ON STABLE SOLUTIONS OF THE FRACTIONAL HENON-LANE-EMDEN EQUATION

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Abstract. We derive a monotonicity formula for solutions of the fractional Hénon-Lane-Emden equation

$$(-\Delta)^s u = |x|^a |u|^{p-1} u \quad \mathbb{R}^n$$

where $0 < s < 2$, $a > 0$ and $p > 1$. Then we apply this formula to classify stable solutions of the above equation.

1. Introduction and Main Results

We study the classification stable solutions of the following equation

$$(-\Delta)^s u = |x|^a |u|^{p-1} u \quad \mathbb{R}^n \tag{1.1}$$

where $(-\Delta)^s$ is the fractional Laplacian operator for $0 < s < 2$. Here is what we mean by stability.

Definition 1.1. *We say that a solution u of (1.1) is stable if*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{n+2s}} dxdy - p \int_{\mathbb{R}^n} |x|^a |u|^{p-1} \phi^2 \geq 0 \tag{1.2}$$

for any $\phi \in C_c^\infty(\mathbb{R}^n)$.

For the local cases $s = 1$ and $s = 2$, the classification of stable solutions is completely known for $a \geq 0$. We refer the interested readers to Farina [14] for the case of $s = 1$ and $a = 0$ and to Cowan-Fazly [6], Wang-Ye [31], Dancer-Du-Guo [7], Du-Guo-Wang [11] for the case $s = 1$ and $a > -2$. Also, for the fourth order Lane-Emden equation that is when $s = 2$ we refer to Davila-Dupaigne-Wang-Wei [10] where $a = 0$ and to Hu [20] where $a > 0$. In this note, we focus on the case of fractional Laplacian operator.

It is by now standard that the fractional Laplacian can be seen as a Dirichlet-to-Neumann operator for a degenerate but *local* diffusion operator in the higher-dimensional half-space $\mathbb{R}^{n+1}_+$. For the case of $0 < s < 1$ this in fact can be seen as the following theorem given by Caffarelli-Silvestre [2]. See also [27].

Theorem 1.1. *Take $s \in (0,1)$, $\sigma > s$ and $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|t|)^{n+2s}dt)$. For $X = (x,y) \in \mathbb{R}^{n+1}_+$, let*

$$u_e(X) = \int_{\mathbb{R}^n} P(X,t) u(t)\, dt,$$

where

$$P(X,t) = p_{n,s}\, t^{2s} |X-t|^{-(n+2s)}$$

and $p_{n,s}$ is chosen so that $\int_{\mathbb{R}^n} P(X,t)\, dt = 1$. Then, $u_e \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$, $y^{1-2s}\partial_y u_e \in C(\overline{\mathbb{R}^{n+1}_+})$ and

$$\begin{cases} \nabla\cdot(y^{1-2s}\nabla u_e) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u_e = u & \text{on } \partial\mathbb{R}^{n+1}_+, \\ -\lim_{y\to 0} y^{1-2s}\partial_t u_e = \kappa_s (-\Delta)^s u & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

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where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}. \tag{1.3}$$

From this theorem for a solution of the fractional Henon-Lane-Emden equation, we get the following equation in the higher-dimensional half-space $\mathbb{R}^{n+1}_+$,

$$\begin{cases} -\nabla\cdot(y^{1-2s}\nabla u_e) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\lim\limits_{y\to 0} y^{1-2s}\partial_t u_e = \kappa_s |x|^a |u_e|^{p-1}u_e & \text{in } \mathbb{R}^n \end{cases} \tag{1.4}$$

There are different ways of defining the fractional operator $(-\Delta)^s$ where $1<s<2$, just like the case of $0<s<1$. Applying the Fourier transform one can define the fractional Laplacian by

$$\widehat{(-\Delta)^s} u(\zeta) = |\zeta|^{2s}\hat{u}(\zeta)$$

or equivalently define this operator inductively by $(-\Delta)^s = (-\Delta)^{s-1} o(-\Delta)$, see [26]. Recently, Yang in [29] gave a characterization of the fractional Laplacian $(-\Delta)^s$, where s is any positive, noninteger number as the Dirichlet-to-Neumann map for a function u_e satisfying a higher order elliptic equation in the upper half space with one extra spatial dimension. This is a generalization of the work of Caffarelli and Silvestre in [2] for the case of $0<s<1$. We first fix the following notation then we present the Yang's characterization. See also Case-Chang [3] and Chang-Gonzales [4] for higher order fractional operators.

Notation 1.1. *Throughout this note set $b := 3-2s$ and define the operator*

$$\Delta_b w := \Delta w + \frac{b}{y} w_y = y^{-b}\,\mathrm{div}(y^b \nabla w).$$

for a function $w \in W^{2,2}(\mathbb{R}^{n+1}, y^b)$.

As it is shown by Yang in [29], if $u(x)$ is a solution of (1.1) then the extended function $u_e(x,y)$ where $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^+$ satisfies

$$\begin{cases} \Delta_b^2 u_e &= 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \lim_{y\to 0} y^b \partial_y u_e &= 0 \text{ in } \partial\mathbb{R}^{n+1}_+, \\ \lim_{y\to 0} y^b \partial_y \Delta_b u_e &= C_{n,s}|x|^a|u|^{p-1}u \text{ in } \mathbb{R}^n \end{cases} \tag{1.5}$$

Moreover,

$$\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u(\xi)}|^2 d\xi = C_{n,s}\int_{\mathbb{R}^{n+1}_+} y^b |\Delta_b u_e(x,y)|^2 dxdy$$

Note that $u(x) = u_e(x,0)$ in $\mathbb{R}^n$.

On the other hand, Herbst in [19] (see also [30]), shoed that when $n>2s$ the following Hardy inequality holds

$$\int_{\mathbb{R}^n} |\xi|^{2s}|\hat{\phi}|^2 d\xi > \Lambda_{n,s}\int_{\mathbb{R}^n} |x|^{-2s}\phi^2 dx$$

for any $\phi\in C_c^\infty(\mathbb{R}^n)$ where the optimal constant given by

$$\Lambda_{n,s} = 2^{2s}\frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

Here we fix a constant that plays an important role in the classification of solutions of (1.1)

$$p_S(n,a) = \begin{cases} +\infty & \text{if } n\le 2s \\ \dfrac{n+2s+2a}{n-2s} & \text{if } n>2s \end{cases} \tag{1.6}$$

Remark 1.1. *Note that for $p > p_S(n,a)$ the function*

$$u_s(x) = A|x|^{-\frac{2s+a}{p-1}} \tag{1.7}$$

where

$$A^{p-1} = \lambda\left(\frac{n-2s}{2} - \frac{2s+a}{p-1}\right)$$

for constant

$$\lambda(\alpha) = 2^{2s}\frac{\Gamma(\frac{n+2s+2\alpha}{4})\Gamma(\frac{n+2s-2\alpha}{4})}{\Gamma(\frac{n-2s-2\alpha}{4})\Gamma(\frac{n-2s+2\alpha}{4})} \tag{1.8}$$

is a singular solution of (1.1) where $0 < s < 2$. For details, we refer the interested readers to [13] for the case of $0 < s < 1$ and to [16] for the case of $1 < s < 2$.

Here is our main result

Theorem 1.2. *Assume that $n \geq 1$ and $0 < s < \sigma < 2$. Let $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|y|)^{n+2s}dy)$ be a stable solution to (1.1).*

- *If $1 < p < p_S(n,a)$ or if $p_S(n,a) < p$ and*

$$p\frac{\Gamma(\frac{n}{2} - \frac{s+\frac{a}{2}}{p-1})\Gamma(s + \frac{s+\frac{a}{2}}{p-1})}{\Gamma(\frac{s+\frac{a}{2}}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s+\frac{a}{2}}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}, \tag{1.9}$$

  then $u \equiv 0$;
- *If $p = p_S(n,a)$, then u has finite energy i.e.*

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |x|^a|u|^{p+1} < +\infty.$$

  If in addition u is stable, then in fact $u \equiv 0$.

Note that the classification of finite Morse index solutions of (1.1) when $a = 0$ is given by Davila-Dupaigne-Wei in [9] when $0 < s < 1$ and by Fazly-Wei in [16] $1 < s < 2$.

Note also that in the absence of stability it is expected that the only nonnegative bounded solution of (1.1) must be zero for the subcritical exponents $1 < p < p_S(n,a)$ where $a \geq 0$. To our knowledge not much is known about the classification of solutions when $a \neq 0$ even for the standard case $s = 1$. For the case of $s = 1$, Phan-Souplet in [23] proved that the only nonnegative bounded solution of (1.1) in three dimensions must be zero for the case of $1 < p < p_S(n,a)$ and $a > -2$. Some partial results are given in [17].

2. The monotonicity formula

Here is the monotonicity formula for the case of $0 < s < 1$.

