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On Squashed Designs

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Abstract. Extremal problems and the existence of designs is investigated in a new type of combinatorial structures, called squashed geometries.

1. Introduction

Suppose X is an n-element set $0<k<n$, $L\subset\{0,1,\ldots,k-1\}$. An (n,k,L)-system is a family $\mathscr{A}$ of k-subsets of X satisfying

$$|A\cap A'|\in L \quad \text{for all distinct} \quad A, A'\in\mathscr{A}. \tag{1.1}$$

The investigation of the maximum size of (n,k,L)-systems has been the subject of a large number of papers. Let us recall:

Theorem 1.1 ([DEF]). *Suppose $\mathscr{A}$ is an (n,k,L)-system, $L=\{l_0,\ldots,l_{s-1}\}$, $l_0<l_1<\cdots<l_{s-1}$. There exists a constant $c=c(k,L)$ such that either $|\mathscr{A}|<cn^{s-1}$ or* (i)–(iii) *hold*:

(i) $$|\mathscr{A}|\le\prod_{l\in L}\frac{n-l}{k-l};$$

(ii) $$(l_1-l_0)|(l_2-l_1)|\cdots|(l_{s-1}-l_{s-2})|(k-l_{s-1});$$

(iii) $$\left|\bigcap_{A\in\mathscr{A}}A\right|=l_0$$

Note that the upper bound in (i) has degree s in n, thus it holds for all $n>n_0(k,s)$. In (i) equality holds (for $n>n_0(k,s)$) if and only if $\mathscr{A}$ is the family of flats of rank s of a *perfect matroid design* (a matroid in which all flats of rank i have size l_i, $0\le i<s$), cf. [De 1]. Special cases include the Erdös–Ko–Rado theorem [EKR]: $L=\{t,t+1,\ldots,k-1)$, t-designs with $\lambda=1$: $L=\{0,1,\ldots,t-1\}$, etc.

In the present paper we investigate the problem for restricted (n, k, L)-systems. Let $\mathscr{F}$ be a family of sets closed under intersection $\mathscr{F} \subset 2^X$, $F, G \in \mathscr{F}$ implies $F \cap G \in \mathscr{F}$.

Definition 1.2. An (n, k, L)-system $\mathscr{A} \subset \mathscr{F}$ is called an *$\mathscr{F}$-squashed* (or shortly) *squashed* family.

From now on $\mathscr{A}$ will always denote an $\mathscr{F}$-squashed (n, k, L)-system. The maximum size of $\mathscr{A}$ is denoted by $m_{\mathscr{F}}(n, k, L)$. Set

$$\mathscr{F}^{(r)} = \{F \in \mathscr{F}: |F| = r\}.$$

Let us define the quantity n_r by

$$n_r = \max_{G \in \mathscr{F}^{(r)}} (r) \left| \bigcup_{G \subset F \in \mathscr{F}} F - G \right|.$$

Note that $n_r = n - r$ holds for $\mathscr{F} = 2^X$.

Theorem 1.3. *Suppose that $\mathscr{A}$ is an $\mathscr{F}$-squashed (n, k, L)-system. Then either $|\mathscr{A}| < c(k, L)n^{s-1}$ or* (i)–(iii) *hold.*

(i) $$|\mathscr{A}| = \leq \prod_{i=0}^{s-1} \frac{n_{l_i}}{k - l_i};$$

(ii) $$(l_1 - l_0)| \cdots |(l_{s-1} - l_{s-2})|(k - l_{s-1});$$

(iii) $$\left| \bigcap_{A \in \mathscr{A}} A \right| = l_0.$$

The investigation of the case of equality leads to the definition of *squashed designs* and *squashed geometries* which we discuss in Section 3, after the proof of Theorem 1.3 given in Section 2. In the case where $N^{(i)} = \{0, 1, \ldots, n^{(i)} - 1\}$ and $X = N^{(1)} \times N^{(2)} \times \cdots \times N^{(d)}$ one calls a subset $B \subset X$ *injective* if all elements of B have distinct entries for each $N^{(i)}$, i.e., if $\mathbf{b} = (b_1, \ldots, b_d)$ and $\mathbf{b}' = (b'_1, \ldots, b'_d)$ are in B then $b_i \neq b'_i$ holds for $1 \leq i \leq d$.

If $\mathscr{F}$ is the family of all injective subsets of $N^{(1)} \times \cdots \times N^{(d)}$ then Theorem 1.3 implies

Corollary 1.4 ([DF]). *Suppose $\mathscr{A}$ is an (n, k, L)-system of injective subsets of $N^{(1)} \times \cdots \times N^{(d)}$ then for* $\min_i n^{(i)} \geq n_0(k, L)$

$$|\mathscr{A}| \leq \prod_{i=0}^{s-1} \frac{\prod_{j=1}^{d} n^{(j)} - l_i}{k - l_i} \quad \textit{holds.} \tag{1.2}$$

In (1.2) equality holds if and only if $\mathscr{A}$ is an *injection design* (cf. [DF]).

Suppose $X = X_1 \cup \cdots \cup X_k$ where the X_i are pairwise disjoint $\tilde{n}$-element sets and $\mathscr{F}$ consists of *partial transversals*, i.e., $\mathscr{F} = \{F \subset X: |F \cap X_i| \leq 1 \text{ for } 1 \leq i \leq k\}$, we have $n_r = (k - r)\tilde{n}$. Then Theorem 1.3 yields

Corollary 1.5. *Suppose $\mathscr{A}$ is an (n, k, L)-system consisting of transversals of $X_1 \cup \cdots \cup X_k$, $\tilde{n} > n_0(k, L)$. Then $|\mathscr{A}| \le \tilde{n}^s$ holds.*

Here equality corresponds to *transversal matroid designs* (cf. [CDF]).

