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On staggered indecomposable Virasoro modules

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In this article, certain indecomposable Virasoro modules are studied. Specifically, the Virasoro mode L_0 is assumed to be nondiagonalizable, possessing Jordan blocks of rank 2. Moreover, the module is further assumed to have a highest weight submodule, the "left module," and that the quotient by this submodule yields another highest weight module, the "right module." Such modules, which have been called *staggered*, have appeared repeatedly in the logarithmic conformal field theory literature, but their theory has not been explored in full generality. Here, such a theory is developed for the Virasoro algebra using rather elementary techniques. The focus centers on two different but related questions typically encountered in practical studies: How can one identify a given staggered module, and how can one demonstrate the existence of a proposed staggered module. Given just the values of the highest weights of the left and right modules, themselves subject to simple necessary conditions, invariants are defined which together with the knowledge of the left and right modules uniquely identify a staggered module. The possible values of these invariants form a vector space of dimension 0, 1, or 2, and the structures of the left and right modules limit the isomorphism classes of the corresponding staggered modules to an affine subspace (possibly empty). The number of invariants and affine restrictions is purely determined by the structures of the left and right modules. Moreover, in order to facilitate applications, the expressions for the invariants and restrictions are given by formulas as explicit as possible (they generally rely on expressions for Virasoro singular vectors). Finally, the text is liberally peppered throughout with examples illustrating the general concepts. These have been carefully chosen for their physical relevance or for the novel features they exhibit. © *2009 American Institute of Physics.*
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I. INTRODUCTION

The successes of conformal field theory, in particular, its applications to condensed matter physics, depended crucially on the theory of highest weight modules of the Virasoro algebra. Such a theory became available in the early eighties, ultimately due to the work of Kac[1] and Feigin and Fuchs.[2] The corresponding conformal field theories, the minimal models of Ref. 3 are constructed from a certain finite collection of irreducible highest weight Virasoro modules and rightly enjoy their position as some of the simplest and most useful of conformal field theories.

In spite of this, the past 15 years have witnessed the construction, in varying degrees, of a different kind of conformal field theory.[4,5] These theories are constructed from certain indecomposable, rather than irreducible, modules and are collectively known as logarithmic conformal field theories (LCFT). Despite a promising beginning, logarithmic theories quickly attained a reputation for being esoteric and technical. Some impressive examples were constructed, but the

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field suffered from a perceived lack of concrete applications. To be sure, there were many attempts to use logarithmic theories to explain discrepancies in models of the fractional quantum Hall effect, Abelian sandpiles, D-brane recoil, and more (see Ref. 6, for references to these), but none of these attempts really left an enduring mark upon their intended field. Nevertheless, condensed matter physicists remained interested in these theories for the simple reason that the standard minimal model description of many of their favorite models was known to be incomplete or even entirely missing.

Recently, there has been something of a resurgence in the study of logarithmic conformal field theories, with the aim of clarifying applications to condensed matter physics and developing the mathematical properties of logarithmic theories so as to more closely mirror those of standard theories. One can isolate several different approaches including free field methods and connections to quantum group theory,[7,8] lattice model constructions,[9,10] and construction through explicit fusion.[11,12] All of these involve exploring the new features of a theory built from indecomposable but reducible modules. Intriguingly, recent developments in random conformally invariant fractals, Schramm–Loewner evolutions (SLE) in particular,[13] have started to bridge the gap between the field-theoretic and probabilistic approaches to the statistical models of condensed matter theory (see Refs. 14–16 for reviews). In particular, the kernel of the infinitesimal generator of the Schramm–Loewner evolution, which consists of local martingales of the stochastic growth process that builds the fractal curve, carries a representation of the Virasoro algebra,[17–19] and it has recently been observed that in certain cases this representation becomes indecomposable, of the type found in logarithmic conformal field theory.[20] This has led to renewed proposals for some sort of SLE-LCFT correspondence.[21–23]

Advances such as these have necessitated a better understanding of the representation theory of the Virasoro algebra beyond highest weight modules. In the corresponding logarithmic theories, the Virasoro element L_0 acts nondiagonalisably, manifestly demonstrating that more general classes of modules are required. One such class consists of the so-called *staggered* modules and it is these which we will study in what follows. More precisely, we will consider indecomposable Virasoro modules on which L_0 acts nondiagonalizably and which generalize highest weight modules by having a submodule isomorphic to a highest weight module such that the quotient by this submodule is again isomorphic to a highest weight module. We refer to the submodule and its quotient as the left and right modules, respectively (the naturality of this nomenclature will become evident in Sec. III). Roughly speaking, these staggered modules can be visualized as two highest weight modules which have been "glued" together by a nondiagonalizable action of L_0. Such staggered modules were first constructed for the Virasoro algebra in Ref. 24.

We mention that staggered Virasoro modules corresponding to gluing more than two highest weight modules together have certainly been considered in the literature,[11,25] but we shall not do so here. Similarly, one could try to develop staggered module theories for other algebras which arise naturally in logarithmic conformal field theories. We will leave such studies for future work, noting only that we expect that the results we are reporting will provide a very useful guide to the eventual form of these generalizations. Here, we restrict ourselves to the simplest case, treating it in as elementary a way as possible. We hope that the resulting clarity will allow the reader to easily apply our results and to build upon them. Our belief is that this simple case will be a correct and important step toward a more complete representation theory applicable to general logarithmic conformal field theories.

No introduction to these representation-theoretic aspects of logarithmic conformal field theory could be complete without mentioning the seminal contributions of Rohsiepe. These appeared 13 years ago as a preprint,[26] which to the best of our knowledge was never published, and a dissertation in German.[27] As far as we are aware, these are the only works which try to systematically develop a representation theory for the Virasoro algebra, keeping in mind applications to logarithmic conformal field theory [specifically the so-called $LM(1,q)$ theories of Ref. 24]. Indeed, it was Rohsiepe who first introduced the term "staggered module," although in a setting rather more general than we use it here. These references contain crucial insights on how to start building the theory and treat explicitly a particular subcase of the formalism we construct. We clearly owe a lot to the ideas and results contained therein.

On the other hand, Rohsiepe's formulation of the problem in Ref. 26 is somewhat different to our own, which in our opinion has made applying his results a little bit inconvenient. Moreover, an unfortunate choice of wording in several of his statements, as well as in the introduction and conclusions, can lead the casual reader to conclude that the results have been proven in a generality significantly exceeding the actuality. Finally, the article seems to contain several inaccuracies and logical gaps which we believe deserve correction and filling (respectively). We depart somewhat from the notation and terminology of Ref. 26 when we feel that it is important for clarity.

We have organized our article as follows. Section II introduces the necessary basics—the Virasoro algebra, some generalities about its representations, and most importantly the result of Feigin and Fuchs describing the structure of highest weight modules. This section also serves to introduce the notation and conventions that we shall employ throughout. In Sec. III, we then precisely define our staggered modules and state the question which we are trying to answer. Here again, we fix notation and conventions. The rest of the section is devoted to observing some simple but important consequences of our definitions. In particular, we derive some basic necessary conditions that must be satisfied by a staggered module and show how to determine when two staggered modules are isomorphic. This gives us a kind of uniqueness result.

Section IV then marks the beginning of our study of the far more subtle question of existence. Here, we prove an existence result by explicitly constructing staggered modules, noting that we succeed precisely when a certain condition is satisfied. This condition is not yet in a particularly amenable form, but it does allow us to deduce two useful results which answer the existence question for certain staggered modules provided that the answer has been found for certain other staggered modules. These results are crucial to the development that follows. In particular, we conclude that if a staggered module exists, then the module obtained by replacing its right module by a Verma module (with the same highest weight) also exists.

We then digress briefly to set up and prove a technical result, the Projection Lemma, which will be used later to reduce the enormous number of staggered module possibilities to the consideration of a finite number of cases. This is the subject of Sec. V. We then turn in Sec. VI to the existence question in the case when the right module is a Verma module, knowing that this case is the least restrictive. Our goal is to reduce the not-so-amenable condition for the existence which we derived in Sec. IV to a problem in linear algebra. This is an admittedly lengthy exercise, with four separate cases of varying difficulty to be considered (thanks to the Projection Lemma). The result is nevertheless a problem that we can solve, and its solution yields a complete classification of staggered modules whose right module is Verma. This is completed in Sec. VI D. We then consider in Sec. VI E how to distinguish different staggered modules within the space of isomorphism classes, when their left and right modules are the same. This is achieved by introducing invariants of the staggered module structure and proving that they completely parametrize this space.

Having solved the case when the right module is Verma, we attack the general case in Sec. VII. We first characterize when one can pass from Verma to general right modules in terms of singular vectors of staggered modules. This characterization is then combined with the Projection Lemma to deduce the classification of staggered modules in all but a finite number of cases. Unhappily, our methods do not allow us to completely settle the outstanding cases, but we outline what we expect in Sec. VII C based on theoretical arguments and studying an extensive collection of examples. Finally, we present our results in Sec. VIII in a self-contained summary. Throughout, we attempt to illustrate the formalism that we are developing with relevant examples, many of which have a physical motivation and are based on explicit constructions in logarithmic conformal field theory or Schramm–Loewner evolution.

II. NOTATION, CONVENTIONS, AND BACKGROUND

Our interest lies in the indecomposable modules of the Virasoro algebra, $\mathfrak{vir}$. These are modules which cannot be written as a direct sum of two (nontrivial) submodules, and therefore generalize the concept of irreducibility. The Virasoro algebra is the infinite-dimensional (complex) Lie algebra spanned by modes L_n ($n \in \mathbb{Z}$) and C, which satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C \quad \text{and} \quad [L_m, C] = 0. \tag{2.1}$$

The mode C is clearly central and, in fact, spans the center of $\mathfrak{vir}$. We will assume from the outset that C can be diagonalized on the modules we consider (this is certainly true of the modules which have been studied by physicists). Its eigenvalue c on an indecomposable module is then well defined and is called the central charge of that module. We will always assume that the central charge is real. Note that under the adjoint action, $\mathfrak{vir}$ is itself an indecomposable $\mathfrak{vir}$-module with central charge $c=0$.

In applications, the central charges of the relevant indecomposable modules usually all coincide. It therefore makes sense to speak of the central charge of a theory. To compare different theories, it is convenient to parametrize the central charge, and a common parametrization is the following:

$$c = 13 - 6(t + t^{-1}). \tag{2.2}$$

This is clearly symmetric under $t \leftrightarrow t^{-1}$. For $c \leqslant 1$, we may take $t \geqslant 1$. For $c \geqslant 25$, we may take $t \leqslant -1$. When $1 < c < 25$, t must be taken complex. Many physical applications correspond to t rational, so we may write $t=q/p$ with $\gcd\{p,q\}=1$. In this case, the above parametrization becomes

$$c = 1 - \frac{6(p-q)^2}{pq}. \tag{2.3}$$

The Virasoro algebra is moreover graded by the eigenvalue of L_0 under the adjoint action. Note, however, that this action on L_n gives $-nL_n$—the index and the grade are opposite one another. This is a consequence of the factor $(m-n)$ on the right hand side of Eq. (2.1). Changing this to $(n-m)$ by replacing L_n by $-L_n$ would alleviate this problem, and, in fact, this is often done in the mathematical literature. However, we shall put up with this minor annoyance as it is this definition which is used, almost universally, by the physics community.

The Virasoro algebra admits a triangular decomposition into subalgebras,

$$\mathfrak{vir} = \mathfrak{vir}^- \oplus \mathfrak{vir}^0 \oplus \mathfrak{vir}^+, \tag{2.4}$$

in which $\mathfrak{vir}^\pm$ is spanned by the modes L_n with n positive or negative (as appropriate) and $\mathfrak{vir}^0$ is spanned by L_0 and C. We note that $\mathfrak{vir}^+$ is generated as a Lie subalgebra by the modes L_1 and L_2 (and similarly for $\mathfrak{vir}^-$). This follows recursively from the fact that commuting L_1 with L_n gives a nonzero multiple of L_{n+1}, for $n \geqslant 2$. The corresponding Borel subalgebras will be denoted by $\mathfrak{vir}^{\leqslant 0} = \mathfrak{vir}^- \oplus \mathfrak{vir}^0$ and $\mathfrak{vir}^{\geqslant 0} = \mathfrak{vir}^0 \oplus \mathfrak{vir}^+$. We mention that this triangular decomposition respects the standard anti-involution of the Virasoro algebra which is given by

$$L_n^\dagger = L_{-n} \quad \text{and} \quad C^\dagger = C, \tag{2.5}$$

extended linearly to the whole algebra. We shall often refer to this as the adjoint.[28]

We will frequently find it more convenient to work within the universal enveloping algebra of the Virasoro algebra. As we are assuming that C always acts as $c\mathbf{1}$ on representations, we find it convenient to make this identification from the outset. In other words, we quotient the universal enveloping algebra of $\mathfrak{vir}$ by the ideal generated by $C - c\mathbf{1}$. We denote this quotient by $\mathcal{U}$ and will henceforth abuse terminology by referring to it as the universal enveloping algebra of $\mathfrak{vir}$. Similarly, the universal enveloping algebras of $\mathfrak{vir}^-$, $\mathfrak{vir}^+$, $\mathfrak{vir}^{\leqslant}$, and $\mathfrak{vir}^{\geqslant}$ will be denoted by $\mathcal{U}^-$, $\mathcal{U}^+$, $\mathcal{U}^{\leqslant}$, and $\mathcal{U}^{\geqslant}$, respectively. The latter two are also to be understood as quotients in which C and $c\mathbf{1}$ are identified.