Theorem 2.1. *Suppose that $0 < s < 1$. Let $u_e \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$ be a solution of (1.1) such that $y^{1-2s}\partial_y u_e \in C(\overline{\mathbb{R}^{n+1}_+})$. For $x_0 \in \partial\mathbb{R}^{n+1}_+$, $\lambda > 0$, let*

$$\begin{aligned} E(u_e,\lambda) \quad := \quad & \lambda^{\frac{2s(p+1)+2a}{p-1}-n}\left(\frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_\lambda} y^{1-2s}|\nabla u_e|^2\,dx\,dy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_\lambda}|x|^a|u_e|^{p+1}\,dx\right) \\ & +\lambda^{\frac{2s(p+1)+2a}{p-1}-n-1}\frac{s+\frac{a}{2}}{p+1}\int_{\partial B_\lambda\cap\mathbb{R}^{n+1}_+} y^{1-2s}u_e^2\,d\sigma. \end{aligned}$$

Then, E is a nondecreasing function of λ. Furthermore,

$$\frac{dE}{d\lambda} = \lambda^{\frac{2s(p+1)+a}{p-1}-n+1}\int_{\partial B(x_0,\lambda)\cap\mathbb{R}^{n+1}_+} y^{1-2s}\left(\frac{\partial u_e}{\partial r} + \frac{2s+a}{p-1}\frac{u_e}{r}\right)^2\,d\sigma$$

Proof. Let

$$I(u_e,\lambda)=\lambda^{2s\frac{p+1}{p-1}-n}\left(\int_{\mathbb{R}^{n+1}_+\cap B_\lambda} y^{1-2s}\frac{|\nabla u_e|^2}{2}dx\,dy-\frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_\lambda}|x|^a|u_e|^{p+1}dx\right) \tag{2.1}$$

Now for $X\in\mathbb{R}^{n+1}_+$, define

$$u_e^\lambda(X)=\lambda^{\frac{2s+a}{p-1}}u_e(\lambda X). \tag{2.2}$$

Then, u_e^λ solves (1.5) and in addition

$$I(u_e,\lambda)=I(u_e^\lambda,1). \tag{2.3}$$

Taking partial derivatives we get

$$\lambda\partial_\lambda u_e^\lambda=\frac{2s+a}{p-1}u_e^\lambda+r\partial_r u_e^\lambda. \tag{2.4}$$

Differentiating the operator (2.1) w.r.t. λ, we find

$$\partial_\lambda I(u_e,\lambda)=\int_{\mathbb{R}^{n+1}_+\cap B_1} y^{1-2s}\nabla u_e^\lambda\cdot\nabla\partial_\lambda u_e^\lambda dx\,dy-\kappa_s\int_{\partial\mathbb{R}^{n+1}_+\cap B_1}|x|^a|u_e^\lambda|^{p-1}\partial_\lambda u_e^\lambda dx.$$

Integrating by parts and then using (2.4),

$$\begin{aligned}\partial_\lambda I(u_e,\lambda)&=\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} y^{1-2s}\partial_r u_e^\lambda\partial_\lambda u_e^\lambda d\sigma\\&=\lambda\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} y^{1-2s}(\partial_\lambda u_e^\lambda)^2d\sigma-\frac{2s+a}{p-1}\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} y^{1-2s}u_e^\lambda\partial_\lambda u_e^\lambda d\sigma\\&=\lambda\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} y^{1-2s}(\partial_\lambda u_e^\lambda)^2d\sigma-\frac{s+\frac{a}{2}}{p-1}\partial_\lambda\left(\int_{\partial B_1\cap\mathbb{R}^{n+1}_+} y^{1-2s}(u_e^\lambda)^2\ d\sigma\right)\end{aligned}$$

Scaling finishes the proof.

□

We now consider the case of $1<s<2$ and $a>0$. Note that a monotonicity formula is given for the case of $a=0$ and $s=2$ and $1<s<2$ by Davila-Dupaigne-Wang-Wei in [10] and Fazly-Wei in [16], respectively. We define the energy functional

$$\begin{aligned}E(u_e,r)\ \ :=\ \ & r^{2s\frac{p+1}{p-1}-n}\left(\int_{\mathbb{R}^{n+1}_+\cap B_r}\frac{1}{2}y^{3-2s}|\Delta_b u_e|^2-\frac{C_{n,s}}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_r}|x|^a u_e^{p+1}\right)\\&-\frac{s+\frac{a}{2}}{p-1}\left(\frac{p+2s+a-1}{p-1}-n-b\right)r^{-3+2s+\frac{4s+2a}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_r}y^{3-2s}u_e^2\\&-\frac{s+\frac{a}{2}}{p-1}\left(\frac{p+2s+a-1}{p-1}-n-b\right)\frac{d}{dr}\left[r^{\frac{4s+2a}{p-1}+2s-2-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_r}y^{3-2s}u_e^2\right]\\&+\frac{1}{2}r^3\frac{d}{dr}\left[r^{\frac{4s+2a}{p-1}+2s-3-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_r}y^{3-2s}\left(\frac{2s+a}{p-1}r^{-1}u+\frac{\partial u_e}{\partial r}\right)^2\right]\\&+\frac{1}{2}\frac{d}{dr}\left[r^{\frac{2s(p+1)+2a}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_r}y^{3-2s}\left(|\nabla u_e|^2-\left|\frac{\partial u_e}{\partial r}\right|^2\right)\right]\\&+\frac{1}{2}r^{\frac{2s(p+1)+2a}{p-1}-n-1}\int_{\mathbb{R}^{n+1}_+\cap\partial B_r}y^{3-2s}\left(|\nabla u_e|^2-\left|\frac{\partial u_e}{\partial r}\right|^2\right)\end{aligned}$$

Theorem 2.2. *Assume that* $n > \frac{p+4s+2a-1}{p+2s+a-1} + \frac{2s+a}{p-1} - b$. *Then,* $E(u_e, \lambda)$ *is a nondecreasing function of* $\lambda > 0$. *Furthermore,*

$$\frac{dE(\lambda, u_e)}{d\lambda} \geq C(n,s,p)\ \lambda^{\frac{4s+2a}{p-1}+2s-2-n} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_\lambda} y^{3-2s} \left(\frac{2s+a}{p-1} r^{-1} u + \frac{\partial u_e}{\partial r} \right)^2 \tag{2.5}$$

where $C(n,s,p)$ *is independent from* λ.

Proof: Set,

$$\bar{E}(u_e, \lambda) := \lambda^{\frac{2s(p+1)+2a}{p-1}-n} \left(\int_{\mathbb{R}^{n+1}_+ \cap B_\lambda} \frac{1}{2} y^b |\Delta_b u_e|^2 dxdy - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_\lambda} |x|^a u_e^{p+1} \right) \tag{2.6}$$

Define $v_e := \Delta_b u_e$, $u_e^\lambda(X) := \lambda^{\frac{2s+a}{p-1}} u_e(\lambda X)$, and $v_e^\lambda(X) := \lambda^{\frac{2s+a}{p-1}+2} v_e(\lambda X)$ where $X = (x,y) \in \mathbb{R}^{n+1}_+$. Therefore, $\Delta_b u_e^\lambda(X) = v_e^\lambda(X)$ and

$$\begin{cases} \Delta_b v_e^\lambda &= 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \lim_{y\to 0} y^b \partial_y u_e^\lambda &= 0 \text{ in } \partial\mathbb{R}^{n+1}_+, \\ \lim_{y\to 0} y^b \partial_y v_e^\lambda &= C_{n,s}|x|^a (u_e^\lambda)^p \text{ in } \mathbb{R}^n \end{cases} \tag{2.7}$$

In addition, differentiating with respect to λ we have

$$\Delta_b \frac{du_e^\lambda}{d\lambda} = \frac{dv_e^\lambda}{d\lambda}. \tag{2.8}$$

Note that

$$\bar{E}(u_e, \lambda) = \bar{E}(u_e^\lambda, 1) = \int_{\mathbb{R}^{n+1}_+ \cap B_1} \frac{1}{2} y^b (v_e^\lambda)^2 dxdy - \frac{C_{n,s}}{p+1} \int_{\partial\mathbb{R}^{n+1}_+ \cap B_1} |x|^a |u_e^\lambda|^{p+1}$$

Taking derivate of the energy with respect to λ, we have

$$\frac{d\bar{E}(u_e^\lambda, 1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b v_e^\lambda \frac{dv_e^\lambda}{d\lambda}\ dxdy - C_{n,s} \int_{\partial\mathbb{R}^{n+1}_+ \cap B_1} |x|^a |u_e^\lambda|^p \frac{du_e^\lambda}{d\lambda} \tag{2.9}$$