2. The Proof of Theorem 1.3

We apply induction on k. We distinguish two cases.

(a) $l_0 = 0$

Set $X_0 = \bigcup_{F \in \mathscr{F}} F$. By definition $|X_0| = n_0$. For all $x \in X_0$ the family $\mathscr{A}(x) = \{A - \{x\}: x \in A \in \mathscr{A}\}$ is an $(n-1, k-1, \{l_1 - 1, \ldots, l_{s-1} - 1\})$-system which is $\mathscr{F}(x)$-squashed for $\mathscr{F}(x) = \{F - \{x\}: x \in F \in \mathscr{F}\}$. By induction we infer

$$|\mathscr{A}(x)| \le \prod_{i=1}^{s-1} \frac{n_{l_i}}{k \times l_i} \qquad \text{for} \quad n > n_0(k, L).$$

Moreover, $(l_2 - l_1)|\cdots|(k - l_{s-1})$ and $|\bigcap_{B \in \mathscr{A}(x)} B| = l_1 - 1$ follow unless $|\mathscr{A}(x)| < c(k, L)n^{s-2}$. Using $\sum_{x \in X_0} |\mathscr{A}(x)| = k|\mathscr{A}|$, (i) follows.

To prove (ii) we have to show $l_1 = (l_1 - l_0) \,|\, (l_2 - l_1)$. Define $\mathscr{D}$ as the collection of l_1-subsets D such that for some $x \in D$ we have $D - \{x\} = \cap \mathscr{A}(x)$. By definition $\mathscr{D}$ consists of pairwise disjoint sets only. Omit all $A \in \mathscr{A}$ which contain some x with $|\bigcap \mathscr{A}(x)| \ne l_1 - 1$. By the induction hypotheses we omitted at most $c(k-1, \{l_1 - 1, \ldots, l_{s-1} - 1\})n^{s-1}$ sets. The remaining sets are all the disjoint union of k/l_1 members of D. If two of them, say A, A' intersect in l_2 elements, then $A \cap A'$ must be the disjoint union of members of $\mathscr{D}$, yielding $l_1 | l_2$ and thus $l_1 | (l_2 - l_1|)$. If not, by induction the number of remaining members of $\mathscr{A}$ is bounded by $m(n, k, L - \{l_2\})$. Thus $|\mathscr{A}| < c(k, L)n^{s-1}$, as desired.

(iii) follows in a trivial way.

(b) $l_0 > 0$

Consider $A_0 = \bigcap \mathscr{A}$. If $|A_0| \ne l_0$, then Theorem 1.1 implies $|\mathscr{A}| < c(k, L)n^{s-1}$. If $|A_0| = l_0$, then (i)-(ii) follow by the induction hypothesis applied to $\mathscr{A}(A_0) = \{A - A_0: A \in \mathscr{A}\}$. □

Remark 2.1. Looking carefully at the proof of Theorem 1.3, we see that for $|\mathscr{A}| > c(k, L)n^{s-1}$ equality in (i) implies that there is a family $\mathscr{D}$ of pairwise disjoint $(l_1 - l_0)$-sets D such that $|\bigcup \mathscr{D}| = n_{l_0}$ and every $A \in \mathscr{A}$ is the disjoint union of $(k - l_0)/(l_1 - l_0)$ members of $\mathscr{D}$ along with $A_0 = \bigcap \mathscr{A}$, having cardinality l_0.

Remark 2.2. Note that for the case $\mathscr{F}$ a perfect matroid design, Theorem 1.3 was proved in [De 1].

Remark 2.3. The proof of Theorem 1.3 shows that if $n_{l_{s-1}}$ is sufficiently large and $\mathscr{A}$ attains equality in (i) then $\mathscr{A}$ verifies $|A_1 \cap \cdots \cap A_b| \in L$ for all $b \geq 2$, moreover, the meet semilattice generated by $\mathscr{A}$ is *short*, i.e., for all $A_1, \ldots, A_b \in \mathscr{A}$ there exist $A, A' \in \mathscr{A}$ with $A_1 \cap \cdots \cap A_b = A \cap A'$.

Remark 2.4. It is easy to see that if one strengthens the assumptions of Theorem 1.3 to $|A_1 \cap \cdots \cap A_b| \in L$ for all $b \geq 2$, then (i) and (iii) always hold (i.e., for all n). Moreover, equality in (i) implies that $\mathscr{A}$ is a *squashed design* (cf. the definition in the next section).

3. Squashed Designs and Geometries

An *$\mathscr{F}$-squashed geometry* of rank s is a family $\mathscr{G}$ partitioned into $\mathscr{G} = \mathscr{G}_0 \cup \mathscr{G}_1 \cup \cdots \cup \mathscr{G}_s$ satisfying:

(i) $\mathscr{G} \subset \mathscr{F}$, $\mathscr{G}$ is closed under intersection, i.e., $G, G' \in \mathscr{G}$ implies $(G \cap G') \in \mathscr{G}$.
(ii) If $G \subset H$, $G \in \mathscr{G}_i$, $H \in \mathscr{G}_j$; then $i < j$.
(iii) If $G \in \mathscr{G}_i$, $i < s$ $x \notin G$, $G \cup \{x\}$ is contained in some member of $\mathscr{F}$, then there exists an $H \in \mathscr{G}_{i+1}$ containing $G \cup \{x\}$.

