(\mathbf{x}_j - \mathbf{x}) \big\}. \quad (4.6)$$

Also (cf. (2.9))

$$\mathbf{j}_w^b(\mathbf{x}) := \sum_j Z_j\, e_j \mathbf{v}_j\, w(\mathbf{x}_j - \mathbf{x}) + \sum_j \sum_{j_k=1}^{N_j} (-e) \mathbf{v}_{j_k} w(\mathbf{x}_{j_k} - \mathbf{x})$$

$$= \sum_j \left\{ (Z_j - N_j) e \mathbf{v}_j + \sum_{j_k=1}^{N_j} e(\mathbf{v}_j - \mathbf{v}_{j_k}) \right\} w(\mathbf{x}_j - \mathbf{x})$$

$$+ \sum_j \sum_{j_k=1}^{N_j} e \mathbf{v}_{j_k} \big\{ w(\mathbf{x}_j - \mathbf{x}) - w(\mathbf{x}_{j_k} - \mathbf{x}) \big\}. \quad (4.7)$$

If P_j and its bound charges lie in $S_\epsilon(\mathbf{x})$, then from $(3.5)_1$ and (3.12) their contribution to $\rho_w^b(\mathbf{x})$ is essentially $(Z_j - N_j)e/V_\epsilon$ and to $\mathbf{j}_w^b$ is $\{(Z_j - N_j)e\mathbf{v}_j + \sum_{j_k=1}^{N_j} e(\mathbf{v}_j - \mathbf{v}_{j_k})\}/V_\epsilon$. If this subsystem remains in $S_\epsilon(\mathbf{x})$ for the time interval $(t - \Delta, t)$, then its contribution to $(\rho_w^b)_\Delta(\mathbf{x}, t)$ is $(Z_j - N_j)e/V_\epsilon$ and to $(\mathbf{j}_w^b)_\Delta(\mathbf{x}, t)$ is $(Z_j - N_j)e(\mathbf{v}_j)_\Delta/V_\epsilon$. Here the factor $(\mathbf{v}_j - \mathbf{v}_{j_k})_\Delta = (\dot{\mathbf{x}} - \dot{\mathbf{x}}_{j_k})_\Delta = d/dt\{(\mathbf{x}_j - \mathbf{x}_{j_k})_\Delta\}$ has been regarded to be negligible as a consequence of the bounded nature of P_{j_k}. In a solid macroscopically at rest, nuclei vibrate erratically on time scales of order 10^{-13} s about fixed locations. Accordingly, if $\Delta \gg 10^{-13}$ s (say $\Delta \sim 10^{-6}$ s) then $(\mathbf{v}_j)_\Delta = (\dot{\mathbf{x}}_j)_\Delta = d/dt\{(\mathbf{x}_j)_\Delta\} = \mathbf{0}$. In such case (and neglecting nuclei which cross or straddle the mollifying region $S_{\epsilon+\delta}(\mathbf{x}) - S_\epsilon(\mathbf{x})$) $(\mathbf{j}_w^b)_\Delta(\mathbf{x}, t) = \mathbf{0}$ and $(\rho_w^b)_\Delta(\mathbf{x}, t)$ does not change with time. Thus both terms in the time-averaged version of $(4.4)_1$ vanish and, from (4.5),

$$(\rho_w)_\Delta = \big(\rho_w^b\big)_\Delta + \big(\rho_w^f\big)_\Delta, \qquad (\mathbf{j}_w)_\Delta = \big(\mathbf{j}_w^f\big)_\Delta \quad (4.8)$$

and

$$\partial/\partial t \big\{ \big(\rho_w^f\big)_\Delta \big\} + \operatorname{div} \big\{ \big(\mathbf{j}_w^f\big)_\Delta \big\} = 0 \quad (4.9)$$

for a conductor macroscopically at rest. More generally it is time averages of terms $(Z_j - N_j)e/V_\epsilon$ and $(Z_j - N_j)e\mathbf{v}_j/V_\epsilon$, taken over nuclei instantaneously in $S_\epsilon(\mathbf{x})$, which dominate contributions to $(\rho_w^b)_\Delta(\mathbf{x}, t)$ and $(\mathbf{j}_w^b)_\Delta(\mathbf{x}, t)$. The latter is an ionic current density which might be significant in an electrolytic context.

Remark 4.2 Consider an assembly of neutral atoms. In such case all electrons are bound, so $N_j = Z_j$ for all atoms. The contribution to $\mathbf{P}_w(\mathbf{x}, t)$ from such an atom, all of whose charges lie in $S_\epsilon(\mathbf{x})$ at time t, is from $(3.36)_2$ closely approximated (here r is neglected, so $k = V_\epsilon^{-1}$ from (3.12)) by

$$-(1/3V_\epsilon) \left\{ Z_j\, e(\mathbf{x}_j - \mathbf{x}) + \sum_{j_k=1}^{Z_j} (-e)(\mathbf{x}_{j_k} - \mathbf{x}) \right\} = -\mathbf{p}_j/4\pi\epsilon^3. \quad (4.10)$$

Here

$$\mathbf{p}_j := \sum_{j_k=1}^{Z_j} (-e)\mathbf{r}_{j_k} \quad \text{with } \mathbf{r}_{j_k} := \mathbf{x}_{j_k} - \mathbf{x}_j. \tag{4.11}$$

Term $-e\mathbf{r}_{j_k}$ represents an instantaneous dipole moment associated with the displacement of electron P_{j_k} from its parent nucleus. Such moment will vary erratically on an atomic timescale ($\sim 10^{-13}$ s). However, if $\Delta \sim 10^{-6}$ s, then $-e(\mathbf{r}_{j_k})_\Delta$ represents a time-averaged dipole moment which is a measure of time-averaged orbital asymmetry. Correspondingly, $(\mathbf{p}_j)_\Delta$ is a measure of the averaged total charge distribution about nucleus P_j, namely the polarisation of this atom. From (4.10) the contribution to $(\mathbf{P}_w)_\Delta(\mathbf{x}, t)$ from atoms which remain in $S_\epsilon(\mathbf{x})$ during time averaging constitutes a polarisation density associated with these atoms: strictly speaking, $-1/3$ multiplied by the sum of the time-averaged dipole moments and divided by V_ϵ. In general there will also be contributions both from atoms which migrate into and out of $S_\epsilon(\mathbf{x})$, and atoms which straddle the mollifying region, at any time in the interval $(t - \Delta, t)$. If the system constitutes a solid at macroscopic rest ($(\mathbf{v}_j)_\Delta = \mathbf{0}$), then only 'straddling' atoms are involved and, since $\delta \ll \epsilon$, their contribution may be expected to be negligible.

Remark 4.3 The contribution to $\mathbf{M}_w(\mathbf{x}, t)$ from a neutral atom within $S_\epsilon(\mathbf{x})$ at time t is (cf. $(3.38)_2$, $(3.5)_1$ and (3.12)) essentially

$$\begin{aligned}
&(1/3V_\epsilon)\left\{ Z_j e\mathbf{u}_j \times \mathbf{v}_j + \sum_{j_k=1}^{Z_j} (-e)\mathbf{u}_{j_k} \times \mathbf{v}_{j_k} \right\} \\
&= (1/3V_\epsilon)\left\{ Z_j\, e\, \mathbf{u}_j \times \mathbf{v}_j + \sum_{j_k=1}^{Z_j} (-e)(\mathbf{r}_{j_k} + \mathbf{u}_j) \times (\dot{\mathbf{r}}_{j_k} + \mathbf{v}_j) \right\} \\
&= (1/3V_\epsilon)\left\{ \sum_{j_k=1}^{Z_j} \mathbf{m}_{j_k} + \mathbf{u}_j \times \dot{\mathbf{p}}_j + \mathbf{p}_j \times \mathbf{v}_j \right\},
\end{aligned} \tag{4.12}$$

where

$$\mathbf{m}_{j_k} := (-e)\mathbf{r}_{j_k} \times \dot{\mathbf{r}}_{j_k}. \tag{4.13}$$

Term $\mathbf{m}_{j_k}$ is the instantaneous magnetic moment of electron P_{j_k} about its parent nucleus. Erratic variation in $\mathbf{m}_{j_k}$ is smoothed by time averaging, and

$$(\mathbf{m}_j)_\Delta := \sum_{j_k=1}^{Z_j} (\mathbf{m}_{j_k})_\Delta \tag{4.14}$$

is a measure of averaged orbital angular momentum of electrons about their parent nucleus (on noting $\mathbf{m}_{j_k} = -(e/m)\mathbf{r}_{j_k} \times m\dot{\mathbf{r}}_{j_k}$, where m denotes electron mass). From (4.12), (4.14) and (3.38), the contribution to $(\mathbf{M}_w)_\Delta(\mathbf{x}, t)$, from atoms which remain in $S_\epsilon(\mathbf{x})$ during time averaging, is a density ($\times 1/3$) of net time-averaged atomic moments $(\mathbf{m}_j)_\Delta$ together with averages $(\mathbf{u}_j \times \dot{\mathbf{p}}_j + \mathbf{p}_j \times \mathbf{v}_j)_\Delta$. For a solid at macroscopic rest $\mathbf{M}_w(\mathbf{x}, t)$ is characterised by such contributions upon neglecting 'straddling' atoms (cf. Remarks 4.1 and 4.2).