Using (2.7) we end up with

$$\frac{d\bar{E}(u_e^\lambda, 1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b v_e^\lambda \frac{dv_e^\lambda}{d\lambda}\ dxdy - \int_{\partial\mathbb{R}^{n+1}_+ \cap B_1} \lim_{y\to 0} y^b \partial_y v_e^\lambda \frac{du_e^\lambda}{d\lambda} \tag{2.10}$$

From (2.8) and by integration by parts we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b v_e^\lambda \frac{dv_e^\lambda}{d\lambda} &= \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \Delta_b u_e^\lambda \Delta_b \frac{du_e^\lambda}{d\lambda} \\ &= -\int_{\mathbb{R}^{n+1}_+ \cap B_1} \nabla \Delta_b u_e^\lambda \cdot \nabla \left(\frac{du_e^\lambda}{d\lambda} \right) y^b + \int_{\partial(\mathbb{R}^{n+1}_+ \cap B_1)} \Delta_b u_e^\lambda y^b \partial_\nu \left(\frac{du_e^\lambda}{d\lambda} \right) \end{aligned}$$

Note that

$$\begin{aligned} -\int_{\mathbb{R}^{n+1}_+ \cap B_1} \nabla \Delta_b u_e \cdot \nabla \frac{du_e^\lambda}{d\lambda} y^b &= \int_{\mathbb{R}^{n+1}_+ \cap B_1} \operatorname{div}(\nabla \Delta_b u_e^\lambda y^b) \frac{du_e^\lambda}{d\lambda} - \int_{\partial(\mathbb{R}^{n+1}_+ \cap B_1)} y^b \partial_\nu (\Delta_b u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\ &= \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \Delta_b^2 u_e^\lambda \frac{du_e^\lambda}{d\lambda} - \int_{\partial(\mathbb{R}^{n+1}_+ \cap B_1)} y^b \partial_\nu (\Delta_b u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \\ &= -\int_{\partial(\mathbb{R}^{n+1}_+ \cap B_1)} y^b \partial_\nu (\Delta_b u_e^\lambda) \frac{du_e^\lambda}{d\lambda} \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b v_e^\lambda \frac{dv_e^\lambda}{d\lambda} = \int_{\partial(\mathbb{R}^{n+1}_+ \cap B_1)} \Delta_b u_e^\lambda y^b \partial_\nu \left(\frac{du_e^\lambda}{d\lambda} \right) - \int_{\partial(\mathbb{R}^{n+1}_+ \cap B_1)} y^b \partial_\nu (\Delta_b u_e^\lambda) \frac{du_e^\lambda}{d\lambda}$$

Boundary of $\mathbb{R}^{n+1}_+ \cap B_1$ consists of $\partial\mathbb{R}^{n+1}_+ \cap B_1$ and $\mathbb{R}^{n+1}_+ \cap \partial B_1$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\cap B_1} y^b v_e^\lambda \frac{dv_e^\lambda}{d\lambda} &= \int_{\partial\mathbb{R}^{n+1}_+\cap B_1} -v_e^\lambda \lim_{y\to 0} y^b \partial_y \left(\frac{du_e^\lambda}{d\lambda}\right) + \lim_{y\to 0} y^b \partial_y v_e^\lambda \frac{du_e^\lambda}{d\lambda} \\ &+ \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} y^b v_e^\lambda \partial_r \left(\frac{du_e^\lambda}{d\lambda}\right) - y^b \partial_r v_e^\lambda \frac{du_e^\lambda}{d\lambda}\end{aligned}$$

where $r = |X|$, $X = (x, y) \in \mathbb{R}^{n+1}_+$ and $\partial_r = \nabla \cdot \frac{X}{r}$ is the corresponding radial derivative. Note that the first integral in the right-hand side vanishes since $\partial_y \left(\frac{du_e^\lambda}{d\lambda}\right) = 0$ on $\partial\mathbb{R}^{n+1}_+$. From (2.10) we obtain

$$\frac{d\bar{E}(u_e^\lambda, 1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} y^b \left(v_e^\lambda \partial_r \left(\frac{du_e^\lambda}{d\lambda}\right) - \partial_r \left(v_e^\lambda\right) \frac{du_e^\lambda}{d\lambda}\right) \tag{2.11}$$

Now note that from the definition of u_e^λ and v_e^λ and by differentiating in λ we get the following for $X \in \mathbb{R}^{n+1}_+$

$$\frac{du_e^\lambda(X)}{d\lambda} = \frac{1}{\lambda}\left(\frac{2s+a}{p-1} u_e^\lambda(X) + r\partial_r u_e^\lambda(X)\right) \tag{2.12}$$

$$\frac{dv_e^\lambda(X)}{d\lambda} = \frac{1}{\lambda}\left(\frac{2(p+s-1)+a}{p-1} v_e^\lambda(X) + r\partial_r v_e^\lambda(X)\right) \tag{2.13}$$

Therefore, differentiating with respect to λ we get

$$\lambda \frac{d^2 u_e^\lambda(X)}{d\lambda^2} + \frac{du_e^\lambda(X)}{d\lambda} = \frac{2s+a}{p-1}\frac{du_e^\lambda(X)}{d\lambda} + r\partial_r \frac{du_e^\lambda(X)}{d\lambda}$$

So, for all $X \in \mathbb{R}^{n+1}_+ \cap \partial B_1$

$$\partial_r \left(u_e^\lambda(X)\right) = \lambda \frac{du_e^\lambda(X)}{d\lambda} - \frac{2s+a}{p-1} u_e^\lambda(X) \tag{2.14}$$

$$\partial_r \left(\frac{du_e^\lambda(X)}{d\lambda}\right) = \lambda \frac{d^2 u_e^\lambda(X)}{d\lambda^2} + \frac{p-1-2s-a}{p-1}\frac{du_e^\lambda(X)}{d\lambda} \tag{2.15}$$

$$\partial_r \left(v_e^\lambda(X)\right) = \lambda \frac{dv_e^\lambda(X)}{d\lambda} - \frac{2(p+s-1)+a}{p-1} v_e^\lambda(X) \tag{2.16}$$

Substituting (2.15) and (2.16) in (2.11) we get

$$\begin{aligned}\frac{d\bar{E}(u_e^\lambda, 1)}{d\lambda} &= \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} y^b v_e^\lambda \left(\lambda \frac{d^2 u_e^\lambda}{d\lambda^2} + \frac{p-1-2s-a}{p-1}\frac{du_e^\lambda}{d\lambda}\right) - y^b \left(\lambda \frac{dv_e^\lambda}{d\lambda} - \frac{2(p+s-1)+a}{p-1} v_e^\lambda\right)\frac{du_e^\lambda}{d\lambda} \\ &= \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} y^b \left(\lambda v_e^\lambda \frac{d^2 u_e^\lambda}{d\lambda^2} + 3 v_e^\lambda \frac{du_e^\lambda}{d\lambda} - \lambda \frac{dv_e^\lambda}{d\lambda}\frac{du_e^\lambda}{d\lambda}\right)\end{aligned} \tag{2.17}$$

Taking derivative of (2.12) in r we get

$$r\frac{\partial^2 u_e^\lambda}{\partial r^2} + \frac{\partial u_e^\lambda}{\partial r} = \lambda \frac{\partial}{\partial r}\left(\frac{du_e^\lambda}{d\lambda}\right) - \frac{2s+a}{p-1}\frac{\partial u_e^\lambda}{\partial r}$$

So, from (2.15) for all $X \in \mathbb{R}^{n+1}_+ \cap \partial B_1$ we have

$$\begin{aligned}\frac{\partial^2 u_e^\lambda}{\partial r^2} &= \lambda \frac{\partial}{\partial r}\left(\frac{du_e^\lambda}{d\lambda}\right) - \frac{p+2s+a-1}{p-1}\frac{\partial u_e^\lambda}{\partial r} \\ &= \lambda\left(\lambda \frac{d^2 u_e^\lambda}{d\lambda^2} + \frac{p-2s-1-a}{p-1}\frac{du_e^\lambda}{d\lambda}\right) - \frac{p+2s+a-1}{p-1}\left(\lambda \frac{du_e^\lambda}{d\lambda} - \frac{2s+a}{p-1} u_e^\lambda\right) \\ &= \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} - \frac{4s+2a}{p-1}\lambda \frac{du_e^\lambda}{d\lambda} + \frac{(2s+a)(p+2s+a-1)}{(p-1)^2} u_e^\lambda\end{aligned} \tag{2.18}$$

Note that

$$v_e^\lambda = \Delta_b u_e^\lambda = y^{-b}\operatorname{div}(y^b \nabla u_e^\lambda)$$

and on $\mathbb{R}^{n+1}_+ \cap \partial B_1$, we have

$$\operatorname{div}(y^b \nabla u_e^\lambda) = (u_{rr} + (n+b)u_r)\theta_1^b + \operatorname{div}_{\mathcal{S}^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)$$

where $\theta_1 = \frac{y}{r}$. From the above, (2.14) and (2.18) we get

$$v_e^\lambda \;=\; \lambda^2 \frac{d^2 u_e^\lambda}{d\lambda^2} + \lambda \frac{du_e^\lambda}{d\lambda}(n+b-\frac{4s+2a}{p-1}) + u_e^\lambda(\frac{2s+a}{p-1})(\frac{p+2s+a-1}{p-1} - n - b) + \theta_1^{-b}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b \nabla_{S^n} u_e^\lambda)$$

