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ON STABILITY OF THE P_n^{mod}/P_n ELEMENT FOR INCOMPRESSIBLE FLOW PROBLEMS*

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Abstract. It is well known that finite element spaces used for approximating the velocity and the pressure in an incompressible flow problem have to be stable in the sense of the inf-sup condition of Babuška and Brezzi if a stabilization of the incompressibility constraint is not applied. In this paper we consider a recently introduced class of triangular nonconforming finite elements of nth order accuracy in the energy norm called P_n^{mod} elements. For $n \leqslant 3$ we show that the stability condition holds if the velocity space is constructed using the P_n^{mod} elements and the pressure space consists of continuous piecewise polynomial functions of degree n.

Keywords: nonconforming finite element method, inf-sup condition, incompressible flow problem

MSC 2000: 65N30, 65N12, 76D05

1. Introduction

In computational fluid dynamics, nonconforming finite elements are often used for discretizing incompressible flow problems. One advantage of nonconforming finite elements in comparison to conforming ones is that they usually satisfy inf-sup conditions with more convenient pressure spaces and that discretely divergence-free bases can often be more easily constructed for this type of finite elements. Another reason for the application of nonconforming finite elements may be that they are more suitable for parallel implementation since their degrees of freedom are associated with edges (or with interior points of the elements of the triangulation), which leads to a cheap local communication between processors. In addition, nonconforming finite elements often exhibit nice stability properties and lead to very efficient procedures

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for numerical solution of partial differential equations. We refer to [9], [10], [15], [18] for more details on the properties of nonconforming finite elements applied to incompressible flow problems.

In the two-dimensional case, the typical feature of nonconforming finite elements is that, on edges between neighbouring elements, the finite element functions are continuous at some points only. Often, this property is equivalent to the fact that the jump $[|v_h|]_E$ of any finite element function v_h across any inner edge E is L^2 orthogonal to the space $P_k(E)$ of polynomials on E of degree k:

$$\int_E [|v_h|]_E q \,\mathrm{d}\sigma = 0 \quad \forall\, q \in P_k(E). \tag{1}$$

If a nonconforming finite element is locally of approximation order n with respect to the energy norm, the usual requirement is the validity of (1) with $k = n - 1$. Then optimal error estimates in the energy norm can be proved for second order elliptic problems (see e.g. Ciarlet [5] and Crouzeix and Raviart [6]).

However, it was observed that nonconforming finite elements sometimes do not lead to the expected accuracy if they are applied to the numerical solution of convection dominated problems. This phenomenon was thoroughly investigated by Knobloch and Tobiska [14] and by Knobloch [12] for a scalar convection-diffusion equation discretized by means of the streamline diffusion method. It was shown that, in the convection dominated case, the validity of (1) with $k = n - 1$ for finite elements of local approximation order n is not sufficient for proving optimal convergence results uniform with respect to the perturbation parameter. Therefore, it was suggested to use $k = n+1$, for which optimal convergence results were proved. In [14], Knobloch and Tobiska developed a new triangular nonconforming finite element of first order accuracy called the P_1^{mod} element which satisfies (1) with $k = 2$. Later, using the ideas of [14], Knobloch [12] introduced a class of triangular nonconforming finite elements of an arbitrary order n of accuracy satisfying (1) with any prescribed $k \geqslant n$. These finite elements were named P_n^{mod}. Numerical experiments in [14] for convection-diffusion equations and in [11] for the Stokes equations show that the P_1^{mod} element leads to a considerable improvement of the accuracy in comparison with the nonconforming piecewise linear Crouzeix-Raviart element [6].

The main reason for developing the P_n^{mod} elements was to have nonconforming finite elements suitable for approximating the velocity $\boldsymbol{u}$ in incompressible flow problems, e.g., in the incompressible Navier-Stokes equations

$$-\nu\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{4}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain with a polygonal boundary $\partial\Omega$, $\nu > 0$ is the kinematic viscosity, $\boldsymbol{f}$ is an outer volume force and p is a second unknown function, the pressure. In view of the incompressibility constraint, finite element spaces $\mathbf{V}_h$ and Q_h for approximating the velocity $\boldsymbol{u}$ and pressure p, respectively, cannot be chosen arbitrarily if one wants to obtain a stable discretization with respect to $h \to 0$ and no additional stabilization of the continuity equation (3) is used (see e.g. Brezzi and Fortin [4] or Girault and Raviart [8]). A sufficient requirement on the spaces $\mathbf{V}_h$, Q_h is the validity of the inf-sup condition

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geqslant \beta \|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h, \tag{5}$$

where $\beta > 0$ is independent of the discretization parameter h,

$$b_h(\boldsymbol{v}_h, q_h) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}x, \qquad |\boldsymbol{v}_h|_{1,h} = \Big(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2\Big)^{1/2}$$

and $\mathcal{T}_h$ is a triangulation of Ω consisting of elements K used for constructing the spaces $\boldsymbol{V}_h$ and Q_h. The notation $|\boldsymbol{v}_h|_{1,K}$ is used for the seminorm of $\boldsymbol{v}_h|_K$ in the space $H^1(K)^2$. The inf-sup condition (5) makes it possible to establish optimal error estimates for the discrete solution of (2)–(4), see e.g. Girault and Raviart [8].

Let us consider spaces $\mathbf{V}_h$ defined using the P_n^{mod} element ($n \geqslant 1$) and denote these spaces by $\mathbf{V}_h^{\mathrm{mod},n}$. We assume that these spaces approximate the homogeneous Dirichlet boundary condition (4). It follows from the results of Knobloch [11], [13] and Scott and Vogelius [16] that, for any $n \in \mathbb{N}$, the inf-sup condition (5) holds if we use the space $\mathbf{V}_h^{\mathrm{mod},n}$ together with a pressure space $\bar{Q}_h^{n-1} \subset L_0^2(\Omega)$ consisting of discontinuous piecewise polynomial functions of degree $n-1$.

The use of a velocity space of approximation order n together with a pressure space of approximation order $n-1$ is optimal with respect to the asymptotic convergence rate. Moreover, a pressure space consisting of piecewise discontinuous functions guarantees local mass conservation, which is sometimes of importance. Therefore, the combination $\mathbf{V}_h^{\mathrm{mod},n}/\bar{Q}_h^{n-1}$ mentioned in the preceding paragraph is often satisfactory. However, as explained by Bernardi and Hecht [2], who investigated the validity of the inf-sup condition (5) for a velocity space defined using the nonconforming Crouzeix-Raviart element and a pressure space consisting of certain continuous piecewise polynomial functions, it is sometimes important to use continuous pressures. This is essential when working with geophysical flows, where the Coriolis acceleration must be taken into account. The use of continuous pressures is then necessary to respect the geostrophic equilibrium. It is also necessary for optimizing the geometry of the fluid domain when the optimization criterion involves local gradients of the pressure. Moreover, as mentioned by Ainsworth and Coggins [1], also in other cases many practitioners prefer the use of continuous pressures. The reasons cited for this preference include the superior stability of methods based on continuous pressures in general, the reduced number of degrees of freedom associated with continuous pressure spaces and the more aesthetically pleasing pressure contours arising from continuous pressure spaces.

Therefore, in this paper, we consider pressure spaces

$$Q_h^n = \{q_h \in C(\bar{\Omega}) \cap L_0^2(\Omega);\ q_h|_K \in P_n(K)\ \forall K \in \mathcal{T}_h\} \tag{6}$$

consisting of continuous piecewise polynomial functions of degree n having zero mean value on Ω. Of course, it follows from the above-mentioned results that, for any $n \in \mathbb{N}$, the inf-sup condition (5) is satisfied if we use the velocity space $\mathbf{V}_h^{\mathrm{mod},n}$ together with the pressure space Q_h^{n-1}. However, the discussion in Knobloch [11] shows that, in some cases, a significant improvement of the velocity approximation can be achieved if velocity and pressure spaces of the same approximation order are applied. Thus, the aim of this paper is to investigate whether the inf-sup condition (5) holds if we use the velocity space $\mathbf{V}_h^{\mathrm{mod},n}$ together with the pressure space Q_h^n. This will be proved for $n \leqslant 3$.