Remark 4.4 From (3.37) and $(3.33)_2$, only charges further than ϵ from $\mathbf{x}$ at time t contribute to $\boldsymbol{\mathcal{E}}_w(\mathbf{x},t)$. Any charge further than $\epsilon+\delta$ at this time yields (cf. $(3.34)_2$) a contribution $-q_i\mathbf{u}_i/4\pi u_i^3$. The net contribution from a neutral atom outside $S_{\epsilon+\delta}(\mathbf{x})$ is

$$-(1/4\pi)\left\{Z_j e\mathbf{u}_j/u_j^3+\sum_{j_k=1}^{Z_j}(-e)\mathbf{u}_{j_k}/u_{j_k}^3\right\}. \tag{4.15}$$

Now

$$\begin{aligned} u_{j_k}^{-3} &= (\mathbf{u}_{j_k}\,.\,\mathbf{u}_{j_k})^{-3/2}=\big[(\mathbf{u}_j+\mathbf{r}_{j_k})\,.\,(\mathbf{u}_j+\mathrm{r}_{j_k})\big]^{-3/2}\\ &= u_j^{-3}\big[1+2\mathbf{r}_{j_k}\,.\,\mathbf{u}_j/u_j^2+(r_{j_k}/u_j)^2\big]^{-3/2}\\ &= u_j^{-3}\big[1-3(\mathbf{r}_{j_k}\,.\,\mathbf{u}_j)/u_j^2+O\big((r_{j_k}/u_j)^2\big)\big]. \end{aligned} \tag{4.16}$$

Contribution (4.15) may thus (on neglect of $O((r_{j_k}/u_j)^2)$ terms) be written as

$$\begin{aligned} &-(1/4\pi)\sum_{j_k=1}^{Z_j}e\big\{\mathbf{u}_j/u_j^3-(\mathbf{u}_j+\mathbf{r}_{j_k})\big[u_j^{-3}-3(\mathbf{r}_{j_k}\,.\,\mathbf{u}_j)u_j^{-5}\big]\big\}\\ &=-(1/4\pi)\big\{\mathbf{p}_j/u_j^3-3(\mathbf{u}_j\otimes\mathbf{u}_j)\mathbf{p}_j\,u_j^{-5}\big\}=-\mathbf{A}_j\mathbf{p}_j, \end{aligned} \tag{4.17}$$

where (cf. $(3.43)_2$)

$$\mathbf{A}_j:=\big(1/4\pi u_j^3\big)(\mathbf{1}-3\hat{\mathbf{u}}_i\otimes\hat{\mathbf{u}}_i). \tag{4.18}$$

Fluctuations in the location of an atomic nucleus in a solid at rest are much less than a typical atomic radius of 10^{-10} m. Thus if $\epsilon>10^{-6}$ m, then (noting here $u_j>\epsilon+\delta$) $(\mathbf{A}_j\,\mathbf{p}_j)_\Delta\sim(1/4\pi u_j^3)(\hat{\mathbf{A}}_j\,\mathbf{p}_j)_\Delta$, where $\hat{\mathbf{A}}_j:=(\mathbf{1}-3\hat{\mathbf{u}}_i\otimes\hat{\mathbf{u}}_i)$.

Remark 4.5 From (3.39) and $(3.33)_2$, only charges further than ϵ from $\mathbf{x}$ at time t contribute to $\boldsymbol{\mathcal{B}}_w(\mathbf{x},t)$. Any charge for which $u_i>\epsilon+\delta$ at this time yields (cf. $(3.34)_2$) a contribution $q_i\mathbf{u}_i\times\mathbf{v}_i/4\pi u_i^3$. Thus the contribution from a neutral atom outside $S_{\epsilon+\delta}(\mathbf{x})$ is (neglecting terms as in (4.17))

$$\begin{aligned} &(1/4\pi)\sum_{j_k=1}^{Z_j}e\big\{\mathbf{u}_j\times\mathbf{v}_j/u_j^3-(\mathbf{u}_j+\mathbf{r}_{j_k})\times(\mathbf{v}_j+\dot{\mathbf{r}}_{j_k})\big[u_j^{-3}-3(\mathbf{r}_{j_k}\,.\,\mathbf{u}_j)u_j^{-5}\big]\big\}\\ &=\big(1/4\pi u_j^3\big)\left\{\mathbf{u}_j\times\dot{\mathbf{p}}_j+\mathbf{m}_j+3\hat{\mathbf{u}}_j\times e\left(\sum_{j_k=1}^{Z_j}\dot{\mathbf{r}}_{j_k}\otimes\mathbf{r}_{j_k}\right)\hat{\mathbf{u}}_j\right\}+\mathbf{A}_j\,\mathbf{p}_j\times\mathbf{v}_j. \end{aligned} \tag{4.19}$$

In a solid at macroscopic rest $\mathbf{u}_j$ varies negligibly in $[t-\Delta,t]$ and the contribution to $(\boldsymbol{\mathcal{B}}_w)_\Delta(\mathbf{x},t)$ is essentially

$$\big(1/4\pi u_j^3\big)\left\{\mathbf{u}_j\times d/dt\big\{(\mathbf{p}_j)_\Delta\big\}+(\mathbf{m}_j)_\Delta+3\hat{\mathbf{u}}_j\times e\left(\sum_{j_k=1}^{Z_j}\dot{\mathbf{r}}_{j_k}\otimes\mathbf{r}_{j_k}\right)_\Delta\hat{\mathbf{u}}_j+(\hat{\mathbf{A}}_j\,\mathbf{p}_j\times\mathbf{v}_j)_\Delta\right\}. \tag{4.20}$$

The foregoing remarks emphasise the rôle of time averaging in interpreting the subatomic contributions to macroscopic measures of atomic kinetics for simple systems. The methodology extends to molecular systems modulo context-dependent considerations: for example, book-keeping appropriate to (valence) electrons which are shared by several nuclei.

5 Consequences of the Coulomb and Biot-Savart Laws

If $\boldsymbol{\mathcal{E}}_i$ denotes the contribution to $\boldsymbol{\mathcal{E}}_w$ of a charge q_i for which $u_i > \epsilon + \delta$, then (cf. $(3.37)_1$ and $(3.34)_2$) for any charge q

$$q\boldsymbol{\mathcal{E}}_i = -qq_i\,\mathbf{u}_i/4\pi u_i^3. \tag{5.1}$$

This may be compared with Coulomb's law for the force $\mathbf{f}^{es}_{qq_i}$ exerted *in vacuo* on a stationary charge q at location $\mathbf{x}$ by a stationary charge q_i at $\mathbf{x}_i$, namely (in SI units: cf., e.g., Griffiths [11], (2.1))

$$\mathbf{f}^{es}_{qq_i} = -qq_i\mathbf{u}_i/4\pi\epsilon_0 u_i^3. \tag{5.2}$$

Thus, for *in vacuo* separations in excess of $\epsilon + \delta$,

$$\mathbf{f}^{es}_{qq_i} = q\boldsymbol{\mathcal{E}}_i/\epsilon_0. \tag{5.3}$$

Accordingly, for any set $\{q_i\}$ of stationary charges all of which are distant at least $\epsilon + \delta$ from $\mathbf{x}$, the assumption of linear superposition (cf. [11], 2.4; Zangwill [12], (2.18); Jackson [13], 24–26; Elliott [14], Sect. 3.2) yields a *static* electric field $\mathbf{E}^s_w$ at $\mathbf{x}$ for which

$$q\mathbf{E}^s_w := \sum_i \mathbf{f}^{es}_{qq_i} = q\epsilon_0^{-1}\sum_i \boldsymbol{\mathcal{E}}_i. \tag{5.4}$$

Hence, for such a collection of charges

$$\mathbf{E}^s_w = \epsilon_0^{-1}\boldsymbol{\mathcal{E}}_w. \tag{5.5}$$

Similarly, if $u_i > \epsilon + \delta$, then the contribution of charge q_i to $\boldsymbol{\mathcal{B}}_w$ is (cf. $(3.39)_1$ and $(3.34)_2$)

$$\boldsymbol{\mathcal{B}}_i := q_i\mathbf{u}_i \times \mathbf{v}_i/4\pi u_i^3. \tag{5.6}$$

Consider a localised set of charges P_i which lie within a sphere of radius ϵ centred at point $\mathbf{X}$. If $\mathbf{R} = \mathbf{X} - \mathbf{x}$ and $R := \|\mathbf{R}\| \gg \epsilon$, then $\mathbf{u}_i = (\mathbf{x}_i - \mathbf{X}) + (\mathbf{X} - \mathbf{x}) \sim \mathbf{R}$ and

$$\begin{aligned}\sum_i \boldsymbol{\mathcal{B}}_i &\sim \sum_i q_i\mathbf{R} \times \mathbf{v}_i/4\pi R^3 \\ &= \left(\mathbf{R}/4\pi R^3\right) \times \left(\sum_i q_i\mathbf{v}_i/V_\epsilon\right)V_\epsilon \sim \left(\mathbf{R}/4\pi R^3\right) \times \mathbf{j}_w(\mathbf{X})V_\epsilon.\end{aligned} \tag{5.7}$$

This may be compared with the Biot-Savart law in which the contribution $\Delta\mathbf{B}^s$ to the net magnetic field $\mathbf{B}^s$ at $\mathbf{x}$ arising from such a collection of charges is (cf., e.g., [12], (10.15)) essentially

$$\Delta\mathbf{B}^s = \left(\mu_0\,\mathbf{R}/4\pi R^3 \times \mathbf{j}^s\right)V_\epsilon. \tag{5.8}$$

Here $\mathbf{j}^s$ is the current density associated with the charges and is *steady* (emphasised by superscript 's'). Comparison of (5.7) with (5.8) suggests the natural identification (for the charges considered)

$$\mathcal{B}_w = \sum_i \mathcal{B}_i =: \mu_0^{-1}\mathbf{B}_w^s. \tag{5.9}$$

Remark 5.1 Relation (5.5) and identification (5.9) pertain to experimental results *in vacuo* (otherwise described as 'free space') based upon force relation (1.9). Specifically, no charges within $\epsilon + \delta$ of $\mathbf{x}$ are present. More generally, (5.5) and (5.9) motivate definitions

$$\mathbf{E}_w^s := \epsilon_0^{-1}\mathcal{E}_w, \qquad \mathbf{B}_w^s := \mu_0\mathcal{B}_w, \tag{5.10}$$

which add contributions (if any) from charges for which $\epsilon \le u_i \le \epsilon + \delta$ (cf. $(3.37)_1$, $(3.39)_1$, $(3.33)_2$, $(3.31)_2$ and (3.16)). Since $\delta \ll \epsilon$ is arbitrarily small, definitions (5.10) in general represent precise book-keeping rather than a physical issue.