From this and (2.17) we get

$$\frac{d\bar{E}(u_e^\lambda,1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b \lambda \left(\lambda^2 \frac{d^2u_e^\lambda}{d\lambda^2} + \alpha\lambda\frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda\right)\frac{d^2u_e^\lambda}{d\lambda^2} \tag{2.19}$$

$$+\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b 3 \left(\lambda^2 \frac{d^2u_e^\lambda}{d\lambda^2} + \alpha\lambda\frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda\right)\frac{du_e^\lambda}{d\lambda} \tag{2.20}$$

$$-\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b \lambda \frac{du_e^\lambda}{d\lambda}\frac{d}{d\lambda}\left(\lambda^2 \frac{d^2u_e^\lambda}{d\lambda^2} + \alpha\lambda\frac{du_e^\lambda}{d\lambda} + \beta u_e^\lambda\right) \tag{2.21}$$

$$+\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b \lambda \frac{d^2u_e^\lambda}{d\lambda^2}\theta_1^{-b}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda) \tag{2.22}$$

$$+\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} 3\theta_1^b \frac{du_e^\lambda}{d\lambda}\theta_1^{-b}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda) \tag{2.23}$$

$$-\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b \lambda \frac{d}{d\lambda}\left(\theta_1^{-b}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda)\right)\frac{du_e^\lambda}{d\lambda} \tag{2.24}$$

where $\alpha := n+b-\frac{4s+2a}{p-1}$ and $\beta := \frac{2s+a}{p-1}\left(\frac{p+2s+a-1}{p-1} - n - b\right)$. Simplifying the integrals we get

$$\begin{aligned}\frac{d\bar{E}(u_e^\lambda,1)}{d\lambda} &= \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b\left(2\lambda^3\left(\frac{d^2u_e^\lambda}{d\lambda^2}\right)^2 + 4\lambda^2\frac{d^2u_e^\lambda}{d\lambda^2}\frac{du_e^\lambda}{d\lambda} + 2(\alpha-\beta)\lambda\left(\frac{du_e^\lambda}{d\lambda}\right)^2\right)\\ &+\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b\left(\frac{\beta}{2}\frac{d^2}{d\lambda^2}\left(\lambda(u_e^\lambda)^2\right) - \frac{1}{2}\frac{d}{d\lambda}\left(\lambda^3\frac{d}{d\lambda}\left(\frac{du_e^\lambda}{d\lambda}\right)^2\right) + \frac{\beta}{2}\frac{d}{d\lambda}(u_e^\lambda)^2\right)\\ &+\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \lambda\frac{d^2u_e^\lambda}{d\lambda^2}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda) + 3\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda)\frac{du_e^\lambda}{d\lambda} - \lambda\frac{d}{d\lambda}\left(\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda)\right)\frac{du_e^\lambda}{d\lambda}\end{aligned} \tag{2.25}$$

Note that from the assumptions we have $\alpha-\beta-1>0$, therefore the first term in the RHS of (2.25) is positive that is

$$2\lambda^3\left(\frac{d^2u_e^\lambda}{d\lambda^2}\right)^2 + 4\lambda^2\frac{d^2u_e^\lambda}{d\lambda^2}\frac{du_e^\lambda}{d\lambda} + 2(\alpha-\beta)\lambda\left(\frac{du_e^\lambda}{d\lambda}\right)^2 = 2\lambda\left(\lambda\frac{d^2u_e^\lambda}{d\lambda^2} + \frac{du_e^\lambda}{d\lambda}\right)^2 + 2(\alpha-\beta-1)\lambda\left(\frac{du_e^\lambda}{d\lambda}\right)^2 > 0$$

From this we have

$$\begin{aligned}\frac{d\bar{E}(u_e^\lambda,1)}{d\lambda} &\geq \int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \theta_1^b\left(\frac{\beta}{2}\frac{d^2}{d\lambda^2}\left(\lambda(u_e^\lambda)^2\right) - \frac{1}{2}\frac{d}{d\lambda}\left(\lambda^3\frac{d}{d\lambda}\left(\frac{du_e^\lambda}{d\lambda}\right)^2\right) + \frac{\beta}{2}\frac{d}{d\lambda}(u_e^\lambda)^2\right)\\ &+\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} \lambda\frac{d^2u_e^\lambda}{d\lambda^2}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda) + 3\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda)\frac{du_e^\lambda}{d\lambda} - \lambda\frac{d}{d\lambda}\left(\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{S^n}u_e^\lambda)\right)\frac{du_e^\lambda}{d\lambda}\\ &=: R_1 + R_2.\end{aligned}$$

Note that the terms appeared in R_1 are of the following form

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b\frac{d^2}{d\lambda^2}\left(\lambda(u_e^\lambda)^2\right) &= \frac{d^2}{d\lambda^2}\left(\lambda^{\frac{4s+2a}{p-1}+2(s-1)-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_\lambda} y^b u_e^2\right)\\
\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b\frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\frac{du_e^\lambda}{d\lambda}\right)^2\right] &= \frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\lambda^{\frac{4s+2a}{p-1}+2s-3-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_\lambda} y^b\left[\frac{2s+a}{p-1}\lambda^{-1}u_e+\frac{\partial u_e}{\partial r}\right]^2\right)\right]\\
\int_{\mathbb{R}^{n+1}_+\cap\partial B_1} y^b\frac{d}{d\lambda}(u_e^\lambda)^2 &= \frac{d}{d\lambda}\left(\lambda^{2s-3+\frac{4s+2a}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_\lambda} y^b u_e^2\right)
\end{aligned}$$

We now apply integration by parts to simplify the terms appeared in R_2.

$$\begin{aligned}
R_2 &= \int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\lambda\frac{d^2u_e^\lambda}{d\lambda^2}\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{\mathcal{S}^n}u_e^\lambda)+3\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{\mathcal{S}^n}u_e^\lambda)\frac{du_e^\lambda}{d\lambda}-\lambda\frac{d}{d\lambda}\left(\operatorname{div}_{\mathcal{S}^n}(\theta_1^b\nabla_{\mathcal{S}^n}u_e^\lambda)\right)\frac{du_e^\lambda}{d\lambda}\\
&= \int_{\mathbb{R}^{n+1}_+\cap\partial B_1}-\theta_1^b\lambda\nabla_{\mathcal{S}^n}u_e^\lambda\cdot\nabla_{\mathcal{S}^n}\frac{d^2u_e^\lambda}{d\lambda^2}-3\theta_1^b\nabla_{\mathcal{S}^n}u_e^\lambda\cdot\nabla_{\mathcal{S}^n}\frac{du_e^\lambda}{d\lambda}+\theta_1^b\lambda\left|\nabla_{\mathcal{S}^n}\frac{du_e^\lambda}{d\lambda}\right|^2\\
&= -\frac{\lambda}{2}\frac{d^2}{d\lambda^2}\left(\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right)-\frac{3}{2}\frac{d}{d\lambda}\left(\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right)+2\lambda\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b\left|\nabla_\theta\frac{du_e^\lambda}{d\lambda}\right|^2\\
&= -\frac{1}{2}\frac{d^2}{d\lambda^2}\left(\lambda\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right)-\frac{1}{2}\frac{d}{d\lambda}\left(\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right)+2\lambda\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b\left|\nabla_\theta\frac{du_e^\lambda}{d\lambda}\right|^2\\
&\geq -\frac{1}{2}\frac{d^2}{d\lambda^2}\left(\lambda\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right)-\frac{1}{2}\frac{d}{d\lambda}\left(\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right)
\end{aligned}$$

Note that the two terms that appear as lower bound for R_3 are of the form

$$\begin{aligned}
\frac{d^2}{d\lambda^2}\left(\lambda\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right) &= \frac{d^2}{d\lambda^2}\left[\lambda^{\frac{2s(p+1)+2a}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_\lambda} y^b\left(|\nabla u|^2-\left|\frac{\partial u}{\partial r}\right|^2\right)\right]\\
\frac{d}{d\lambda}\left(\int_{\mathbb{R}^{n+1}_+\cap\partial B_1}\theta_1^b|\nabla_\theta u_e^\lambda|^2\right) &= \frac{d}{d\lambda}\left[\lambda^{\frac{2s(p+1)+2a}{p-1}-n-1}\int_{\mathbb{R}^{n+1}_+\cap\partial B_\lambda} y^b\left(|\nabla u|^2-\left|\frac{\partial u}{\partial r}\right|^2\right)\right]
\end{aligned}$$

□

Remark 2.1. *It is straightforward to show that* $n>\frac{2s(p+1)+2a}{p-1}$ *implies* $n>\frac{p+4s+2a-1}{p+2s+a-1}+\frac{2s+a}{p-1}-b$.