The universal enveloping algebra is a $\mathfrak{vir}$-module under left multiplication. Moreover, it is also an L_0-graded $\mathfrak{vir}$-module with central charge 0 under the (induced) adjoint action, and it is convenient to have a notation for the homogeneous subspaces. We let $\mathcal{U}_n$ denote the elements $U \in \mathcal{U}$ for which

$$L_0 U - U L_0 = nU. \tag{2.6}$$

Note that Eq. (2.1) forces $L_n \in \mathcal{U}_{-n}$. We moreover remark that the adjoint (2.5) extends to an adjoint on $\mathcal{U}$ in the obvious fashion: $(L_{n_1} \cdots L_{n_k})^\dagger = L_{-n_k} \cdots L_{-n_1}$.

The most important fact about universal enveloping algebras is the Poincaré–Birkhoff–Witt theorem which states, for $\mathfrak{vir}$, that the set

$$\{\cdots L_{-m}^{a_{-m}} \cdots L_{-1}^{a_{-1}} L_0^{a_0} L_1^{a_1} \cdots L_n^{a_n} \cdots : a_i \in \mathbb{N} \text{ with only finitely many } a_i \neq 0\}$$

constitutes a basis of $\mathcal{U}$. Similar results are valid for $\mathcal{U}^-$, $\mathcal{U}^+$, $\mathcal{U}^{\leqslant}$, and $\mathcal{U}^{\geqslant}$ (a proof valid for quite general universal enveloping algebras may be found in Ref. 32). Two simple but useful consequences of this are that $\mathcal{U}$ and its variants have no zero divisors and that

$$\dim \mathcal{U}^{\pm}_{\mp n} = p(n), \tag{2.7}$$

where $p(n)$ denotes the number of partitions of $n \in \mathbb{N}$.

As we have a triangular decomposition, we can define highest weight vectors and Verma modules. A highest weight vector for $\mathfrak{vir}$ is an eigenvector of $\mathfrak{vir}^0$ which is annihilated by $\mathfrak{vir}^+$. To construct a Verma module, we begin with a vector v. We make the space $\mathbb{C}v$ into a $\mathfrak{vir}^{\geqslant 0}$-module (hence a $\mathcal{U}^{\geqslant 0}$-module) by requiring that v is an eigenvector of $\mathfrak{vir}^0$ which is annihilated by $\mathfrak{vir}^+$ (v is then a highest weight vector for $\mathfrak{vir}^{\geqslant 0}$). Finally, the Verma module is then the $\mathfrak{vir}$-module,

$$\mathbb{C}v \underset{\mathcal{U}^{\geqslant 0}}{\otimes} \mathcal{U},$$

in which the Virasoro action is just by left multiplication on the second factor. This is an example of the induced module construction. Roughly speaking, it just amounts to letting $\mathfrak{vir}^-$ act freely on the highest weight vector v. In particular, we may identify this Verma module with $\mathcal{U}^- v$.

It follows that Verma modules are completely characterized by their central charge c and the eigenvalue h of L_0 on their highest weight vector. We will therefore denote a Verma module by $\mathcal{V}_{h,c}$ (although we will frequently omit the c-dependence when this is clear from the context). Its highest weight vector will be similarly denoted by $v_{h,c}$ (so $\mathcal{V}_{h,c} = \mathcal{U}^- v_{h,c}$). The Poincaré–Birkhoff–Witt theorem for $\mathcal{U}^-$ then implies that $\mathcal{V}_{h,c}$ has the following basis:

$$\{L_{-n_1} L_{-n_2} \cdots L_{-n_k} v_{h,c} : k \geqslant 0 \text{ and } n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1\}.$$

L_0 is thus diagonalizable on $\mathcal{V}_{h,c}$, so $\mathcal{V}_{h,c}$ may be graded by the L_0-eigenvalues relative to that of the highest weight vector. These eigenvalues are called the conformal dimensions of the corresponding eigenstates. The homogeneous subspaces $(\mathcal{V}_{h,c})_n = \mathrm{Ker}(L_0 - h - n)$ are finite dimensional and, in fact,

$$\dim (\mathcal{V}_{h,c})_n = p(n), \tag{2.8}$$

by Eq. (2.7). Finally, each $\mathcal{V}_{h,c}$ admits a unique symmetric bilinear form $\langle \cdot , \cdot \rangle_{\mathcal{V}_{h,c}}$, contravariant with respect to the adjoint (2.5), $\langle u', Uu \rangle = \langle U^\dagger u', u \rangle$, and normalized by $\langle v_{h,c}, v_{h,c} \rangle = 1$ (we will usually neglect to specify the module with a subscript index when this causes no confusion). This is referred to as the Shapovalov form of $\mathcal{V}_{h,c}$.[29] We will also refer to it as the scalar product. Note that distinct homogeneous subspaces are orthogonal with respect to this form.

A useful alternative construction of the Verma module $\mathcal{V}_{h,c}$ is to instead regard it as the quotient of $\mathcal{U}$ (regarded now as a $\mathfrak{vir}$-module under left multiplication) by the left ideal (left submodule) $\mathcal{I}$ generated by $L_0-h\mathbf{1}$, L_1, and L_2 (recall that L_1 and L_2 generate $\mathfrak{vir}^+$ hence $\mathcal{U}^+$). It is easy to check that the equivalence class of the unit $[\mathbf{1}]$ is a highest weight vector of $\mathcal{V}_{h,c}$ with the correct conformal dimension and central charge. We will frequently use the consequence that any element $U \in \mathcal{U}$ which annihilates the highest weight vector of $\mathcal{V}_{h,c}$ must belong to $\mathcal{I}$: If $Uv_{c,h}=0$, then

$$U = U_0(L_0 - h\mathbf{1}) + U_1 L_1 + U_2 L_2 \quad \text{for some } U_0, U_1, U_2 \in \mathcal{U}. \tag{2.9}$$

As Verma modules are cyclic (generated by acting upon a single vector), they are necessarily indecomposable. However, they need not be irreducible. If the Verma module $\mathcal{V}_{h,c}$ is reducible, then it can be shown that there exists another L_0-eigenvector, not proportional to $v_{h,c}$, which is annihilated by $\mathfrak{vir}^+$. Such vectors are known as *singular vectors*. If there is a singular vector $w \in \mathcal{V}_{h,c}$ at grade n, then it generates a submodule isomorphic to $\mathcal{V}_{h+n,c}$. Conversely, every submodule of a Verma module is generated by singular vectors. Any quotient of a Verma module by a proper submodule is said to be a highest weight module. It follows that such a quotient also has a cyclic highest weight vector (in fact, this is the usual definition of a highest weight module) with the same conformal dimension and central charge as that of the Verma module. Moreover, it inherits the obvious L_0-grading. Finally, factoring out the maximal proper submodule gives an irreducible highest weight module, which we will denote by $\mathcal{L}_{h,c}$ (or $\mathcal{L}_h$ when c is contextually clear).

We pause here to mention that in the physics literature, the term "singular vector" is often used to emphasize that the highest weight vector in question is not the one from which the entire highest weight module is generated (that is, it is not the cyclic highest weight vector). This is rather inconvenient from a mathematical point of view, but is natural because of the following calculation: If $w \in (\mathcal{V}_{h,c})_n$ is a singular vector (with $n>0$), then for all $w' = Uv_{h,c} \in (\mathcal{V}_{h,c})_n$ (so $U \in \mathcal{U}_n^-$), hence for all $w' \in \mathcal{V}_{h,c}$,

$$\langle w, w' \rangle = \langle w, Uv_{h,c} \rangle = \langle U^\dagger w, v_{h,c} \rangle = 0, \tag{2.10}$$

as $U^\dagger \in \mathcal{U}_{-n}^+$ with $n>0$. We will, however, follow the definition used in mathematics in which a singular vector is precisely a highest weight vector, qualifying those which are not generating as *proper*. We will also frequently express a singular vector in the form $w = Xv_{h,c}$, $X \in \mathcal{U}^-$, in which case we will also refer to X as being singular.[30]

Let us further define a descendant of a singular vector w to be an element of $\mathcal{U}^- w$. The above calculation then states that proper singular vectors and their descendants have vanishing scalar product with all of $\mathcal{V}_{h,c}$, including themselves.[31] It now follows that the maximal proper submodule of $\mathcal{V}_{h,c}$ is precisely the subspace of vectors which are orthogonal to $\mathcal{V}_{h,c}$. The Shapovalov form $\langle \cdot , \cdot \rangle_{\mathcal{V}_{h,c}}$ therefore descends to a well-defined symmetric bilinear form $\langle \cdot , \cdot \rangle_{\mathcal{K}}$ on any highest weight module $\mathcal{K}$ (also called the Shapovalov form). It is nondegenerate if and only if $\mathcal{K}$ is irreducible.

Through a cleverly arranged computation,[33] it is not hard to show the following facts: In a Virasoro Verma module, there can only exist one singular vector $w = Xv_{h,c}$, up to constant multipliers, at any given grade n (that is, with $X \in \mathcal{U}_n^-$). Moreover, the coefficient of L_{-1}^n when X is written in the Poincaré–Birkhoff–Witt-basis is never zero. If this coefficient is unity, we will say that X is *normalized*, and by association, that w is also normalized. This particular normalization is convenient because it does not depend on whether we choose to represent X as a sum of monomials ordered in our standard Poincaré–Birkhoff–Witt manner or with respect to some other ordering. We note explicitly that $v_{h,c}$ is a normalized singular vector. This normalization also extends readily to cover general highest weight modules: A (nonzero) singular vector of such a module will be said to be normalized if it is the projection of a normalized singular vector of the corresponding Verma module.

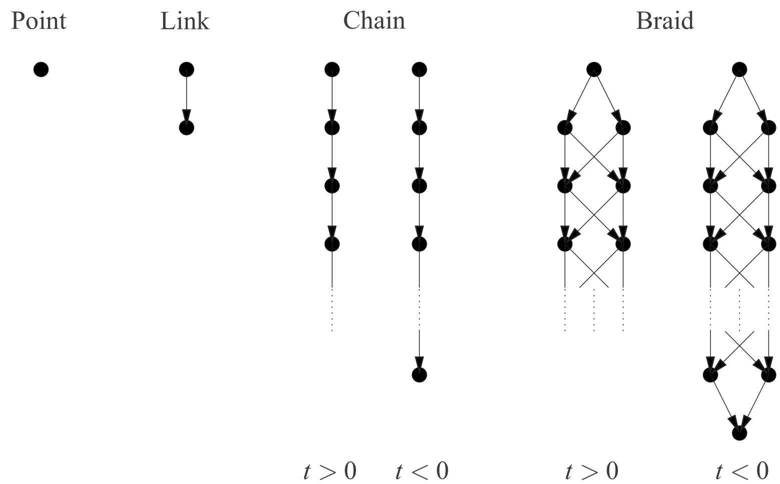

FIG. 1. The singular vector structure (including the highest weight vector), marked by black circles, of Virasoro Verma modules. The arrows from one vector to another indicate that the latter is a descendant of the former and not vice versa. Point and link-type Verma modules occur for all central charges. Chain and braid-type modules occur only when t is rational and nonzero. Note that $t>0$ corresponds to $c\leqslant 1$ and $t<0$ corresponds to $c\geqslant 25$.

A far more difficult, but nevertheless fundamental, result in Virasoro algebra representation theory concerns the explicit evaluation of the determinant of the Shapovalov form, restricted to $(\mathcal{V}_{h,c})_n$. The vanishing of this determinant indicates the existence of proper singular vectors (and their descendants), so understanding the submodule structure of highest weight modules reduces, to a large extent, to finding the zeros of the *Kac determinant formula*,

$$\det \langle \cdot , \cdot \rangle|_{(\mathcal{V}_{h,c})_n} = \alpha_n \prod_{\substack{r,s\in\mathbb{Z}_+ \\ rs\leqslant n}} (h-h_{r,s})^{p(n-rs)}. \tag{2.11}$$

Here, α_n is a nonzero constant independent of h and c, and the $h_{r,s}$ vary with c according to

$$h_{r,s} = \frac{r^2-1}{4}t - \frac{rs-1}{2} + \frac{s^2-1}{4}t^{-1} = \frac{(ps-qr)^2-(p-q)^2}{4pq}, \tag{2.12}$$

when c is parametrized as in Eqs. (2.2) and (2.3) (respectively). This determinant vanishes when $h=h_{r,s}$ for some $r,s\in\mathbb{Z}_+$ with $rs\leqslant n$. Given such an $h=h_{r,s}$, then it can be shown that there exists a (proper) singular vector at grade rs.