From (3.35) and (5.10),

$$\mathbf{D}_w = \mathbf{P}_w + \epsilon_0\mathbf{E}_w^s, \qquad \mathbf{H}_w = -\mathbf{M}_w + \mu_0^{-1}\mathbf{B}_w^s. \tag{5.11}$$

Further, from (3.52), (3.53) and (5.10),

$$\mathbf{E}_w^s = -\nabla\psi_w^s, \qquad \mathbf{B}_w^s = \operatorname{curl}\mathbf{A}_w^s, \tag{5.12}$$

where

$$\psi_w^s := \epsilon_0^{-1}\psi_w = \epsilon_0^{-1}\sum_i q_i\, f(u_i), \qquad \mathbf{A}_w^s := \mu_0\mathcal{A}_w = \mu_0\sum_i q_i\, f(u_i)\mathbf{v}_i. \tag{5.13}$$

Relation (3.57) may be written as

$$\partial\psi_w^s/\partial t + (1/\epsilon_0\,\mu_0)\operatorname{div}\mathbf{A}_w^s = 0. \tag{5.14}$$

Remark 5.2 From $(5.10)_1$, $(3.63)_1$ and $(3.64)_1$,

$$\dim\epsilon_0 = \dim\mathcal{E}_w/\dim\mathbf{E}_w^s = \dim\mathcal{E}_w/\dim\mathbf{E} = M^{-1}L^{-3}T^2Q^2. \tag{5.15}$$

From $(5.10)_2$, $(3.63)_2$ and $(3.64)_2$,

$$\dim\mu_0 = \dim\mathbf{B}_w^s/\dim\mathcal{B}_w = \dim\mathbf{B}/\dim\mathcal{B}_w = ML\,Q^{-2}. \tag{5.16}$$

Hence

$$\dim(\epsilon_0\mu_0) = L^{-2}T^2 = \left(LT^{-1}\right)^{-2}, \tag{5.17}$$

$\epsilon_0\mu_0$ has the dimension of (speed)$^{-2}$, and so $\epsilon_0\mu_0 = \alpha c^{-2}$ for some dimensionless constant α. Experimentation yields $\alpha = 1$: that is,

$$\epsilon_0\mu_0 = c^{-2}. \tag{5.18}$$

From (5.12), (5.14) and (5.18),

$$\operatorname{curl}\mathbf{E}_w^s = \mathbf{0}, \qquad \operatorname{div}\mathbf{B}_w^s = 0, \tag{5.19}$$

and

$$\partial \psi_w^s / \partial t + c^2 \operatorname{div} \mathbf{A}_w^s = 0. \tag{5.20}$$

Remark 5.3 The experimental laws of Coulomb and Biot-Savart relate to macroscopically stationary situations and hold at points in the neighbourhoods of which no charges are to be found. Specifically, the stationary requirement yields (cf. (2.8), (2.33) and (5.14))

$$\partial \rho_w / \partial t = 0, \qquad \partial \mathbf{D}_w / \partial t = \mathbf{0}, \qquad \partial \psi_w^s / \partial t = 0, \tag{5.21}$$

while the free space condition (at scale $\epsilon + \delta$) implies that (cf. (2.5), (2.9), $(3.36)_2$ and $(3.38)_2$)

$$\rho_w = 0, \qquad \mathbf{j}_w = \mathbf{0}, \qquad \mathbf{P}_w = \mathbf{0}, \qquad \mathbf{M}_w = \mathbf{0}. \tag{5.22}$$

Accordingly, from (2.20), $(5.22)_1$, $(5.11)_1$ and $(5.22)_3$,

$$\operatorname{div} \mathbf{E}_w^s = \mathbf{0}, \tag{5.23}$$

while from (2.33), $(5.21)_2$, $(5.22)_2$, $(5.11)_2$ and $(5.22)_4$,

$$\operatorname{curl} \mathbf{B}_w^s = \mathbf{0}. \tag{5.24}$$

It follows from $(5.12)_1$ and (5.23) that

$$\Delta \psi_w^s = 0, \tag{5.25}$$

while (5.24) and $(5.12)_2$ yield

$$\mathbf{0} = \operatorname{curl} \operatorname{curl} \mathbf{A}_w^s = \nabla \left\{ \operatorname{div} \mathbf{A}_w^s \right\} - \Delta \mathbf{A}_w^s.$$

Accordingly, from $(5.21)_3$ and (5.14),

$$\Delta \mathbf{A}_w^s = \mathbf{0}. \tag{5.26}$$

Of course, relations (5.25) and (5.26) are immediately evident on noting that here $u_i > \epsilon + \delta$ and thus $f(u_i) = 1/4\pi u_i$ (cf. $(3.62)_2$).

As they stand, relations (5.12) and (5.13) do not of themselves indicate any restriction to macroscopically-static situations, and admit *formal* generalisation to dynamical contexts without change. In particular, (2.20) may be written, via $(5.11)_1$, as

$$\epsilon_0 \operatorname{div} \mathbf{E}_w^s = \rho', \tag{5.27}$$

where

$$\rho' := \rho_w - \operatorname{div} \mathbf{P}_w. \tag{5.28}$$

Further, (2.33) may be expressed, via $(5.11)_1$ and $(5.11)_2$, as

$$\epsilon_0 \mu_0 \, \partial \mathbf{E}_w^s / \partial t + \mu_0 \mathbf{J}' = \operatorname{curl} \mathbf{B}_w^s, \tag{5.29}$$

where

$$\mathbf{J}' := \mathbf{j}_w + \partial \mathbf{P}_w / \partial t + \operatorname{curl} \mathbf{M}_w. \tag{5.30}$$

Thus

$$\partial\rho'/\partial t = \partial\rho_w/\partial t - \partial/\partial t\{\operatorname{div}\mathbf{P}_w\} = \partial\rho_w/\partial t - \operatorname{div}\{\partial\mathbf{P}_w/\partial t\}, \tag{5.31}$$

$$\operatorname{div}\mathbf{J}' = \operatorname{div}\mathbf{j}_w + \operatorname{div}\{\partial\mathbf{P}_w/\partial t\}, \tag{5.32}$$

and hence, via (2.8),

$$\partial\rho'/\partial t + \operatorname{div}\mathbf{J}' = 0. \tag{5.33}$$

While (5.27) and (5.29) are two often-cited versions of Maxwell relations (cf., e.g., Griffiths [11], §10.1.1) the foregoing approach does not yield a corresponding version of (1.8). Guided by (1.7), suppose

$$\tilde{\mathbf{E}}_w := \mathbf{E}^s_w - \partial\mathbf{A}^s_w/\partial t = -\nabla\psi^s_w - \partial\mathbf{A}^s_w/\partial t. \tag{5.34}$$

Then clearly

$$\operatorname{curl}\tilde{\mathbf{E}}_w = -\operatorname{curl}\left\{\partial\mathbf{A}^s_w/\partial t\right\} = -\partial/\partial t\left\{\operatorname{curl}\mathbf{A}^s_w\right\}, \tag{5.35}$$

whence, from $(5.12)_2$,

$$\operatorname{curl}\tilde{\mathbf{E}}_w = -\partial\mathbf{B}^s_w/\partial t. \tag{5.36}$$

While this relation has been an immediate consequence of (5.34), relation (5.29), with $\mathbf{E}^s_w$ equated with $\tilde{\mathbf{E}}_w + \partial\mathbf{A}^s_w/\partial t$, introduces an extra term $\epsilon_0\mu_0\,\partial^2\mathbf{A}^s_w/\partial t^2$. Rather than adopting this approach, consider the direct consequence of definition (5.34) which is summarised in the following result.

Theorem 5.1

$$\partial\mathbf{B}^s_w/\partial t = -\operatorname{curl}\tilde{\mathbf{E}}_w, \qquad c^{-2}\partial\tilde{\mathbf{E}}_w/\partial t + \mu_0\tilde{\mathbf{J}}_w = \operatorname{curl}\mathbf{B}^s_w, \tag{5.37}$$

where

$$-\mu_0\tilde{\mathbf{J}}_w := \Delta\mathbf{A}^s_w - c^{-2}\partial^2\mathbf{A}^s_w/\partial t^2. \tag{5.38}$$

Further,

$$\operatorname{div}\tilde{\mathbf{E}}_w = \tilde{\rho}_w/\epsilon_0, \tag{5.39}$$

where

$$-\epsilon_0^{-1}\tilde{\rho}_w := \Delta\psi^s_w - c^{-2}\partial^2\psi^s_w/\partial t^2, \tag{5.40}$$

and

$$\partial\tilde{\rho}_w/\partial t + \operatorname{div}\tilde{\mathbf{J}}_w = 0. \tag{5.41}$$

Proof Result $(5.37)_1$ was derived above. From $(5.12)_2$, (5.20), $(5.12)_1$, $(5.34)_1$ and (5.38),

$$\begin{aligned}\operatorname{curl}\mathbf{B}^s_w &= \operatorname{curl}\operatorname{curl}\mathbf{A}^s_w = \nabla\left\{\operatorname{div}\mathbf{A}^s_w\right\} - \Delta\mathbf{A}^s_w = \nabla\left\{-c^{-2}\partial\psi^s_w/\partial t\right\} - \Delta\mathbf{A}^s_w\\ &= c^{-2}\partial/\partial\left\{-\nabla\psi^s_w\right\} - \Delta\mathbf{A}^s_w = c^{-2}\partial\mathbf{E}^s_w/\partial t - \Delta\mathbf{A}^s_w\\ &= c^{-2}\left\{\partial\tilde{\mathbf{E}}_w/\partial t + \partial^2\mathbf{A}^s_w/\partial t^2\right\} - \Delta\mathbf{A}^s_w = c^{-2}\partial\tilde{\mathbf{E}}_w/\partial t + \mu_0\tilde{\mathbf{J}}_w.\end{aligned}$$