3. Homogeneous Solutions

Theorem 3.1. *Suppose that* $u=r^{-\frac{2s+a}{p-1}}\psi(\theta)$ *is a stable solution of (1.1) then* $\psi=0$ *provided* $p>\frac{n+2s+2a}{n-2s}$ *and*

$$p\frac{\Gamma(\frac{n}{2}-\frac{s+\frac{a}{2}}{p-1})\Gamma(s+\frac{s+\frac{a}{2}}{p-1})}{\Gamma(\frac{s+\frac{a}{2}}{p-1})\Gamma(\frac{n-2s}{2}-\frac{s+\frac{a}{2}}{p-1})}>\frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}, \tag{3.1}$$

Proof. Since u satisfies (1.1), the function ψ satisfies

$$\begin{aligned} |x|^{ap}|x|^{-\frac{2ps+ap}{p-1}}\psi^p(\theta) &= \int \frac{|x|^{-\frac{2s+a}{p-1}}\psi(\theta)-|y|^{-\frac{2s+a}{p-1}}\psi(\sigma)}{|x-y|^{n+2s}}dy \\ &= \int \frac{|x|^{-\frac{2s+a}{p-1}}\psi(\theta)-r^{-\frac{2s+a}{p-1}}t^{-\frac{2s+a}{p-1}}\psi(\sigma)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}|x|^{n+2s}}|x|^n t^{n-1}dtd\sigma \quad \text{where} \quad |y|=rt \\ &= |x|^{-\frac{2ps+a}{p-1}}[\int \frac{\psi(\theta)-t^{-\frac{2s+a}{p-1}}\psi(\theta)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}dtd\sigma \\ &+ \int \frac{t^{-\frac{2s+a}{p-1}}(\psi(\theta)-\psi(\sigma)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}dtd\sigma] \end{aligned}$$

We now drop $|x|^{-\frac{2ps+a}{p-1}}$ and get

$$\psi(\theta)A_{n,s,a}(\theta)+\int_{\mathbb{S}^{n-1}}K_{\frac{2s+a}{p-1}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))d\sigma=\psi^p(\theta) \tag{3.2}$$

where

$$A_{n,s,a}:=\int_0^\infty\int_{\mathbb{S}^{n-1}}\frac{1-t^{-\frac{2s+a}{p-1}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}d\sigma dt$$

and

$$K_{\frac{2s+a}{p-1}}(<\theta,\sigma>):=\int_0^\infty\frac{t^{n-1-\frac{2s}{p-1}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt$$

Note that

$$\begin{aligned} K_{\frac{2s+a}{p-1}}(<\theta,\sigma>) &= \int_0^1\frac{t^{n-1-\frac{2s+a}{p-1}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt+\int_1^\infty\frac{t^{n-1-\frac{2s+a}{p-1}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt \\ &= \int_0^1\frac{t^{n-1-\frac{2s+a}{p-1}}+t^{2s-1+\frac{2s+a}{p-1}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt \end{aligned}$$

We now set $K_\alpha(<\theta,\sigma>)=\int_0^1\frac{t^{n-1+\alpha}+t^{2s-1+\alpha}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt$. The most important property of the K_α is that K_α is decreasing in α. This can be seen by the following elementary calculations

$$\begin{aligned} \partial_\alpha K_\alpha &= \int_0^1\frac{-t^{n-1-\alpha}\ln t+t^{2s-1+\alpha}\ln t}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt \\ &= \int_0^1\frac{\ln t(-t^{n-1-\alpha}+t^{2s-1+\alpha})}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dt<0 \end{aligned}$$

For the last part we have used the fact that for $p>\frac{n+2s+2a}{n-2s}$ we have $2s-1+\alpha<n-1-\alpha$.

From (3.2) we get the following

$$\int_{\mathbb{S}^{n-1}}\psi^2(\theta)A_{n,s,a}+\int_{\mathbb{S}^{n-1}}K_{\frac{2s+a}{p-1}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2d\theta d\sigma=\int_{\mathbb{S}^{n-1}}\psi^{p+1}(\theta)d\theta \tag{3.3}$$

We set a standard cut-off function $\eta_\epsilon\in C_c^1(\mathbb{R}_+)$ at the origin and at infinity that is $\eta_\epsilon=1$ for $\epsilon<r<\epsilon^{-1}$ and $\eta_\epsilon=0$ for either $r<\epsilon/2$ or $r>2/\epsilon$. We test the stability (1.2) on the function $\phi(x)=r^{-\frac{n-2s}{2}}\psi(\theta)\eta_\epsilon(r)$.

Note that

$$\int_{\mathbb{R}^n}\frac{\phi(x)-\phi(y)}{|x-y|^{n+2s}}dy=\int\int_{\mathbb{S}^{n-1}}\frac{r^{-\frac{n-2s}{2}}\psi(\theta)\eta_\epsilon(r)-|y|^{-\frac{n-2s}{2}}\psi(\sigma)\eta_\epsilon(|y|)}{(r^2+|y|^2-2r|y|<\theta,\sigma>)^{\frac{n+2s}{2}}}d\sigma d(|y|)$$

Now set $|y| = rt$ then

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\phi(x)-\phi(y)}{|x-y|^{n+2s}}dy &= r^{-\frac{n}{2}-s}\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{\psi(\theta)\eta_\epsilon(r) - t^{-\frac{n-2s}{2}}\psi(\sigma)\eta_\epsilon(rt)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}dtd\sigma \\
&= r^{-\frac{n}{2}-s}\int\int_{\mathbb{S}^{n-1}} \frac{\psi(\theta)\eta_\epsilon(r) - t^{-\frac{n-2s}{2}}\psi(\sigma)\eta_\epsilon(r) + t^{-\frac{n-2s}{2}}(\eta(r)\psi(\theta)-\eta_\epsilon(rt)\psi(\sigma))}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}dtd\sigma \\
&= r^{-\frac{n}{2}-s}\eta_\epsilon(r)\psi(\theta)\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1-t^{\frac{n-2s}{2}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}dtd\sigma \\
&\quad + r^{-\frac{n}{2}-s}\eta_\epsilon(r)\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{t^{n-1-\frac{n-2s}{2}}(\psi(\theta)-\psi(\sigma))}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dtd\sigma \\
&\quad + r^{-\frac{n}{2}-s}\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{t^{n-1-\frac{n-2s}{2}}(\eta_\epsilon(r)-\eta_\epsilon(rt))\psi(\sigma)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dtd\sigma
\end{aligned}$$

Define $\Lambda_{n,s} := \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1-t^{\frac{n-2s}{2}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}t^{n-1}d\sigma dt$. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\phi(x)-\phi(y)}{|x-y|^{n+2s}}dy &= r^{-\frac{n}{2}-s}\eta_\epsilon(r)\psi(\theta)\Lambda_{n,s} \\
&\quad + r^{-\frac{n}{2}-s}\eta_\epsilon(r)\int_{\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))d\sigma \\
&\quad + r^{-\frac{n}{2}-s}\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{t^{-\frac{n-2s}{2}}(\eta_\epsilon(r)-\eta_\epsilon(rt))\psi(\sigma)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}dtd\sigma
\end{aligned}$$

Applying the above, we compute the left-hand side of the stability inequality (1.2),

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{n+2s}}dxdy &= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))\phi(x)}{|x-y|^{n+2s}}dxdy \\
&= 2\int_0^\infty r^{-1}\eta_\epsilon^2(r)dr \int_{\mathbb{S}^{n-1}} \psi^2\Lambda_{n,s}d\theta \\
&\quad +2\int_0^\infty r^{-1}\eta_\epsilon^2(r)dr\int_{\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2 d\sigma d\theta \\
&\quad +2\int_0^\infty \left[\int_0^\infty r^{-1}\eta_\epsilon(r)(\eta_\epsilon(r)-\eta_\epsilon(rt))dr\right]\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \frac{t^{n-1-\frac{n-2s}{2}}\psi(\sigma)\psi(\theta)}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}d\sigma d\theta dt
\end{aligned} \tag{3.4}$$

We now compute the second term in the stability inequality (1.2) for the test function $\phi(x) = r^{-\frac{n-2s}{2}}\psi(\theta)\eta_\epsilon(r)$ and $u = r^{-\frac{2s}{p-1}}\psi(\theta)$,

$$\begin{aligned}
p\int_0^\infty r^a|u|^{p-1}\phi^2 &= p\int_0^\infty r^a r^{-(2s+a)}r^{-(n-2s)}\psi^{p+1}\eta_\epsilon^2(r)dr \\
&= p\int_0^\infty r^{-1}\eta_\epsilon^2(r)dr\int_{\mathbb{S}^{n-1}}\psi^{p+1}(\theta)d\theta
\end{aligned} \tag{3.5}$$