The paper is organized in the following way. First, in Sect. 2, we introduce some assumptions and summarize the notation which will be used in the subsequent sections. Then, in Sect. 3, we recall the definition of the P_n^{mod} element and mention some of its properties. In Sect. 4, we introduce the macroelement technique which will be the basic tool for proving the inf-sup condition (5) for the combination $\mathbf{V}_h^{\mathrm{mod},n}/Q_h^n$ with $n = 1, 2, 3$ in Sect. 5–7. The paper will be finished with conclusions in Sect. 8.

2. Assumptions and notation

We assume that we are given a bounded domain $\Omega \subset \mathbb{R}^2$ with a polygonal boundary $\partial\Omega$ and a family $\{\mathcal{T}_h\}$ of triangulations of Ω. The triangulations are assumed to consist of closed triangular elements K, to possess the usual compatibility properties (see e.g. Ciarlet [5]) and to satisfy $h_K \equiv \operatorname{diam}(K) \leqslant h$ for any $K \in \mathcal{T}_h$. We assume that the family of triangulations is regular, i.e., there exists a constant σ independent of h such that

$$\frac{h_K}{\varrho_K} \leqslant \sigma \quad \forall K \in \mathcal{T}_h,\ \ h > 0, \tag{7}$$

where ϱ_K is the maximum diameter of circles inscribed into K. Finally, we assume that any element $K \in \mathcal{T}_h$ has at least one vertex in Ω.

We denote by $\mathcal{E}_h$ the set of edges E of $\mathcal{T}_h$ and by $\mathcal{E}_h^i$ the subset of $\mathcal{E}_h$ consisting of inner edges. Further, for any edge E, we denote by h_E the length of E, by $x_{E,1}$,

$x_{E,2}$ the end points of E and by $\lambda_{E,1}$, $\lambda_{E,2}$ the linear functions on E satisfying $\lambda_{E,i}(x_{E,j}) = \delta_{ij}$, $i,j = 1,2$, where δ_{ij} denotes the Kronecker symbol. For any edge E, we denote by $\boldsymbol{t}_E = (t_{E1}, t_{E2})$ the unit tangent vector to E which points from $x_{E,1}$ to $x_{E,2}$ and by $\boldsymbol{n}_E \equiv (-t_{E2}, t_{E1})$ a normal vector to E. For any inner edge $E \in \mathcal{E}_h^i$, we define the jump of a function v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\tilde{K}})|_E,$$

where K, $\tilde{K}$ are the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\tilde{K}$. If an edge $E \in \mathcal{E}_h$ lies on the boundary of Ω, then we set $[|v|]_E = v|_E$.

Throughout the paper we use standard notation $L^2(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $P_k(\Omega)$ etc. for the usual function spaces, see e.g. Ciarlet [5]. We only mention that we denote by $L^2_0(\Omega)$ the space of functions from $L^2(\Omega)$ having zero mean value on Ω. The norm and seminorm in the Sobolev space $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. Further, we use the notation $|G|$ to denote the d-dimensional Lebesgue measure of a set $G \subset \mathbb{R}^d$, $d = 1,2$. Finally, we denote by $(\cdot,\cdot)_G$ the inner product in the space $L^2(G)$.

3. Definition and properties of the P_n^{mod} element

In this section we recall the definition of the P_n^{mod} element introduced in Knobloch [12] and mention some of its basic properties.

We start with describing the space $P_n^{\mathrm{mod}}(\hat{K})$ of shape functions on the standard reference element $\hat{K}$. Given an integer $n \geqslant 1$, we set

$$P_n^{\mathrm{mod}}(\hat{K}) = P_n(\hat{K}) \oplus \mathrm{span}\{\hat{b}_1, \hat{b}_2, \hat{b}_3\},$$

where $\hat{b}_1$, $\hat{b}_2$ and $\hat{b}_3$ are functions on $\hat{K}$ associated respectively with the edges $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_3$ of the element $\hat{K}$. We assume that

$$\hat{b}_1 \in H^1(\hat{K}), \quad \hat{b}_1|_{\partial\hat{K}\setminus\hat{E}_1} = 0, \tag{8}$$

$$\hat{b}_1|_{\hat{E}_1} \text{ is odd with respect to the midpoint of } \hat{E}_1, \tag{9}$$

$$\int_{\hat{E}_1} [(1 - 2\hat{\lambda}_2) + \hat{b}_1]\hat{q}\,\mathrm{d}\hat{\sigma} = 0 \quad \forall\,\hat{q} \in P_k(\hat{E}_1), \tag{10}$$

where $k \geqslant n$ and $\hat{\lambda}_2$ is the barycentric coordinate on $\hat{K}$ with respect to the vertex $\hat{x}_2$ which is the vertex of $\hat{K}$ opposite the edge $\hat{E}_2$ (the remaining vertices $\hat{x}_1$ and $\hat{x}_3$ and the barycentric coordinates $\hat{\lambda}_1$ and $\hat{\lambda}_3$ are defined analogously). The functions $\hat{b}_2$ and $\hat{b}_3$ are simply defined by affine transformations of $\hat{b}_1$, i.e., $\hat{b}_i = \hat{b}_1 \circ \hat{F}_i$, $i = 2,3$, where

$\hat{F}_2$ and $\hat{F}_3$ are affine regular mappings on $\mathbb{R}^2$ such that $\hat{F}_i(\hat{K}) = \hat{K}$, $\hat{F}_i(\hat{E}_i) = \hat{E}_1$, $i = 2, 3$.

The following functions are examples of the function $\hat{b}_1$ satisfying (8)–(10) for various values of k:

$$\begin{aligned}
k = 2: &\quad \hat{b}_1 = 10\,\hat{\lambda}_2\hat{\lambda}_3(\hat{\lambda}_2 - \hat{\lambda}_3),\\
k = 4: &\quad \hat{b}_1 = (28\,\hat{\lambda}_2\hat{\lambda}_3 - 126\,\hat{\lambda}_2^2\hat{\lambda}_3^2)(\hat{\lambda}_2 - \hat{\lambda}_3),\\
k = 6: &\quad \hat{b}_1 = (54\,\hat{\lambda}_2\hat{\lambda}_3 - 594\,\hat{\lambda}_2^2\hat{\lambda}_3^2 + 1716\,\hat{\lambda}_2^3\hat{\lambda}_3^3)(\hat{\lambda}_2 - \hat{\lambda}_3).
\end{aligned}$$

Generally, to satisfy the assumptions (8)–(10) with $k = 2l$, where l is a given positive integer, we can set

$$\hat{b}_1 = (\hat{\lambda}_2 - \hat{\lambda}_3)\sum_{i=1}^{l} a_i(\hat{\lambda}_2\hat{\lambda}_3)^i, \tag{11}$$

where $a_1, \ldots, a_l \in \mathbb{R}$ are uniquely determined numbers.

For introducing a finite element space we have to specify a set of the so-called nodal functionals which have to be unisolvent with the space $P_n^{\rm mod}(\hat{K})$. For example, we can use $n+1$ moments on each edge, i.e.,

$$\frac{1}{|\hat{E}_i|}\int_{\hat{E}_i} \hat{v}\hat{\lambda}_{i+1}^j \,\mathrm{d}\hat{\sigma}, \quad j = 0, \ldots, n, \;\; i = 1, 2, 3,$$

where $\hat{\lambda}_4 \equiv \hat{\lambda}_1$. If $n > 2$, then we further add a set of nodal functionals unisolvent with the space $P_n(\hat{K}) \cap H_0^1(\hat{K})$. These nodal functionals may be e.g. some suitable integrals over $\hat{K}$ or values at the inner points of the principal lattice of order n of the triangle $\hat{K}$.