If $\mathscr{F} = 2^X$, then a squashed geometry is just a matroid. For $G \in \mathscr{G}_i$ the family $\mathscr{G}_G = \{G \cap G' \colon G' \in \mathscr{G}\}$ is the family of flats of a matroid of rank i on G. An *$\mathscr{F}$-squashed design* is an $\mathscr{F}$-squashed geometry in which flats of equal rank have equal size. The sizes $l_0, l_1, \ldots, l_s$ along with $l_{s+1} = n = |\bigcup \mathscr{G}|$ are called the *parameters* of the squashed design. Clearly, setting $l_s = k$, $\mathscr{G}_s$ is an $(n, k, \{l_0, \ldots, l_{s-1}\})$-system. For $G \in \mathscr{G}_s$, $\mathscr{G}_G = \{G \cap G' \colon G' \in \mathscr{G}\}$ is the family of flats of a perfect matroid design.

Proposition 3.1. *If $\mathscr{G}$ is a squashed design of rank s with parameters $l_0, \ldots, l_{s-1}, k, n$ then*

$$|\mathscr{G}_s| \leq \prod_{i=0}^{s-1} \frac{n_{l_i}}{k - l_i} \tag{3.1}$$

with equality holding if and only if

$$\Big|\bigcup (H - G) \colon G \subset H \in \mathscr{F}\Big| = n_{l_i} \quad \textit{holds for all} \quad G \in \mathscr{G}_i, \qquad 1 \leq i < s. \tag{3.2}$$

Proof. Let us note first that in view of (i) and (ii) $\mathscr{G}_0$ consists of a single set, say $G_0, |G_0| = l_0$. Consequently every $G \in \mathscr{G}$ contains G_0. Defining $\mathscr{G}(G_0) = \{G - G_0 \colon G \in \mathscr{G}\}$, $\mathscr{F}(G_0) = \{F - G_0 \colon G_0 \subset F \in \mathscr{F}\}$, $\mathscr{G}(G_0)$ is an $\mathscr{F}(G_0)$-squashed design with parameters $(0, l_1 - l_0, \ldots, l_{s-1} - l_0, k - l_0, n - l_0)$, thus it is sufficient to consider the case $l_0 = 0$, i.e., $\mathscr{G}_0 = \{\emptyset\}$. Now (i) and (iii) imply that $\bigcup \mathscr{G} = \bigcup \mathscr{F}$.

We apply induction on s. If $s = 1$, then $\mathscr{G}_s = \mathscr{G}_1$ is a family of pairwise disjoint $l_1 - l_0 = l_1$-element sets. In view of (i) and (iii) we have $\bigcup \mathscr{G}_1 = \bigcup \mathscr{F}$. Consequently $|\mathscr{G}_1| = n_0 / l_1$ holds. Next consider the general case, $s \geq 2$. For an arbitrary $x \in \bigcup \mathscr{F}$

$\mathscr{G}(x)=\{G-x: x\in G\in\mathscr{G}\}$ is an $\mathscr{F}(x)$-squashed design with parameters $(l_1-1,\dots,l_{s-1}, k-1, n(\mathscr{F}(x))$, where

$$n(\mathscr{F}(x))=\left|\bigcup_{H\in\mathscr{F}(x)} H\right|=l_1-1+\left|\bigcup(F-G_0(x)): G_0(x)\subseteq F\in\mathscr{F}\right|\le l_1-1+n_{l_1}.$$

Consequently

$$|\mathscr{G}_s(x)|\le\prod_{i=1}^{s-1}\frac{n_{l_i}}{k-l_i}$$

holds with equality iff $n(\mathscr{F}(x))=l_1-1+n_{l_1}$ and $\mathscr{F}(x)$ satisfies the last conclusion of Proposition 3.1. Now (3.1) follows from $\sum_x|\mathscr{G}_s(x)|=k|\mathscr{G}_s|$, and the condition for the equality is derived via the above observation. □

If (3.2) holds for all i, $1\le i\le s$, then $\mathscr{G}$ is called a *perfect squashed design.* Let us note that every $\mathscr{F}$-squashed design $\mathscr{G}$ is a $\mathscr{G}$-squashed design.

Proposition 3.2. *Suppose $\mathscr{G}=\mathscr{F}$ is a $\mathscr{F}$-squashed design. Then the following conditions are equivalent:*

(i) *$\mathscr{G}$ is a perfect $\mathscr{F}$-squashed design of rank s;*
(ii) *there exist integers $r_0,\dots,r_s$ so that each $H\in\mathscr{G}_i$ is contained in exactly r_i members of $\mathscr{G}_s$.*