Due to the definition of the η_ϵ, we have $\int_0^\infty r^{-1}\eta_\epsilon^2(r)dr = \ln(2/\epsilon) + O(1)$. Note that this term appears in both terms of the stability inequality that we computed in (3.4) and (3.6). We now claim that

$$f_\epsilon(t) := \int_0^\infty r^{-1}\eta_\epsilon(r)(\eta_\epsilon(r)-\eta_\epsilon(rt))dr = O(\ln t)$$

Note that $\eta_\epsilon(rt)=1$ for $\frac{\epsilon}{t}<r<\frac{1}{t\epsilon}$ and $\eta_\epsilon(rt)=0$ for either $r<\frac{\epsilon}{2t}$ or $r>\frac{2}{t\epsilon}$. Now consider various ranges of value of $t\in(0,\infty)$ to compare the support of $\eta_\epsilon(r)$ and $\eta_\epsilon(rt)$. From the definition of η_ϵ, we have

$$f_\epsilon(t)=\int_{\frac{\epsilon}{2}}^{\frac{2}{\epsilon}} r^{-1}\eta_\epsilon(r)(\eta_\epsilon(r)-\eta_\epsilon(rt))dr$$

In what follows we consider a few cases to explain the claim. For example when $\epsilon<\frac{\epsilon}{t}<\frac{1}{\epsilon}$ then

$$f_\epsilon(t)\approx\int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{t}} r^{-1}dr+\int_{\frac{1}{\epsilon}}^{\frac{2}{\epsilon t}} r^{-1}dr\approx\ln t$$

Now consider the case $\frac{1}{\epsilon}<\frac{\epsilon}{t}<\frac{1}{\epsilon}$ then $t\approx\epsilon^2$. So,

$$f_\epsilon(t)\approx\int_{\frac{\epsilon}{2}}^{\frac{\epsilon}{t}} r^{-1}dr+\int_{\frac{\epsilon}{t}}^{\frac{2}{\epsilon}} r^{-1}dr\approx\ln t+\ln\epsilon\approx\ln t$$

Other cases can be treated similarly. From this one can see that

$$\int_0^\infty\left[\int_0^\infty r^{-1}\eta(r)(\eta(r)-\eta(rt))dr\right]\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\frac{t^{n-1-\frac{n-2s}{2}}}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}\psi(\sigma)\psi(\theta)d\sigma d\theta dt \tag{3.6}$$

$$\approx\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty\frac{t^{n-1-\frac{n-2s}{2}}\ln t}{(t^2+1-2t<\theta,\sigma>)^{\frac{n+2s}{2}}}\psi(\sigma)\psi(\theta)dtd\sigma d\theta \tag{3.7}$$

$$=O(1) \tag{3.8}$$

Collecting higher order terms of the stability inequality we get

$$\Lambda_{n,s}\int_{\mathbb{S}^{n-1}}\psi^2+\int_{\mathbb{S}^{n-1}}K_{\frac{n-2s}{2}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2d\sigma\geq p\int_{\mathbb{S}^{n-1}}\psi^{p+1} \tag{3.9}$$

From this and (3.3) we obtain

$$(\Lambda_{n,s}-pA_{n,s,a})\int_{\mathbb{S}^{n-1}}\psi^2+\int_{\mathbb{S}^{n-1}}(K_{\frac{n-2s}{2}}-pK_{\frac{2s+a}{p-1}})(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2d\sigma\geq 0$$

Note that K_α is decreasing in α. This implies $K_{\frac{n-2s}{2}}<K_{\frac{2s+a}{p-1}}$ for $p>\frac{n+2s+2a}{n-2s}$. So, $K_{\frac{n-2s}{2}}-pK_{\frac{2s+a}{p-1}}<0$. On the other hand the assumption of the theorem implies that $\Lambda_{n,s}-pA_{n,s,a}<0$. Therefore, $\psi=0$.

□

4. Energy Estimates

In this section, we provide some estimates for solutions of (1.1). These estimates are needed in the next section when we perform a blow-down analysis argument. The methods and ideas provided in this section are strongly motivated by [9, 10].

Lemma 4.1. *Let u be a stable solution to* (1.1). *Let also $\eta\in C_c^\infty(\mathbb{R}^n)$ and for $x\in\mathbb{R}^n$, define*

$$\rho(x)=\int_{\mathbb{R}^n}\frac{(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}}\,dy \tag{4.1}$$

Then,

$$\int_{\mathbb{R}^n}|x|^a|u|^{p+1}\eta^2dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)\eta(x)-u(y)\eta(y)|^2}{|x-y|^{n+2s}}dxdy\leq C\int_{\mathbb{R}^n}u^2\rho dx \tag{4.2}$$

Proof. Proof is quite similar to Lemma 2.1 in [9] and we omit it here. □

Lemma 4.2. *Let $m > n/2$ and $x \in \mathbb{R}^n$. Set*

$$\rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy \quad where \quad \eta(x) = (1+|x|^2)^{-m/2} \tag{4.3}$$

Then there is a constant $C = C(n, s, m) > 0$ such that

$$C^{-1}(1+|x|^2)^{-n/2-s} \le \rho(x) \le C(1+|x|^2)^{-n/2-s} \tag{4.4}$$

Proof. Proof is quite similar to Lemma 2.2 in [9] and we omit it here. □

Corollary 4.1. *Suppose that $m > n/2$, η given by (4.3) and $R > 1$. Define*

$$\rho_R(x) = \int_{\mathbb{R}^n} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{n+2s}} dy \quad where \quad \eta_R(x) = \eta(x/R)\phi(x) \tag{4.5}$$

where $\phi \in C^\infty(\mathbb{R}^n) \cap [0,1]$ is a cut-off function. Then there exists a constant $C > 0$ such that

$$\rho_R(x) \le C\eta\left(\frac{x}{R}\right)^2 |x|^{-n-2s} + R^{-2s}\rho\left(\frac{x}{R}\right)$$

Lemma 4.3. *Suppose that u is a stable solution of (1.1). Consider ρ_R that is defined in Corollary 4.1 for $n/2 < m < n/2 + s(p+1)/2$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} u^2 \rho_R \le CR^{n-\frac{2s(p+1)+2a}{p-1}}$$

for any $R > 1$

Proof. Note that

$$\int_{\mathbb{R}^n} u^2 \rho_R dx \le \left(\int_{\mathbb{R}^n} |x|^a ||u|^{p+1} \eta_R^2 dx\right)^{\frac{2}{p+1}} \left(\int_{\mathbb{R}^n} |x|^{-\frac{2a}{p-1}} \rho_R^{\frac{p+1}{p-1}} \eta_R^{-\frac{4}{p-1}} dx\right)^{\frac{p-1}{p+1}}$$

From Lemma 4.1 we get

$$\int_{\mathbb{R}^n} u^2 \rho_R dx \le \int_{\mathbb{R}^n} |x|^{-\frac{2a}{p-1}} \rho_R^{\frac{p+1}{p-1}} \eta_R^{-\frac{4}{p-1}} dx$$

Now applying Corollary 4.1 for two different cases $|x| > R$ and $|x| < R$ one can get $\rho_R(x) \le C(|x|^{-n-2s} + R^{-2s})$ and $\rho(x) \le CR^{-2s}\left(1 + \frac{|x|^2}{R^2}\right)^{-n2/-s}$. This finishes the proof.

Note that □

We are now ready to state the essential estimate on stable solutions. Since the proofs are similar to the ones given in [9], for the case of $0 < s < 1$, and in [16], for the case of $1 < s < 2$, we omit them here.

Lemma 4.4. *Suppose that $p \ne \frac{n+2s+2a}{n-2s}$. Let u be a stable solution of (1.1) and u_e satisfies (1.5). Then there exists a constant $C > 0$ such that*

(i) *for $0 < s < 1$*

$$\int_{B_R} y^{1-2s} u_e^2 \le CR^{n+2-\frac{2s(p+1)+2a}{p-1}}$$

and

(ii) *for $1 < s < 2$*

$$\int_{B_R} y^{3-2s} u_e^2 \le CR^{n+4-\frac{2s(p+1)+2a}{p-1}}$$

Lemma 4.5. *Let u be a stable solution of (1.1) and u_e satisfies (1.5). Then there exists a positive constant C such that*

(i) *for* $0 < s < 1$

$$\int_{B_R\cap\partial\mathbb{R}^{n+1}_+} |x|^a|u_e|^{p+1}dx + \int_{B_R\cap\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla u_e|^2 dxdy \le CR^{n-\frac{2s(p+1)+2a}{p-1}} \tag{4.6}$$

and

(ii) *for* $1 < s < 2$

$$\int_{B_R\cap\partial\mathbb{R}^{n+1}_+} |x|^a|u_e|^{p+1}dx + \int_{B_R\cap\mathbb{R}^{n+1}_+} y^{3-2s}|\Delta_b u_e|^2 dxdy \le CR^{n-\frac{2s(p+1)+2a}{p-1}} \tag{4.7}$$

5. Blow-down analysis

This section is devoted to the proof of Theorem 1.2. The methods and ideas are strongly motivated by the ones given in [9, 10].