For any element $K \in \mathcal{T}_h$, we introduce a regular affine mapping $F_K\colon \hat{K} \to K$ such that $F_K(\hat{K}) = K$ and transform the space $P_n^{\rm mod}(\hat{K})$ to the space

$$P_n^{\rm mod}(K) = \{\hat{v} \circ F_K^{-1};\; \hat{v} \in P_n^{\rm mod}(\hat{K})\}.$$

Similarly, we transform the nodal functionals defined on the space $P_n^{\rm mod}(\hat{K})$ to analogous nodal functionals defined on the spaces $P_n^{\rm mod}(K)$. Now, the $P_n^{\rm mod}$ finite element space approximating the space $H_0^1(\Omega)$ consists of functions which belong to $P_n^{\rm mod}(K)$ on each element K and for which the nodal functionals assigned to boundary edges vanish and the corresponding pairs of nodal functionals assigned to inner edges give the same value. Thus, we obtain the space

$$\begin{aligned}
V_h^{{\rm mod},n} = \Big\{ & v_h \in L^2(\Omega);\; v_h|_K \in P_n^{\rm mod}(K)\;\forall\, K \in \mathcal{T}_h,\\
& \int_E [|v_h|]_E\, q \,\mathrm{d}\sigma = 0 \;\forall\, q \in P_n(E),\;\; E \in \mathcal{E}_h \Big\}.
\end{aligned}$$

As we see, the space $V_h^{\mathrm{mod},n}$ is of approximation order n with respect to $|\cdot|_{1,h}$. It is a nonconforming finite element space which is edge- and element-oriented, i.e., one can use basis functions which are all associated with edges or with elements. Precisely, for any inner edge, we have $n+1$ basis functions associated with this edge and having their supports in the two elements adjacent to this edge. Further, on each element, we have $\frac{1}{2}(n-1)(n-2)$ basis functions having their supports in this element. An important property of the P_n^{mod} element is that (see Knobloch [12])

$$\int_E [|v_h|]_E q \,\mathrm{d}\sigma = 0 \quad \forall\, v_h \in V_h^{\mathrm{mod},n}, \;\; q \in P_k(E), \;\; E \in \mathcal{E}_h, \tag{12}$$

where k is the integer introduced in (10). Thus, choosing the function $\hat{b}_1$ in a suitable way, we can enforce the validity of (12) with an arbitrarily high k. The relation (12) together with the Gauss integral theorem implies that

$$b_h(\boldsymbol{v}_h, q_h) = \int_\Omega \boldsymbol{v}_h \cdot \nabla q_h \,\mathrm{d}x \quad \forall\, \boldsymbol{v}_h \in [V_h^{\mathrm{mod},n}]^2, \;\; q_h \in Q_h^n, \;\; n \in \mathbb{N}. \tag{13}$$

To describe a basis of the space $V_h^{\mathrm{mod},n}$, let us first introduce some notation. Consider any $K \in \mathcal{T}_h$ and any $E \in \mathcal{E}_h$ such that $E \subset \partial K$. Let $i \in \{1,2,3\}$ be such that $E = F_K(\hat{E}_i)$. Then we set

$$b_{K,E} = \begin{cases} \pm \hat{b}_i \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

where the sign is chosen in such a way that

$$\int_E (b_{K,E}|_K)\lambda_{E,1} \,\mathrm{d}\sigma > 0. \tag{14}$$

Thus, the space $P_n^{\mathrm{mod}}(K)$ of shape functions on K can be written as

$$P_n^{\mathrm{mod}}(K) = P_n(K) \oplus \mathrm{span}\{b_{K,E}|_K\}_{E \in \mathcal{E}_h,\, E \subset \partial K}.$$

Now let us consider any inner edge $E \in \mathcal{E}_h^i$ and denote by K, $\tilde{K}$ the two elements adjacent to E, by E, E_1, E_2 the edges of K, and by E, E_3, E_4 the edges of $\tilde{K}$. Let ζ_E be the standard nonconforming piecewise linear basis function associated with the edge E (i.e., ζ_E is piecewise linear, equals 1 on E and vanishes at the midpoints of all edges different from E). Then we define functions ψ_E, $\chi_E \in V_h^{\mathrm{mod},n}$ by

$$\psi_E = \zeta_E + \beta_{E,1} b_{K,E_1} + \beta_{E,2} b_{K,E_2} + \beta_{E,3} b_{\tilde{K},E_3} + \beta_{E,4} b_{\tilde{K},E_4}, \tag{15}$$

$$\chi_E = \begin{cases} b_{K,E} & \text{in } K, \\ b_{\tilde{K},E} & \text{in } \Omega \setminus K, \end{cases} \tag{16}$$

where $\beta_{E,i} = -1$ if $x_{E_i,1} \in E$ and $\beta_{E,i} = 1$ if $x_{E_i,1} \notin E$, $i = 1, \dots, 4$. In view of the assumptions on $\hat{b}_1$, we have $\chi_E \in H^1_0(\Omega)$ and hence the functions χ_E generate a conforming subspace of $V_h^{\mathrm{mod},n}$. The functions ψ_E are purely nonconforming functions since they have jumps across the edges $E_1, \dots, E_4$. They can be viewed as modifications of the basis functions ζ_E, which gave rise to the notation P_n^{mod}. The functions $\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i}$ represent a basis of the space $V_h^{\mathrm{mod},n}$ for $n = 1$. If $n > 1$, then

$$\begin{aligned} V_h^{\mathrm{mod},n} = \operatorname{span}\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i} \oplus \{v \in H^1_0(\Omega);\ v|_K \in P_n(K)\ \forall K \in \mathcal{T}_h, \\ v(x) = 0 \text{ at any vertex } x \text{ of } \mathcal{T}_h\}. \end{aligned}$$

Thus, a basis of the space $V_h^{\mathrm{mod},n}$ is formed by the functions $\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i}$ and by the standard conforming basis functions of the right-hand space in the above direct sum. In such a basis the only nonconforming functions are the functions ψ_E and the functions generated by the function $\hat{b}_1$ appear only in the functions ψ_E and χ_E.

Note also that there is no difference between defining the P_n^{mod} elements of odd and even approximation order whereas standard triangular nonconforming finite elements of even degree are difficult to handle. A further difference from standard nonconforming finite elements is that the spaces $V_h^{\mathrm{mod},n}$ with various values of n can be made nested by defining them using the same function $\hat{b}_1$.

Now let us turn our attention to the case $n \leqslant 3$ for which we will investigate the validity of inf-sup conditions in the next sections. For any inner edge $E \in \mathcal{E}_h^i$ we introduce functions $\varrho_E, \zeta_E \in C(\bar{\Omega})$ vanishing outside the two elements adjacent to E and defined by

$$\varrho_E|_K = \lambda_1\lambda_2, \quad \zeta_E|_K = \lambda_1\lambda_2(\lambda_1 - \lambda_2)$$

for any element K adjacent to E, where λ_1, λ_2 are the barycentric coordinates on K with respect to $x_{E,1}$, $x_{E,2}$, respectively. Further, for any element $K \in \mathcal{T}_h$ we define the bubble function π_K by

$$\pi_K|_K = \lambda_1\lambda_2\lambda_3, \quad \pi_K|_{\Omega\setminus K} = 0,$$

where λ_1, λ_2, λ_3 are the barycentric coordinates on K. Then

$$\begin{aligned} V_h^{\mathrm{mod},1} &= \operatorname{span}\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i}, \\ V_h^{\mathrm{mod},2} &= \operatorname{span}\{\psi_E, \chi_E, \varrho_E\}_{E\in\mathcal{E}_h^i}, \\ V_h^{\mathrm{mod},3} &= \operatorname{span}\big\{\{\psi_E, \chi_E, \varrho_E, \zeta_E\}_{E\in\mathcal{E}_h^i} \cup \{\pi_K\}_{K\in\mathcal{T}_h}\big\}. \end{aligned}$$

Since all these basis functions can be obtained by affine transformations of functions defined on the reference element $\hat{K}$, we have (cf. Ciarlet [5, Sect. 15])

$$(17)\quad |\psi_E|_{1,h} + |\chi_E|_{1,\Omega} + |\varrho_E|_{1,\Omega} + |\zeta_E|_{1,\Omega} \leqslant C\ \ \forall E \in \mathcal{E}_h^i, \quad |\pi_K|_{1,\Omega} \leqslant C\ \ \forall K \in \mathcal{T}_h,$$

where the constant C depends only on $\hat{b}_1$ and σ from (7).