The Kac determinant formula was conjectured by Kac in Ref. 1 and proven by Feigin and Fuchs in Ref. 2. Reasonably accessible treatments may be found in Refs. 34 and 35. Feigin and Fuchs then used this formula to find all the homomorphisms between Verma modules, effectively determining the singular vector structure of any Verma module.[36] It turns out to be convenient to distinguish four different types of structures which we illustrate in Fig. 1. We will refer to these as "point," "link," "chain," or "braid-type" Verma modules (hopefully this notation is self-explanatory). These correspond to the cases I, II_0 and II_- (point), II_+ (link), $\text{III}_{\pm}^{0}$ and $\text{III}_{\pm}^{00}$ (chain), and $\text{III}_{\pm}$ (braid), in the notation of Feigin and Fuchs. We will also say that more general highest weight modules are of the above types, defined through inheriting their type from the corresponding Verma module.

We take this opportunity to describe when each of these cases occurs (see Ref. 36 for further details) and to introduce some useful notation for each. Recall that each $h_{r,s}$ depends on t and that t parametrizes the central charge via Eq. (2.2).

Point: If t and h are such that $h\neq h_{r,s}$ for every $r,s\in\mathbb{Z}_+$, then $\mathcal{V}_h$ is irreducible and there are no highest weight vectors besides the multiples of the cyclic highest weight vector v_h.

Link: Suppose that $t\notin\mathbb{Q}$ (recall that t may be complex) and that there exist $r,s\in\mathbb{Z}_+$ (unique since t is not rational) such that $h=h_{r,s}$. Then $\mathcal{V}_h$ possesses a singular vector at grade rs which generates the maximal proper submodule of $\mathcal{V}_h$. This maximal proper submodule, itself isomorphic to $\mathcal{V}_{h+rs}$, is then of point type, so there are no other nontrivial singular vectors. We denote the normalized singular vector at grade rs by $X_1 v_h$ ($X_1 \in \mathcal{U}_{rs}^-$ is therefore also normalized), and for compatibility with the chain case, we will denote the grade of this singular vector by $\ell_1 = rs$.

Chain: Suppose that $t=q/p$ with $p \in \mathbb{Z}_+$ and $q \in \mathbb{Z}\setminus\{0\}$ relatively prime, and that $h=h_{r,s}$ for some $r,s \in \mathbb{Z}_+$ with $p|r$ or $q|s$. Then, choosing r and s such that $h=h_{r,s}$ and $rs>0$ is minimal, $\mathcal{V}_h$ has a singular vector at grade rs which generates the maximal proper submodule, itself isomorphic to $\mathcal{V}_{h+rs}$. In contrast to the link case, this maximal proper submodule is also of chain type, except in the degenerate case where $t<0$, $r \leqslant p$, and $s \leqslant |q|$, in which case it is of point type. Thus, we iteratively find a sequence of singular vectors as in Fig. 1. This sequence is infinite if t is positive and finite if t is negative (terminating with a degenerate case). We write the normalized singular vectors of $\mathcal{V}_h$ as $v_h = X_0 v_h, X_1 v_h, X_2 v_h, \ldots$, and denote their respective grades by $0=\ell_0<\ell_1<\ell_2<\cdots$ (so $X_k \in \mathcal{U}_{\ell_k}^-$).

Braid: Suppose that $t=q/p$ with $p \in \mathbb{Z}_+$ and $q \in \mathbb{Z}\setminus\{0\}$ relatively prime, and that $h=h_{r,s}$ for some $r,s \in \mathbb{Z}_+$ with $p \nmid r$ and $q \nmid s$. Choose r, s, r', and s' such that $h=h_{r,s}=h_{r',s'}$, $rs>0$ is minimal and $r's'>rs$ is minimal but for rs (such r',s' always exist except in certain degenerate cases which we will describe below). Then $\mathcal{V}_h$ has two singular vectors, $X_1^- v_h$ and $X_1^+ v_h$ at grades $h+rs$ and $h+r's'$, respectively. Together they generate the maximal proper submodule (not a highest weight module in this case). The Verma modules generated by these two singular vectors (separately) are again of braid type (except in the degenerate cases), and their intersection is the maximal proper submodule of either. One therefore finds a double sequence of singular vectors in this case, as illustrated in Fig. 1. As in the chain case, these sequences are infinite if t is positive and finite if t is negative.

The degenerate cases referred to above occur when $t<0$, $r<p$, and $s<|q|$. Then, there are no labels r',s' to be found, the maximal proper submodule is generated by a single singular vector and is, in fact, of point type. In the nondegenerate cases, we write the normalized singular vectors of $\mathcal{V}_h$ as $v_h = X_0^+ v_h, X_1^- v_h, X_1^+ v_h, X_2^- v_h, X_2^+ v_h, \ldots$, denoting their respective grades by $0=\ell_0^+<\ell_1^-<\ell_1^+<\ell_2^-<\ell_2^+<\cdots$ (so $X_k^{\pm} \in \mathcal{U}_{\ell_k^{\pm}}^-$). When $t<0$, the double sequence of singular vectors terminates because of the above degenerate cases, so for some k, there is no X_k^+ and the singular vector $X_k^- v_h$ generates an irreducible Verma module.

Note that when it comes to the submodule structure, the link case is identical to the degenerate cases of both the chain and braid cases. However, we emphasize that chain- and braid-type modules only exist when t is rational. With this proviso in mind, we can (and often will) treat the link case as a subcase of the chain case.

Suppose that for a (normalized) singular vector $w = X v_{h,c}$, we can factor $X \in \mathcal{U}^-$ nontrivially as $X'X''$, where $X'' v_{h,c}$ is again (normalized and) singular. We will then say that w (and X) is composite. Otherwise, w (and X) is said to be prime. A composite singular vector is then just one which is a proper descendant of another (proper) singular vector. We can generalize this by further factoring X as $X^{(1)} X^{(2)} \cdots X^{(\rho)}$, where $X^{(i)} X^{(i+1)} \cdots X^{(\rho)} v_{h,c}$ is (normalized and) singular for all i. Such factorizations will not be unique, but when they cannot be further refined, we will say that each $X^{(i)}$ is prime. Such prime factorizations need not be unique either when the Verma module is of braid type, but it is easy to check from the above classification that for these factorizations the number of factors ρ is constant. We will refer to ρ as the rank of the singular vector $w = X v_{h,c}$. Rank-1 singular vectors are therefore prime, and we may regard the cyclic highest weight vector as the (unique) rank-0 singular vector. In our depiction of Verma modules (Fig. 1), the singular vector rank corresponds to the vertical axis (pointing down).

III. STAGGERED MODULES

The central objects of our study are the so-called *staggered modules* of Rohsiepe.[26] The simplest nontrivial case, which is all that will concern us, is the following: A staggered module $\mathcal{S}$ is an indecomposable $\mathfrak{vir}$-module for which we have a short exact sequence,

$$0 \to \mathcal{H}^L \xrightarrow{\iota} \mathcal{S} \xrightarrow{\pi} \mathcal{H}^R \to 0, \tag{3.1}$$

in which it is understood that $\mathcal{H}^L$ and $\mathcal{H}^R$ are highest weight modules, ι and π are module homomorphisms, and L_0 is *not* diagonalizable on $\mathcal{S}$, possessing instead Jordan cells of rank at most 2. When we refer to a module as being staggered, we have these restrictions in mind. In particular, our staggered modules are extensions of one highest weight module by another. As we are assuming that indecomposable modules such as $\mathcal{S}$ have a well-defined central charge, those of $\mathcal{H}^L$ and $\mathcal{H}^R$ must coincide. More generally, one could consider indecomposable modules constructed from more than two highest weight modules, and with higher-rank Jordan cells for L_0, but we shall not do so here.

We call $\mathcal{H}^L$ and $\mathcal{H}^R$ the *left* and *right* modules (of $\mathcal{S}$) and denote their highest weight vectors by x^L and x^R, with (real) conformal dimensions h^L and h^R, respectively. $\mathcal{H}^L$ is then a submodule of $\mathcal{S}$ [we will frequently forget to distinguish between $\mathcal{H}^L$ and $\iota(\mathcal{H}^L)$], whereas $\mathcal{H}^R$ is not (in general). We remark that Rohsiepe uses similar nomenclature in this case, defining "lower" and "upper modules" such that the latter is the quotient of the staggered module by the former. However, we stress that these do not, in general, coincide with our left and right modules. In particular, Rohsiepe defines his lower module to be the submodule of all L_0-eigenvectors, which need not be a highest weight module (a concrete illustration of this will be given in Example 2 and the remark following it—the general phenomenon will be discussed after Proposition 7.2).

Our question is the following:
Given two highest weight modules $\mathcal{H}^L$ and $\mathcal{H}^R$*, can we classify the (nonisomorphic) staggered modules* $\mathcal{S}$ *corresponding to the short exact sequence (3.1)?*

Abstractly, if we dropped the requirement that L_0 has nontrivial Jordan cells, then we would be asking for a computation of $\mathrm{Ext}^1_{\mathcal{U}}(\mathcal{H}^R,\mathcal{H}^L)$ in an appropriate category,[37] a difficult task. As we shall see, however, requiring nondiagonalisability leads to a reasonably tractable problem for which we do not need the abstract machinery of homological algebra.

An answer to our question will be given in the following sections. For convenience, we summarize our results in Sec. VIII (Theorem 8.1). This section is largely self-contained, and so may be read independently of most of what follows. However, we suggest that an appreciation of the role of the beta invariants (Secs. III and VI E) represents a minimal prerequisite for this result.

As staggered modules necessarily have vectors which are not L_0-eigenvectors, we cannot grade the module by the eigenvalue of L_0 relative to that of some reference vector. However, L_0 can still be put in Jordan normal form, so we may decompose it into commuting diagonalizable and nilpotent operators: $L_0=\mathsf{L}_0^d+\mathsf{L}_0^n$. A staggered module may then be consistently graded by the eigenvalues of its vectors under L_0^d, relative to the minimal eigenvalue of L_0^d. We will refer to L_0^d-eigenvalues as conformal dimensions, even when the corresponding eigenvector is not an L_0-eigenvector. Note that the maps L_m are still consistent with this more general grading—one easily checks that $L_m \in \mathcal{U}_m$ maps the L_0^d eigenspace of eigenvalue h to that of eigenvalue $h-m$.

A submodule of a (graded) Virasoro module can be assigned a grading in at least two distinct ways. First, it can inherit the grading from its parent, so that homogeneous states have the same grade in both modules. The inclusion map is then a graded homomorphism. Second, a grading may be defined as the conformal dimension of the states relative to the minimal conformal dimension of the submodule. Both have their uses, but unless otherwise specified, we will always assume that a submodule inherits its grading from its parent.

We introduce some more notation. Let $x=\iota(x^L)$ denote the highest weight vector of the submodule $\iota(\mathcal{H}^L)\subset\mathcal{S}$ and choose an L_0^d-eigenvector y in the preimage $\pi^{-1}(x^R)\subset\mathcal{S}$. The vector x is then an eigenvector of L_0, while y is not (if it were, its descendants would also be, hence L_0 would be diagonalizable on $\mathcal{S}$). Their conformal dimensions are h^L and h^R, respectively. We now define the auxiliary vectors,

$$\omega_0 = (L_0 - h^R)y, \quad \omega_1 = L_1 y, \quad \text{and } \omega_2 = L_2 y. \tag{3.2}$$

Since L_1 and L_2 generate $\mathcal{U}^+$, ω_1, and ω_2 determine the action of $\mathcal{U}^+$ on y.

Proposition 3.1: $\omega_0, \omega_1, \omega_2 \in \mathcal{H}^L$, and ω_0 is a nonzero singular vector of $\mathcal{H}^L \subset \mathcal{S}$.

Proof: Since $L_0 - h^R$, L_1 and L_2 annihilate $x^R = \pi(y) \in \mathcal{H}^R = \mathcal{S}/\mathcal{H}^L$, their action on y must yield elements of $\mathcal{H}^L$. If ω_0 vanished then y would be an eigenvector of L_0, hence $\omega_0 \neq 0$. Moreover,

$$L_n \omega_0 = L_n (L_0 - h^R) y = (L_0 - h^R + n) L_n y, \tag{3.3}$$

hence $L_n \omega_0 = 0$ for all $n > 0$, as y has L_0^d-eigenvalue h^R, so $L_n y \in \mathcal{H}^L$ has L_0-eigenvalue $h^R - n$. ∎

Define $\ell = h^R - h^L$. It follows that ℓ is then the grade of the singular vector ω_0 and its Jordan partner y in the staggered module $\mathcal{S}$. The grades of ω_1 and ω_2 are therefore $\ell - 1$ and $\ell - 2$, respectively. One immediate consequence is that ℓ is a non-negative integer. Exact sequences (3.1) with $\ell < 0$ certainly exist, but cannot describe staggered modules.[38] When $\ell = 0$, we must have $\omega_0 = x$ up to a nonzero multiplicative constant. When $\ell > 0$, $\mathcal{H}^L$ has a proper singular vector, hence the Kac determinant formula (2.11) has a zero. We thereby obtain our first necessary conditions on the existence of staggered modules.

