On Stabilization of Linear Systems With Limited Information

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***Abstract*—We consider the problem of stabilizing a linear time-invariant system using sampled encoded measurements of its state or output. We derive a relationship between the number of values taken by the encoder and the norm of the transition matrix of the open-loop system over one sampling period, which guarantees that global asymptotic stabilization can be achieved. A coding scheme and a stabilizing control strategy are described explicitly.**

***Index Terms*—Asymptotic stabilization, coding, limited information, linear system, sampling.**

I. Introduction

Suppose that we are given a stabilizable linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n. \tag{1}$$

In this note we study the problem of designing a controller that asymptotically stabilizes the system (1) using limited information about its state x. This problem arises, for example, when the state measurements are to be passed to the controller via a limited capacity communication channel. We specify what we mean by limited information as follows.

Sampling: The measurements are to be received by the controller at discrete times $0, \tau, 2\tau, \ldots$, where $\tau > 0$ is a fixed *sampling period*.

Encoding: At each of the aforementioned times, the measurement received by the controller must be a number in the set $\{1, 2, \ldots, N\}$, where N is a fixed positive integer.

In other words, the data available to the controller consists of the stream of integers

$$q_0(x(0)), q_1(x(\tau)), q_2(x(2\tau)), \ldots$$

where $q_k(\cdot) : \mathbb{R}^n \to \{1, 2, \ldots, N\}$ is, for each k, some *encoding function*. For different values of k we can use different encoding functions. As we will see, it is natural to use the previous values $q_i(x(i\tau))$, $i = 0, \ldots, k-1$ to define the function $q_k(\cdot)$. We assume that the controller knows the initial encoding function $q_0(\cdot)$ as well as the rule that defines $q_k(\cdot)$ on the basis of the previously received encoded measurements, so that for each k the function q_k is known to the controller. In other words, there is a communication protocol satisfying the above constraints upon which the process (encoder) and the controller (decoder) agree in advance.

Instead of sending to the controller the sampled and encoded measurements of the entire state x, we can work with an output $y = Cx \in \mathbb{R}^p$, $p \leq n$. In this case, the data available to the controller will consist of the stream of integers

$$q_0(y(0)), q_1(y(\tau)), q_2(y(2\tau)), \ldots$$

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Here the matrix C can be viewed either as a design choice or as a given constraint in the problem. Of course, we need to ensure that the output contains enough information for the controller to be able to stabilize the system. Since the system (1) is assumed to be stabilizable by state feedback—but not assumed to be stabilizable by output feedback for any particular output—a reasonable requirement to impose in this regard is that (C, A) be an observable pair. Transmitting fewer variables over the limited capacity communication channel, one reduces the errors introduced by the encoder. The price to pay, however, is that the decoder needs to recover state information. The resulting overall state estimation error is a product of the output encoding error and a quantity that characterizes observability of the system.

To summarize, we are given system (1) and three numbers: a positive real number τ (the sampling period), a positive integer N (the number of values of each encoding function), and a positive integer $p \leq n$ (the dimension of the transmitted output). The problem under consideration is to choose a communication protocol of the kind previously described and a controller so that the closed-loop system is globally asymptotically stable. Our main goal is to derive precise relationships between τ, N, and p which guarantee that this task can be accomplished. Solutions in situations where some of these quantities are fixed, while others need to be minimized or maximized, are then easy to obtain.

To solve the aforementioned problem, we build on ideas from the work on quantized feedback stabilization reported in [1] and [2] (the latter reference essentially contains in implicit form some of the results given here). Recent references that describe related developments (although in settings different from ours) include [3]–[8] and the articles in [9]. Loosely speaking, we will show that if the amount by which the open-loop system can expand during one sampling period is not too large compared to N, then it is possible to obtain an asymptotically correct estimate of x and use it to stabilize the system. More precisely, we will give a constructive proof of the following statement (the notation is clarified at the beginning of Section II).