To make the proof of the inf-sup condition possible, we introduce additional assumptions on the function $\hat{b}_1$. First, we assume that, for any function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $g(x,y) = g(y,x)$ for all $x, y \in \mathbb{R}$, the function $\hat{b}_1$ satisfies

$$\int_{\hat{K}} \hat{b}_1 g(\hat{\lambda}_2, \hat{\lambda}_3)\,\mathrm{d}\hat{x} = 0, \tag{18}$$

which is satisfied for any function $\hat{b}_1$ of the type (11). We denote

$$A = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{b}_1 \hat{\lambda}_2\,\mathrm{d}\hat{x}, \quad B = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{b}_1 \hat{\lambda}_2^2\,\mathrm{d}\hat{x}, \quad D = 6A - 7B. \tag{19}$$

If $n = 2$, we assume that

$$A \neq 0, \quad A \neq \frac{1}{30}.$$

If $n = 3$, we assume that

$$D \neq 0, \quad D \neq -\frac{1}{30}, \quad A \neq B, \quad 3B - 2A \neq \frac{1}{45}.$$

These assumptions are satisfied for the examples of the function $\hat{b}_1$ given above.

Finally, let us mention a few useful relations which can be readily verified. Let $E \in \mathcal{E}_h^i$ be any inner edge of the triangulation $\mathcal{T}_h$ and let $K \in \mathcal{T}_h$ be any element adjacent to this edge. Let λ_1 and λ_2 be the barycentric coordinates on K with respect to $x_{E,1}$, $x_{E,2}$, respectively, and let λ_3 be the remaining barycentric coordinate on K. Then

$$\int_K \psi_E \lambda_1\,\mathrm{d}x = \int_K \psi_E \lambda_2\,\mathrm{d}x = \Big(\frac{1}{6} - A\Big)|K|, \quad \int_K \psi_E \lambda_3\,\mathrm{d}x = 2A|K|, \tag{20}$$

$$\int_K \psi_E \lambda_1^2\,\mathrm{d}x = \int_K \psi_E \lambda_2^2\,\mathrm{d}x = \Big(\frac{1}{10} - B\Big)|K|, \tag{21}$$

$$\int_K \psi_E \lambda_3^2\,\mathrm{d}x = \Big(-\frac{1}{30} + 2B\Big)|K|,$$

$$\int_K \chi_E \lambda_1\,\mathrm{d}x = -\int_K \chi_E \lambda_2\,\mathrm{d}x = A|K|, \quad \int_K \chi_E \lambda_3\,\mathrm{d}x = 0, \tag{22}$$

$$\int_K \chi_E \lambda_1^2\,\mathrm{d}x = -\int_K \chi_E \lambda_2^2\,\mathrm{d}x = B|K|, \quad \int_K \chi_E \lambda_3^2\,\mathrm{d}x = 0. \tag{23}$$

4. Macroelement technique

The proof of the inf-sup conditions will be related to the macroelement technique of Boland and Nicolaides [3] and Stenberg [17] which was extended to the nonconforming case by Crouzeix and Falk [7] and Knobloch [11].

We denote by $\{x_i\}_{i=1}^{N_h}$ the inner vertices of the triangulation $\mathcal{T}_h$ and, for any vertex x_i, we introduce a macroelement

$$\Delta_i = \bigcup_{K\in\mathcal{T}_h,\, x_i\in K} K$$

consisting of elements grouped around x_i. Further, we set

$$\bar{Q}_h = \{q_h \in L^2_0(\Omega);\ q_h|_K \in P_0(K)\ \ \forall\, K \in \mathcal{T}_h\}$$

and, for any $m, n \in \mathbb{N}$, denote

$$\begin{aligned}\tilde{Q}_h^m &= \{q_h \in Q_h^m \oplus \bar{Q}_h;\ q_h|_K \in L^2_0(K)\ \ \forall\, K \in \mathcal{T}_h\},\\ \mathbf{V}_h^{i,n} &= \{\boldsymbol{v}_h \in \mathbf{V}_h^{\mathrm{mod},n};\ \boldsymbol{v}_h = \mathbf{0} \text{ in } \Omega\setminus\Delta_i\}, \quad i = 1,\dots,N_h,\end{aligned}$$

where Q_h^m is the space defined in (6) and $\mathbf{V}_h^{\mathrm{mod},n} \equiv [V_h^{\mathrm{mod},n}]^2$. Then the following theorem holds.

Theorem 1. *Let $m, n \in \mathbb{N}$ be such that, for any $\tilde{q}_h \in \tilde{Q}_h^m$ and any $i \in \{1,\dots,N_h\}$, there exists $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,n}$ satisfying*

$$b_h(\boldsymbol{v}_h^i, \bar{q}_h) = 0 \quad \forall\, \bar{q}_h \in \bar{Q}_h, \tag{24}$$

$$b_h(\boldsymbol{v}_h^i, \tilde{q}_h) = \|\tilde{q}_h\|^2_{0,\Delta_i}, \tag{25}$$

$$|\boldsymbol{v}_h^i|_{1,h} \leqslant \overline{C}\|\tilde{q}_h\|_{0,\Delta_i}, \tag{26}$$

where $\overline{C}$ is a constant independent of h. Then there exists a constant $\beta > 0$ depending only on $\overline{C}$, σ, $\hat{b}_1$ and Ω such that

$$\sup_{\boldsymbol{v}_h\in\mathbf{V}_h^{\mathrm{mod},n}\setminus\{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geqslant \beta\|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h^m.$$

P r o o f. See the proof of Theorem 2 in Knobloch [11]. □

R e m a r k 1. The assumption that any element $K \in \mathcal{T}_h$ has at least one vertex in Ω (cf. Sect. 2) ensures that any element $K \in \mathcal{T}_h$ is contained in at least one macroelement, which is crucial for the proof of Theorem 1.

In the forthcoming sections we shall show that the assumptions of Theorem 1 are satisfied for $m = n \leqslant 3$.

5. Inf-sup condition for the $P_1^{\rm mod}$ element

In this section we prove that the spaces $\mathbf{V}_h^{\rm mod,1} \equiv [V_h^{\rm mod,1}]^2$ and Q_h^1 satisfy the inf-sup condition (5). First, let us prove a simple auxiliary result.

Lemma 1. *Let E_1 and E_2 be two edges of an arbitrary element $K \in \mathcal{T}_h$. Let $P = E_1 \cap E_2$ and let Q be the vertex of K not contained in E_1. Let $\boldsymbol{t}$ be the unit tangent vector to E_2 which points from P to Q. Then*

$$\nabla \lambda_Q = -\frac{h_{E_1}}{2|K|}\boldsymbol{n}_{\partial K}|_{E_1}, \quad \boldsymbol{t}\cdot\nabla\lambda_Q = \frac{1}{h_{E_2}}, \tag{27}$$

where λ_Q is the barycentric coordinate on K with respect to Q and $\boldsymbol{n}_{\partial K}$ is the unit outer normal vector to the boundary of K.

Proof. In view of the Gauss integral theorem, we have

$$-2|K|\nabla\lambda_Q = \int_K \nabla(1-2\lambda_Q)\,\mathrm{d}x = \int_{\partial K}(1-2\lambda_Q)\boldsymbol{n}_{\partial K}\,\mathrm{d}\sigma = h_{E_1}\boldsymbol{n}_{\partial K}|_{E_1}.$$

The second relation is obvious. □

Lemma 2. *For any $\tilde{q}_h \in \tilde{Q}_h^1$ and any $i \in \{1,\dots,N_h\}$ there exists $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,1}$ satisfying (24)–(26), where the constant $\overline{C}$ depends only on σ and $\hat{b}_1$.*

Proof. Consider any $i \in \{1,\dots,N_h\}$ and let Δ_i consist of elements $K_1,\dots,K_n$, i.e.,

$$\Delta_i = \bigcup_{j=1}^n K_j,$$

and let K_{j-1} and K_j have a common edge E_j, $j=1,\dots,n$, see Fig. 1. Here and in the sequel, the index 0 is considered as the index n and the index $n+1$ is considered as the index 1. Without loss of generality, we may assume that $x_{E_j,1} = x_i$ for $j=1,\dots,n$. Then, for any $j \in \{1,\dots,n\}$, the normal vector $\boldsymbol{n}_{E_j}$ points into K_j and the tangent vector $\boldsymbol{t}_{E_j}$ points from x_i to the other vertex of E_j (cf. Fig. 1). Finally, we denote by φ_j the usual continuous piecewise linear basis function which equals 1 at the vertex of E_j different from x_i and vanishes at all other vertices of the triangulation.