From $(5.34)_1$, $(5.12)_1$, (5.20) and (5.40),

$$\begin{aligned}\operatorname{div}\tilde{\mathbf{E}}_w &= \operatorname{div}\mathbf{E}^s_w - \operatorname{div}\{\partial\mathbf{A}^s_w/\partial t\} = -\Delta\psi^s_w - \partial/\partial t\{\operatorname{div}\mathbf{A}^s_w\}\\ &= -\Delta\psi^s_w + c^{-2}\partial^2\psi^s_w/\partial t^2 = \tilde{\rho}/\epsilon_0.\end{aligned}$$

From (5.38), (5.20), (5.40) and (5.18),

$$\begin{aligned}-\operatorname{div}\{\mu_0\tilde{\mathbf{J}}_w\} &= \operatorname{div}\{\Delta\mathbf{A}^s_w\} - c^{-2}\operatorname{div}\{\partial^2\mathbf{A}^s_w/\partial t^2\}\\ &= \{\Delta - c^{-2}\partial^2/\partial t^2\}\{\operatorname{div}\mathbf{A}^s_w\} = \{\Delta - c^{-2}\partial^2/\partial t^2\}\{-c^{-2}\partial\psi^s_w/\partial t\}\\ &= -c^{-2}\partial/\partial t\{(\Delta - c^{-2}\partial^2/\partial t^2)\psi^s_w\} = c^{-2}\epsilon_0^{-1}\partial\tilde{\rho}_w/\partial t = \mu_0\partial\tilde{\rho}_w/\partial t. \qquad \square\end{aligned}$$

Corollary 5.1.1

$$\Delta\mathbf{B}^s_w - c^{-2}\partial^2\mathbf{B}^s_w/\partial t^2 = \mathbf{s}_1, \tag{5.42}$$

and

$$\Delta\tilde{\mathbf{E}}_w - c^{-2}\partial^2\tilde{\mathbf{E}}_w/\partial t^2 = \mathbf{s}_2, \tag{5.43}$$

where

$$\mathbf{s}_1 := -\mu_0\operatorname{curl}\tilde{\mathbf{J}}_w, \qquad \mathbf{s}_2 := \epsilon_0^{-1}\nabla\tilde{\rho}_w + \mu_0\partial\tilde{\mathbf{J}}_w/\partial t. \tag{5.44}$$

Proof Relation (5.42) with $(5.44)_1$ follow upon taking the curl of (5.38) and noting $(5.12)_2$. Taking the gradient of (5.40) and noting (5.34),

$$\begin{aligned}\epsilon_0^{-1}\nabla\tilde{\rho}_w &= \{\Delta - c^{-2}\partial^2/\partial t^2\}\{\tilde{\mathbf{E}}_w + \partial\mathbf{A}^s_w/\partial t\}\\ &= \Delta\tilde{\mathbf{E}}_w - c^{-2}\partial^2\tilde{\mathbf{E}}_w/\partial t^2 + \partial/\partial t\{-\mu_0\tilde{\mathbf{J}}_w\}\end{aligned}$$

via the time derivative of (5.38). $\square$

Remark 5.4 The foregoing definitions and manipulations require that $\tilde{\mathbf{E}}_w$ and $\mathbf{B}^s_w$ be of class C^2 in space and time. In particular, $(5.10)_{1,2}$ require that $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w$ be spatially of class C^2. Accordingly, from (3.37), (3.39) and (3.31), w must be of class C^2. However, simplest choice (3.8) of C^1 mollifier φ implies that w is only of class C^1. The simplest choice of C^2 mollifier is (cf. [5], (4.3.59))

$$\varphi_5(\lambda) := -6\lambda^5 + 15\lambda^4 - 10\lambda^3 + 1. \tag{5.45}$$

The corresponding analysis is straightforward, delivers an appropriate polynomial in place of $F(\lambda)$ (cf. (3.16) and (3.17)), and affects only separations in the range $\epsilon < u_i < \epsilon + \delta$.

Remark 5.5 Since $f(u_i) = 0$ if $u_i < \epsilon$, from (5.13), (5.12), (5.34), (5.42) and (5.43) only charges further than ϵ from $\mathbf{x}$ contribute to the values of ψ^s_w, $\mathbf{A}^s_w$, $\tilde{\mathbf{E}}_w$, $\mathbf{B}^s_w$, $\mathbf{s}_1$ and $\mathbf{s}_2$. If all charges are further than a distance d from $\mathbf{x}$ (so that $\mathbf{x}$ lies in free space), then one can choose a scale ϵ and mollifying interval δ such that $\epsilon + \delta < d$. In such case relations (5.22) hold at any such scale. Further, (5.42) and (5.43) hold with sources given by $(5.44)_{1,2}$ in which $\tilde{\rho}_w$ and $\tilde{\mathbf{J}}_w$ satisfy (5.40) and (5.38). Here ψ^s_w and $\mathbf{A}^s_w$ are defined by $(5.13)_{1,2}$ with $f(u_i) = 1/4\pi u_i$. Specifically, we have

Corollary 5.1.2 *If all charges are further than $\epsilon+\delta$ from location $\boldsymbol{x}$, then at $\boldsymbol{x}$*

$$\tilde{\rho}_w = \big(1/4\pi c^2\big)\sum_i q_i\big\{\alpha_i - (\mathbf{u}_i\,.\,\dot{\mathbf{v}}_i)u_i^{-3}\big\} \tag{5.46}$$

and

$$\tilde{\mathbf{J}}_w = \big(1/4\pi c^2\big)\sum_i q_i\big\{\alpha_i\mathbf{v}_i - \mathbf{L}_i\dot{\mathbf{v}}_i + u_i^{-1}\ddot{\mathbf{v}}_i\big\}, \tag{5.47}$$

where

$$\alpha_i := \big[3(\mathbf{u}_i\,.\,\mathbf{v}_i)^2 - u_i^2v_i^2\big]u_i^{-5} \tag{5.48}$$

and

$$\mathbf{L}_i := \big[2(\mathbf{u}_i\,.\,\mathbf{v}_i)\mathbf{1} + \mathbf{v}_i\otimes\mathbf{u}_i\big]u_i^{-3}. \tag{5.49}$$

Further,

$$\tilde{\mathbf{E}}_w = -(1/4\pi\epsilon_0)\sum_i q_i u_i^{-3}\big\{\mathbf{u}_i + c^{-2}\big[u_i^2\dot{\mathbf{v}}_i - (\mathbf{u}_i\,.\,\mathbf{v}_i)\mathbf{v}_i\big]\big\} \tag{5.50}$$

and

$$\mathbf{B}^s_w = \big(1/4\pi\epsilon_0c^2\big)\sum_i q_i u_i^{-3}(\mathbf{u}_i\times\mathbf{v}_i). \tag{5.51}$$

Proof If $u_i > \epsilon+\delta$, then (cf. (3.62)$_2$) $f(u_i) = 1/4\pi u_i$. Straightforward calculations yield

$$\Delta\big\{u_i^{-1}\big\} = 0, \qquad \Delta\big\{u_i^{-1}\mathbf{v}_i\big\} = \mathbf{0}, \qquad \partial^2/\partial t^2\big\{u_i^{-1}\big\} = \alpha_i - (\mathbf{u}_i\,.\,\dot{\mathbf{v}}_i)u_i^{-3}, \tag{5.52}$$

$$\partial^2/\partial t^2\big\{u_i^{-1}\mathbf{v}_i\big\} = \big[\alpha_i - (\mathbf{u}_i\,.\,\dot{\mathbf{v}}_i)u_i^{-3}\big]\mathbf{v}_i - 2(\mathbf{u}_i\,.\,\mathbf{v}_i)u_i^{-3}\dot{\mathbf{v}}_i + u_i^{-1}\ddot{\mathbf{v}}_i. \tag{5.53}$$

Relations (5.46) and (5.47) follow from (5.40), (5.38) and definitions (5.13)$_{1,2}$.

Fields $\tilde{\mathbf{E}}_w$ and $\mathbf{B}^s_w$ are given by (5.34) and (5.12)$_2$ with ψ^s_w and $\mathbf{A}^s_w$ defined by (5.13)$_{1,2}$. Relations (5.50) and (5.51) follow on noting that

$$\nabla\big\{u_i^{-1}\big\} = u_i^{-3}\mathbf{u}_i, \qquad \partial/\partial t\big\{u_i^{-1}\mathbf{v}_i\big\} = -u_i^{-3}(\mathbf{u}_i\,.\,\mathbf{v}_i)\mathbf{v}_i + u_i^{-1}\dot{\mathbf{v}}_i, \tag{5.54}$$

and

$$\mathrm{curl}\big\{u_i^{-1}\mathbf{v}_i\big\} = \nabla\big\{u_i^{-1}\big\}\times\mathbf{v}_i = u_i^{-3}\mathbf{u}_i\times\mathbf{v}_i. \tag{5.55}$$

□

Remark 5.6 At this point a complete set of Maxwell equations has been obtained, at any spatial scale, from a Newtonian perspective. While relations (2.8), (2.20) and (2.33) are entirely general (and of purely kinematic character) relations (5.37) are associated with forces in free space, via the results of Coulomb and Biot-Savart, and hence have been shown to hold *in vacuo*. The analytic key to the theory is function f: this determines ψ^s_w and $\mathbf{A}^s_w$ (cf. (5.13)) and thereby $\tilde{\mathbf{E}}_w$ and $\mathbf{B}^s_w$ (cf. (5.12) and (5.34)). The form of f is determined by choice of scale ϵ and mollifier φ. Departure from Newtonian dynamics stems from consideration of the argument u_i of f in respect of any charge. Determination of $u_i(\mathbf{x},t)$ requires knowledge of the location of P_i at time t. In a dynamic context this cannot be known at $\mathbf{x}$ since information transfer is not instantaneous. Indeed, the inhomogeneous wave equations (5.42) and

(5.43) indicate that the transfer speed in free space is c. Information available at $\mathbf{x}$ at time t consists of the *apparent* displacement $\mathbf{u}_i^a(\mathbf{x},t)$ of P_i from $\mathbf{x}$ at time t, and time derivatives thereof: in particular, the apparent velocity $\mathbf{v}_i^a := \partial \mathbf{u}_i^a/\partial t$. In generalising the discussion to take account of information delay ('retardation') the key is to find analogues of potentials ψ_w^s and $\mathbf{A}_w^s$ which depend upon available knowledge, namely $\mathbf{u}_i^a$ and $\mathbf{v}_i^a$.