Proof. (i)⇒(ii): $H\in\mathscr{G}_i$, it is easily seen that $\mathscr{G}(H)=\{G-H: H\in G\in\mathscr{G}\}$ is a perfect $\mathscr{F}(H)$-squashed design. Thus (ii) holds with

$$r_i=\prod_{i\le j<s}\frac{n_{l_j}}{k-l_j}.$$

(ii)⇒(i): For $H\in\mathscr{G}_i$ define $X(H)=\bigcup_{H\subset F\in\mathscr{F}}F-H$. By definition $n_{l_i}=\max_{H\in\mathscr{G}_j}|X(H)|$. For $i=s-1$ we obtain $r_{s-1}=|X(H)|/(k-l_{s-1})$, i.e., $|X(H)|$ is constant: $n_{l_{s-1}}=r_{s-1}(k-l_{s-1})$ proving (3.2) for $i=s-1$. Applying induction backwards we see that r_i, the number of $G\in\mathscr{G}_s$ containing $H\in\mathscr{G}_i$, is

$$r_i=\frac{|X(H)|\prod_{i<j<s}n_{l_j}}{\prod_{i\le j<s}(k-l_j)}.$$

This yields that $|X(H)|$ is the same for all $H\in\mathscr{G}_i$ as desired. □

In view of Proposition 3.2 a meet semilattice $\mathscr{F}$ is a perfect $\mathscr{F}$-squashed design if and only if it is an M_s-design in the sense of Neumaier [Ne]. We refer the reader for some interesting results on M_s-designs to [Ne].

Definition 3.3. An $\mathscr{F}$-squahsed geometry $\mathscr{G}$ of rank s is called *regular* if it enjoys the following two properties:

(i) Given $G \in \mathscr{G}_s$, $H \in \mathscr{G}_t$, $H \subset G$, the number of sets $E \in \mathscr{G}_r$ with $H \subset E \subset G$ is a constant $m(t, r)$.

(ii) Given $H \in \mathscr{G}_t$, the number of sets $E \in \mathscr{G}_r$ with $H \subset E$ is a constant $c(t, r)$.

Note that conditions (i) and (ii) are in complete analogy with those of [D 1], however, Delsarte in his definition of regular semilattices puts a third, stronger condition which ensures that $\mathscr{G}_s$ is an association scheme where $G, G' \in \mathscr{G}_s$ are i-associated if $G \cap G' \in \mathscr{G}_i$.

4. Examples of Squashed Designs

The simplest case is $\mathscr{F} = 2^X$. Then $\mathscr{G} = \mathscr{F}$ is a perfect squashed design. So are the truncations of $\mathscr{F}$: $\mathscr{F}^{(s)}$: $\{F \subset X: |F| \leq s\} \cup \{X\}$. In general, given an $\mathscr{F}$-squashed design $\mathscr{G}$ with parameters $(l_0, l_1, \ldots, l_{s-1}, k, n)$ one may define its tth *truncation* $\mathscr{G}^{(t)} = \mathscr{G}_0 \cup \cdots \cup \mathscr{G}_t$, $1 \leq t < s$. It is an $\mathscr{F}$-squashed design with parameters $(l_0, l_1, \ldots, l_t, n)$. Analogously one can define *derived designs*: $\mathscr{G}(G) = \{H - G: G \subseteq H \in \mathscr{G}\}$ is an $\mathscr{F}(G)$-squashed design for all $G \in \mathscr{G}_i$, $i < s$.

Instead of 2^X one can consider $\mathscr{F}$ as all subspaces of V, an n-dimensional vector space, projective or affine space over $GF(q)$. By analogy we denote it by 2^V. In general, for $\mathscr{F} = 2^X$, a perfect squashed design is simply a perfect matroid design. In most cases $\mathscr{G}$ is uniquely determined by $\mathscr{G}_s$, therefore in most examples we define $\mathscr{G}_s$ only.

Example 4.1. (a) Suppose X is the Cartesian product $X_1 \times \cdots \times X_d$ where $|X_i| = n_i$ and $\mathscr{F}$ is the family of all *injective* subsets ($F \subset X$ is injective if for all distinct elements $(f_1, \ldots, f_d)$, $(f'_1, \ldots, f'_d) \in F$, $f_i \neq f'_i$ for $1 \leq i \leq d$ (cf. [DF])). We set $\mathscr{F} = \mathscr{J}(X_1, \ldots, X_d)$.

(b) Suppose now $V = V_1 \times V_2 \times \cdots \times V_d$ where V_i is a vector space of dimension n_i, If W is an affine subspace of V, one can define $\pi_i(W)$ as its natural projection on V_i. Define $\mathscr{J}_A(V_1, \ldots, V_d)$ as the set of all injective affine subspaces, $\mathscr{J}_A(V_1, \ldots, V_d) = \{A \leq V: \dim(\pi_i(A)) = \dim(A) \text{ for } i = 1, \ldots, d\}$. Similarly, for vector subspaces

$$\mathscr{J}_V(V_1, \ldots, V_d) = \{W \leq V: \dim(\pi_i(W)) = \dim(W) \text{ for } i = 1, \ldots, d\}.$$

Example 4.2. Suppose Y and Z are both either finite sets or finite-dimensional vector spaces over GF(q). Let $\mathscr{T}(Y, Z)$ denote the set of all partial transversals (transverse subspaces), i.e., all $W < Y \times Z$ such that $|W \cap (Y \times \{z\})| \leq 1$ holds for all $z \in Z$. Note that the definition is not symmetric in Y and Z. (This example is due to Delsarte [D 1].)

Let us note that Example 4.2 is a regular semilattice in the sense of Delsarte [D 1]), but Example 4.1(a), (b) is not.

Example 4.3. Suppose V is an n-dimensional vector space and $f(x, y)$ is a sesquilinear form on V (cf. [Di]). Let $\mathscr{F} = \mathscr{V}_f$ be the collection of singular subspaces, i.e.,

$$\mathscr{V}_f = \{W \leq V: f(w, w') = 0 \text{ for all } w, w' \in W\}.$$

Note that $2^V = \mathscr{V}_f$ for f the identically zero form. Note also that if f is nondegenerate then $\mathscr{V}_f$ is a perfect squashed design. This is true for the other examples and their truncations as well.