Now, let us consider any function $\tilde{q}_h \in \tilde{Q}_h^1$ and denote

$$\boldsymbol{v}_h^i = \sum_{k=1}^n \alpha_k \psi_{E_k}\boldsymbol{t}_{E_k}$$

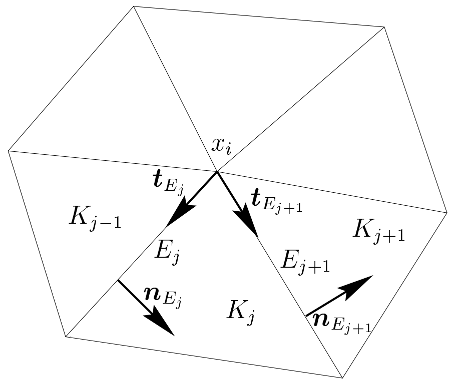

Figure 1. Notation inside a macroelement Δ_i.

with constants α_k, $k = 1, \ldots, n$, to be determined later. Then $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,1}$ and (24) holds. Moreover, $b_h(\boldsymbol{v}_h^i, 1) = 0$. For any $j = 1, \ldots, n$, we derive using (13), (20) and (27)

$$b_h(\boldsymbol{v}_h^i, \varphi_j) = \int_{K_{j-1}\cup K_j} \boldsymbol{v}_h^i \cdot \nabla\varphi_j \,\mathrm{d}x = \alpha_j \int_{K_{j-1}\cup K_j} \psi_{E_j} \boldsymbol{t}_{E_j} \cdot \nabla\varphi_j \,\mathrm{d}x$$
$$= \alpha_j \frac{|K_{j-1}| + |K_j|}{3h_{E_j}}.$$

Thus, setting

$$\alpha_j = \frac{3h_{E_j}}{|K_{j-1}| + |K_j|} (\tilde{q}_h, \varphi_j)_{\Delta_i}, \quad j = 1, \ldots, n,$$

we have

$$b_h(\boldsymbol{v}_h^i, q_h) = (\tilde{q}_h, q_h)_{\Delta_i} \quad \forall\, q_h \in \tilde{Q}_h^1. \tag{28}$$

For $q_h = \tilde{q}_h$ we obtain (25). Using (7), we deduce that

$$h_{E_j}^2 \leqslant \sigma h_{E_j} \varrho_{K_{j-1}} \leqslant 2\sigma |K_{j-1}|, \quad |K_{j-1}| \leqslant \frac{1}{2} h_{E_j} h_{K_{j-1}} \leqslant \frac{\sigma}{2} h_{E_j}^2. \tag{29}$$

Similarly, we get

$$h_{E_j}^2 \leqslant 2\sigma |K_j|, \quad |K_j| \leqslant \frac{\sigma}{2} h_{E_j}^2. \tag{30}$$

Thus, we derive that $|\alpha_j| \leqslant 3\sqrt{\sigma} \|\tilde{q}_h\|_{0,K_{j-1}\cup K_j}$, $j = 1, \ldots, n$, which implies (26) in view of (17) and the fact that the maximum number of elements $K \in \mathcal{T}_h$ having a common vertex depends only on σ. $\square$

Lemma 2 and Theorem 1 immediately imply the following result.

Corollary 1. *There exists a constant $\beta > 0$ depending only on σ, $\hat{b}_1$ and Ω such that*

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h^{\mathrm{mod},1} \setminus \{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geqslant \beta \|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h^1.$$

R e m a r k 2. The above result also follows from Knobloch [11] where an inf-sup condition for the P_1^{mod} element used with discontinuous piecewise linear pressure was proved. Since the proof of Lemma 2 is very easy and this lemma as well as most of the material introduced in this section will be needed in the sequel, we included the proof for completeness.

6. Inf-sup condition for the P_2^{mod} element

This section is devoted to the proof of the inf-sup condition (5) for the spaces $\mathbf{V}_h^{\mathrm{mod},2} \equiv [V_h^{\mathrm{mod},2}]^2$ and Q_h^2. We will use the notation introduced in the preceding section.

Lemma 3. *For any $\tilde{q}_h \in \tilde{Q}_h^2$ and any $i \in \{1, \dots, N_h\}$, there exists $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,2}$ satisfying* (24)–(26), *where the constant $\overline{C}$ depends only on σ and $\hat{b}_1$.*

P r o o f. Consider any $\tilde{q}_h \in \tilde{Q}_h^2$ and any $i \in \{1, \dots, N_h\}$. Then, according to the proof of Lemma 2, there exists a function $\tilde{\boldsymbol{v}}_h^i \in \mathbf{V}_h^{i,2}$ satisfying (24), (26) and (28). Let us denote

$$\bar{\boldsymbol{v}}_h^i = \sum_{k=1}^{n} \{(\alpha_k \tilde{\psi}_{E_k} + \beta_k \chi_{E_k}) \boldsymbol{t}_{E_k} + \gamma_k \chi_{E_k} \boldsymbol{n}_{E_k}\}$$

with

$$\tilde{\psi}_{E_k} = \psi_{E_k} - 4\varrho_{E_k}, \quad k = 1, \dots, n,$$

and constants α_k, β_k, γ_k, $k = 1, \dots, n$, to be determined later. Then $\bar{\boldsymbol{v}}_h^i$ belongs to $\mathbf{V}_h^{i,2}$ and satisfies (24). For any $K \in \mathcal{T}_h$ and any $k \in \{1, \dots, n\}$, we derive using (20) and (22) that $\int_K \tilde{\psi}_{E_k}\,\mathrm{d}x = \int_K \chi_{E_k}\,\mathrm{d}x = 0$ and hence, in view of (13), we have $b_h(\bar{\boldsymbol{v}}_h^i, q_h) = 0$ for any $q_h \in Q_h^1$. Thus, setting $\boldsymbol{v}_h^i = \tilde{\boldsymbol{v}}_h^i + \bar{\boldsymbol{v}}_h^i$, we obtain a function from $\mathbf{V}_h^{i,2}$ satisfying

$$b_h(\boldsymbol{v}_h^i, q_h) = (\tilde{q}_h, q_h)_{\Delta_i} \quad \forall\, q_h \in \tilde{Q}_h^1. \tag{31}$$

We shall show that the constants α_k, β_k, γ_k, $k = 1, \dots, n$, can be chosen in such a way that

$$b_h(\boldsymbol{v}_h^i, q_h) = (\tilde{q}_h, q_h)_{\Delta_i} \quad \forall\, q_h \in \tilde{Q}_h^2. \tag{32}$$

In view of (31), it suffices to fulfil

$$b_h(\boldsymbol{v}_h^i, \varphi_j^2) = (\tilde{q}_h, \varphi_j^2)_{\Delta_i}, \qquad j = 1, \dots, n, \tag{33}$$

$$b_h(\boldsymbol{v}_h^i, \varphi_j\varphi_{j+1}) = (\tilde{q}_h, \varphi_j\varphi_{j+1})_{\Delta_i}, \quad j = 1, \dots, n. \tag{34}$$