Corollary 3.2: A staggered module cannot exist unless $\ell \in \mathbb{N}$. Moreover, if $\ell > 0$, then $h^L = h_{r,s}$ for some $r, s \in \mathbb{Z}_+$ [where $h_{r,s}$ is given in Eq. (2.12)].

We will assume from here on that $\omega_0 = Xx$, where $X \in \mathcal{U}_\ell^-$ is normalized (and singular). Since y is related to the normalized singular vector ω_0 by Eq. (3.2), this also serves to normalize y (equivalently, we rescale π). However, there is still some residual freedom in the choice of y. Indeed, y was only chosen to be an L_0^d-eigenvector in $\pi^{-1}(x^R)$, so we are still free to make the redefinitions,

$$y \longrightarrow y + u \quad \text{for any } u \in \mathcal{H}_\ell^L, \tag{3.4}$$

without affecting the defining property (or normalization) of y. Following Ref. 21, we shall refer to such redefinitions as gauge transformations. These transformations obviously do not change the abstract structure of the staggered module (for a more formal statement see Proposition 3.6).

It is natural then to enquire about gauge-invariant quantities as one expects that it is these, and only these, which characterize the staggered module. When $\ell > 0$, a simple but important example is given by[12]

$$\beta = \langle x, X^\dagger y \rangle \quad (\text{recall } \omega_0 = Xx). \tag{3.5}$$

This β is obviously gauge invariant, as $\langle x, X^\dagger u \rangle = \langle \omega_0, u \rangle = 0$ for all $u \in \mathcal{H}_\ell^L$. In the physics literature, this has been called the logarithmic coupling for field-theoretic reasons.[39] Here, we shall just refer to it as the *beta invariant*. Note that since $\langle x, x \rangle = 1$ and $\dim \mathcal{H}_0^L = 1$,

$$X^\dagger y = \beta x \quad (\ell > 0). \tag{3.6}$$

We further note that the numerical value of this invariant depends on the chosen normalizations of ω_0 and y (which is why we have specified these normalizations explicitly). It is worth pointing out that if X were composite, $X = X^{(1)} X^{(2)}$ with both $X^{(j)}$ nontrivial, then

$$\beta = \langle x, (X^{(2)})^\dagger (X^{(1)})^\dagger y \rangle = \langle X^{(2)} x, (X^{(1)})^\dagger y \rangle = 0 \tag{3.7}$$

because $X^{(2)} x \in \mathcal{H}^L$ is singular and $(X^{(1)})^\dagger y \in \mathcal{H}^L$. The beta invariant is therefore always trivial in such cases. Nontrivial invariants can still be defined when X is composite, although their properties necessarily require a little more background. We will defer a formal discussion of such invariants until Sec. VI E.

Consider now the right module $\mathcal{H}^R = \mathcal{V}_{h^R}/\mathcal{J}$. If $\mathcal{J}$ is nontrivial, then it will be generated as a submodule of $\mathcal{V}_{h^R}$ by one or two singular vectors of the same rank (Fig. 1). When one generator suffices, we denote it by $\bar{X} v_{h^R}$; when two generators are required, they will be denoted by $\bar{X}^- v_{h^R}$

and $\bar{X}^{+}v_{h^R}$. As usual, we take all of these to be normalized. The corresponding grades are $\bar{\ell}$ or $\bar{\ell}^{-}<\bar{\ell}^{+}$, respectively. However, unless we are explicitly discussing the case of two independent generators, we shall suppress the superscript indices for clarity.

We have introduced ω_0, ω_1, and ω_2 to specify the action of $\mathcal{U}^{\geqslant 0}$ on y. When $\mathcal{J}$ is nontrivial, the action of $\mathcal{U}^{-}$ on y will not be free. Instead, we have $\bar{X}x^R=0$ in $\mathcal{H}^R$, hence

$$\bar{X}y = \varpi \quad (\text{in } \mathcal{S}) \tag{3.8}$$

defines a vector $\varpi \in \mathcal{H}^L$ (two vectors $\varpi^{\pm}$ when $\mathcal{J}$ is generated by two singular vectors). The grade of ϖ is then $\ell+\bar{\ell}$. Recalling that $\mathcal{S}$ as a vector space is just the direct sum of $\mathcal{H}^L$ and $\mathcal{H}^R$, and considering a vector space basis of $\mathcal{V}_{h^R}$ that extends a basis for the submodule $\mathcal{J}$, it is easy to see that the Virasoro module structure of $\mathcal{S}$ is completely determined by ω_0, ω_1, ω_2, and ϖ.

The existence of ϖ also leads to the following important structural observation.

Proposition 3.3: When $\mathcal{H}^R$ is not Verma, so $\bar{X}$ is defined, we have $\bar{X}\omega_0=0$.

Proof: Since $\bar{X} \in \mathcal{U}^{-}_{\bar{\ell}}$,

$$\bar{X}\omega_0 = \bar{X}(L_0 - h^R)y = (L_0 - h^R - \bar{\ell})\bar{X}y = (L_0 - h^R - \bar{\ell})\varpi = 0, \tag{3.9}$$

as ϖ is an L_0-eigenvector of dimension $h^R+\bar{\ell}$. ∎

We remark that the vanishing of $\bar{X}\omega_0$ implies that there are no nonzero singular vectors in $\mathcal{H}^L_{\ell+\bar{\ell}}$. Indeed, the normalized singular vector of this degree is $\bar{X}Xx$ (which is composite if $\ell>0$). Thus we may interpret Proposition 3.3 as saying that if a singular vector of $\mathcal{V}_{h^R}$ is set to zero in $\mathcal{H}^R$, then the singular vector of $\mathcal{V}_{h^L}$ of the same conformal dimension must also be set to zero in $\mathcal{H}^L$. Otherwise, the module $\mathcal{S}$ cannot be staggered. Contrapositively, if $\mathcal{H}^L$ has a nontrivial singular vector (of rank greater than that of ω_0), then $\mathcal{H}^R$ must have a nontrivial singular vector of the same conformal dimension. More formally, there is a module homomorphism $\mathcal{H}^R \to \mathcal{H}^L$ which maps $x^R \mapsto \omega_0$. In particular, if $\mathcal{H}^L$ is a Verma module, then $\mathcal{H}^R$ must likewise be Verma.

It turns out that there is some redundancy inherent in describing a staggered module in terms of the vectors ω_0, ω_1, ω_2, and ϖ.

Proposition 3.4: The vector ϖ is determined by the knowledge of $\mathcal{H}^L$, $\mathcal{H}^R$, ω_1, and ω_2.

Proof: We consider the action of L_n on $\varpi=\bar{X}y$ for $n>0$, recalling that $\bar{X} \in \mathcal{U}^{-}_{\bar{\ell}}$. First note that $L_n\bar{X} \in \mathcal{U}$ annihilates $v_{h^R} \in \mathcal{V}_{h^R}$, since $\bar{X}v_{h^R}$ is singular. Hence, we may write

$$L_n\bar{X} = U_0(L_0 - h^R) + U_1L_1 + U_2L_2 \tag{3.10}$$

for some U_0, U_1, $U_2 \in \mathcal{U}$ (depending on n). Such U_j can clearly be computed, for example, by Poincaré–Birkhoff–Witt ordering $L_n\bar{X}$ and in each resulting term, rewriting the rightmost L_m (if $m>2$) in terms of L_1 and L_2. It follows that

$$L_n\varpi = U_0\omega_0 + U_1\omega_1 + U_2\omega_2, \tag{3.11}$$

so it remains to demonstrate that knowing $L_n\varpi$ for all $n>0$ is equivalent to knowing $\varpi \in \mathcal{H}^L_{\ell+\bar{\ell}}$. But, the intersection of the kernels of the L_n with $n>0$ on $\mathcal{H}^L_{\ell+\bar{\ell}}$ is just the set of singular vectors of this subspace. The only candidate for such a singular vector is $\bar{X}\omega_0$, and this vanishes by Proposition 3.3. ∎

We recall that ω_0 is already determined by $\mathcal{H}^L$ and $\mathcal{H}^R$, which is why it was not referred to explicitly in Proposition 3.4. We will therefore refer to the pair

$$(\omega_1,\omega_2) \in \mathcal{H}^L_{\ell-1} \oplus \mathcal{H}^L_{\ell-2}$$

as the *data* of a given staggered module. That is not to say that ω_0 and the ϖ will not play an important role in what follows. Rather, it just notes that ω_1 and ω_2 are sufficient to describe $\mathcal{S}$

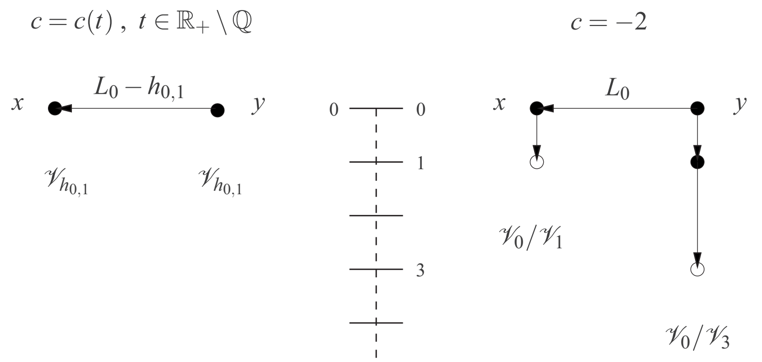

FIG. 2. An illustration of the staggered modules of Examples 1 (left) and 2 (right). We have indicated the singular vector structure of the respective left and right modules by using black circles for the generating states and singular vectors, and white circles to indicate singular vectors of the corresponding Verma modules which have been set to zero. The dividing scale gives the grades. It should be understood that singular vectors of the right module need not "lift" to singular vectors of the staggered module and are indicated purely to facilitate the discussion. (Technically, these lifts are *subsingular vectors* of the staggered module—they become singular upon taking an appropriate quotient.)

completely. One simple consequence arises when $\ell=0$, for then there is only one possible choice of data, $\omega_1=\omega_2=0$.

Corollary 3.5: If $\ell=0$, there exists at most one staggered module (up to isomorphism) for any given choice of left and right modules.

Example 1: In Ref. 20, staggered modules with $\ell=0$ were identified in the context of the Schramm–Loewner evolution curve with parameters $\kappa=4t>0$ and $\rho=\frac{1}{2}(\kappa-4)$.[40] More precisely, at these parameters a staggered module $\mathcal{S}$ with $h^L=h^R=h_{0,1}=\frac{1}{4}(2-t)$ is realized as a space of local martingales of the SLE$_\kappa(\rho)$ growth process. The central charge of this module is $c=c(t)=c(\kappa/4)$. The computations do not, in general, identify the left and right modules, but from the Feigin–Fuchs classification, we may, for example, conclude that in the case of irrational κ, $\mathcal{H}^L=\mathcal{H}^R=\mathcal{V}_{h_{0,1}}$ (these Verma modules are of point type). In other words, the short exact sequence has the form

$$0 \to \mathcal{V}_{h_{0,1}} \to \mathcal{S} \to \mathcal{V}_{h_{0,1}} \to 0 \quad (t>0, t\notin \mathbb{Q}). \tag{3.12}$$

We illustrate these staggered modules in Fig. 2 (left). By Corollary 3.5, such staggered modules are unique when they exist. But this concrete construction demonstrates existence, so we can conclude that at least one staggered module exists for any $t\in\mathbb{R}_+$, hence two for any central charge $-\infty<c<1$ (one for $c=t=1$).

Example 2: In Ref. 5 it was shown that the logarithmic singularity in a certain $c=-2$ ($t=2$) conformal field theory correlation function implied the existence of a staggered module $\mathcal{S}$ with $h^L=h^R=0$. This module was constructed explicitly in Ref. 24 by fusing the irreducible module $\mathcal{L}_{-1/8}$ with itself. The resulting structure is summarized by the short exact sequence,

$$0 \to \mathcal{V}_0/\mathcal{V}_1 \to \mathcal{S} \to \mathcal{V}_0/\mathcal{V}_3 \to 0, \tag{3.13}$$

and illustrated in Fig. 2 (right). In fact, this example is also related to the SLE construction of Example 1. For $\kappa=8$, the weight $h_{0,1}$ vanishes and the left and right modules can be computed explicitly to be those given in (3.13).[20]

We remark that in Example 2, the vector $L_{-1}y$ is an eigenvector of L_0 which does not belong to $\mathcal{H}^L$. This shows that the submodule of L_0-eigenvectors need not coincide with the left module and, in fact, need not be a highest weight module, in general.