Theorem 1: In the state encoding case, global asymptotic stabilization is possible if $N \geq 2^n$ and

$$\max_{0 \leq t \leq \tau} \|e^{At}\|_\infty < \lfloor \sqrt[n]{N} \rfloor. \tag{2}$$

In the output encoding case, global asymptotic stabilization is possible if $N \geq 2^p$ and

$$\|W^\dagger\|_\infty \|C\|_\infty \max_{0 \leq t \leq \tau} \|e^{At}\|_\infty^{2\eta - 1} < \lfloor \sqrt[p]{N} \rfloor \tag{3}$$

where η is the observability index of the pair $(C, e^{A\tau})$ and $W^\dagger$ is a left inverse of the matrix

$$W := \begin{pmatrix} C \\ Ce^{A\tau} \\ \vdots \\ Ce^{A(\eta-1)\tau} \end{pmatrix}. \tag{4}$$

Note that in this note we are only concerned with sufficient conditions for stabilization. The paper [3], which studies asymptotic stabilization of discrete-time linear systems with encoded state measurements and a known bound on the initial state, shows that a similar condition is necessary and sufficient for stabilization in that context (provided that the system matrix A is diagonalizable).

II. Semiglobal Asymptotic Stabilization

In what follows, we find it convenient to use the norm $\|x\|_\infty := \max\{|x_i| : 1 \le i \le n\}$ on $\mathbb{R}^n$ and the induced matrix norm $\|A\|_\infty := \max\{\sum_{j=1}^n |A_{ij}| : 1 \le i \le n\}$ on $\mathbb{R}^{n\times n}$. We define

$$\Lambda := \max_{0\le t\le\tau} \|e^{At}\|_\infty \ge 1. \tag{5}$$

The largest integer smaller than or equal to a given number z is denoted by $\lfloor z \rfloor$. We let $B_\infty^n(x_0, r)$ denote the square box in $\mathbb{R}^n$ centered at x_0 with edges $2r$, i.e.,

$$B_\infty^n(x_0, r) := \{x \in \mathbb{R}^n : \|x - x_0\|_\infty \le r\}.$$

Assume for the moment that an upper bound on the size of the initial state is known. Namely, let us assume that for some known constant $E_0 > 0$ we have

$$\|x(0)\|_\infty \le E_0. \tag{6}$$

Such a bound may be given to us in advance or may be obtained on the basis of prior measurements (see Section III). The purpose of this section is to describe a coding scheme and a dynamic feedback control law that achieve asymptotic stabilization for this situation.

A. State Encoding

We begin by considering the case when sampled encoded measurements of the entire state x are transmitted. The inequality (6) means that the state of the system at time $t = 0$ lies in $B_\infty^n(0, E_0)$. Assume for notational convenience that $\sqrt[n]{N}$ is an integer, so that $\lfloor \sqrt[n]{N} \rfloor = \sqrt[n]{N}$. (Otherwise, replace N by the largest integer $N' \le N$ such that $\sqrt[n]{N'}$ is an integer.) We also require that $\sqrt[n]{N} \ge 2$. Let us define the encoding function q_0 as follows: divide $B_\infty^n(0, E_0)$ into N equal square boxes, numbered from 1 to N in some specific way, and let $q_0(x)$ be the number of the box that contains x. In case x lies on the boundary of several boxes, the value $q_0(x)$ can be chosen arbitrarily among the candidates.

We have thus singled out a square box with edges at most $2E_0/\sqrt[n]{N}$ which contains $x(0)$. Denoting the center of this box by $\hat{x}(0)$, we obtain

$$\|x(0) - \hat{x}(0)\|_\infty \le \frac{E_0}{\sqrt[n]{N}}. \tag{7}$$

For $t \in [0, \tau)$, let

$$u(t) = K\hat{x}(t) \tag{8}$$

where

$$\hat{x}(t) := e^{(A+BK)t}\hat{x}(0)$$

and K is chosen so that the eigenvalues of $A + BK$ have negative real parts. From the equations $\dot{\hat{x}} = A\hat{x} + Bu$ and $\dot{x} = Ax + Bu$, and (5) and (7), we conclude that

$$\|x(t) - \hat{x}(t)\|_\infty \le \Lambda\|x(0) - \hat{x}(0)\|_\infty \le \frac{\Lambda E_0}{\sqrt[n]{N}}, \qquad 0 \le t < \tau.$$

This means that for $0 \le t < \tau$, the state $x(t)$ belongs to $B_\infty^n(\hat{x}(t), \Lambda E_0/\sqrt[n]{N})$.