Applying (13), (27), (20) and (22), we obtain for any $j \in \{1, \dots, n\}$

$$b_h(\overline{\boldsymbol{v}}_h^i, \varphi_j\varphi_{j+1}) = \Big(2A - \frac{1}{15}\Big)|K_j|\Big(\frac{\alpha_j}{h_{E_j}} + \frac{\alpha_{j+1}}{h_{E_{j+1}}}\Big) + \frac{1}{2}A(\gamma_{j+1}h_{E_{j+1}} - \gamma_j h_{E_j}).$$

Thus, setting

$$\gamma_j = -\frac{(60A-2)|K_{j-1}|}{15Ah_{E_j}^2}\alpha_j, \quad j = 1, \dots, n,$$

we get

$$b_h(\overline{\boldsymbol{v}}_h^i, \varphi_j\varphi_{j+1}) = \Big(2A - \frac{1}{15}\Big)(|K_{j-1}| + |K_j|)\frac{\alpha_j}{h_{E_j}}, \quad j = 1, \dots, n.$$

Then α_j, $j = 1, \dots, n$, are uniquely determined by (34). Since, due to (13), (27) and (22),

$$b_h(\overline{\boldsymbol{v}}_h^i, \varphi_j^2) = -2A(|K_{j-1}| + |K_j|)\frac{\beta_j}{h_{E_j}} + b_h(\alpha_j\tilde{\psi}_{E_j}\boldsymbol{t}_{E_j} + \gamma_j\chi_{E_j}\boldsymbol{n}_{E_j}, \varphi_j^2),$$

there are uniquely determined constants β_j, $j = 1, \dots, n$, such that (33) holds. Using (17), (29) and (30), it is easy to show that $|\overline{\boldsymbol{v}}_h^i|_{1,h} \leqslant C\|\tilde{q}_h\|_{0,\Delta_i}$ with C depending only on σ and $\hat{b}_1$. Finally, setting $q_h = \tilde{q}_h$ in (32), we obtain (25). $\square$

Corollary 2. *There exists a constant $\beta > 0$ depending only on σ, $\hat{b}_1$ and Ω such that*

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h^{\mathrm{mod},2}\setminus\{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geqslant \beta\|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h^2.$$

P r o o f. The corollary again immediately follows from Theorem 1 and Lemma 3. $\square$

7. Inf-sup condition for the P_3^{mod} element

In this section we finally prove that the inf-sup condition (5) holds also for the spaces $\mathbf{V}_h^{\mathrm{mod},3} \equiv [V_h^{\mathrm{mod},3}]^2$ and Q_h^3. We shall proceed in a similar way as above but the proof will meet with more difficulties.

It is convenient to replace the basis functions ψ_E, χ_E, ϱ_E, ζ_E and π_K introduced in Sect. 3 by functions having some special properties. Consider any inner edge $E \in \mathcal{E}_h^i$ and let K, $\tilde{K}$ be the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\tilde{K}$. Then, using the constants A and D from (19), we define

$$
\begin{aligned}
\pi_E &= \begin{cases} \pi_K & \text{in } K, \\ \pi_{\tilde{K}} & \text{in } \tilde{K}, \\ 0 & \text{in } \Omega \setminus \{K \cup \tilde{K}\}, \end{cases} \\
\overline{\psi}_E &= \psi_E + (180\,A - 10)\varrho_E + (30 - 900\,A)\pi_E, \\
\bar{\chi}_E &= \chi_E - 180\,A\zeta_E, \\
\bar{\varrho}_E &= \varrho_E - 5\pi_E, \\
\bar{\zeta}_E &= \bar{\chi}_E - 180\,D\zeta_E, \\
\xi_E &= \psi_E - 6\varrho_E + 10\,\pi_E, \\
\bar{\pi}_E &= \frac{\pi_K}{|K|} - \frac{\pi_{\tilde{K}}}{|\tilde{K}|}.
\end{aligned}
$$

Some properties of these functions are summarized in the next two lemmas which can be easily proved using the relations from Sect. 3 and the standard relations for barycentric coordinates (cf. Ciarlet [5]).

Lemma 4. *Consider any $E \in \mathcal{E}_h^i$ and let $K \in \mathcal{T}_h$ be an element adjacent to E. Let λ_1 and λ_2 be the barycentric coordinates on K with respect to $x_{E,1}$ and $x_{E,2}$, respectively. Further, let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ be any functions satisfying $f(x,y) = -f(y,x)$ and $g(x,y) = g(y,x)$ for all $x, y \in \mathbb{R}$. Then*

$$\int_K \overline{\psi}_E f(\lambda_1,\lambda_2)\,\mathrm{d}x = \int_K \bar{\varrho}_E f(\lambda_1,\lambda_2)\,\mathrm{d}x = 0, \tag{35}$$

$$\int_K \bar{\pi}_E f(\lambda_1,\lambda_2)\,\mathrm{d}x = \int_K \xi_E f(\lambda_1,\lambda_2)\,\mathrm{d}x = 0, \tag{36}$$

$$\int_K \bar{\chi}_E g(\lambda_1,\lambda_2)\,\mathrm{d}x = \int_K \bar{\zeta}_E g(\lambda_1,\lambda_2)\,\mathrm{d}x = 0. \tag{37}$$

R e m a r k 3. Let λ_1, λ_2 be as in Lemma 4 and let λ_3 be the remaining barycentric coordinate on K. Then $\lambda_3 = g(\lambda_1,\lambda_2)$ with $g(x,y) = 1 - x - y$.

Lemma 5. *Consider any $E \in \mathcal{E}_h^i$ and let $K \in \mathcal{T}_h$ be an element adjacent to E. Let λ_1 and λ_2 be the barycentric coordinates on K with respect to $x_{E,1}$, $x_{E,2}$, respectively, and let λ_3 be the remaining barycentric coordinate on K. Then*

$$\int_K \bar{\varrho}_E \,\mathrm{d}x = \int_K \bar{\zeta}_E \,\mathrm{d}x = \int_K \xi_E \,\mathrm{d}x = \int_E \xi_E \,\mathrm{d}\sigma = 0, \tag{38}$$

$$\int_K \overline{\psi}_E q \,\mathrm{d}x = \int_K \bar{\chi}_E q \,\mathrm{d}x = 0 \quad \forall\, q \in P_1(K),\ \ K \in \mathcal{T}_h, \tag{39}$$

$$\int_K \bar{\varrho}_E \lambda_3 \,\mathrm{d}x = -\frac{|K|}{90}, \quad \int_K \bar{\zeta}_E \lambda_2 \,\mathrm{d}x = D|K|, \quad \int_K \bar{\zeta}_E \lambda_2^2 \,\mathrm{d}x = D|K|, \tag{40}$$

$$\int_K \overline{\psi}_E \lambda_2^2 \,\mathrm{d}x = \Big(\frac{1}{210} + \frac{D}{7}\Big)|K|, \quad \int_K \bar{\chi}_E \lambda_2^2 \,\mathrm{d}x = \frac{D}{7}|K|, \tag{41}$$

$$\int_K \overline{\psi}_E \lambda_3^2 \,\mathrm{d}x = -2\int_K \overline{\psi}_E \lambda_2\lambda_3 \,\mathrm{d}x = \Big(-\frac{11}{630} - \frac{8}{7}A + 2B\Big)|K|, \tag{42}$$

$$\int_K \bar{\chi}_E \lambda_1\lambda_3 \,\mathrm{d}x = -\int_K \bar{\chi}_E \lambda_2\lambda_3 \,\mathrm{d}x = \frac{D}{7}|K|, \quad \int_K \pi_K \lambda_2\lambda_3 \,\mathrm{d}x = \frac{|K|}{630}, \tag{43}$$

$$\int_K \bar{\zeta}_E \lambda_1\lambda_2 \,\mathrm{d}x = \int_K \bar{\zeta}_E \lambda_1\lambda_3 \,\mathrm{d}x = \int_K \bar{\zeta}_E \lambda_2\lambda_3 \,\mathrm{d}x = 0, \tag{44}$$

$$\int_K \xi_E (\lambda_1 - \lambda_3)\lambda_2 \,\mathrm{d}x = 3(B - A)|K|, \tag{45}$$

$$\int_K \pi_K \lambda_i \,\mathrm{d}x = \frac{|K|}{180}, \quad \int_K \pi_K \lambda_i^2 \,\mathrm{d}x = \frac{|K|}{420}, \quad i = 1, 2, 3. \tag{46}$$

Further, we shall need the following simple result.