There is one obvious deficiency inherent in describing staggered modules by their data (ω_1,ω_2). This is the fact that neither ω_1 nor ω_2 are gauge invariant, in general. Under the gauge transformations (3.4), the data transform as follows:

$$(\omega_1, \omega_2) \to (\omega_1 + L_1 u, \omega_2 + L_2 u) \quad (u \in \mathcal{H}_\ell^L). \tag{3.14}$$

This suggests introducing maps g_u for each $u \in \mathcal{H}_\ell^L$ which take $\mathcal{H}_{\ell-1}^L \oplus \mathcal{H}_{\ell-2}^L$ into itself via

$$g_u(w_1, w_2) = (w_1 + L_1 u, w_2 + L_2 u) \quad (u \in \mathcal{H}_\ell^L). \tag{3.15}$$

We will also refer to these maps as gauge transformations. Clearly the composition of gauge transformations is the vector space addition of $\mathcal{H}_\ell^L$. It is then natural to lift the scalar multiplication of $\mathcal{H}_\ell^L$ to the set of gauge transformations, making the latter into a vector space itself. We denote this vector space by $G = \{g_u : u \in \mathcal{H}_\ell^L\}$. We further note that the kernel of the map $u \mapsto g_u$ is one dimensional, spanned by the singular vector ω_0. Thus, G may be identified with $\mathcal{H}_\ell^L / \mathbb{C}\omega_0$. In particular, its dimension is

$$\dim G = \dim \mathcal{H}_\ell^L - 1. \tag{3.16}$$

Because the gauge-transformed data describe the same staggered module as the original data, we will say that the data (ω_1, ω_2) and its transforms $g_u(\omega_1, \omega_2)$ *are equivalent* for all $u \in \mathcal{H}_\ell^L$. The following result now characterizes isomorphic staggered modules completely.

Proposition 3.6: Let $\mathcal{S}$ and $\mathcal{S}'$ be staggered modules with the same left and right modules $\mathcal{H}^L$ and $\mathcal{H}^R$ and with respective data (ω_1, ω_2) and (ω_1', ω_2'). Then, upon identifying the two left modules via $x' = x$, we have $\mathcal{S}' \cong \mathcal{S}$ if and only if the data (ω_1, ω_2) and (ω_1', ω_2') are equivalent.

Proof: If $(\omega_1', \omega_2') = g_u(\omega_1, \omega_2)$ for some $u \in \mathcal{H}_\ell^L$, then $y' = y + u$ defines the isomorphism $\mathcal{S}' \cong \mathcal{S}$. Conversely, suppose that $\psi : \mathcal{S}' \to \mathcal{S}$ is an isomorphism extending the identification of the respective left modules [that is, such that $\psi(x') = x$]. Then,

$$L_0 y = h^R y + \omega_0 \quad \text{and} \quad L_0 \psi(y') = \psi(h^R y' + \omega_0') = h^R \psi(y') + \omega_0, \tag{3.17}$$

so $\psi(y') - y$ is an L_0-eigenvector of dimension h^R. We may therefore take $u = \psi(y') - y \in \mathcal{H}_\ell^L$, hence

$$\psi(\omega_i') = L_i \psi(y') = L_i(y + u) = \omega_i + L_i u \quad (i = 1, 2), \tag{3.18}$$

as required. ∎

This completes the analysis of when two staggered modules are isomorphic. It remains, however, to study the existence question. The question of which data (ω_1, ω_2) actually correspond to staggered modules is quite subtle, and we will address it in the following sections. First, however, we present two motivating examples from the literature to illustrate this subtlety.

Example 3: In Ref. 24, it was shown that fusing the two $c = -2$ ($t = 2$) irreducible modules $\mathcal{L}_{-1/8}$ and *$\mathcal{L}_{3/8}$ results in a staggered module $\mathcal{S}$ given by the short exact sequence,*

$$0 \to \mathcal{V}_0 / \mathcal{V}_3 \to \mathcal{S} \to \mathcal{V}_1 / \mathcal{V}_6 \to 0. \tag{3.19}$$

We illustrate $\mathcal{S}$ in Fig. 3 (left). In our notation, $\ell = 1$, $\omega_0 = L_{-1} x$, $\omega_1 = L_1 y = \beta x$, where β is the beta invariant of Eq. (3.5) and $\omega_2 = L_2 y = 0$. The explicit calculation shows that $\beta = -1$.

It seems reasonable to suppose that because the data $(\omega_1 = \beta x, \omega_2 = 0)$ of the staggered module (3.19) are fixed by the beta invariant, there should exist a continuum of such modules, one for each value of β. This was suggested in Ref. 24, referring to Rohsiepe,[27] but we are not aware of any proof of this fact. Indeed, one of our aims (see Examples 10 and 11 in Sec. VII) is to prove and understand why this is indeed the case.

Example 4: A $c = 0$ $\left(t = \frac{3}{2}\right)$ staggered module with the short exact sequence,

$$0 \to \mathcal{V}_0 / \mathcal{V}_2 \to \mathcal{S} \to \mathcal{V}_1 / \mathcal{V}_5 \to 0, \tag{3.20}$$

has appeared several times in the physics literature.[11,12,41] We again have $\ell = 1$, $\omega_0 = L_{-1} x$, $\omega_1 = \beta x$, and $\omega_2 = 0$. This time β turns out to be $-\frac{1}{2}$. This module is also illustrated in Fig. 3 (right).

One could be forgiven for thinking that because of the similarity of this example and the last, there will be a continuum of staggered modules with the exact sequence (3.20), parametrized by β. But surprisingly, this is not the case. It was argued in Ref. 21 that $\beta = -\frac{1}{2}$ is the only possible value

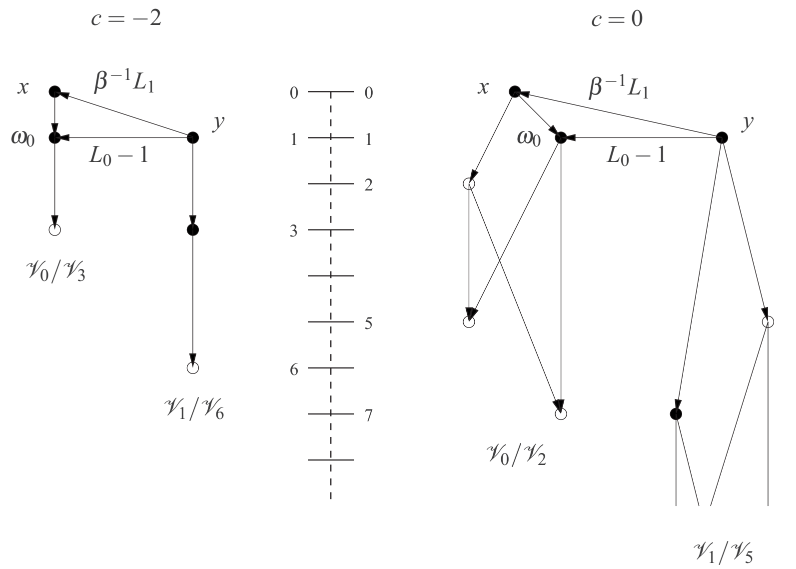

FIG. 3. An illustration of the staggered modules presented in Examples 3 (left) and 4 (right). The structure is to be interpreted as in Fig. 2. We remark that when $\beta=0$, which is possible for the module on the left, the label $\beta^{-1}L_1$ should be interpreted as saying that x cannot be obtained from y under the action of L_1, that is, $L_1 y=0$.

for such a staggered module, and hence that such a staggered module is unique (up to isomorphism). We shall prove this in Sec. VII (Examples 10 and 11).

There are some obvious structural differences between Examples 3 and 4, but it is not immediately clear what causes the observed restriction on the isomorphism classes of staggered modules. In fact, the desire to understand this mechanism is precisely the original motivation for the research reported here.

Example 5: The above two examples may, in fact, be regarded as members of another family of staggered modules parametrized by t. *For* $t\in\mathbb{R}_+\setminus\{1\}$, *this family can again be realized concretely as a module of local martingales of SLEs, with* $\kappa=4t$ *and* $\rho=-2$.[20] *Each member has* $h^L=0$ and $h^R=1$, *but as in Example 1, determining the precise identity of* $\mathcal{H}^L$ and $\mathcal{H}^R$ *requires nontrivial calculations, in general. However, when* κ *is irrational, these identities are settled automatically because then* $\mathcal{V}_{h^L}$ *is of link type and* $\mathcal{V}_{h^R}$ *is of point type (irreducible). By Proposition 3.1,* $\omega_0\in\mathcal{H}^L$ *is nonvanishing, so* $\mathcal{H}^L=\mathcal{V}_{h^L}$. *The exact sequence is therefore*

$$0\to\mathcal{V}_0\to\mathcal{S}\to\mathcal{V}_1\to 0 \quad (t>0, t\notin\mathbb{Q}). \tag{3.21}$$

The beta invariant was computed in Ref. 20 (see also Ref. 21) for all $t\in\mathbb{R}_+\setminus\{1\}$ *to be* $\beta=1-t$, *which coincides with the values in Examples 3 and 4 (when* $t=2$ *and* $t=\frac{3}{2}$, *respectively). For these two rational values, the left and right modules were also computed explicitly in the SLE picture, finding agreement with the fusion computations above. Thus this family of examples shows an interesting interplay of continuously varying beta invariant, but discontinuously varying left and right modules.*

IV. CONSTRUCTING STAGGERED MODULES: GENERALITIES

In the previous section, we have introduced staggered modules and determined some simple necessary conditions for their existence. We now turn to the more subtle question of sufficient conditions for existence. As we have seen in Example 4, it is not true that given left and right modules, every possible choice of data (ω_1,ω_2) describes a staggered module. We are therefore faced with the task of having to determine which data give rise to staggered modules. Such data will be termed *admissible*.

One simple reason[26] why a given set of data (ω_1,ω_2) might fail to correspond to any staggered module is that there could exist an element $U\in\mathcal{U}$ such that[42]

$$U=U_1L_1=-U_2L_2,\quad \text{but } U_1\omega_1+U_2\omega_2\neq 0. \tag{4.1}$$

For then, $Uy=U_1\omega_1\neq -U_2\omega_2=Uy$, a contradiction. We mention that given any $U=U_1L_1=-U_2L_2\in\mathcal{U}L_1\cap\mathcal{U}L_2$, the elements U_1 and U_2 are uniquely determined because $\mathcal{U}$ has no zero divisors.

We therefore define the subset

$$\Omega=\{(w_1,w_2)\in\mathcal{H}^L_{\ell-1}\oplus\mathcal{H}^L_{\ell-2}: U_1w_1+U_2w_2=0 \text{ for all } U=U_1L_1=-U_2L_2\in\mathcal{U}L_1\cap\mathcal{U}L_2\}. \tag{4.2}$$

With this notation, our necessary condition on the data becomes as follows.

Lemma 4.1: If a staggered module with data (ω_1,ω_2) *exists, then* $(\omega_1,\omega_2)\in\Omega$.

We can obtain a useful simplification of this condition through Poincaré–Birkhoff–Witt ordering the $U\in\mathcal{U}L_1\cap\mathcal{U}L_2$.

Lemma 4.2: $\mathcal{U}L_1\cap\mathcal{U}L_2=\mathcal{U}^{\leq 0}(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)$.