Let $\hat{x}(\tau^-) := \lim_{t\to\tau^-} \hat{x}(t)$. At the time τ we divide $B_\infty^n(\hat{x}(\tau^-), \Lambda E_0/\sqrt[n]{N})$ into N equal square boxes and let $q_1(x)$ be the number of the box that contains x. Denoting the center of this box by $\hat{x}(\tau)$, we have

$$\|x(\tau) - \hat{x}(\tau)\|_\infty \le \frac{\Lambda E_0}{(\sqrt[n]{N})^2}.$$

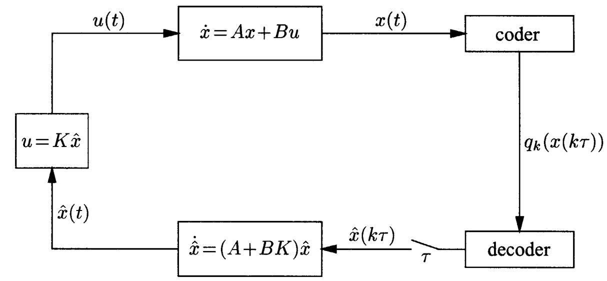

Fig. 1. Closed-loop system.

For $t \in [\tau, 2\tau)$, define the control by (8), where

$$\hat{x}(t) := e^{(A+BK)(t-\tau)}\hat{x}(\tau).$$

We have

$$\|x(t) - \hat{x}(t)\|_\infty \le \Lambda\|x(\tau) - \hat{x}(\tau)\|_\infty \le \frac{\Lambda^2 E_0}{(\sqrt[n]{N})^2} \qquad \tau \le t < 2\tau.$$

Continuing this process, we obtain a system that can be represented by the block diagram in Fig. 1. We see that the upper bound on $\|x(t) - \hat{x}(t)\|_\infty$ is divided by $\sqrt[n]{N}$ at the times $\tau, 2\tau, \ldots$ and grows by a factor of Λ on every interval between these times. This clearly implies that if $\Lambda < \sqrt[n]{N}$, which is equivalent to (2) since $\sqrt[n]{N}$ is taken to be an integer, then $\|x(t) - \hat{x}(t)\|_\infty$ converges to 0 as $t \to \infty$. We assume from now on that (2) holds. The closed-loop system can, thus, be written as

$$\dot{x} = (A + BK)x + e \tag{9}$$

where $e := BK(\hat{x} - x) \to 0$. It follows at once that $x(t) \to 0$ as $t \to 0$.

Having established asymptotic convergence to the origin, we only need to show stability in the sense of Lyapunov. In the following argument, we assume that $\sqrt{[n]N}$ is odd so that the equilibrium at the origin is preserved; to have Lyapunov stability when $\sqrt{[n]N}$ is even, a slight modification to the above strategy is needed. Let $V(x) = x^T Px$ be a quadratic Lyapunov function for the system $\dot{x} = (A + BK)x$, and denote by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the smallest and the largest eigenvalue of P, respectively. Take an arbitrary $\varepsilon > 0$. It is straightforward to show that there exists a $\gamma > 0$ such that solutions of the system (9) starting in the region

$$\mathcal{R} := \{x : V(x) \le \varepsilon^2\lambda_{\min}(P)\}$$

remain in this region as long as $\|e\|_\infty \le \gamma$. Choose a sufficiently large integer $k \ge 0$ such that

$$\|BK\|_\infty E_0 \left(\frac{\Lambda}{\sqrt[n]{N}}\right)^{k+1} \le \gamma.$$

Then, our previous analysis implies that $\|e(t)\|_\infty \le \gamma$ for all $t \ge k\tau$. Now, choose a sufficiently small $\delta > 0$ such that

$$\Lambda^k\delta < \min\left\{\frac{E_0\Lambda^{k-1}}{(\sqrt[n]{N})^k}, \frac{\varepsilon}{\sqrt{n}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\right\}.$$