Lemma 6. *Let $n \in \mathbb{N}$ be any positive integer and $f_1, \ldots, f_n \in \mathbb{R}$ any real numbers. Let us consider the linear system*

$$x_j + x_{j+1} = f_j, \quad j = 1, \ldots, n, \tag{47}$$

with $x_{n+1} \equiv x_1$. If n is even, we assume that

$$\sum_{j=1}^{n} (-1)^j f_j = 0, \tag{48}$$

which is a necessary condition for the solvability of (47). Then the linear system (47) has a solution which satisfies

$$x_j = (-1)^{j+1} x_1 + \sum_{k=1}^{j-1} (-1)^{j+k+1} f_k, \quad j = 2, \ldots, n.$$

If n is odd, then x_1 is uniquely determined by

$$x_1 = \frac{1}{2} \sum_{j=1}^{n} (-1)^{j+1} f_j.$$

If n is even, then x_1 can be chosen arbitrarily.

Proof. The proof is straightforward. $\square$

The following lemma shows that the assumptions of Theorem 1 hold true for $m = n = 3$.

Lemma 7. *For any $\tilde{q}_h \in \tilde{Q}_h^3$ and any $i \in \{1, \ldots, N_h\}$ there exists $\boldsymbol{v}_h^i \in \mathbf{V}_h^{i,3}$ satisfying* (24)–(26)*, where the constant $\overline{C}$ depends only on σ and $\hat{b}_1$.*

Proof. Consider any $\tilde{q}_h \in \tilde{Q}_h^3$ and any $i \in \{1, \ldots, N_h\}$. Then, according to the proof of Lemma 3, there exists a function $\tilde{\boldsymbol{v}}_h^i \in \mathbf{V}_h^{i,3}$ satisfying (24), (26) and (32). Let us denote

$$\overline{\boldsymbol{v}}_h^i = \sum_{k=1}^{n} \{(\alpha_k \overline{\psi}_{E_k} + \beta_k \bar{\chi}_{E_k} + \gamma_k \bar{\varrho}_{E_k} + \delta_k \bar{\bar{\zeta}}_{E_k} + \varepsilon_k \overline{\pi}_{E_k}) \boldsymbol{t}_{E_k} + (\kappa_k \xi_{E_k} + \mu_k \bar{\zeta}_{Ek}) \boldsymbol{n}_{E_k}\}$$

with constants α_k, β_k, γ_k, δ_k, ε_k, κ_k, μ_k, $k = 1, \ldots, n$, to be determined later. Then $\overline{\boldsymbol{v}}_h^i$ belongs to $\mathbf{V}_h^{i,3}$ and satisfies (24). Moreover, due to (13), (27), (38) and (39), it is easy to verify that

$$b_h(\overline{\boldsymbol{v}}_h^i, q_h) = 0 \quad \forall\, q_h \in Q_h^1.$$

Our aim is to choose the above introduced constants in such a way that the function $\boldsymbol{v}_h^i \equiv \tilde{\boldsymbol{v}}_h^i + \overline{\boldsymbol{v}}_h^i$ satisfies

$$b_h(\boldsymbol{v}_h^i, q_h) = (\tilde{q}_h, q_h)_{\Delta_i} \quad \forall\, q_h \in \tilde{Q}_h^3. \tag{49}$$

To this end, it is sufficient to fulfil

$$b_h(\overline{\boldsymbol{v}}_h^i, \varphi_j \varphi_{j+1}) = 0, \tag{50}$$

$$b_h(\overline{\boldsymbol{v}}_h^i, \varphi_j^2) = 0, \tag{51}$$

$$b_h(\overline{\boldsymbol{v}}_h^i, 2\varphi_j^3 - 3\varphi_j^2) = r_h(\tilde{\boldsymbol{v}}_h^i, 2\varphi_j^3), \tag{52}$$

$$b_h(\overline{\boldsymbol{v}}_h^i, \varphi_j \varphi_{j+1}(\varphi_j - \varphi_{j+1})) = r_h(\tilde{\boldsymbol{v}}_h^i, \varphi_j \varphi_{j+1}(\varphi_j - \varphi_{j+1})), \tag{53}$$

$$b_h(\overline{\boldsymbol{v}}_h^i, \pi_{K_j}) = r_h(\tilde{\boldsymbol{v}}_h^i, \pi_{K_j}), \tag{54}$$

where $j = 1, \ldots, n$ and

$$r_h(\boldsymbol{v}, q) = (\tilde{q}_h, q)_{\Delta_i} - b_h(\boldsymbol{v}, q).$$

Let φ_i be the continuous piecewise linear basis function equal to 1 at the vertex x_i and vanishing at all other vertices. Then

$$\nabla \pi_{K_j}|_{K_j} = (\varphi_i - \varphi_j)\varphi_{j+1}\nabla\varphi_j + (\varphi_i - \varphi_{j+1})\varphi_j \nabla\varphi_{j+1}.$$

Therefore, we obtain using (13), (27), (35), (36) and (43)–(45)

$$b_h(\bar{\boldsymbol{v}}_h^i, \pi_{K_j}) = \frac{3}{2}(A-B)(\kappa_{j+1}h_{E_{j+1}} - \kappa_j h_{E_j}) + \frac{2D}{7}|K_j|\Big(\frac{\beta_j}{h_{E_j}} + \frac{\beta_{j+1}}{h_{E_{j+1}}}\Big),$$

which shows that we can fulfil (54) using Lemma 6. First we choose the constants κ_j to satisfy (48) if n is even. This can be done in many ways and we set

$$s_j = \sum_{k=1}^{j}(-1)^k \frac{r_h(\tilde{\boldsymbol{v}}_h^i, \pi_{K_k})}{|K_k|}, \quad j = 1,\dots,n,$$
$$\kappa_1 = \frac{2|K_1|\,|K_n|}{3(A-B)h_{E_1}(|K_1|+|K_n|)}\, s_n, \quad \kappa_j = 0, \;\; j = 2,\dots,n.$$

Now, denoting

$$\overline{\beta}_1 = -\frac{7h_{E_1}}{2D}\frac{|K_1|}{|K_1|+|K_n|}\, s_n, \quad \overline{\beta}_j = (-1)^j \frac{7h_{E_j}}{2D}(s_n - s_{j-1}), \quad j = 2,\dots,n,$$

the equation (54) holds if

$$\beta_j = (-1)^j \theta h_{E_j} + \overline{\beta}_j, \quad j = 1,\dots,n, \tag{55}$$

where $\theta = 0$ if n is odd. If n is even, then the value of θ can be chosen arbitrarily and will be fixed later.

In what follows, we use notation of the type

$$\bar{\boldsymbol{v}}_h^{i,\kappa} = \sum_{k=1}^{n} \kappa_k \xi_{E_k} \boldsymbol{n}_{E_k}, \quad \bar{\boldsymbol{v}}_h^{i,\gamma} = \sum_{k=1}^{n} \gamma_k \bar{\varrho}_{E_k} \boldsymbol{t}_{E_k}, \quad \bar{\boldsymbol{v}}_h^{i,\overline{\beta}} = \sum_{k=1}^{n} \overline{\beta}_k \bar{\chi}_{E_k} \boldsymbol{t}_{E_k}$$

and set $\bar{\boldsymbol{v}}_h^{i,\gamma+\kappa} = \bar{\boldsymbol{v}}_h^{i,\gamma} + \bar{\boldsymbol{v}}_h^{i,\kappa}$, $\bar{\boldsymbol{v}}_h^{i,\overline{\beta}+\gamma+\kappa} = \bar{\boldsymbol{v}}_h^{i,\overline{\beta}} + \bar{\boldsymbol{v}}_h^{i,\gamma} + \bar{\boldsymbol{v}}_h^{i,\kappa}$, etc. Then we derive using (13), (27), (37), (39), (40) and (46)

$$\begin{aligned} b_h(\bar{\boldsymbol{v}}_h^i - \bar{\boldsymbol{v}}_h^{i,\kappa}, \varphi_j\varphi_{j+1}) = & -\frac{|K_j|}{90}\Big(\frac{\gamma_j}{h_{E_j}} + \frac{\gamma_{j+1}}{h_{E_{j+1}}}\Big) + \frac{1}{180}\Big(\frac{\varepsilon_{j+1}}{h_{E_{j+1}}} - \frac{\varepsilon_j}{h_{E_j}}\Big) \\ & + \frac{D}{2}(\mu_j h_{E_j} - \mu_{j+1}h_{E_{j+1}}). \end{aligned}$$