Proof: If $U\in\mathcal{U}L_1\cap\mathcal{U}L_2$, we may write $U=U_1L_1=-U_2L_2$ with the U_i Poincaré–Birkhoff–Witt ordered: $U_i=\sum_n U^{\leq 0}_{i,n}U^+_{i,n}$, with $U^{\leq 0}_{i,n}\in\mathcal{U}^{\leq 0}$ and $U^+_{i,n}\in\mathcal{U}^+$. Thus,

$$U=\sum_n U^{\leq 0}_{1,n}U^+_{1,n}L_1=-\sum_n U^{\leq 0}_{2,n}U^+_{2,n}L_2. \tag{4.3}$$

Since similarly ordering U in its entirety will not affect the $U^{\leq 0}_{i,n}$ factors, the linear independence of Poincaré–Birkhoff–Witt monomials implies that (with an appropriate shuffling of the index n) we may take $U^{\leq 0}_{1,n}=U^{\leq 0}_{2,n}$. It follows, again from linear independence, that $U^+_{1,n}L_1=-U^+_{2,n}L_2$. This proves that $\mathcal{U}L_1\cap\mathcal{U}L_2\subseteq\mathcal{U}^{\leq 0}(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)$ and the reverse inclusion is trivial. ■

We apply Lemma 4.2 to the conditions of Eq. (4.1) as follows. The first of these just states that $U\in\mathcal{U}L_1\cap\mathcal{U}L_2$, hence Lemma 4.2 lets us write $U=\sum_n U^{\leq 0}_n U^+_{1,n}L_1=-\sum_n U^{\leq 0}_n U^+_{2,n}L_2$ for some $U^{\leq 0}_n\in\mathcal{U}^{\leq 0}$ and $U^+_{i,n}\in\mathcal{U}^+$, where

$$U^+_{1,n}L_1+U^+_{2,n}L_2=0 \tag{4.4}$$

for all n. Moreover, the second condition of (4.1) is now $\sum_n U^{\leq 0}_n U^+_{1,n}\omega_1+\sum_n U^{\leq 0}_n U^+_{2,n}\omega_2\neq 0$, which implies that

$$U^+_{1,n}\omega_1+U^+_{2,n}\omega_2\neq 0 \tag{4.5}$$

for some n. It follows that in Eq. (4.1), we may suppose that U_1 and U_2 belong to $\mathcal{U}^+$, without any loss of generality. In other words, if an element $U\in\mathcal{U}L_1\cap\mathcal{U}L_2$ spoils the admissibility of (ω_1,ω_2), then there is an element spoiling admissibility in $\mathcal{U}^+L_1\cap\mathcal{U}^+L_2$.

This somewhat lengthy argument then allows us to conclude that Ω may be equivalently defined as

$$\Omega=\{(w_1,w_2)\in\mathcal{H}^L_{\ell-1}\oplus\mathcal{H}^L_{\ell-2}: U_1w_1+U_2w_2=0 \text{ for all } U=U_1L_1=-U_2L_2\in\mathcal{U}^+L_1\cap\mathcal{U}^+L_2\}. \tag{4.6}$$

The value of this slight simplification lies in the fact that the homogeneous subspaces of $\mathcal{U}^+L_1\cap\mathcal{U}^+L_2$ are finite dimensional.

Lemma 4.3: For $m>0$, *the dimension of* $(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)_{-m}=\mathcal{U}^+_{1-m}L_1\cap\mathcal{U}^+_{2-m}L_2$ *is equal to* $d(m)=p(m-1)+p(m-2)-p(m)$. *When* $m=0$, *this dimension is 0.*

Proof: As L_1 and L_2 generate $\mathfrak{vir}^+$, we have $(\mathcal{U}^+L_1+\mathcal{U}^+L_2)_{-m}=\mathcal{U}^+_{-m}$ for $m>0$. Taking dimensions of this equality we get $\dim\mathcal{U}^+_{1-m}+\dim\mathcal{U}^+_{2-m}-\dim(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)_{-m}=\dim\mathcal{U}^+_{-m}$, which leads to the asserted formula. ■

As an aside to the advanced reader, we mention that by treating $\mathcal{U}^+$ as a Virasoro module with $h=c=0$ (we set $\mathfrak{vir}^{\leq 0}\mathbf{1}=0$), $\mathcal{U}^+L_1\cap\mathcal{U}^+L_2$ may be identified as the submodule generated by the singular vectors at grades -5 and -7.[43] Indeed, thinking of $\mathcal{U}^{+}$ as a *lowest* weight Verma module, our intersection corresponds to the intersection of the submodules generated by the rank 1 singular vectors at grades -1 and -2. The Feigin–Fuchs classification for lowest weight Verma modules states that this is generated by the rank 2 singular vectors, which turn out to have grades -5 and -7 (as stated).

We tabulate the first few of these dimensions for convenience:

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
$d(m)$	0	0	0	0	0	1	1	3	4	7	10	16	21	32	43	60	$\cdots$

Note that if $U=U_1L_1=-U_2L_2 \in \mathcal{U}^{+}_{1-m}L_1 \cap \mathcal{U}^{+}_{2-m}L_2$ with $m>\ell$, then U_1w_1 and U_2w_2 both vanish for all $(w_1,w_2)\in \mathcal{H}^L_{\ell-1}\oplus\mathcal{H}^L_{\ell-2}$ (for dimensional reasons). We therefore need $\ell \geqslant 5$ to find examples where $\Omega \neq \mathcal{H}^L_{\ell-1}\oplus\mathcal{H}^L_{\ell-2}$. We also point out that Ω is not necessarily equal to the set of admissible data. Example 4 provides an illustration of this fact: The dimension of Ω is $\dim(\mathcal{H}^L_0\oplus\mathcal{H}^L_{-1})=1$ in this case, but the set of admissible data is a singleton.

Example 6: A staggered module $\mathcal{S}$ with $c=0$ $\left(t=\frac{3}{2}\right)$ and short exact sequence,

$$0 \to \mathcal{V}_1/\mathcal{V}_5 \to \mathcal{S} \to \mathcal{V}_7/\mathcal{V}_{15} \to 0, \tag{4.7}$$

was constructed in Ref. 11. Note that $\ell=6$. Its beta invariant was shown in Ref. 12 to be $\beta = -10\,780\,000/243$ (with our normalization for ω_0), where it was also argued to be the unique such value. What is interesting here is that the authors noted that this example presents some subtlety upon trying to "fix the gauge" before computing β. It is this subtlety which we want to explain here.

With our notation, the problem arose when the authors tried to determine $\omega_1 \in \mathcal{H}^L_5$ and $\omega_2 \in \mathcal{H}^L_4$ in terms of the (unknown) β. Since $\dim \mathcal{H}^L_5=6$, $\dim \mathcal{H}^L_4=4$, and there are $\dim G=\dim \mathcal{H}^L_6 - 1 = 8$ independent gauge transformations, they could assume that $\omega_1=0$ and $\omega_2=(aL_{-4}+bL_{-2}^2)x$. There were therefore two unknowns a and b. The definition of the beta invariant then gave a single linear relation connecting it with a and b.

While the authors of Ref. 21 were able to divine another linear relation between a and b, thereby determining them in terms of β and completing the gauge fixing, we can understand this problem as arising from the existence of nontrivial elements of $\mathcal{U}^{+}L_1\cap\mathcal{U}^{+}L_2$. Indeed, $(\mathcal{U}^{+}L_1\cap\mathcal{U}^{+}L_2)_{-5}$ is spanned by

$$(L_1^2L_2+6L_2^2-L_1L_3+2L_4)L_1=(L_1^3+6L_1L_2+12L_3)L_2, \tag{4.8}$$

and left multiplying by L_1 gives a spanning element of $(\mathcal{U}^{+}L_1\cap\mathcal{U}^{+}L_2)_{-6}$. It follows that the assumed data $(\omega_1=0,\omega_2=(aL_{-4}+bL_{-2}^2)x)$ are not in Ω (and hence not admissible) unless

$$L_1(L_1^2L_2+6L_2^2-L_1L_3+2L_4)\omega_1=L_1(L_1^3+6L_1L_2+12L_3)\omega_2. \tag{4.9}$$

Evaluating this constraint gives the second relation found in Ref. 21 through other, less canonical, means.

To attack the question of which (ω_1,ω_2) can arise as the data of a staggered module $\mathcal{S}$, given left and right modules $\mathcal{H}^L$ and $\mathcal{H}^R$, we consider the following explicit construction (generalizing that of Rohsiepe[26]). We start with the Virasoro module $\mathcal{H}^L\oplus\mathcal{U}$, where $\mathfrak{vir}$ is understood to act on $\mathcal{U}$ by left multiplication. We let $\mathcal{N}$ be the submodule of $\mathcal{H}^L\oplus\mathcal{U}$ generated by

$$\begin{gathered}(\omega_0,h^R-L_0),\quad (\omega_1,-L_1),\quad (\omega_2,-L_2),\\ \text{and } (\varpi,-\bar{X}),\quad \text{or } (\varpi^{\pm},-\bar{X}^{\pm}),\quad \text{when appropriate}.\end{gathered} \tag{4.10}$$

Here, we understand that when required, ϖ (or $\varpi^{\pm}$) is deduced from the ω_j as in the proof of Proposition 3.4. The idea is that $\mathbf{1}\in\mathcal{U}$ will project onto $y\in\mathcal{S}$ upon quotienting by $\mathcal{N}$. More specifically, we will attempt to construct $\mathcal{S}$ as $(\mathcal{H}^L\oplus\mathcal{U})/\mathcal{N}$, requiring then only a precise analysis of when this succeeds. Denote by $\pi^R: \mathcal{H}^L\oplus\mathcal{U}\to\mathcal{U}$ the projection onto the second component.

The question of whether this construction recovers $\mathcal{S}$ turns out to boil down to whether the submodule $\mathcal{N}^{\circ}=\mathcal{N}\cap\operatorname{Ker}\pi^{R}$ is trivial or not.

Theorem 4.4: *Given* $\mathcal{H}^L$, $\mathcal{H}^R$, $\omega_1\in\mathcal{H}^L_{\ell-1}$, and $\omega_2\in\mathcal{H}^L_{\ell-2}$, *we have the following.*

(i) *If* $\mathcal{N}^{\circ}=\{0\}$ *then* $(\mathcal{H}^L\oplus\mathcal{U})/\mathcal{N}$ *is a staggered module with the desired short exact sequence,*

$$0\to\mathcal{H}^L\xrightarrow{\iota}\frac{(\mathcal{H}^L\oplus\mathcal{U})}{\mathcal{N}}\xrightarrow{\pi}\mathcal{H}^R\to 0 \tag{4.11}$$

and data (ω_1,ω_2).

(ii) *If* $\mathcal{N}^{\circ}\neq\{0\}$ *then a staggered module with the desired exact sequence and data does not exist.*

Proof: Denote by $\pi_{\mathcal{N}}:\mathcal{H}^L\oplus\mathcal{U}\to(\mathcal{H}^L\oplus\mathcal{U})/\mathcal{N}$ the canonical projection, and assume (at first) that $\mathcal{N}^{\circ}=\{0\}$. We will construct the required homomorphisms ι and π by imposing commutativity of the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{H}^L & \xrightarrow{\iota^L} & \mathcal{H}^L\oplus\mathcal{U} & \xrightarrow{\pi^R} & \mathcal{U} & \to & 0\\
 & & \| & & \downarrow\pi_{\mathcal{N}} & & \downarrow\pi_{\mathcal{I}} & & \\
0 & \to & \mathcal{H}^L & \xrightarrow{\iota} & \dfrac{\mathcal{H}^L\oplus\mathcal{U}}{\mathcal{N}} & \xrightarrow{\pi} & \mathcal{H}^R & \to & 0
\end{array} \tag{4.12}$$

Here, ι^L denotes the obvious injection $u\mapsto(u,0)$ (the top row is therefore exact) and $\pi_{\mathcal{I}}$ denotes the canonical projection onto the quotient of $\mathcal{U}$ by the submodule (left ideal) $\mathcal{I}$ generated by L_0-h^R, L_1, L_2, and $\bar{X}$.

Observe then that $\iota=\pi_{\mathcal{N}}\circ\iota^L$ has kernel $\operatorname{Im}\iota^L\cap\mathcal{N}=\mathcal{N}^{\circ}=\{0\}$, hence is injective. On the other hand, the map π satisfies $\pi\circ\pi_{\mathcal{N}}=\pi_{\mathcal{I}}\circ\pi^R$, which, in fact, defines it as $\pi_{\mathcal{I}}\circ\pi^R$ maps $\mathcal{N}=\operatorname{Ker}\pi_{\mathcal{N}}$ to zero by construction. The map π is clearly surjective as both π^R and $\pi_{\mathcal{I}}$ are. It remains to check that the bottom row is exact in the middle. From the exactness of the top row we get

$$\pi\circ\iota=\pi_{\mathcal{I}}\circ\pi^R\circ\iota^L=0,\quad\text{hence }\operatorname{Im}\iota\subseteq\operatorname{Ker}\pi. \tag{4.13}$$

On the other hand, if $\pi\circ\pi_{\mathcal{N}}(w,U)=0$ for some $(w,U)\in\mathcal{H}^L\oplus\mathcal{U}$, then $U\in\mathcal{I}$ by commutativity of (4.12). By definition of $\mathcal{I}$ and $\mathcal{N}$, $(w,U)=(w',0)\ (\mathrm{mod}\,\mathcal{N})$ for some $w'\in\mathcal{H}^L$, hence

$$\pi_{\mathcal{N}}(w,U)=\pi_{\mathcal{N}}\circ\iota^L(w')=\iota(w'),\quad\text{hence }\operatorname{Ker}\pi\subseteq\operatorname{Im}\iota. \tag{4.14}$$

The module $(\mathcal{H}^L\oplus\mathcal{U})/\mathcal{N}$ is then staggered and the data are correct because

$$(L_0-h^R)y=(\omega_0,0)=\iota(\omega_0)\quad\text{and}\quad L_jy=(\omega_j,0)=\iota(\omega_j)\quad(\mathrm{mod}\,\mathcal{N}), \tag{4.15}$$

where $y=(0,1)$ and $x=(x^L,0)\ (\mathrm{mod}\,\mathcal{N})$. This proves (i).