This inequality ensures that if

$$\|x(0)\|_\infty \le \delta \tag{10}$$

then $\hat{x}(t) \equiv 0$ on the time interval $[0, k\tau)$ and

$$\|x(t)\|_\infty < \frac{\varepsilon}{\sqrt{n}}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$$

on the same time interval. This implies that $V(x(t)) \le \varepsilon^2 \lambda_{\min}(P)$ for all $t \in [0, k\tau]$. In light of the analysis given before for $t \ge k\tau$, we conclude that $\mathcal{R}$ is an invariant region for the system (9) with initial conditions satisfying (10). It remains to notice that $\mathcal{R}$ is contained in the ball $\{x : |x| \le \varepsilon\}$. We proved the following result.

Proposition 2: If the inequalities (2) and $N \ge 2^n$ hold, then the previous state coding/feedback strategy makes the origin an asymptotically stable equilibrium of the closed-loop system, with a region of attraction containing all initial conditions that satisfy the bound (6).

The inequality (2) characterizes the tradeoff between the amount of information provided by the encoder at each sampling time and the required sampling frequency. This relationship depends explicitly on a measure of instability of the open-loop system, expressed by Λ which is defined via (5). We see, for instance, that if τ is given, then N needs to be sufficiently large for asymptotic stabilization to be possible.

Remark 1: It is not hard to see from the above proof that the state of the system actually converges to zero exponentially fast. This follows from the fact that the evolution of x is described by the system (9), in which the autonomous part is exponentially stable and the rate of convergence of the error signal e to zero is exponential.

B. Output Encoding

We now turn to the case when sampled encoded measurements of the output $y = Cx$ are transmitted, where C is some $p \times n$ matrix such that the pair (C, A) is observable. There is no loss of generality in assuming that the pair $(C, e^{A\tau})$ is also observable (see, e.g, [10, Ch. 6]). Denote by η the corresponding observability index (the largest integer between 1 and n for which the matrix (4) has rank n).

Assume again that the initial state satisfies the bound (6). This implies that $\|y(0)\|_\infty \le \|C\|_\infty E_0$, i.e., $y(0) \in B_\infty^p(0, \|C\|_\infty E_0)$. For $t \in [0, \eta\tau)$, let $u(t) \equiv 0$. Then, we have

$$y(k\tau) \in B_\infty^p(0, \|C\|_\infty \Lambda^k E_0), \qquad k = 0, 1, \ldots, \eta - 1$$

where Λ is defined by (5) as before. Let us suppose for convenience that $\sqrt[p]{N}$ is an integer, which is required to be greater than or equal to 2 (cf. the remarks at the beginning of Section II-A). For each $k \in \{0, 1, \ldots, \eta - 1\}$, define $q_k(y(k\tau))$ in the following way: divide $B_\infty^p(0, \|C\|_\infty \Lambda^k E_0)$ into N equal square boxes, and let $q_k(y(k\tau))$ be the number of the box that contains y. Denote the center of this box by $\hat{y}(k\tau)$. We have

$$\|y(k\tau) - \hat{y}(k\tau)\|_\infty \le \frac{\|C\|_\infty \Lambda^k E_0}{\sqrt[p]{N}}, \qquad k = 0, 1, \ldots, \eta - 1. \quad (11)$$

We know that

$$x(0) = W^\dagger \begin{pmatrix} y(0) \\ \vdots \\ y((\eta - 1)\tau) \end{pmatrix}$$

where $W^\dagger$ is a left inverse[1] of the matrix W defined by (4). Let

$$\hat{x}(\eta\tau) := e^{A\eta\tau} W^\dagger \begin{pmatrix} \hat{y}(0) \\ \vdots \\ \hat{y}((\eta - 1)\tau) \end{pmatrix}.$$

In view of (11) and the equality $x(\eta\tau) = e^{A\eta\tau} x(0)$, we obtain

$$\|x(\eta\tau) - \hat{x}(\eta\tau)\|_\infty \le \frac{\|W^\dagger\|_\infty \|C\|_\infty \Lambda^{2\eta - 1} E_0}{\sqrt[p]{N}}.$$

[1]This can be defined as $W^\dagger := (W^T W)^{-1} W^T$.