In the examples considered so far $\mathscr{G}$ was simply a truncation of $\mathscr{F}$. There are several other examples of this type, related to polar spaces and buildings (cf. [Ne]).

Example 4.4. Suppose $\mathscr{F} = 2^X, 2^A, 2^V, 2^P$, or more generally a squashed geometry. A family $\mathscr{G} \subset \mathscr{F}_s$ is called a t-design with repetition λ in $\mathscr{F}_s$ if every $F \in \mathscr{F}_t$ is contained in exactly λ members of $\mathscr{G}$. For $\lambda = 1$ we call $\mathscr{G}$ shortly a t-design in $\mathscr{F}_s$. If $\mathscr{F}_s$ is a perfect squashed design then a t-design $\mathscr{G}$ defines a perfect squashed design.

Note that for regular semilattices our definition of t-designs coincides with the notion of combinatorial t-design in [D 1] or combinatorial relative t-designs in [D 2]. If $\mathscr{F}_0 \cup \cdots \cup \mathscr{F}_s$ is a regular squashed geometry then a t-design with repetition λ is also a t'-design with some repetition λ' for $0 < t' < t$.

Let us remark that a t-design $\mathscr{T}$ in a regular squashed geometry is a packing of maximal size of members of $\mathscr{F}_s$ such that any two of them have intersection of rank less than t. Similarly, it is a minimal covering of the members of $\mathscr{F}_t$.

Let us note that $|\mathscr{T}| = |\mathscr{F}_t|/m(0, t)$. It follows from a much more general result of [FR] that for an arbitrary positive ε and $n_{t_{s-1}}$ sufficiently large with respect to $m(0, t)$ and ε there exists a packing of size at least $(1-\varepsilon)|\mathscr{F}_t|/m(0, t)$.

Note that if $d = 2$ and $|X_1| = |X_2| = n$ then there is a natural correspondence between the symmetric group S_n and the flats of maximal rank (i.e., of rank n) in $\mathscr{I}(X_1, X_2)$. Namely, a permutation $(i_1, i_2, \ldots, i_n)$ is associated with $\{(1, i_1), \ldots, (n, i_n)\}$. In this case any t-fold transitive permutation group Γ is a t-design with repetition $\lambda = |\Gamma|/n(n-1)\cdots(n-t+1)$. A general t-design with repetition λ in $\mathscr{I}(X_1, X_2)$ is called a λ-uniform t-transitive permutation set (cf. [De 2]) or orthogonal permutation array (cf. [It]).

The existence problem of t-designs is a hopelessly difficult one in general, even for $\mathscr{F} = 2^X$. In this case 1-designs are just partitions of X into subsets of size s. In the case $\mathscr{F} = 2^X$, $t = 2$ Wilson [Wi] proved that t-designs with repetition λ exist in $\binom{X}{2}$ whenever the trivial necessary conditions and $n > n_0(\mathscr{F}, \lambda)$ are fulfilled. The existence problem for t-designs in 2^V was raised in [Ray].

Example 4.5. Suppose A is an n-dimensional affine space over $\mathrm{GF}(q)$ and $\mathscr{G}_1$ is the class of subspaces of dimension d, parallel to a fixed d-dimensional subspace. Then $\mathscr{G}_1$ is a 2^A-squashed design.

Example 4.6. Suppose P is an n-dimensional projective space, $(d+1)|(n+1)$ and $\mathscr{G}_1$ is a *spread* of d-dimensional subspaces (i.e., the members of $\mathscr{G}_1$ are pairwise disjoint and their union is P, cf. [Dem]). Then $\mathscr{G}_1$ is a 2^P-squashed design.

Example 4.7. Let V be an $(n+1)$-dimensional vector space corresponding to P from the preceding example, and A the corresponding $(n+1)$-dimensional affine space. Let $\mathscr{G}_2$ consist of all affine translates of all $(d+1)$-dimensional subspaces whose projection in P is in the spread. Then $\mathscr{G}_2$ is a 2^A-squashed design with parameters $(0, 1, q^{d+1}, q^{n+1})$.

Example 4.8 (Laurent [La]). Suppose $k|n_1 n_2 \dots n_d$, $k \le \min_i n_i$. Then there exists a 1-design in $\mathscr{I}^{(k)}(X_1, X_2, \dots, X_d)$.

Example 4.9. In $\mathscr{T}(Y, Z)$ and Y, Z finite sets t-designs with repetition λ correspond to orthogonal arrays of strength t (cf. [Rao]). In the vector space case t-designs always exist (cf. [D 1]).

Example 4.10. If a family $\mathscr{A}$ attains equality in Theorem 1.3(i) for $\mathscr{F} = \mathscr{F}(Y, Z)$, it is called a *transversal matroid design* (cf. [CDF] for some examples).

Example 4.11. A family $\mathscr{A}$ attaining equality in Theorem 1.3(i) for $\mathscr{F} = \mathscr{I}(X_1, \dots, X_d)$ is called an *injection design* (cf. [DF]). The special case $d=2$, $|X_1| = |X_2| = k$ corresponds to *permutation geometries* (cf. [CD]). In [CD], [DF], and [DCF] several examples are given and we describe several new constructions in Sections 5 and 6.