Proof of Theorem 1.2: Let u be a stable solution of (1.1) and let u_e be its extension solving (1.5). For the case $1 < p \le p_S(n, a)$ the conclusion follows from the Pohozaev identity. Note that for the subcritical case Lemma 4.5 implies that $u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Multiplying (1.1) with u and doing integration, we obtain

$$\int_{\mathbb{R}^n} |x|^a u|^{p+1} = ||u||^2_{\dot{H}^s(\mathbb{R}^n)} \tag{5.1}$$

in addition multiplying (1.1) with $u^\lambda(x) = u(\lambda x)$ yields

$$\int_{\mathbb{R}^n} |x|^a|u|^{p-1}u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2}u(-\Delta)^{s/2}u^\lambda = \lambda^s \int_{\mathbb{R}^n} ww_\lambda$$

where $w = (-\Delta)^{s/2}u$. Following ideas provided in [10, 26] and using the change of variable $z = \sqrt{\lambda}x$ one can get the following Pohozaev identity

$$-\frac{n+a}{p+1}\int_{\mathbb{R}^n} |x|^a|u|^{p+1} = \frac{2s-n}{2}\int_{\mathbb{R}^n} w^2 + \frac{d}{d\lambda}|_{\lambda=1}\int_{\mathbb{R}^n} w^{\sqrt{\lambda}}w^{1/\sqrt{\lambda}}dz = \frac{2s-n}{2}||u||^2_{\dot{H}^s(\mathbb{R}^n)}$$

This equality together and (5.1) proves the theorem for the subcritical case.

Now suppose that $p > p_S(n, a)$.

Case 1: $0 < s < 1$**.** We perform the proof in a few steps.

Step 1. $\lim_{\lambda\to+\infty} E(u_e, \lambda) < +\infty$. From the fact that E is nondecreasing in λ, it suffices to show that $E(u_e, \lambda)$ is bounded. Write $E = I + J$, where I is given by (2.1) and

$$J(u_e, \lambda) = \lambda^{\frac{2s(p+1)+2a}{p-1}-n-1}\frac{s+a}{p+1}\int_{\partial B_\lambda\cap\mathbb{R}^{n+1}_+} y^{1-2s}u_e^2\, d\sigma$$

Note that Lemma 4.5 implies that I is bounded. To show that E is bounded we state the following argument. The nondecreasing property of E yields

$$E(u_e, \lambda) \le \frac{1}{\lambda}\int_\lambda^{2\lambda} E(u, t)dt \le C + \lambda^{\frac{2s(p+1)+2a}{p-1}-n-1}\int_{B_{2\lambda}\cap\mathbb{R}^{n+1}_+} y^{1-2s}u_e^2.$$

From Lemma 4.4 we conclude that E is bounded.

Step 2. There exists a sequence $\lambda_i \to +\infty$ such that $(u_e^{\lambda_i})$ converges weakly in $H^1_{loc}(\mathbb{R}^{n+1}_+; y^{1-2s}dydx)$ to a function u_e^∞.

This follows from the fact that $(u_e^{\lambda_i})$ is bounded in $H^1_{loc}(\mathbb{R}^{n+1}_+; y^{1-2s}dxdy)$ by Lemma 4.5.

Step 3. u_e^∞ is homogeneous.

To see this, apply the scale invariance of E, its finiteness and the monotonicity formula: given $R_2 > R_1 > 0$,

$$\begin{aligned} 0 &= \lim_{n\to+\infty} E(u_e,\lambda_i R_2) - E(u_e,\lambda_i R_1) \\ &= \lim_{n\to+\infty} E(u_e^{\lambda_i}, R_2) - E(u_e^{\lambda_i}, R_1) \\ &\geq \liminf_{n\to+\infty} \int_{(B_{R_2}\setminus B_{R_1})\cap\mathbb{R}^{n+1}_+} y^{1-2s} r^{2-n+\frac{4s+2a}{p-1}} \left(\frac{2s+a}{p-1}\frac{u_e^{\lambda_i}}{r} + \frac{\partial u_e^{\lambda_i}}{\partial r}\right)^2 dxdy \\ &\geq \int_{(B_{R_2}\setminus B_{R_1})\cap\mathbb{R}^{n+1}_+} y^{1-2s} r^{2-n+\frac{4s+2a}{p-1}} \left(\frac{2s+a}{p-1}\frac{u_e^{\infty}}{r} + \frac{\partial u_e^{\infty}}{\partial r}\right)^2 dxdy \end{aligned}$$

Note that in the last inequality we only used the weak convergence of $(u_e^{\lambda_i})$ to u_e^∞ in $H^1_{loc}(\mathbb{R}^{n+1}_+; y^{1-2s}dxdy)$. So,

$$\frac{2s+a}{p-1}\frac{u_e^{\infty}}{r} + \frac{\partial u_e^{\infty}}{\partial r} = 0 \quad a.e. \text{ in } \mathbb{R}^{n+1}_+.$$

And so, u_e^∞ is homogeneous.

Step 4. $u_e^\infty \equiv 0$. This is a direct consequence of Theorem 3.1.

Step 5. $(u_e^{\lambda_i})$ converges strongly to zero in $H^1(B_R\setminus B_\varepsilon; y^{1-2s}dxdy)$ and (u^{λ_i}) converges strongly to zero in $L^{p+1}(B_R\setminus B_\varepsilon)$ for all $R>\epsilon>0$.

From Step 2 and Step 3, we have $(u_e^{\lambda_i})$ is bounded in $H^1_{loc}(\mathbb{R}^{n+1}_+; y^{1-2s}dxdy)$ and converges weakly to 0. Therefore, $(u_e^{\lambda_i})$ converges strongly to zero in $L^2_{loc}(\mathbb{R}^{n+1}_+; y^{1-2s}dxdy)$. By the standard Rellich-Kondrachov theorem and a diagonal argument, passing to a subsequence, for any $B_R = B_R(0)\subset\mathbb{R}^{n+1}$ and A of the form $A=\{(x,t)\in\mathbb{R}^{n+1}_+ : 0<t<r/2\}$, where $R,r>0$ we obtain

$$\lim_{i\to\infty}\int_{\mathbb{R}^{n+1}_+\cap(B_R\setminus A)} y^{1-2s}|u_e^{\lambda_i}|^2\,dxdy \to 0.$$

By [12, Theorem 1.2],

$$\int_{\mathbb{R}^{n+1}_+\cap B_r(x)} y^{1-2s}|u_e^{\lambda_i}|^2\,dxdy \leq Cr^2 \int_{\mathbb{R}^{n+1}_+\cap B_r(x)} y^{1-2s}|\nabla u_e^{\lambda_i}|^2\,dxdy$$

for any $x\in\partial\mathbb{R}^{n+1}_+$, $|x|\leq R$, with a uniform constant C. Applying similar arguments as [9] one can get $(u_e^{\lambda_i})$ converges strongly to 0 in $H^1_{loc}(\mathbb{R}^{n+1}_+\setminus\{0\}; y^{1-2s}dxdy)$ and the convergence also holds in $L^{p+1}_{loc}(\mathbb{R}^n\setminus\{0\})$.

Step 6. $u_e\equiv 0$.

$$\begin{aligned} I(u_e,\lambda) &= I(u_e^\lambda,1) \\ &= \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1} y^{1-2s}|\nabla u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1} |x|^a|u_e^\lambda|^{p+1}dx \\ &= \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_\epsilon} y^{1-2s}|\nabla u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_\epsilon} |x|^a|u_e^\lambda|^{p+1}dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} y^{1-2s}|\nabla u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} |x|^a|u_e^\lambda|^{p+1}dx \\ &= \varepsilon^{n-\frac{2s(p+1)+2a}{p-1}} I(u_e,0,\lambda\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} y^{1-2s}|\nabla u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} |x|^a|u_e^\lambda|^{p+1}dx \\ &\leq C\varepsilon^{n-\frac{2s(p+1)+2a}{p-1}} + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} y^{1-2s}|\nabla u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} |x|^a|u_e^\lambda|^{p+1}dx \end{aligned}$$

Letting $\lambda\to+\infty$ and then $\varepsilon\to 0$, we deduce that $\lim_{\lambda\to+\infty} I(u_e,\lambda)=0$. Using the monotonicity of E,

$$E(u_e,\lambda) \leq \frac{1}{\lambda}\int_\lambda^{2\lambda} E(t)\,dt \leq \sup_{[\lambda,2\lambda]} I + C\lambda^{-n-1+\frac{2s(p+1)+2s}{p-1}} \int_{B_{2\lambda}\setminus B_\lambda} u_e^2 \tag{5.2}$$

and so $\lim_{\lambda\to+\infty} E(u_e,\lambda)=0$. Since u is smooth, we also have $E(u_e,0)=0$. Since E is monotone, $E\equiv 0$ and so u_e must be homogeneous, a contradiction unless $u_e\equiv 0$.