If $\mathcal{N}^{\circ}\neq\{0\}$, then (given $\mathcal{H}^L$) there exists $U_0,U_1,U_2,\bar{U}\in\mathcal{U}$, such that

$$U_0(L_0-h^R)+U_1L_1+U_2L_2+\bar{U}\bar{X}=0,\quad\text{but }U_0\omega_0+U_1\omega_1+U_2\omega_2+\bar{U}\varpi\neq 0. \tag{4.16}$$

Suppose that $\mathcal{S}$ was a staggered module with the desired exact sequence and data, and choose $y\in\mathcal{S}$ such that $\pi(y)=x^R$ and $L_jy=\omega_j$. Now applying the first of these equations to y would give zero, contradicting the second. This proves (ii). ∎

The role that $\mathcal{N}^{\circ}$ plays in this construction of a staggered module is best seen by regarding $\mathcal{N}^{\circ}=\mathcal{N}\cap\operatorname{Im}\iota^L$ as a submodule of $\mathcal{H}^L$. If nontrivial, $\mathcal{N}^{\circ}$ is generated by singular vectors of $\mathcal{H}^L$. The quotient of $\mathcal{H}^L\oplus\mathcal{U}$ by $\mathcal{N}$ will then no longer have a left module isomorphic to $\mathcal{H}^L$, but will be some quotient thereof. For example, if $x\in\mathcal{N}^{\circ}$, then all of $\mathcal{H}^L$ is "quotiented away" and the above construction gives a highest weight module, not a staggered module. Similarly, if $\omega_0\in\mathcal{N}^{\circ}$ but

$x \notin \mathcal{N}^{\circ}$, then the construction results in an indecomposable module on which L_0 is diagonalizable. It is only when $\mathcal{N}^{\circ}=\{0\}$ that $\mathcal{H}^L$ is preserved, and then Theorem 4.4 tells us that we do indeed obtain a staggered module with the correct left and right modules and data.

Before concluding this section, let us first make two brief observations relating to the above construction arguments. These allow us to answer the question of the existence or nonexistence of a staggered module, assuming we have already answered the question for another related staggered module. Roughly speaking, the existence becomes easier if we take a smaller left module or a bigger right module. The precise statement for the left module is as follows.

Proposition 4.5: Suppose that there exists a staggered module $\mathcal{S}$ with exact sequence,

$$0 \to \mathcal{H}^L \to \mathcal{S} \to \mathcal{H}^R \to 0, \tag{4.17}$$

and data $(\omega_1,\omega_2) \in \mathcal{H}^L_{\ell-1} \oplus \mathcal{H}^L_{\ell-2}$. If $\hat{\mathcal{J}}$ is a submodule of $\mathcal{H}^L$ not containing ω_0, then there exists a staggered module $\hat{\mathcal{S}}$ with exact sequence,

$$0 \to \hat{\mathcal{H}}^L \to \hat{\mathcal{S}} \to \mathcal{H}^R \to 0 \quad (\hat{\mathcal{H}}^L = \mathcal{H}^L/\hat{\mathcal{J}}), \tag{4.18}$$

and data $([\omega_1],[\omega_2]) \in \hat{\mathcal{H}}^L_{\ell-1} \oplus \hat{\mathcal{H}}^L_{\ell-2}$. Indeed, we may identify $\hat{\mathcal{S}}$ with $\mathcal{S}/\hat{\mathcal{J}}$.

This follows from the fact that $\mathcal{H}^L$ is a submodule of $\mathcal{S}$. We only require $\omega_0 \notin \hat{\mathcal{J}}$ to ensure that the quotient $\mathcal{S}/\hat{\mathcal{J}}$ is still staggered.

For the right module we have instead the following, somewhat less trivial, result.

Proposition 4.6: Suppose that there exists a staggered module $\mathcal{S}$ with exact sequence,

$$0 \to \mathcal{H}^L \to \mathcal{S} \to \mathcal{H}^R \to 0, \tag{4.19}$$

and data $(\omega_1,\omega_2) \in \mathcal{H}^L_{\ell-1} \oplus \mathcal{H}^L_{\ell-2}$. If $\mathcal{H}^R$ is a quotient of the highest weight module $\check{\mathcal{H}}^R$, then there exists a staggered module $\check{\mathcal{S}}$ with exact sequence,

$$0 \to \mathcal{H}^L \to \check{\mathcal{S}} \to \check{\mathcal{H}}^R \to 0, \tag{4.20}$$

and the same data $(\omega_1,\omega_2) \in \mathcal{H}^L_{\ell-1} \oplus \mathcal{H}^L_{\ell-2}$. Moreover, we may identify $\mathcal{S}$ as a quotient of $\check{\mathcal{S}}$.

Proof. We will show that the submodules of $\mathcal{H}^L \oplus \mathcal{U}$ used in the construction of Theorem 4.4 satisfy $\check{\mathcal{N}} \subseteq \mathcal{N}$, so $\check{\mathcal{N}}^{\circ} \subseteq \mathcal{N}^{\circ}=\{0\}$ (identifying the left modules of $\check{\mathcal{S}}$ and $\mathcal{S}$ in the obvious way). As $\mathcal{H}^R$ is a (nonzero) quotient of $\check{\mathcal{H}}^R$, $\check{h}^R=h^R$, and we see that $\check{\omega}_0=\omega_0$. The proposition states that the data of $\check{\mathcal{S}}$ and $\mathcal{S}$ are likewise identified, so the only difference between the generators (4.10) of $\check{\mathcal{N}}$ and $\mathcal{N}$ is that the former includes $(\check{\varpi},-\check{\bar{X}})$, whereas in the latter we have instead $(\varpi,-\bar{X})$.[44] But, as $\mathcal{H}^R$ is a quotient of $\check{\mathcal{H}}^R$, we may write $\check{\bar{X}}=\chi\bar{X}$ for some singular $\chi \in \mathcal{U}^-$, so if we can show that $\check{\varpi}=\chi\varpi$, then $\check{\mathcal{N}} \subseteq \mathcal{N}$ follows and we are done. Moreover, this would allow us to write

$$\mathcal{S} = \frac{\mathcal{H}^L \oplus \mathcal{U}}{\mathcal{N}} = \frac{\mathcal{H}^L \oplus \mathcal{U}}{\check{\mathcal{N}}} \Bigg/ \frac{\mathcal{N}}{\check{\mathcal{N}}} = \frac{\check{\mathcal{S}}}{\mathcal{N}/\check{\mathcal{N}}}, \tag{4.21}$$

realizing $\mathcal{S}$ as a quotient of $\check{\mathcal{S}}$.

It remains then to prove that $\check{\varpi}=\chi\varpi$. This is a straightforward check based on Proposition 3.4. To whit, the proof of this proposition tells us that $\check{\varpi}$ is completely determined by the conditions (one for each $n>0$),

$$L_n\check{\varpi} = U_0\omega_0 + U_1\omega_1 + U_2\omega_2, \quad \text{where } L_n\check{\bar{X}} = U_0(L_0-h^R) + U_1L_1 + U_2L_2. \tag{4.22}$$

By hypothesis, $\mathcal{S}$ exists, so there is a $y \in \mathcal{S}$ defining the ω_j as in Eq. (3.2). Now, $\check{\bar{X}}=\chi\bar{X}$ implies that

$$L_n\check{\varpi} = (U_0(L_0 - h^R) + U_1L_1 + U_2L_2)y = L_n\check{\check{X}}y = L_n\chi\varpi \quad \text{for all } n > 0. \tag{4.23}$$

Since $\mathcal{H}^L$ has no (nonzero) singular vectors at the grade of $\check{\varpi}$ (Proposition 3.3), we conclude that $\check{\varpi} = \chi\check{\varpi}$, as required. The proof is therefore complete. ∎

Corollary 4.7: Every staggered module can be realized as a quotient of a staggered module whose right module is Verma.

To summarize, Theorem 4.4 shows that the data (ω_1, ω_2) are admissible if and only if the module $\mathcal{N}^\circ$ (whose definition depends on ω_1 and ω_2) is trivial. This construction is therefore fundamental for the question of the existence of staggered modules, but as such is it not yet completely transparent. What is missing are easily checked sufficient conditions to guarantee that $\mathcal{N}^\circ = \{0\}$. The best way to proceed is to first analyze the case in which the right module $\mathcal{H}^R$ is a Verma module. By Proposition 4.6, this case is the least restrictive, and we devote Sec. VI to this task, which is decidedly nontrivial in itself. The treatment of general $\mathcal{H}^R$ can then be reduced to the analysis of certain submodules of the $\mathcal{H}^R$ Verma case, by Corollary 4.7. This is the subject of Sec. VII. First, however, we must briefly digress in order to introduce an important auxiliary result which will be used in both Secs. VI and VII.

V. THE PROJECTION LEMMA

This section is devoted to an auxiliary result which we call the *Projection Lemma* (Lemma 5.1). This will be used at several key places in the sequel, in particular, Secs. VI B and VII B, but in slightly different contexts. We will therefore present it in a somewhat general form. The relevance to the development thus far should, however, be readily apparent.

Recall that we defined a set Ω in Eq. (4.6). We generalize this definition slightly,

$$\Omega_m = \{(w_1, w_2) \in \mathcal{H}^L_{m-1} \oplus \mathcal{H}^L_{m-2} : U_1w_1 + U_2w_2 = 0$$
$$\text{for all } U = U_1L_1 = -U_2L_2 \in \mathcal{U}^+L_1 \cap \mathcal{U}^+L_2\}. \tag{5.1}$$

We will always take m to be the grade of a singular vector, $m = \ell_r$ or $m = \ell_r^\pm$. Thus Ω coincides with Ω_ℓ. Similarly, we defined a vector space G that acts on Ω, in fact, on $\mathcal{H}^L_{\ell-1} \oplus \mathcal{H}^L_{\ell-2}$, by Eq. (3.15). We also generalize this, defining G_m to be the vector space of transformations g_u of $\mathcal{H}^L_{m-1} \oplus \mathcal{H}^L_{m-2}$ which take the form

$$g_u(w_1, w_2) = (w_1 + L_1u, w_2 + L_2u) \quad (u \in \mathcal{H}^L_m). \tag{5.2}$$

Again, G coincides with G_ℓ.

We next define a filtration of Ω_m which is induced by the singular vector structure of $\mathcal{H}^L$. Recall that at the end of Sec. II, we discussed the Feigin–Fuchs classification of Virasoro Verma modules and introduced notation for their singular vectors. The structure and notation differed according to whether the Verma module was of chain (and link) or braid type, and so the explicit forms of our filtration must also differ according to these two cases.

Chain case: Define subspaces of Ω_m in which both w_i are descendants of the singular vector X_kx,

$$\Omega_m^{(k)} = \{(w_1, w_2) \in \Omega_m : w_1, w_2 \in \mathcal{U}X_kx\}. \tag{5.3}$$

When $m = \ell_r$, this gives a filtration of Ω_m, of the form

$$\Omega_m = \Omega_m^{(0)} \supseteq \Omega_m^{(1)} \supseteq \Omega_m^{(2)} \supseteq \cdots \supseteq \Omega_m^{(r-2)} \supseteq \Omega_m^{(r-1)}. \tag{5.4}$$

Clearly, $\Omega_m^{(k)} = \{0\}$ for all $k \geqslant r$. An obvious remark that is nevertheless worth keeping in mind is that the spaces $\Omega_m^{(k)}$ may be trivial even when $k < r$, for example, if $X_kx = 0$.

Braid case: We define subspaces of Ω_m similarly,[45]

$$\Omega_m^{(k;+)} = \{(w_1, w_2) \in \Omega_m | w_j \in \mathcal{U}X_k^+x\}, \tag{5.5a}$$

$$\Omega_m^{(k;-)} = \{(w_1,w_2) \in \Omega_m | w_j \in \mathcal{U}X_k^- x + \mathcal{U}X_k^+ x\}. \tag{5.5b}$$

When m is the grade of a rank r singular vector ($m=\ell_r^{\pm}$), these subspaces are nested as

$$\Omega_m = \Omega_m^{(0;+)} \supseteq \Omega_m^{(1;-)} \supseteq \Omega_m^{(1;+)} \supseteq \Omega_m^{(2;-)} \supseteq \cdots \supseteq \Omega_m^{(r-2;+)} \supseteq \Omega_m^{(r-1;-)}. \tag{5.6}$$

We note again that if $\mathcal{H}^L$ contains no (nonzero) singular vectors of rank k, then $\Omega_m^{(k;\pm)}=\{0\}$. However, this case differs from the chain case in that there is the possibility that for a certain rank, one of the singular vectors of $\mathcal{H}^L$ is present while the other is not.