For $t \in [\eta\tau, 2\eta\tau)$, let $u(t) = K\hat{x}(t)$, where

$$\hat{x}(t) := e^{(A+BK)(t - \eta\tau)} \hat{x}(\eta\tau)$$

and K is chosen so that the eigenvalues of $A + BK$ have negative real parts. Using the same arguments as before, we can show that

$$\begin{aligned} y(k\tau) \in & B_\infty^p \left(C\hat{x}(k\tau), \frac{\|W^\dagger\|_\infty \|C\|_\infty^2 \Lambda^{\eta - 1 + k} E_0}{\sqrt[p]{N}} \right) \\ k = & \eta, \eta + 1, \ldots, 2\eta - 1. \end{aligned}$$

For each $k \in \{\eta, \eta + 1, \ldots, 2\eta - 1\}$, divide $B_\infty^p(C\hat{x}(k\tau), \|W^\dagger\|_\infty \|C\|_\infty^2 \Lambda^{\eta - 1 + k} E_0 / \sqrt[p]{N})$ into N equal square boxes, and let $q_k(y(k\tau))$ be the number of the box that contains y. Denoting the center of that box by $\hat{y}(k\tau)$, we obtain

$$\begin{aligned} \|y(k\tau) - \hat{y}(k\tau)\|_\infty \le & \frac{\|W^\dagger\|_\infty \|C\|_\infty^2 \Lambda^{\eta - 1 + k} E_0}{(\sqrt[p]{N})^2} \\ k = & \eta, \eta + 1, \ldots, 2\eta - 1. \end{aligned}$$

Now, we know that

$$x(\eta\tau) = W^\dagger \begin{pmatrix} y(\eta\tau) \\ \vdots \\ y((2\eta - 1)\tau) \end{pmatrix} + v$$

where v is a known vector (computed from the variation of constants formula). Thus, we define

$$\begin{aligned} \hat{x}(2\eta\tau) := & e^{A\eta\tau} W^\dagger \begin{pmatrix} \hat{y}(\eta\tau) \\ \vdots \\ \hat{y}((2\eta - 1)\tau) \end{pmatrix} \\ & + e^{A\eta\tau} v + \int_{\eta\tau}^{2\eta\tau} e^{A(2\eta\tau - t)} Bu(t)dt \end{aligned}$$

which leads to the inequality

$$\|x(2\eta\tau) - \hat{x}(2\eta\tau)\|_\infty \le \frac{\|W^\dagger\|_\infty^2 \|C\|_\infty^2 \Lambda^{4\eta - 2} E_0}{(\sqrt[p]{N})^2}.$$

Repeating this procedure, we arrive at an upper bound on $\|Kx(t) - u(t)\|_\infty$ which is multiplied by $\|W^\dagger\|_\infty \|C\|_\infty \Lambda^{\eta - 1} / \sqrt[p]{N}$ at the times $\eta\tau, 2\eta\tau, \ldots$ and grows by a factor of Λ^η on every interval between these times. Thus, if the inequality (3) is satisfied, then asymptotic stability follows as in the state encoding case. We established the following result.

Proposition 3: If the inequalities (3) and $N \ge 2^p$ hold, then the previous output coding/feedback strategy makes the origin an asymptotically stable equilibrium of the closed-loop system, with a region of attraction containing all initial conditions that satisfy the bound (6).

Note that in the generic case when A is a cyclic matrix, i.e., a matrix with exactly one Jordan block for each distinct eigenvalue, it is possible to find a scalar output through which the system is observable (see, e.g., [10, Ch. 8]). An interesting optimization problem, directly motivated by the previous result, consists in minimizing the left-hand side of (3) over all integers p between 1 and n and all $p \times n$ matrices C (observability is needed to ensure that this expression is well defined).

This problem can be studied numerically using tools from semidefinite programming.