5. Some Examples of t-Designs

Proposition 5.1. *Suppose $\mathscr{F}_0 = \mathscr{F}_0 \cup \cdots \cup \mathscr{F}_s$ is a (regular) squashed design and $\mathscr{T} = \mathscr{F}_s$ is a t-design in $\mathscr{F}$. Suppose that for some $k \le s$ and all $T \in \mathscr{T}$ there exists a t-design with repetition λ in $(\mathscr{F}_0 \cup \cdots \cup \mathscr{F}_k)_T = \{F \in \mathscr{F}_i \colon F \subset T, i \le k\}$. That is, for each $T \in \mathscr{T}$ there exists $\mathscr{I}(T) \subset (\mathscr{F}_k \cap 2^T)$ with the property that each $F \in \mathscr{F}_t$ with $F \subset T$ is contained in exactly λ members of $\mathscr{I}(T)$. Then $\bigcup_{T \in \mathscr{T}} \mathscr{I}(T)$ is a t-design with repetition λ in the kth truncation, $\mathscr{F}_0 \cup \cdots \cup \mathscr{F}_k$ of $\mathscr{F}$.*

Proof. By inspection, it is left to the reader. □

Example 5.2. Suppose $|X| = p^\alpha$, p a prime and consider the elements of X as the elements of the finite field of order p^α. Let $A = \mathrm{AL}(1, p^\alpha)$ be the group of all affine linear transformations $x \to ax + b$. Then $|A| = p^\alpha(p^\alpha - 1)$ and A is sharply 2-transitive. Thus (see the remark after Example 4.4) A corresponds to a 2-design $\mathscr{T}(A)$ in $\mathscr{I}(X, X)$. If for some $2 < k < p^\alpha$ there exists a 2-design in $\binom{X}{k}$ with repetition λ then there exists a 2-design with repetition λ in the kth truncation of $\mathscr{I}(X, X)$. In view of Example 1.6 in [DF] the same is true in $\mathscr{I}(X, X, \dots, X)$.

Corollary 5.3. *Suppose* $p^\alpha \equiv 1$ *or* 3 (mod 6). *Then there exists a Steiner triple-system in* $\mathcal{I}(X_1, \ldots, X_d)$ *for* $|X_1| = \cdots = |X_d| = p^\alpha$.

Proof. Use Example 5.2 and the fact that Steiner triple-systems exist in $\binom{X}{3}$ whenever $p^\alpha \equiv 1$ or 3 (mod 6) (cf. [Ki]) □

Example 5.4. Using the general projective group PGL$(2, p^\alpha)$ which acts sharply 3-transitively on the $p^\alpha + 1$ points of the projective line, it was shown in [DF] that there exists a 3-design in $\mathcal{I}(X_1, X_2, \ldots, X_d)$ for $|X_1| = \cdots = |X_d| = p^\alpha + 1$, $d \geq 2$. Since there exist 3-designs in $\binom{X}{p^\beta+1}$ for $|X| = p^{2\beta} + 1$, in the above way, we infer the existence of 3-designs in the $(p^\beta + 1)$th trunction of $\mathcal{I}(X_1, \ldots, X_d)$ for $|X_1| = \cdots = |X_d| = p^{2\beta} + 1$.

Let us introduce the notation $\mathcal{I}(n_1, \ldots, n_d)$ for $\mathcal{I}(\{1, \ldots, n_1\}, \ldots, \{1, \ldots, n_d\})$. For given $k \geq 3$ and $d \geq 1$ define $M(k, d)$ as the set of those values of $(n_1, \ldots, n_d)$ for which there exists a 2-design of k-element injective sets in $\mathcal{I}(n_1, \ldots, n_d)$.

Proposition 5.5. *If* $(n_1, \ldots, n_d) \in M(k, d)$ *then* $\min\{n_1, \ldots, n_d\} \geq k$; *moreover* (i) *and* (ii) *hold*:

$$\text{(i)} \qquad k(k-1) \Big| \prod_{1 \leq j \leq d} (n_j(n_j - 1),$$

$$\text{(ii)} \qquad (k-1) \Big| \prod_{1 \leq j \leq d} (n_j - 1).$$

Proof. Clearly the number of 2-element injective subsets in $\mathcal{I}(n_1, \ldots, n_d)$ must be a multiple of $\binom{k}{2}$, this implies (i). To prove (ii), one considers the derived designs, e.g., the $(k-1)$-element injective sets left over after removing the point $(n_1, \ldots, n_d)$ from all sets containing it. These sets partition $\mathcal{I}(n_1 - 1, \ldots, n_d - 1)$ yielding (ii).

Conjecture 5.6. *If* $\min_{1 \leq j \leq d} n_j$ *is sufficiently large with respect to* k *and* (i) *and* (ii) *are fulfilled, then* $(n_1, \ldots, n_d) \in M(k, d)$.

Let us note that for $d = 1$, Conjecture 5.6 follows from the existence results of Wilson [Wi].

Recall that a pairwise balanced design $\mathcal{B} \subset 2^X$ is just a collection of subsets of X so that $|B| \geq 2$ for all $B \in \mathcal{B}$ and every 2-subset of X is contained in exactly one member of $\mathcal{B}$.

Proposition 5.7. *Suppose there exist pairwise balanced designs* $\mathcal{B}_i \subset 2^{\{1,2,\ldots,n_i\}}$, $i = 1, \ldots, d$ *so that* $(|B_1|, \ldots, |B_d|) \in M(k, d)$ *holds for all* $B_i \in \mathcal{B}_i$, $i = 1, \ldots, d$. *Then* $(n_1, \ldots, n_d) \in M(k, d)$.