Case 2: $1<s<2$. Proof of this case is very similar to Case 1. We perform the proof in a few steps.
Step 1. $\lim_{\lambda\to\infty} E(u_e,\lambda)<\infty$.

From Theorem 2.2, E is nondecreasing. So, we only need to show that $E(u_e,\lambda)$ is bounded. Note that

$$E(u_e,\lambda)\le \frac{1}{\lambda}\int_\lambda^{2\lambda} E(u_e,t)dt \le \frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda} E(u_e,\gamma)d\gamma dt$$

From Lemma 4.5 we conclude that

$$\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\gamma^{2s\frac{p+1}{p-1}-n}\left(\int_{\mathbb{R}^{n+1}_+\cap B_\gamma}\frac{1}{2}y^{3-2s}|\Delta_b u_e|^2 dydx-\frac{C_{n,s}}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_\gamma}|x|^a u_e^{p+1}dx\right)d\gamma dt\le C$$

where $C>0$ is independent from λ. For the next term in the energy we have

$$\begin{aligned}\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\left(\gamma^{-3+2s+\frac{4s+2a}{p-1}-n}\int_{\mathbb{R}^{n+1}_+\cap\partial B_\gamma} y^{3-2s}u_e^2dydx\right)d\gamma dt &\le \frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{-3+2s+\frac{4s+2a}{p-1}-n}\int_{B_{t+\lambda}\setminus B_t} y^{3-2s}u_e^2 dydxdt\\ &\le \frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{-3+2s+\frac{4s+2a}{p-1}-n}\left(\int_{B_{3\lambda}} y^{3-2s}u_e^2dydx\right)dt\\ &\le \lambda^{n+4-\frac{2s(p+1)+2a}{p-1}}\frac{1}{\lambda^2}\int_\lambda^{2\lambda} t^{-3+2s+\frac{4s+2a}{p-1}-n}dt\\ &\le C\end{aligned}$$

where $C>0$ is independent from λ. In the above estimates we have applied Lemma 4.4. For the next term we have

$$\begin{aligned}&\frac{1}{\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\frac{\gamma^3}{2}\frac{d}{d\gamma}\left[\gamma^{2s-3-n+\frac{4s+2a}{p-1}}\int_{\partial B_\gamma} y^{3-2s}\left(\frac{2s+a}{p-1}\gamma^{-1}u_e+\frac{\partial u_e}{\partial r}\right)^2\right]d\gamma dt\\ =\;&\frac{1}{2\lambda^2}\int_\lambda^{2\lambda}[(t+\lambda)^{2s-n+\frac{4s+2a}{p-1}}\int_{\partial B_{t+\lambda}} y^{3-2s}\left(\frac{2s+a}{p-1}(t+\lambda)^{-1}u_e+\frac{\partial u_e}{\partial r}\right)^2\\ &-t^{2s-n+\frac{4s+2a}{p-1}}\int_{\partial B_\lambda} y^{3-2s}\left(\frac{2s+a}{p-1}\gamma^{-1}u_e+\frac{\partial u_e}{\partial r}\right)^2]dt\\ &-\frac{3}{2\lambda^2}\int_\lambda^{2\lambda}\int_t^{t+\lambda}\left[\gamma^{2s-1-n+\frac{4s+2a}{p-1}}\int_{\partial B_\gamma} y^{3-2s}\left(\frac{2s+a}{p-1}\gamma^{-1}u_e+\frac{\partial u_e}{\partial r}\right)^2\right]d\gamma dt\\ \le\;&\lambda^{-2+2s-n+\frac{4s+2a}{p-1}}\int_{B_{3\lambda}\setminus B_\lambda} y^{3-2s}\left(\frac{2s+a}{p-1}\lambda^{-1}u_e+\frac{\partial u_e}{\partial r}\right)^2\le C\end{aligned}$$

where $C>0$ is independent from λ. The rest of the terms can be treated similarly.
Step 2. There exists a sequence $\lambda_i\to\infty$ such that $(u_e^{\lambda_i})$ converges weakly in $H^1_{loc}(\mathbb{R}^n,y^{3-2s}dxdy)$ to a function u_e^∞.

Note that this is a direct consequence of Lemma 4.5.
Step 3. u_e^∞ is homogeneous and therefore $u_e^\infty=0$.

To prove this claim, apply the scale invariance of E, its finiteness and the monotonicity formula; given $R_2 > R_1 > 0$,

$$\begin{aligned} 0 &= \lim_{i\to\infty} \left(E(u_e, R_2\lambda_i) - E(u_e, R_1\lambda_i)\right) \\ &= \lim_{i\to\infty} \left(E(u_e^{\lambda_i}, R_2) - E(u_e^{\lambda_i}, R_1)\right) \\ &\geq \liminf_{i\to\infty} \int_{(B_{R_2}\setminus B_{R_1})\cap\mathbb{R}^{n+1}_+} y^{3-2s} r^{\frac{4s+2a}{p-1}+2s-2-n} \left(\frac{2s+a}{p-1} r^{-1} u_e^{\lambda_i} + \frac{\partial u_e^{\lambda_i}}{\partial r}\right)^2 dydx \\ &\geq \int_{(B_{R_2}\setminus B_{R_1})\cap\mathbb{R}^{n+1}_+} y^{3-2s} r^{\frac{4s+2a}{p-1}+2s-2-n} \left(\frac{2s+a}{p-1} r^{-1} u_e^{\infty} + \frac{\partial u_e^{\infty}}{\partial r}\right)^2 dydx \end{aligned}$$

In the last inequality we have used the weak convergence of $(u_e^{\lambda_i})$ to u_e^∞ in $H^1_{loc}(\mathbb{R}^n, y^{3-2s}dydx)$. This implies

$$\frac{2s+a}{p-1} r^{-1} u_e^\infty + \frac{\partial u_e^\infty}{\partial r} = 0 \ \text{ a.e. } \ \text{ in } \ \mathbb{R}^{n+1}_+.$$

Therefore, u_e^∞ is homogeneous. Apply Theorem 3.1 we get $u_e^\infty = 0$.
Step 5. $(u_e^{\lambda_i})$ converges strongly to zero in $H^1(B_R \setminus B_\epsilon, y^{3-2s}dydx)$ and $(u_e^{\lambda_i})$ converges strongly to zero in $L^{p+1}(B_R \setminus B_\epsilon)$ for all $R > \epsilon > 0$.
Step 6. $u_e \equiv 0$.

$$\begin{aligned} I(u_e,\lambda) &= I(u_e^\lambda, 1) \\ &= \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1} y^{3-2s}|\Delta_b u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1} |x|^a |u_e^\lambda|^{p+1} dx \\ &= \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_\epsilon} y^{3-2s}|\Delta_b u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_\epsilon} |x|^a |u_e^\lambda|^{p+1} dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} y^{3-2s}|\Delta_b u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} |x|^a |u_e^\lambda|^{p+1} dx \\ &= \varepsilon^{n-\frac{2s(p+1)+2a}{p-1}} I(u_e, \lambda\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} y^{3-2s}|\Delta_b u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} |x|^a |u_e^\lambda|^{p+1} dx \\ &\leq C\varepsilon^{n-\frac{2s(p+1)+2a}{p-1}} + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} y^{3-2s}|\Delta u_e^\lambda|^2 dxdy - \frac{\kappa_s}{p+1}\int_{\partial\mathbb{R}^{n+1}_+\cap B_1\setminus B_\epsilon} |x|^a |u_e^\lambda|^{p+1} dx \end{aligned}$$

Letting $\lambda \to +\infty$ and then $\varepsilon \to 0$, we deduce that $\lim_{\lambda\to+\infty} I(u_e,\lambda) = 0$. Using the monotonicity of E,

$$E(u_e,\lambda) \leq \frac{1}{\lambda}\int_\lambda^{2\lambda} E(t)\, dt \leq \sup_{[\lambda,2\lambda]} I + C\lambda^{-n-1+\frac{2s(p+1)+2a}{p-1}} \int_{B_{2\lambda}\setminus B_\lambda} u_e^2 \tag{5.3}$$

and so $\lim_{\lambda\to+\infty} E(u_e,\lambda) = 0$. Since u is smooth, we also have $E(u_e, 0) = 0$. Since E is monotone, $E \equiv 0$ and so u_e must be homogeneous, a contradiction unless $u_e \equiv 0$.

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