III. Obtaining a State Bound

The developments of Section II relied on an upper bound on the size of the state. We now explain how such a bound can be obtained, for an arbitrary initial state. It turns out that $N = 2$ (binary encoding) is sufficient for this task. Since the requirement $N \geq 2$ is already incorporated in the hypotheses of Propositions 2 and 3, no additional assumptions need to be imposed. This will therefore complete the proof of Theorem 1.

We consider the state encoding case first. Set the control u equal to 0. Pick a sequence $\mu_0, \mu_1, \mu_2, \ldots$ that increases fast enough to dominate the rate of growth of $\|e^{At}\|_\infty$ at the times $0, \tau, 2\tau, \ldots$; for example, we can let $\mu_0 = 1$, $\mu_1 = \tau e^{2\|A\|_\infty \tau}$, $\mu_2 = 2\tau e^{2\|A\|_\infty 2\tau}$, and so on. Then there exists an integer $k_0 \geq 0$ such that $\|x(k_0\tau)\|_\infty \leq \mu_{k_0}$. For $k = 0, 1, \ldots$, define the encoding function q_k by the formula

$$q_k(x) := \begin{cases} 0, & \text{if } x \in B_\infty^n(0, \mu_k) \\ 1, & \text{otherwise.} \end{cases}$$

We have $q_{k_0}(x(k_0\tau)) = 0$, so that k_0 can be determined on the basis of the encoded state measurements. Therefore, the procedure described in Section II-A can be applied starting at the time $t = (k_0 + 1)\tau$ with $E_0 := \Lambda \mu_{k_0}$.

Let us now turn to the output encoding case. Set u equal to 0, and take the same sequence $\{\mu_k\}$ as before. There exists an integer $\bar{k}_0 \geq 0$ such that we have

$$\|x(k\tau)\|_\infty \leq \mu_k, \qquad k = \bar{k}_0, \bar{k}_0 + 1, \bar{k}_0 + \eta - 1.$$

For $k = 0, 1, \ldots$, define the encoding function q_k by the formula

$$q_k(y) := \begin{cases} 0, & \text{if } y \in B_\infty^p(0, \|C\|_\infty \mu_k) \\ 1, & \text{otherwise.} \end{cases}$$

We have

$$q_k(y(k\tau)) = 0, \qquad k = \bar{k}_0, \bar{k}_0 + 1, \bar{k}_0 + \eta - 1 \tag{12}$$

so that $\bar{k}_0$ can be determined on the basis of the encoded output measurements. Formula (12) implies that $\|x(\bar{k}_0\tau)\|_\infty \leq \|W^\dagger\|_\infty \|C\|_\infty \mu_{\bar{k}_0 + \eta - 1}$. Therefore, the procedure described in Section II-B can be applied starting at the time $t = (\bar{k}_0 + \eta)\tau$ with $E_0 := \|W^\dagger\|_\infty \|C\|_\infty \Lambda^\eta \mu_{\bar{k}_0 + \eta - 1}$.

Finally, it is not hard to see that stability in the sense of Lyapunov is preserved when the two stages (obtaining a state bound and achieving asymptotic convergence) are combined.

IV. Concluding Remarks

We studied the problem of stabilizing a linear system using sampled encoded measurements of its state or output. Our main result (Theorem 1) describes a relationship between the number of values taken by the encoder and the norm of the transition matrix of the open-loop system over one sampling period, which is sufficient for global asymptotic stabilization. The stabilizing control law takes the form of a "certainty equivalence" feedback.

Theorem 1 provides a sufficient but not necessary condition for stabilizability. Although very simple, our encoding scheme is not claimed to be optimal in any sense. Without proper modifications, it is also not robust with respect to disturbances or modeling errors.

From conditions (2) and (3), it is not clear whether in the present context there is any advantage to be gained by taking $p < n$. However, the additional flexibility of working with an output may be useful in applications where some sensors are more reliable than others.

It is of interest to extend the techniques presented here to nonlinear systems. One ingredient in achieving this goal is the requirement of input-to-state stabilizability of the given system with respect to measurement errors (cf. [2] and [11]).

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