6 Signal Transmission Times, Retardation and Classical Maxwellian Electrodynamics

Consider the behaviour of a moving point charge P_i as monitored by an observer O located at a point $\mathbf{x}$ in an inertial frame $\mathcal{F}$. Information available to O at time t can involve only data that has reached O at, or before, this time. Since transmission of information is not instantaneous, it is necessary to examine the consequences of transmission time delay. Examined here are the relationship between the apparent location of P_i at time t, the delay such datum of information takes to reach O, and the actual trajectory of P_i.

It is instructive to consider how, at least in principle, information about the motion of P_i could be obtained. Suppose that O has a radar device capable of detecting a reflected segment (from P_i) of any signal that it has transmitted. Suppose further that any such segment, both during its outward and inward paths, travels in a straight line at constant speed c, and is instantaneously reflected by P_i. Such a signal, emitted from $\mathbf{x}$ at time t' and received back at time t, will have travelled a total distance $c(t-t')$ and been reflected at time $t'+(t-t')/2 = (t+t')/2$ when at a distance $c(t-t')/2$ from $\mathbf{x}$. If $\mathbf{x}_i(\tau)$ denotes the location of P_i at time τ, then the distance travelled by the signal between its reflection and reception is

$$\left\|\mathbf{x}_i\left(\left(t+t'\right)/2\right)-\mathbf{x}\right\| = c\left(t-t'\right)/2. \tag{6.1}$$

Changing notation, if $\tau_i(\mathbf{x},t)$ denotes the time at which the signal reaching $\mathbf{x}$ at time t was reflected from P_i, then

$$\tau_i(\mathbf{x},t) = \left(t+t'\right)/2, \tag{6.2}$$

a detectable quantity. Accordingly, (6.1) may be written as

$$\left\|\mathbf{x}_i\left(\tau_i(\mathbf{x},t)\right)-\mathbf{x}\right\| = c\left(t-\tau_i(\mathbf{x},t)\right). \tag{6.3}$$

Assuming that the device can detect the direction of the incoming signal, such information, together with (6.3), determines location $\mathbf{x}_i(\tau_i(\mathbf{x},t))$. This is the *apparent* location of P_i at time t as monitored at location $\mathbf{x}$, $\mathbf{x}_i^a(\mathbf{x},t)$ say. That is,

$$\mathbf{x}_i^a(\mathbf{x},t) := \mathbf{x}_i\left(\tau_i(\mathbf{x},t)\right) = (\mathbf{x}_i \circ \tau_i)(\mathbf{x},t). \tag{6.4}$$

Remark 6.1 In any motion of P_i, apparent location $\mathbf{x}_i^a(\mathbf{x},t)$ is unique. Indeed, suppose that a signal emitted from $\mathbf{x}$ gives rise to segments which are reflected by P_i at times τ and τ', and are both received at $\mathbf{x}$ at time t. Thus τ and τ' are two values of $\tau_i(\mathbf{x},t)$, and $\mathbf{x}_i(\tau)$ and $\mathbf{x}_i(\tau')$ are both candidates for $\mathbf{x}_i^a(\mathbf{x},t)$. From (6.3) it follows that[5]

$$\left\|\mathbf{x}_i(\tau)-\mathbf{x}_i\left(\tau'\right)\right\| = \left\|\left(\mathbf{x}_i(\tau)-\mathbf{x}\right)-\left(\mathbf{x}_i\left(\tau'\right)-\mathbf{x}\right)\right\|$$

[5]For any pair of vectors $\mathbf{v}$ and $\mathbf{v}'$, $\|\mathbf{v}-\mathbf{v}'\| \geq |\,\|\mathbf{v}\| - \|\mathbf{v}'\|\,|$.

$$\geq \left| \left\| \mathbf{x}_i(\tau) - \mathbf{x} \right\| - \left\| \mathbf{x}_i(\tau') - \mathbf{x} \right\| \right|$$
$$= \left| c(t-\tau) - c\big(t-\tau'\big) \right| = c\left|\tau - \tau'\right|. \tag{6.5}$$

Hence the average speed of P_i over a time interval of duration $|\tau - \tau'|$ is at least c, a physical impossibility. Thus the hypothesis of two times τ and τ' is incorrect, $\tau_i(\mathbf{x}, t)$ is unique, and so $\mathbf{x}_i^a(\mathbf{x}, t)$ is unique.

The *a*pparent displacement of P_i from $\mathbf{x}$ at time t is

$$\mathbf{u}_i^a(\mathbf{x}, t) := \mathbf{x}_i^a(\mathbf{x}, t) - \mathbf{x} = \mathbf{x}_i\big(\tau_i(\mathbf{x}, t)\big) - \mathbf{x}. \tag{6.6}$$

Writing

$$u_i^a := \|\mathbf{u}_i^a\|, \tag{6.7}$$

relation (6.3) becomes

$$u_i^a(\mathbf{x}, t) = c\big(t - \tau_i(\mathbf{x}, t)\big). \tag{6.8}$$

Time τ_i, apparent location $\mathbf{x}_i^a$, and apparent displacement $\mathbf{u}_i^a$ are functions of $\mathbf{x}$ and t, and are thus *fields*; τ_i is termed the *retarded time* field. In particular, $\tau_i(\mathbf{x}, t)$ and $\mathbf{x}_i^a(\mathbf{x}, t)$ constitute the basic information about any motion of P_i which is available at $\mathbf{x}$ at time t. Such information is necessary in computation of velocities. Here a distinction must be made between the *actual* velocity of P_i at any time τ, namely

$$\mathbf{v}_i(\tau) := d/d\tau\big\{\mathbf{x}_i(\tau)\big\} = \dot{\mathbf{x}}_i(\tau), \tag{6.9}$$

and the *apparent velocity*

$$\mathbf{v}_i^a(\mathbf{x}, t) := \partial/\partial t\big\{\mathbf{x}_i^a(\mathbf{x}, t)\big\} = \partial/\partial t\big\{\mathbf{u}_i^a(\mathbf{x}, t)\big\} \tag{6.10}$$

corresponding to P_i as monitored at $\mathbf{x}$ and time t. In particular, $\mathbf{v}_i$ is a vector-valued function of time, whereas $\mathbf{v}_i^a$ is a vector-valued field. From (6.4) and the chain rule,

$$\mathbf{v}_i^a(\mathbf{x}, t) = \partial/\partial t\big\{\mathbf{x}_i^a(\mathbf{x}, t)\big\} = \partial/\partial t\big\{\mathbf{x}_i\big(\tau_i(\mathbf{x}, t)\big)\big\}$$
$$= \dot{\mathbf{x}}_i\big(\tau_i(\mathbf{x}, t)\big)\partial\tau_i/\partial t. \tag{6.11}$$

That is,

$$\mathbf{v}_i^a(\mathbf{x}, t) = \mathbf{v}_i\big(\tau_i(\mathbf{x}, t)\big)\partial\tau_i/\partial t. \tag{6.12}$$

To determine $\partial\tau_i/\partial t$, note that from (6.6) and (6.8)

$$\mathbf{u}_i^a \,.\, \mathbf{u}_i^a = \big(u_i^a\big)^2 = c^2(t-\tau_i)^2. \tag{6.13}$$

Differentiation with respect to t yields, via $(6.10)_2$ and (6.8),

$$2\mathbf{u}_i^a \,.\, \mathbf{v}_i^a = 2c^2(t-\tau_i)(1 - \partial\tau_i/\partial t) = 2cu_i^a(1 - \partial\tau_i/\partial t). \tag{6.14}$$

Accordingly,

$$\partial\tau_i/\partial t = 1 - c^{-1}\hat{\mathbf{u}}_i^a \,.\, \mathbf{v}_i^a =: \beta_i, \tag{6.15}$$

where

$$\hat{\mathbf{u}}_i^a := \mathbf{u}_i^a/u_i^a. \tag{6.16}$$

Thus $\hat{\mathbf{u}}_i^a(\mathbf{x},t)$ denotes a unit vector in the direction of P_i from $\mathbf{x}$ at the latest time $\tau_i(\mathbf{x},t)$ that information about P_i reaches $\mathbf{x}_i$ at time t. From (6.12) and (6.15),

$$\mathbf{v}_i\big(\tau_i(\mathbf{x},t)\big) = \big(\beta_i(\mathbf{x},t)\big)^{-1}\mathbf{v}_i^a(\mathbf{x},t). \tag{6.17}$$

Equivalently,

$$\mathbf{v}_i \circ \tau_i = \beta_i^{-1}\mathbf{v}_i^a. \tag{6.18}$$

This relation delivers the actual velocity at time $\tau_i(\mathbf{x},t)$ in terms of information available at $\mathbf{x}$ and time t.