Proof. For each choice of $B_1 \in \mathcal{B}_1, \ldots, B_d \in \mathcal{B}_d$ replace the brick $B_1 \times \cdots \times B_d$ by an injection 2-design. It is left to the reader to verify that in this way we obtain an injection 2-design in $\mathcal{I}(n_1, \ldots, n_d)$. □

Finally, let us mention that for $k=3$, $d=2$ an injection 2-design in $\mathcal{J}(3, n)$ is equivalent to a latin square with $(1, 2, \ldots, n)$ as main diagonal (i.e., multiplication table of idempotent quasigroup). This latter exists for all n except $n=2$. To see this equivalence, first assume $\mathcal{G}$ is an injection 2-design in $\mathcal{J}(3, n)$. Then $|\mathcal{G}| = n(n-1)$. If $G \in \mathcal{G}$, $G = \{(1, a), (2, b), (3, c)\}$ then put in the latin square a in position (b, c). This way the diagonal will remain unfilled. Write a in position (a, a). This leads to a latin square since $\mathcal{G}$ is an injection design and clearly, one can reverse the steps. Note that, in view of Proposition 5.7, the existence of an injection 2-design in $\mathcal{J}(3, n)$ implies that it exists in $\mathcal{J}(m, n)$ whenever $m \equiv 1$ or $3 \pmod 6$. The first case for which we could not decide whether an injection 2-design (with $k=3$) exists is in $\mathcal{J}(4, 5)$.

6. A Recurrent Construction for Injection Designs

Let us recall that an injection design $\mathcal{D}$ with parameters $(l_0, \ldots, l_{s-1}, k, n)$ and of dimension d is a perfect $\mathcal{J}(X_1, \ldots, X_d)$-squashed design of rank s in which the size of the flats are $l_0, \ldots, l_{s-1}$, k. For a set $D = \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(d)}), \ldots, (x_k^{(1)}, \ldots, x_k^{(d)})\}$ we define the first projection of D by $\pi(D) = \{x_1^{(1)}, \ldots, x_k^{(1)}\}$. We know (cf. [DF]) that $\pi(D, \mathcal{D}) = \{\pi(D \cap D') \colon D' \in \mathcal{D}\}$ defines a perfect matroid design for all $D \in \mathcal{D}$, in particular for $D \in \mathcal{D}_s$. $\mathcal{D}$ is called *concentrated* if for all $D, D' \in \mathcal{D}$, $\pi(D) \cap \pi(D') \in \pi(D, \mathcal{D})$ holds.

Proposition 6.1. *Suppose $\mathcal{D}$ is a concentrated injection design with parameters $(l_0, \ldots, l_{s-1}, k, n)$ then for all i, $0 < i < s$ (and even for $i = s$ if $k < n$) the family $\pi(\mathcal{D}_i) = \{\pi(D) \colon D \in \mathcal{D}_i\}$ is a perfect matroid design with parameters $(l_0, \ldots, l_{i-1}, l_i, n)$.*

Proof. Clearly $\pi(\mathcal{D}_i)$ is an $(n, l_i, \{l_0, \ldots, l_{i-1}\})$-system which verifies the assumptions of Remark 2.4, i.e., the intersection of any number of distinct members of $\pi(\mathcal{D}_i)$ has size belonging to L. The proof can be concluded as that of Theorem 1.3. □

For $A \subset X_1$ clearly $A = \pi(D)$ implies that D is an injective subset in $A \times X_2 \times \cdots \times X_d$. Thus, if $|A| = l_i$, $|X_j| = n^{(j)}$, then the number of $D \in \mathcal{D}_i$ with $A = \pi(D)$ is at most $\prod_{0 \le a \le i-1} \prod_{j=1}^{d} (n^{(j)} - l_a)$.

Proposition 6.2. *Suppose $\mathcal{D}^{(\nu)}$ is a concentrated injection design with parameters $(l_0, l_1, \ldots, l_{s-1}, k, n)$ and of dimension $d^{(\nu)}$ in $\mathcal{J}(X_1^{(\nu)}, \ldots, X_{d_\nu}^{(\nu)})$, $\nu = 1, 2$. Suppose further $X_1^{(1)} = X_1^{(2)}$ and for all i, $0 \le i \le s$, $\pi(\mathcal{D}_i^{(1)}) = \pi(\mathcal{D}_i^{(2)})$ holds. Then there exists a concentrated injection design with parameters $(l_0, \ldots, l_{s-1}, k, n)$ and of dimension $d^{(1)} + d^{(2)} - 1$ in $\mathcal{J}(X_1^{(1)}, \ldots, X_{d_1}^{(1)}, X_2^{(2)}, \ldots, X_{d_2}^{(2)})$.*

Proof. With $\mathbf{x} = (a, x_2^{(1)}, \ldots, x_{d_1}^{(1)}) \in X_1^{(1)} \times \cdots \times X_{d_1}^{(1)}$ and $\mathbf{y} = (a, y_2^{(2)}, \ldots, y_{d_2}^{(2)}) \in X_1^{(2)} \times \cdots \times {}_{d_2}^{(2)}$ we associate $\mathbf{xy} = (a, x_2^{(1)}, \ldots, x_{d_1}^{(1)}, y_2^{(2)}, \ldots, y_{d_2}^{(2)})$. Similarly, if $A \in \mathcal{D}^{(1)}$, $B \in \mathcal{D}^{(2)}$ and $\pi(A) = \pi(B)$ then we can define $AB = \{\mathbf{xy} \colon \mathbf{x} \in A, \mathbf{y} \in B, \mathbf{x}$ and $\mathbf{y}$ have the same first coordinate$\}$. This leads to the definition of $\mathcal{D} = \mathcal{D}^{(1)}\mathcal{D}^{(2)} = \{AB \colon A \in \mathcal{D}^{(1)}, B \in \mathcal{D}^{(2)}, \pi(A) = \pi(B)\}$.