Thus, to fulfil (50), we can set

$$\gamma_j = \frac{90\, h_{E_j}}{|K_{j-1}| + |K_j|} b_h(\overline{\boldsymbol{v}}_h^{i,\kappa}, \varphi_j\varphi_{j+1}), \quad j = 1, \dots, n, \tag{56}$$

$$\varepsilon_j - 90\, D h_{E_j}^2 \mu_j = 2|K_{j-1}|\gamma_j, \quad j = 1, \dots, n. \tag{57}$$

Further, due to (37), (39), (40), (41) and (46) we have

$$\begin{aligned} b_h(\overline{\boldsymbol{v}}_h^i - \overline{\boldsymbol{v}}_h^{i,\gamma+\kappa}, \varphi_j^2) &= 2D(|K_{j-1}| + |K_j|)\frac{\delta_j}{h_{E_j}} \\ &\quad + D\boldsymbol{n}_{E_j} \cdot (h_{E_{j-1}}\boldsymbol{n}_{E_{j-1}} - h_{E_{j+1}}\boldsymbol{n}_{E_{j+1}})\mu_j, \end{aligned} \tag{58}$$

$$\begin{aligned} b_h(\overline{\boldsymbol{v}}_h^i - \overline{\boldsymbol{v}}_h^{i,\gamma+\kappa}, \varphi_j^3) &= 3\frac{|K_{j-1}| + |K_j|}{h_{E_j}}\Big[\Big(\frac{1}{210} + \frac{D}{7}\Big)\alpha_j + \frac{D}{7}\beta_j + D\delta_j\Big] \\ &\quad + \frac{3D}{2}\boldsymbol{n}_{E_j} \cdot (h_{E_{j-1}}\boldsymbol{n}_{E_{j-1}} - h_{E_{j+1}}\boldsymbol{n}_{E_{j+1}})\mu_j. \end{aligned} \tag{59}$$

Therefore,

$$b_h(\overline{\boldsymbol{v}}_h^i - \overline{\boldsymbol{v}}_h^{i,\gamma+\kappa}, 2\varphi_j^3 - 3\varphi_j^2) = 6\frac{|K_{j-1}| + |K_j|}{7h_{E_j}}\Big[\Big(\frac{1}{30} + D\Big)\alpha_j + D\beta_j\Big].$$

Denoting

$$\overline{\alpha}_j = -\frac{30\, D}{1 + 30\, D}\overline{\beta}_j + \frac{35\, h_{E_j}}{(1 + 30\, D)(|K_{j-1}| + |K_j|)}\, [r_h(\tilde{\boldsymbol{v}}_h^i, 2\varphi_j^3) - b_h(\overline{\boldsymbol{v}}_h^{i,\gamma+\kappa}, 2\varphi_j^3 - 3\varphi_j^2)],$$

we see that (52) holds if

$$\alpha_j = (-1)^{j+1}\theta\frac{30\, D}{1 + 30\, D}\, h_{E_j} + \overline{\alpha}_j, \quad j = 1, \dots, n. \tag{60}$$

Finally, applying (37), (40), (42), (43), (44) and (46) we derive

$$\begin{aligned} & b_h(\overline{\boldsymbol{v}}_h^i - \overline{\boldsymbol{v}}_h^{i,\gamma+\kappa}, \varphi_j\varphi_{j+1}(\varphi_j - \varphi_{j+1})) \\ &\quad = \frac{D}{2}(\mu_j h_{E_j} + \mu_{j+1}h_{E_{j+1}}) - \frac{1}{1260}\Big(\frac{\varepsilon_j}{h_{E_j}} + \frac{\varepsilon_{j+1}}{h_{E_{j+1}}}\Big) \\ &\quad\quad + |K_j|\Big[\Big(\frac{11}{315} + \frac{16\, A - 28\, B}{7}\Big)\Big(\frac{\alpha_j}{h_{E_j}} - \frac{\alpha_{j+1}}{h_{E_{j+1}}}\Big) - \frac{2D}{7}\Big(\frac{\beta_j}{h_{E_j}} - \frac{\beta_{j+1}}{h_{E_{j+1}}}\Big)\Big]. \end{aligned}$$

This implies in view of (53), (55), (57) and (60) that

$$\begin{aligned} \frac{3D}{7}(\mu_j h_{E_j} + \mu_{j+1}h_{E_{j+1}}) &= (-1)^j\theta\frac{8|K_j|D(1 + 90\, A - 135\, B)}{3 + 90\, D} \\ &\quad + r_h(\tilde{\boldsymbol{v}}_h^i + \overline{\boldsymbol{v}}_h^{i,\overline{\alpha}+\overline{\beta}+\gamma+\kappa}, \varphi_j\varphi_{j+1}(\varphi_j - \varphi_{j+1})) \\ &\quad + \frac{1}{630}\Big(\frac{\gamma_j|K_{j-1}|}{h_{E_j}} + \frac{\gamma_{j+1}|K_j|}{h_{E_{j+1}}}\Big). \end{aligned} \tag{61}$$

If n is odd, then $\theta = 0$ and the constants μ_j, $j = 1, \ldots, n$, are uniquely determined according to Lemma 6. If n is even, we define θ in such a way that the right-hand sides of (61) satisfy (48). Then the constants μ_j satisfying (61) can be again computed using Lemma 6. The values of θ and μ_j, $j = 1, \ldots, n$, now determine the constants α_j, β_j and ε_j, $j = 1, \ldots, n$, by (60), (55) and (57), respectively. Finally, (51) and (58) uniquely determine δ_j, $j = 1, \ldots, n$.

Thus, we have defined all constants in the definition of $\overline{\boldsymbol{v}}_h^i$ in such a way that (50)–(54) hold for $j = 1, \ldots, n$. Consequently, the function $\boldsymbol{v}_h^i$ satisfies (49) and if we set $q_h = \tilde{q}_h$ in (49), we obtain (25). Using (17), (29) and (30), we can easily verify that $\boldsymbol{v}_h^i$ also satisfies (26), which completes the proof. □

Corollary 3. *There exists a constant $\beta > 0$ depending only on σ, $\hat{b}_1$ and Ω such that*

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h^{\mathrm{mod},3} \setminus \{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geqslant \beta \|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h^3.$$

Proof. The corollary again follows directly from Theorem 1 and Lemma 7. □

Remark 4. We conjecture that the validity of the assumptions of Theorem 1 can be also proved for $m = n > 3$ using the same techniques as above. However, due to the increasing complexity of this proceeding for $m = n > 3$, another type of the proof should be developed to show the stability of the P_n^{mod}/P_n element for $n > 3$. This will be a subject of our further research.

8. Conclusions

In this paper we have investigated the triangular nonconforming P_n^{mod} elements which were recently introduced to enhance the accuracy of discrete solutions to convection dominated problems. Applying the macroelement technique, we have proved (for $n \leqslant 3$) that these finite elements are well suited for approximating the velocity in incompressible flow problems since they satisfy the inf-sup condition of Babuška and Brezzi for pressures approximated by continuous piecewise polynomial functions of degree n. This in particular enables us to apply the P_n^{mod}/P_n element to the numerical solution of convection dominated incompressible flow problems. Our preliminary numerical results for the incompressible Navier-Sokes equations are very promising and indicate that the P_n^{mod}/P_n element leads to efficient and accurate procedures for the numerical solution of incompressible flow problems.

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