The spatial derivative (or 'gradient') of (6.13) yields, with (6.8)

$$2\big(\nabla\mathbf{u}_i^a\big)^T\mathbf{u}_i^a = 2c^2(t-\tau_i)(-\nabla\tau_i) = -2cu_i^a\nabla\tau_i. \tag{6.19}$$

From (6.6), the chain rule and (6.9),

$$\nabla\mathbf{u}_i^a = \partial\mathbf{x}_i/\partial\tau\otimes\nabla\tau_i - \mathbf{1} = \dot{\mathbf{x}}_i\otimes\nabla\tau_i - \mathbf{1} = \mathbf{v}_i\otimes\nabla\tau_i - \mathbf{1}, \tag{6.20}$$

whence

$$\big(\nabla\mathbf{u}_i^a\big)^T = \nabla\tau_i\otimes\mathbf{v}_i - \mathbf{1}. \tag{6.21}$$

Hence (6.19) may be written, noting (6.16), as

$$(\nabla\tau_i\otimes\mathbf{v}_i - \mathbf{1})\hat{\mathbf{u}}_i^a = -c\nabla\tau_i, \tag{6.22}$$

and so

$$\big(\mathbf{v}_i\,.\,\hat{\mathbf{u}}_i^a + c\big)\nabla\tau_i = \hat{\mathbf{u}}_i^a. \tag{6.23}$$

Since, from (6.17) and $(6.15)_2$,

$$\mathbf{v}_i\,.\,\hat{\mathbf{u}}_i^a + c = \beta_i^{-1}\mathbf{v}_i^a\,.\,\hat{\mathbf{u}}_i^a + c = \beta_i^{-1}\big(\mathbf{v}_i^a\,.\,\hat{\mathbf{u}}_i^a + \beta_i c\big) = \beta_i^{-1}c, \tag{6.24}$$

it follows from (6.23) that

$$\nabla\tau_i = \beta_i\hat{\mathbf{u}}_i^a/c. \tag{6.25}$$

From $(6.20)_3$, (6.25) and (6.18),

$$\nabla\mathbf{u}_i^a = \mathbf{v}_i\otimes\beta_i\,\hat{\mathbf{u}}_i^a/c - \mathbf{1} = \mathbf{v}_i^a\otimes\hat{\mathbf{u}}_i^a/c - \mathbf{1}, \tag{6.26}$$

and, from (6.8) and (6.25),

$$\nabla u_i^a = -c\nabla\tau_i = -\beta_i\hat{\mathbf{u}}_i^a. \tag{6.27}$$

Further, from (6.26) and $(6.15)_2$,

$$\operatorname{div}\mathbf{u}_i^a := \operatorname{tr}\big\{\nabla\mathbf{u}_i^a\big\} = \mathbf{v}_i^a\,.\,\hat{\mathbf{u}}_i^a/c - 3 = -\beta_i - 2. \tag{6.28}$$

In order to generalise the discussion of Sect. 5 to dynamic situations, taking account of signal transmission delay, it suffices to consider the potential functions: cf. Remark 5.6. A natural choice $\mathbf{A}_w^d$ for the *d*ynamic counterpart of $\mathbf{A}_w^s$ (cf. $(5.13)_2$) would seem to be

$$\mathbf{A}_w^d := \mu_0\sum_i q_i\,f\big(u_i^a\big)\mathbf{v}_i^a = \mu_0\sum_i q_i\,f\big(u_i^a\big)\beta_i\,\mathbf{v}_i. \tag{6.29}$$

Choice $\mathbf{A}_w^d(\mathbf{x},t)$ takes account of the latest information concerning charge displacements and velocities that is available at $\mathbf{x}$ and time t, has the form of $(5.13)_2$, and would coincide with $\mathbf{A}_w^s(\mathbf{x},t)$ were transmission times to be negligible. A choice of the corresponding dynamic electric potential might be

$$\tilde{\psi}_w^d := \epsilon_0^{-1} \sum_i q_i \, f\big(u_i^a\big). \tag{6.30}$$

However, this field does not satisfy the counterpart of the Lorenz gauge relation (cf., e.g., Jackson [13], p. 240), namely

$$\partial \psi_w^d / \partial t + (1/\epsilon_0 \mu_0) \operatorname{div} \mathbf{A}_w^d = 0. \tag{6.31}$$

To obtain an appropriate candidate ψ_w^d, consider (cf. (6.29))

$$\operatorname{div}\big\{ f\big(u_i^a\big)\mathbf{v}_i^a \big\} = \nabla f\big(u_i^a\big) \,.\, \mathbf{v}_i^a + f\big(u_i^a\big) \operatorname{div} \mathbf{v}_i^a. \tag{6.32}$$

From (6.27),

$$\nabla f\big(u_i^a\big) = f'\big(u_i^a\big) \nabla u_i^a = -\beta_i \, f'\big(u_i^a\big) \hat{\mathbf{u}}_i^a, \tag{6.33}$$

while, from (6.28),

$$\operatorname{div} \mathbf{v}_i^a = \operatorname{div}\big\{ \partial \mathbf{u}_i^a / \partial t \big\} = \partial / \partial t \big\{ \operatorname{div} \mathbf{u}_i^a \big\} = -\partial \beta_i / \partial t. \tag{6.34}$$

Accordingly, (6.32), (6.33) and (6.34) imply

$$\operatorname{div}\big\{ f\big(u_i^a\big)\mathbf{v}_i^a \big\} = -\beta_i \, f'\big(u_i^a\big) \hat{\mathbf{u}}_i^a \,.\, \mathbf{v}_i^a - \partial \beta_i / \partial t \, f\big(u_i^a\big). \tag{6.35}$$

From (6.8) and (6.15),

$$\begin{aligned} \partial / \partial t \big\{ f\big(u_i^a\big) \big\} &= f'\big(u_i^a\big) \partial u_i^a / \partial t = f'\big(u_i^a\big) c (1 - \partial \tau_i / \partial t) \\ &= f'\big(u_i^a\big) c c^{-1} \hat{\mathbf{u}}_i^a \,.\, \mathbf{v}_i^a = f'\big(u_i^a\big) \hat{\mathbf{u}}_i^a \,.\, \mathbf{v}_i^a. \end{aligned} \tag{6.36}$$

Hence, from (6.35) and (6.36),

$$\operatorname{div}\big\{ f\big(u_i^a\big)\mathbf{v}_i^a \big\} = -\partial / \partial t \big\{ \beta_i \, f\big(u_i^a\big) \big\}, \tag{6.37}$$

and thus

$$\psi_w^d := \epsilon_0^{-1} \sum_i q_i \beta_i \, f\big(u_i^a\big) \tag{6.38}$$

satisfies (6.31).

The results of Sect. 5 can now be generalised by essentially replacing superscript 's' by 'd'. Specifically, function f is known once weighting function w is chosen: this follows from Theorem 3.1, consequent upon definition (3.5), $(3.31)_2$ and (3.51). In particular (cf. Remark 3.6), $f(u) = 0$ if $u < \epsilon$ and $f(u) = 1/4\pi u$ if $u > \epsilon + \delta$. Knowledge of f, and the latest information concerning displacements and velocities (namely $\mathbf{u}_i^a$ and $\mathbf{v}_i^a$), yield fields $\mathbf{A}_w^d$ and ψ_w^d via (6.29) and (6.38). The corresponding dynamic electric field $\mathbf{E}_w^d$ and magnetic field $\mathbf{B}_w^d$ are

$$\mathbf{E}_w^d := -\nabla \psi_w^d - \partial \mathbf{A}_w^d / \partial t, \qquad \mathbf{B}_w^d := \operatorname{curl} \mathbf{A}_w^d. \tag{6.39}$$

Theorem 6.1

$$\operatorname{div}\mathbf{E}_w^d = \rho_w^d/\epsilon_0, \qquad \operatorname{div}\mathbf{B}_w^d = 0, \tag{6.40}$$

$$\partial\mathbf{B}_w^d/\partial t = -\operatorname{curl}\mathbf{E}_w^d, \qquad c^{-2}\partial\mathbf{E}_w^d/\partial t + \mu_0\,\mathbf{J}_w^d = \operatorname{curl}\mathbf{B}_w^d, \tag{6.41}$$

where

$$-\epsilon_0^{-1}\,\rho_w^d := \Delta\psi_w^d - c^{-2}\partial^2\psi_w^d/\partial t^2, \qquad -\mu_0\,\mathbf{J}_w^d := \Delta\mathbf{A}_w^d - c^{-2}\partial^2\mathbf{A}_w^d/\partial t^2, \tag{6.42}$$

and

$$\partial\rho_w^d/\partial t + \operatorname{div}\mathbf{J}_w^d = 0. \tag{6.43}$$

Proof Relations $(6.40)_2$ and $(6.41)_1$ are immediate consequences of relations (6.39). Proofs of the remaining relations are precise analogues of those in Theorem 5.1, on invoking (6.31) in place of (5.14), writing $\mathbf{E}_w^d$ in place of $\tilde{\mathbf{E}}_w$, and changing superscript 's' to 'd'. □

Remark 6.2 Relations (6.40) and (6.41) constitute the general macroscopic form of Maxwell's equations in free space at scale ϵ in classical electrodynamics when supplemented by the purely kinematic relations (2.8), (2.20) and (2.33), with w given by (3.5). These relations take exactly the same form when additional time averaging is implemented in the manner of Sect. 4, specifically via invocation of properties (4.2) and (4.3). However, although all relations are thus form invariant under changes of length and time scales, the phenomena they represent may not be so obliging. Indeed, fields ρ_w^d and $\mathbf{J}_w^d$, which are the dynamic analogues of charge and current densities ρ_w and $\mathbf{j}_w$, are to be expected to depend upon the spatial and temporal scales associated with measurements of the phenomena of interest. Furthermore, fields ρ_w^d and $\mathbf{J}_w^d$ (or their time-averaged counterparts) can be considered to drive associated electromagnetic phenomena: their prescription gives rise to the inhomogeneous d'Alembert equations (6.42) for the corresponding potentials ψ_w^d and $\mathbf{A}_w^d$ which in turn (via (6.39)) generate the relevant fields $\mathbf{E}_w^d$ and $\mathbf{B}_w^d$ which satisfy (6.40) and (6.41). In particular, suppose the Δ-time averages of ρ_w^d and $\mathbf{J}_w^d$ vanish in some region. Then in this region the corresponding fields $\mathbf{E}_w^d$ and $\mathbf{B}_w^d$ satisfy (cf. $(6.40)_1$ and $(6.41)_2$),

$$\operatorname{div}\mathbf{E}_w^d = 0, \qquad \operatorname{div}\mathbf{B}_w^d = 0, \tag{6.44}$$

$$\partial\mathbf{B}_w^d/\partial t = -\operatorname{curl}\mathbf{E}_w^d, \qquad c^{-2}\partial\mathbf{E}_w^d/\partial t = \operatorname{curl}\mathbf{B}_w^d. \tag{6.45}$$