We want to show that $\mathscr{D}$ is the desired injection design. For $C \subset X_1^{(1)} \subset X_2^{(1)}$ we define $\pi_\nu^{-1}(C) = \{D \in \mathscr{D}^{(\nu)}: \pi(D) = C\}$, $\nu = 1, 2$. Since $\pi(\mathscr{D}^{(1)}) = \pi(\mathscr{D}^{(2)})$ is a PMD on $X_1^{(1)}$, in view of our remarks after Proposition 6.1, $\pi_\nu^{-1}(C)$ is an injection design on $\mathscr{I}(C, X_2^{(\nu)}, \ldots, X_d^{(\nu)})$ for $\nu = 1, 2$ and $C \in \pi(\mathscr{D}^{(1)})$. Define $\pi^{-1}(C) = \{D \in \mathscr{D}: \pi(D) = C\}$. Clearly, $|\pi^{-1}(C)| = |\pi_1^{-1}(C)| \cdot |\pi_2^{-1}(C)|$ holds, i.e., $\pi^{-1}(C)$ has the size of an injection design in $\mathscr{I}(C, X_2^{(1)}, \ldots, X_{d_1}^{(1)}, X_2^{(2)}, \ldots, X_{d_2}^{(2)})$. Thus $\mathscr{D}_i = \{D \in \mathscr{D}: |D| = l_i\}$ satisfies

$$|\mathscr{D}_i| = |\pi(\mathscr{D}_i^{(1)})| \prod_{\nu=1}^{2} \prod_{j=2}^{d^{(\nu)}} \prod_{b=0}^{i-1} (n_j^{(\nu)} - l_b).$$

Using $|\pi(D_i^{(1)})| = \prod_{0 \le b < 1} (n_1^{(1)} - l_b)/(l_i - l_b)$, we see that $\mathscr{D}_i$ has the desired size.

To conclude the proof, in view of our remarks after the proof of Theorem 1.3, we only need to show that $\mathscr{D}$ is closed under intersection. Let $AB, A'B' \in \mathscr{D}$. Since $\mathscr{D}^{(\nu)}$ is concentrated for all $C \in \pi(\mathscr{D}^{(\nu)})$, $D^{(\nu)} \in \mathscr{D}^{(\nu)}$, the set $\{\mathbf{x} \in D^{(\nu)}: \pi(x) \in C\}$ is in $\mathscr{D}^{(\nu)}$. Consequently if $\pi(D^{(1)}) = \pi(D^{(2)})$ then $\{\mathbf{x} \in D^{(1)}D^{(2)}: \pi(\mathbf{x}) \in C\} \in \mathscr{D}$ holds. Since $\pi(\mathscr{D}^{(\nu)}$ is closed intersection, $C = \pi(A \cap A') \cap \pi(B \cap B') \in \pi(\mathscr{D}^{(\nu)})$. Now $AB \cap A'B' = \{\mathbf{x} \in AB: \pi(\mathbf{x}) \in C\}$, thus $(AB \cap A'B') \in \mathscr{D}$, concluding the proof. □

Proposition 6.3. *Suppose $\mathscr{D}^{(\nu)}$ is an injection design with parameters* $(0, 1, \ldots, s-1, k^{(\nu)}, n^{(\nu)})$ in $\mathscr{I}(X_1^{(\nu)}, \ldots, X_{d_\nu}^{(\nu)})$, $\nu = 1, 2$. *Moreover* $X_1^{(1)} = X_1^{(2)}$, $|X_1^{(2)}| = k^{(2)}$. *Then there exists an injection design with parameters* $(0, 1, \ldots, s-1, k^{(2)}, n^{(1)}n^{(2)}/n_1^{(2)})$ *in* $\mathscr{I}(X_1^{(1)}, \ldots, X_{d_1}^{(1)}, X_2^{(2)}, \ldots, X_{d_2}^{(2)})$.

Proof. For $0 \le i < s$ the flats of rank i in the product design $\mathscr{D}$ are all injective subsets of size i in $\mathscr{I}(X_1^{(1)}, \ldots, X_{d_2}^{(2)})$.

As to $\mathscr{D}_s$, for arbitrary $A \in \mathscr{D}_s^{(1)}$ and $B \in \mathscr{D}_s^{(2)}$ we first define B_A as the unique subset of B satisfying $\pi(A) = \pi(B_A)$. Then the corresponding element in $\mathscr{D}_s$ is AB_A. From here on the proof practically coincides with that of the preceding proposition. □

Note that the reason why we do not need concentratedness explicitly in this proposition is that $\pi(\mathscr{D}_i^{(1)}) = \pi(\mathscr{D}_i^{(2)})$ is automatically satisfied for $0 \le i < s$. Also $k^{(2)} = n_1^{(2)}$ implies that $\pi(D) = X_1^{(2)}$ holds for all $D \in \mathscr{D}_s^{(2)}$, i.e., $\mathscr{D}^{(2)}$ is concentrated in a very strong sense.

For more constructions see [DL], [La].

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