Accordingly, from $(6.45)_1$, $(6.45)_2$ and $(6.44)_1$,

$$\begin{aligned}\partial^2\mathbf{B}_w^d/\partial t^2 &= -\partial/\partial t\{\operatorname{curl}\mathbf{E}_w^d\} = -\operatorname{curl}\{\partial\mathbf{E}_w^d/\partial t\}\\ &= -c^2\operatorname{curl}\operatorname{curl}\mathbf{B}_w^d = -c^2\big(\nabla\{\operatorname{div}\mathbf{B}_w^d\} - \Delta\mathbf{B}_w^d\big)\\ &= c^2\Delta\mathbf{B}_w^d,\end{aligned} \tag{6.46}$$

and, similarly,

$$c^{-2}\partial^2\mathbf{E}_w^d/\partial t^2 = \Delta\mathbf{E}_w^d. \tag{6.47}$$

That is, both the electric and magnetic fields constitute wave disturbances travelling at speed c wherever and whenever ρ_w^d and $\mathbf{J}_w^d$ both vanish.

Remark 6.3 Fields ψ_w^d and $\mathbf{A}_w^d$ with $f(u_i^a) = 1/4\pi u_i^a$ are the Liénard-Wiechert potentials. These were derived in connection with solving inhomogeneous d'Alembert equations of form (6.42) using Green's functions (cf., e.g., [13], §6.4 and Jones [15], §3.1).

Remark 6.4 If $u_i^a > \epsilon + \delta$, then (cf. (6.38)) $\beta_i\, f(u_i^a) = \beta_i/4\pi u_i^a$. To discover the physical interpretation of β_i/u_i^a, note that

$$\mathbf{u}_i(\mathbf{x}, t) = \mathbf{x}_i(t) - \mathbf{x} = \mathbf{x}_i(\tau) + \dot{\mathbf{x}}_i(\tau)(t - \tau) + o(t - \tau) - \mathbf{x}. \tag{6.48}$$

With $\tau = \tau_i(\mathbf{x}, t)$ and recalling (6.6), (6.9), (6.8) and (6.17),

$$\begin{aligned}\mathbf{u}_i(\mathbf{x}, t) &= \mathbf{u}_i^a(\mathbf{x}, t) + \mathbf{v}_i\big(\tau_i(\mathbf{x}, t)\big)u_i^a(\mathbf{x}, t)/c + o(t - \tau_i)\\ &= u_i^a(\mathbf{x}, t)\big(\hat{\mathbf{u}}_i^a(\mathbf{x}, t) + \big(\beta_i^{-1}c^{-1}\mathbf{v}_i^a\big)(\mathbf{x}, t)\big) + o\big(u_i^a/c\big).\end{aligned} \tag{6.49}$$

Suppressing arguments $\mathbf{x}$ and t, it follows that

$$\begin{aligned}u_i &= \|\mathbf{u}_i\| = (\mathbf{u}_i\,.\,\mathbf{u}_i)^{1/2}\\ &= u_i^a\big\{\big(\hat{\mathbf{u}}_i^a + \mathbf{v}_i^a/\beta_i\, c\big)\,.\,\big(\hat{\mathbf{u}}_i^a + \mathbf{v}_i^a/\beta_i\, c\big)\big\}^{1/2} + o\big(u_i^a/c\big)\\ &= u_i^a\big(1 + \hat{\mathbf{u}}_i^a\,.\,\mathbf{v}_i^a/\beta_i\, c\big) = u_i^a\big(1 + (1 - \beta_i)/\beta_i\big) = u_i^a/\beta_i,\end{aligned} \tag{6.50}$$

via $(6.15)_2$ and neglecting terms of order $O((v_i^a/c)^2)$ and $o(u_i^a/c)$. Accordingly, if $u_i^a > \epsilon + \delta$, then

$$\beta_i\, f\big(u_i^a\big) = \beta_i/4\pi u_i^a = 1/4\pi u_i, \tag{6.51}$$

upon neglect of terms as in (6.50).

Remark 6.5 Macroscopic steady-state situations were discussed in Sect. 5 and generalised to dynamic contexts via definition (5.34) of a dynamic electric field $\tilde{\mathbf{E}}_w$ together with the consequences exhibited in Theorem 5.1. In order to take account of non-instantaneous information transfer the artifice of hypothetical radar signalling was introduced. However, such monitoring would not only be impracticable but also intrusive. Indeed, radar signals would interact with electrons and nuclei, and constitute a measurement process requiring a quantum mechanical description. Nevertheless, the discussion captures precisely how information concerning a moving charge is communicated: this is transmitted precisely as the signal *after* reflection.

7 The Force on a Moving Charge

Consider the consequences of assuming that the Lorentz force relation (1.9) has general validity. In such case the force acting on a point charge q moving with velocity $\mathbf{v}$ in an inertial frame would be (cf. (6.39))

$$\mathbf{F} = q\big(\mathbf{E}_w^d + \mathbf{v} \times \mathbf{B}_w^d\big) \tag{7.1}$$

$$= q\big(-\nabla\psi_w^d - \partial\mathbf{A}_w^d/\partial t + \mathbf{v} \times \operatorname{curl}\mathbf{A}_w^d\big). \tag{7.2}$$

Here field values are calculated at the instantaneous location $\mathbf{x}$ of q at time t. If $u_i^a(\mathbf{x},t) > \epsilon + \delta$ for all charges P_i, then $f(u_i^a) = 1/4\pi u_i^a$ (cf. Remark 3.6) and the contribution from P_i to $\mathbf{F}$ is

$$\mathbf{F}_i := q\big(\mathbf{E}_i^d + \mathbf{v} \times \mathbf{B}_i^d\big), \tag{7.3}$$

where (cf. $(6.39)_1$, (6.38) and (6.29))

$$\mathbf{E}_i^d := -\nabla\big\{(q_i/4\pi\epsilon_0)\big(\beta_i/u_i^a\big)\big\} - \partial/\partial t\big\{(\mu_0\, q_i/4\pi)\mathbf{v}_i^a/u_i^a\big\} \tag{7.4}$$

and

$$\mathbf{B}_i^d := \operatorname{curl}\big\{(\mu_0 q_i/4\pi)\mathbf{v}_i^a/u_i^a\big\}. \tag{7.5}$$

Recalling (6.18) and writing

$$\gamma_i := \beta_i/u_i^a \tag{7.6}$$

yield (via (5.18))

$$\mathbf{E}_i^d = -(q_i/4\pi\epsilon_0)\big\{\nabla\gamma_i + c^{-2}\partial/\partial t\{\gamma_i\mathbf{v}_i\}\big\} \tag{7.7}$$

and

$$\mathbf{B}_i^d = \big(q_i/4\pi\epsilon_0 c^2\big)\operatorname{curl}\{\gamma_i\,\mathbf{v}_i\}. \tag{7.8}$$

Lemma 7.1

$$\nabla\gamma_i = c^{-1}\gamma_i^2\mathbf{v}_i + \gamma_i^3\big(1 - v_i^2/c^2 - \dot{\mathbf{v}}_i\,.\,\mathbf{u}_i^a/c^2\big)\mathbf{u}_i^a, \tag{7.9}$$

$$\partial/\partial t\{\gamma_i\mathbf{v}_i\} = \gamma_i^2 u_i^a\dot{\mathbf{v}}_i - \gamma_i^3\big\{\mathbf{v}_i\,.\,\mathbf{u}_i^a + \big(u_i^a/c\big)\big(v_i^2 + \dot{\mathbf{v}}_i\,.\,\mathbf{u}_i^a\big)\big\}\mathbf{v}_i, \tag{7.10}$$

$$\operatorname{curl}\{\gamma_i\mathbf{v}_i\} = c^{-2}\gamma_i^3\mathbf{u}_i^a \times \big\{c\gamma_i^{-1}\dot{\mathbf{v}}_i + \big(c^2 - v_i^2 - \dot{\mathbf{v}}_i\,.\,\mathbf{u}_i^a\big)\mathbf{v}_i\big\}. \tag{7.11}$$

For brevity proofs of these results are omitted, as are those of the following theorem.

Theorem 7.1

$$\mathbf{E}_i^d = -\big(q_i/4\pi\epsilon_0 c^2\big)u_i^a\gamma_i^3\big\{\alpha_1\hat{\mathbf{u}}_i^a + \alpha_2\mathbf{v}_i + \alpha_3\dot{\mathbf{v}}_i\big\} \tag{7.12}$$

and

$$\mathbf{B}_i^d = \mathbf{u}_i^a \times \big(q_i/4\pi\epsilon_0 c^3\big)\gamma_i^3\{\alpha_2\mathbf{v}_i + \alpha_3\dot{\mathbf{v}}_i\}, \tag{7.13}$$

where

$$\alpha_1 := c^2 - v_i^2 - \dot{\mathbf{v}}_i\,.\,\mathbf{u}_i^a, \qquad \alpha_2 := c^{-1}\alpha_1, \qquad \alpha_3 := \gamma_i^{-1}. \tag{7.14}$$

Corollary 7.1

$$\mathbf{B}_i^d = -c^{-1}\hat{\mathbf{u}}_i^a \times \mathbf{E}_i^d. \tag{7.15}$$

This is an immediate consequence of (7.12) and (7.13).

Remark 7.1 These results duplicate those given in Griffiths [11]. In particular, (7.12) is expressed (cf. *op. cit.* (10.65)) as

$$\mathbf{E}_i^d = \big(q_i\, r/4\pi\epsilon_0(\mathbf{r}\,.\,\mathbf{u})^3\big)\big\{\big(c^2 - v_i^2\big)\mathbf{u} + \mathbf{r} \times (\mathbf{u} \times \dot{\mathbf{v}}_i)\big\}, \tag{7.16}$$

where

$$\mathbf{r} := -\mathbf{u}_i^a, \qquad r := \|\mathbf{r}\|, \qquad \mathbf{u} := -c\hat{\mathbf{u}}_i^a - \mathbf{v}_i. \tag{7.17}$$

Thus (cf. (6.18), $(6.15)_2$ and (7.6))

$$\mathbf{r}\,.\,\mathbf{u} = cu_i^a + \mathbf{u}_i^a\,.\,\mathbf{v}_i = cu_i^a\big(1 + \hat{\mathbf{u}}_i^a\,.\,\mathbf{v}_i^a/\beta_i\,c\big) = cu_i^a/\beta_i = c\gamma_i^{-1}. \tag{7.18}$$

Remark 7.2 From (7.3) and (7.15),

$$\begin{aligned}\mathbf{F}_i &= q\big\{\mathbf{E}_i^d - c^{-1}\mathbf{v}\times\big(\hat{\mathbf{u}}_i^a\times\mathbf{E}_i^d\big)\big\}\\ &= q\big\{\big(1 + c^{-1}\mathbf{v}\,.\,\hat{\mathbf{u}}_i^a\big)\mathbf{E}_i^d - c^{-1}\big(\mathbf{v}\,.\,\mathbf{E}_i^d\big)\hat{\mathbf{u}}_i^a\big\}.\end{aligned} \tag{7.19}$$

Equivalently,

$$\mathbf{F}_i = \big\{\big(1 + c^{-1}\mathbf{v}\,.\,\hat{\mathbf{u}}_i^a\big)\mathbf{1} - c^{-1}\hat{\mathbf{u}}_i^a\otimes\mathbf{v}\big\}\mathbf{E}_i^d. \tag{7.20}$$

Remark 7.3 The results of Theorem 7.1 furnish individual contributions to the macroscopic fields which satisfy the Maxwell relations (6.40). There were two underlying assumptions involved: (i) separations u_i^a were to exceed $\epsilon + \delta$, where ϵ was associated with any scale at which the Biot-Savart law is valid (cf. identifications (5.7) and (5.8)), and (ii) the potentials given in (5.13) were assumed to have dynamic generalisations (6.38) and (6.29).

8 Summary and Concluding Remarks

The microscopic basis of classical macroscopic electromagnetic relations has been investigated via recognition of the rôle played by spatial and temporal averaging, here implemented in terms of weighting functions. Modelling electrons and nuclei as point charges, any choice w of weighting function yielded definitions ρ_w and $\mathbf{j}_w$ of charge and current densities which satisfy (1.6). Any solution $\mathbf{a}$ to $\operatorname{div}\mathbf{a} = w$ gave rise to purely electrokinetic fields $\mathbf{D}_w$ and $\mathbf{H}_w$ which satisfy (1.1) and (1.2). At any prescribed scale ϵ, choice of a specific and natural weighting function resulted in binary decompositions (3.35) of $\mathbf{D}_w$ and $\mathbf{H}_w$. These decompositions introduced electric and magnetic polarisation densities together with electrokinetic fields $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w$ that were expressible in terms of scalar and vector potentials ψ_w and $\boldsymbol{\mathcal{A}}_w$, respectively. The experimental laws of Coulomb and Biot-Savart enabled the macroscopic time-independent electric and magnetic fields $\mathbf{E}_w^s$ and $\mathbf{B}_w^s$ in free space to be expressed as scalar multiples of $\boldsymbol{\mathcal{E}}_w$ and $\boldsymbol{\mathcal{B}}_w^s$ and hence in terms of potential functions ψ_w^s and $\mathbf{A}_w^s$. Fields $\mathbf{B}_w^s$ and $\mathbf{A}_w^s$ satisfy (1.3) and (1.5), and $\mathbf{E}_w^s$ satisfies (1.7) and (1.8) in this macroscopically stationary context. Generalisation to macroscopically dynamic situations was effected in two stages. In the first stage the dynamic electric field was formally defined via (1.7), with ψ and $\mathbf{A}$ identified with ψ_w^s and $\mathbf{A}_w^s$, and the consequences exhibited in Theorem 5.1. The second stage addressed the consequences of non-instantaneous transmission of information: this involved selection of appropriate modified versions ψ_w^d and $\mathbf{A}_w^d$ of ψ_w^s and $\mathbf{A}_w^s$, and resulted in the final form of the relations under investigation, displayed in Theorem 6.1. The individual contribution of any given charge to the force on another moving charge, as predicted by the Lorentz relation (1.9), was computed as an exercise and for completeness.

Remark 8.1 Relations (2.8), (2.20) and (2.33) are completely general and electrokinetic in nature. However, while Maxwell's equations $(6.40)_{1,2}$ and $(6.41)_{1,2}$ hold *in vacuo*, their

counterparts in regions occupied by matter depend upon the matter in question, and thus must incorporate constitutive relations. The simplest, for homogeneous isotropic media, are

$$\mathbf{P}_w = \chi_e \boldsymbol{\mathcal{E}}_w, \qquad \mathbf{M}_w = \nu_m \boldsymbol{\mathcal{B}}_w, \tag{8.1}$$

where χ_e and ν_m are dimensionless constants. Accordingly, from $(3.35)_{1,2}$,

$$\mathbf{D}_w = (1 + \chi_e)\boldsymbol{\mathcal{E}}_w, \qquad \mathbf{H}_w = (1 - \nu_m)\boldsymbol{\mathcal{B}}_w. \tag{8.2}$$

If the Coulomb and Biot-Savart laws hold within a medium with unchanged values of ϵ_0 and μ_0, then from (5.5) and (5.9)

$$\mathbf{D}_w = \epsilon \mathbf{E}^s_w, \qquad \mathbf{H}_w = \mu^{-1}\mathbf{B}^s_w, \tag{8.3}$$

where, writing $\chi_m := \nu_m/(1 - \nu_m)$,

$$\epsilon := (1 + \chi_e)\epsilon_0, \qquad \mu := (1 + \chi_m)\mu_0. \tag{8.4}$$

Dimensionless constants χ_e and χ_m are termed the electric and magnetic susceptibilities of the medium (cf. [14], p. 354 and p. 412). In such case relations (2.20) and (2.33) yield

$$\operatorname{div} \mathbf{E}^s_w = \epsilon^{-1}\rho_w, \qquad \partial \mathbf{E}^s_w/\partial t + \epsilon^{-1}\mathbf{j}_w = (\epsilon\mu)^{-1}\operatorname{curl} \mathbf{B}^s_w. \tag{8.5}$$

Remark 8.2 Consideration of information transmission in Sect. 6 involved invariance of signal speed, the fundamental phenomenon addressed by special relativity. The final results are accordingly consistent with this theory, but have been derived independently of other relativistic considerations (cf. [9, 14]) and are thus described as *classical*. Of course, the quantum mechanical notion of electron spin and its macroscopic consequences lie outwith a classical formulation.

Remark 8.3 The ubiquitous nature of electromagnetism requires that all charges in the universe be *a priori* considered to contribute to local values of electric and magnetic fields.[6] In examining specific localised behaviour those charges in the region of interest may be distinguished from all others, resulting in internal or localised fields together with their complements, the corresponding external or 'far' fields.

Remark 8.4 As indicated in Remark 6.2, when time-averaged the relations of Theorem 6.1 are form-invariant with respect to changes in the choices of length and time scales associated with the spacetime averaged fields which appear therein. However, if local field values are associated with measurements, then the sensitivity of the measuring devices introduces associated and specific scales of length and time.

[6] For example, the terrestrial magnetic field and its effect upon compass needles, or the effect of solar flares upon long range radio communications.

References

1. Maxwell, J.C.: A Treatise on Electricity and Magnetism, vols. 1 & 2. Clarendon Press, Oxford (1873)
2. Hendry, J.: James Clerk Maxwell and the Theory of the Electromagnetic Field. Hilger, Bristol (1986)
3. Lorentz, H.A.: The fundamental equations for electromagnetic phenomena in ponderable bodies, deduced from the theory of electrons. Proc. K. Akad. Wet. **5**, 254–266 (1902)
4. Lorentz, H.A.: Contributions to the theory of electrons. Simplifications of the fundamental equations by the introduction of new units. Proc. K. Akad. Wet. **5**, 608–628 (1902)
5. Murdoch, A.I.: Physical Foundations of Continuum Mechanics. Cambridge University Press, New York (2012)
6. Van Fleck, J.H.: The Theory of Electric and Magnetic Susceptibilities. Oxford University Press, London (1932)
7. Rosenfeld, L.: Theory of Electrons. North-Holland, Amsterdam (1951)
8. Mazur, P., Nijboer, B.R.A.: On the statistical mechanics of matter in an electromagnetic field. I. Physica **XIX**, 971–986 (1953)
9. de Groot, S.R.: The Maxwell Equations: Non-relativistic and Relativistic Derivations from Electron Theory. North-Holland, Amsterdam (1969)
10. Russakoff, G.: A derivation of the macroscopic Maxwell relations. Am. J. Phys. **38**, 1188–1195 (1970)
11. Griffiths, D.J.: Introduction to Electrodynamics, 3rd edn. Prentice Hall, Upper Saddle River (1999)
12. Zangwill, A.: Modern Electrodynamics. Cambridge University Press, Cambridge (2013)
13. Jackson, J.D.: Classical Electrodynamics, 3rd edn. Wiley, Hoboken (1999)
14. Elliott, R.S.: Electromagnetics. Oxford University Press/IEEE Press, Oxford/Piscataway (1993)
15. Jones, D.S.: The Theory of Electromagnetism. Macmillan, New York (1964)