Lemma 5.1: (The Projection Lemma) *Let $m=\ell_r$ ($m=\ell_r^{\pm}$) be the grade of a singular vector. Then for any $(w_1,w_2)\in\Omega_m$, there exists a $g_u\in G_m$, such that $g_u(w_1,w_2)$ belongs to the subspace $\Omega_m^{(r-1)}$ ($\Omega_m^{(r-1;-)}$).*

Before presenting the proof, let us pause to first describe the idea behind it (in nonrigorous terms). We will prove the required result iteratively. In the chain case, we will show how to take an element of $\Omega_m^{(k)}$ and make a gauge transformation so as to get an (equivalent) element of $\Omega_m^{(k+1)}$. In the braid case, we will do two slightly different alternating steps, showing how to go from $\Omega_m^{(k;-)}$ to $\Omega_m^{(k;+)}$ and from $\Omega_m^{(k;+)}$ to $\Omega_m^{(k+1;-)}$. Composing all of these transformations then gives the required result in each case.

The way in which we transform from one subspace to the next is most transparent when we assume that we are working within a genuine staggered module, with data given by $\omega_j=L_j y$ for $j=1,2$. Under this hypothesis, we will outline the steps required, assuming the chain case for notational simplicity. Suppose then that $(\omega_1,\omega_2)\in\Omega_m^{(k)}$, with $m=\ell$. We first note that we can obtain $X_k x$ from ω_1 or ω_2 by acting with $\mathcal{U}$ if and only if we can obtain it from y. Thus, we take a basis $\{Z_\mu\}$ of $\mathcal{U}^-$ at grade $m-\ell_k$ and consider the complex numbers ζ_μ defined by $Z_\mu^\dagger y=\zeta_\mu X_k x$. By gauge transforming $y\to y'=y+z$ appropriately, it turns out that we can tune all of the ζ_μ to zero. It then follows that we cannot obtain $X_k x$ from y' by acting with $\mathcal{U}$, hence we cannot obtain it from the corresponding $\omega_j'=L_j y'$, $j=1,2$. ω_1' and ω_2' must therefore generate a proper submodule of $\mathcal{U}X_k x$, and so must be descendants of $X_{k+1}x$.

Of course, we cannot assume from the outset that we are working in a staggered module because we want to apply the Projection Lemma to the study of when staggered modules exist! Nevertheless, the outline above serves to motivate the steps in the general proof below. There are a few technicalities to work through, most of which arise because we must make sure that our constructions are well defined in the absence of y. Moreover, we also have to account for the structural, and therefore notational, differences which delineate the chain and braid cases.

Proof. As already stated, there are two cases leading to three steps to consider. The constructions are similar in all three, but because of structural variations, we must split the considerations accordingly. However, we will only provide full details in the chain case, limiting ourselves to describing what is different in the braid cases (Fig. 4).

Chain case $\Omega_m^{(k)}\to\Omega_m^{(k+1)}$: We assume that $(w_1,w_2)\in\Omega_m^{(k)}$ with $k<r-1$, so $m>\ell_{k+1}$. To find a gauge transformation $g_z\in G_m$, such that $g_z(w_1,w_2)\in\Omega_m^{(k+1)}$, we will introduce a basis of $\mathcal{U}_{m-\ell_k}$ with a certain "orthonormality" property. We make this precise as follows.

First, let $X^{(k+1)}\in\mathcal{U}^-$ be defined by $X_{k+1}=X^{(k+1)}X_k$. We choose a basis $\{V_\lambda X^{(k+1)}v_{h^L+\ell_k}\}$ at grade $m-\ell_k$ of the maximal proper submodule of the Verma module $\mathcal{V}_{h^L+\ell_k}$ (whose highest weight vector has conformal dimension equal to that of $X_k x$). Thus, $V_\lambda\in\mathcal{U}^-_{m-\ell_{k+1}}$. We can complete this to a basis of $\mathcal{V}_{h^L+\ell_k}$ at the same grade by adding vectors $Z_\mu v_{h^L+\ell_k}$ with $Z_\mu\in\mathcal{U}^-_{m-\ell_k}$. Since the quotient of a module by its maximal proper submodule has nondegenerate Shapovalov form, we can even choose the Z_μ to be orthonormal,[46]

$$\langle Z_\mu v_{h^L+\ell_k}, Z_\nu v_{h^L+\ell_k}\rangle = \delta_{\mu\nu}, \quad \text{that is } Z_\mu^\dagger Z_\nu v_{h^L+\ell_k} = \delta_{\mu\nu} v_{h^L+\ell_k}. \tag{5.7}$$

This then defines a basis $\{V_\lambda X^{(k+1)}\}\cup\{Z_\mu\}$ of $\mathcal{U}^-_{m-\ell_k}$.

Since the Z_μ are not scalars ($m>\ell_k$), we may write $Z_\mu^\dagger=Z_{\mu;1}^\dagger L_1+Z_{\mu;2}^\dagger L_2$. The choice of $Z_{\mu;1}$ and $Z_{\mu;2}$ is not unique, but if $Z_\mu^\dagger=Z'^\dagger_{\mu;1}L_1+Z'^\dagger_{\mu;2}L_2$ is another choice then

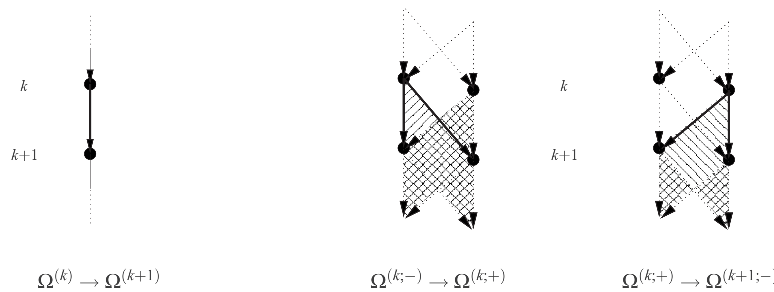

FIG. 4. An illustration of the projections constructed in the proof of Lemma 5.1. On the left we portray the chain case, in which the projection involves taking w_j' from the module $\mathcal{U}X_kx$ (itself a submodule of $\mathcal{H}^L$) to its (maximal) submodule $\mathcal{U}X_{k+1}x$. On the right are the braid cases. We alternate between steps of two types, going from the module $\mathcal{U}X_k^-x+\mathcal{U}X_k^+x$ to its submodule $\mathcal{U}X_k^+x$ (left), and from the module $\mathcal{U}X_k^+x$ to its submodule $\mathcal{U}X_{k+1}^-x+\mathcal{U}X_{k+1}^+x$ (right). The shading indicates schematically the module we start from and the submodule we arrive at, and the emphasised arrows indicate the singular elements $X^{(k+1)}$ and $X^{(k+1;\pm)}$ which are used in the proof.

$$(Z_{\mu;1}^{\dagger} - Z_{\mu;1}'^{\dagger})L_1 = -(Z_{\mu;2}^{\dagger} - Z_{\mu;2}'^{\dagger})L_2 \in \mathcal{U}^+L_1 \cap \mathcal{U}^+L_2. \tag{5.8}$$

It follows that each Z_μ gives rise to a well-defined element $Z_{\mu;1}^{\dagger}w_1+Z_{\mu;2}^{\dagger}w_2$ of $(\mathcal{U}^-X_kx)_{\ell_k}$, as $(w_1,w_2) \in \Omega_m^{(k)}$. We may therefore define $\zeta_\mu \in \mathbb{C}$ by

$$Z_{\mu;1}^{\dagger}w_1 + Z_{\mu;2}^{\dagger}w_2 = \zeta_\mu X_k x. \tag{5.9}$$

We can similarly write $V_\lambda^{\dagger}=V_{\lambda;1}^{\dagger}L_1+V_{\lambda;2}^{\dagger}L_2$, as the V_λ are also not scalars $(m>\ell_{k+1})$. However, the analogs of the ζ_μ all vanish as

$$\langle X_kx, X^{(k+1)\dagger}(V_{\lambda;1}^{\dagger}w_1 + V_{\lambda;2}^{\dagger}w_2)\rangle_{\mathcal{U}X_kx} = \langle X_{k+1}x, V_{\lambda;1}^{\dagger}w_1 + V_{\lambda;2}^{\dagger}w_2\rangle_{\mathcal{U}X_kx} = 0. \tag{5.10}$$

Here, $\langle\cdot,\cdot\rangle_{\mathcal{U}X_kx}$ denotes the Shapovalov form of the submodule $\mathcal{U}X_kx$.

To tune the constants ζ_μ to zero, we set $z=-\Sigma_\nu\zeta_\nu Z_\nu X_kx \in \mathcal{U}X_kx$ and apply the transformation g_z. Letting $w_j'=w_j+L_jz$, for $j=1,2$, explicit computation gives

$$\zeta_\mu X_k x = Z_{\mu;1}^{\dagger}w_1' + Z_{\mu;2}^{\dagger}w_2' = 0 \tag{5.11}$$

for all μ. Here we use the orthonormality of the Z_μ, Eq. (5.7) (which clearly continues to hold upon projecting $\mathcal{V}_{h^L+\ell_k}$ onto $\mathcal{U}X_kx$). We need now only verify that each $w_j' \in \mathcal{U}X_{k+1}x$ (which is the kernel of the Shapovalov form in the submodule $\mathcal{U}X_kx$) by showing that there is no element of $\mathcal{U}$ which takes w_j' to X_kx. We will detail this for $j=1$, the case $j=2$ being entirely analogous.

Clearly, we need only consider elements $U \in \mathcal{U}^+_{-m+1+\ell_k}$. Write $L_{-1}U^{\dagger} \in \mathcal{U}^-_{m-\ell_k}$ in the basis defined above to get

$$\begin{aligned} UL_1 &= \sum_\lambda a_\lambda (X^{(k+1)})^{\dagger}V_\lambda^{\dagger} + \sum_\mu b_\mu Z_\mu^{\dagger} \\ &= \left((X^{(k+1)})^{\dagger}\sum_\lambda a_\lambda V_{\lambda;1}^{\dagger} + \sum_\mu b_\mu Z_{\mu;1}^{\dagger}\right)L_1 + \left((X^{(k+1)})^{\dagger}\sum_\lambda a_\lambda V_{\lambda;2}^{\dagger} + \sum_\mu b_\mu Z_{\mu;2}^{\dagger}\right)L_2, \end{aligned} \tag{5.12}$$

where the a_λ and b_μ denote coefficients. Let U_1' and U_2' be the respective prefactors of L_1 and L_2 appearing in Eq. (5.12). This equation then becomes $(U_1'-U)L_1=-U_2'L_2 \in \mathcal{U}L_1 \cap \mathcal{U}L_2$, so we obtain the equality $Uw_1'=U_1'w_1'+U_2'w_2'$ as $(w_1',w_2') \in \Omega_m^{(k)}$. But, $(X^{(k+1)})^{\dagger}$ annihilates all of $(\mathcal{U}X_kx)_{\ell_{k+1}}$ [compare with Eq. (5.10)], so we see that by tuning the ζ_μ' to zero, we have guaranteed that

$$Uw_1' = U_1'w_1' + U_2'w_2' = \sum_\mu b_\mu(Z_{\mu;1}^\dagger w_1' + Z_{\mu;2}^\dagger w_2') = \sum_\mu b_\mu \zeta_\mu' X_k x = 0, \tag{5.13}$$

by Eq. (5.9). Since this holds for all $U \in \mathcal{U}^+_{-m+1+\ell_k}$, $w_1' \in \mathcal{U}X_{k+1}x$. After repeating this argument for w_2', we have completed the proof: $g_z(w_1,w_2)=(w_1',w_2') \in \Omega_m^{(k+1)}$.

Braid case $\Omega_m^{(k;-)} \to \Omega_m^{(k;+)}$: Suppose that $(w_1,w_2) \in \Omega_m^{(k;-)}$ and $k<r-1$, so $m>\ell_{k+1}^+$. Define $X^{(k+1;\pm)} \in \mathcal{U}^-$ by $X_{k+1}^{\pm}=X^{(k+1;\pm)}X_k^-$. We choose a basis, $\{V_\lambda^- X^{(k+1;-)}v_{h^L+\ell_k^-}\} \cup \{V_\lambda^+ X^{(k+1;+)}v_{h^L+\ell_k^-}\}$, say, of the maximal proper submodule of $\mathcal{V}_{h^L+\ell_k^-}$ at grade $m-\ell_k^-$, and extend it to a basis of $\mathcal{V}_{h^L+\ell_k^-}$ itself, at the same grade, by adding orthonormal elements $Z_\mu v_{h^L+\ell_k^-}$. This defines our basis of $\mathcal{U}^-_{m-\ell_k^-}$ as in the chain case.

Again, $Z_\mu^\dagger = Z_{\mu;1}^\dagger L_1 + Z_{\mu;2}^\dagger L_2$ defines constants ζ_μ by $Z_{\mu;1}^\dagger w_1 + Z_{\mu;2}^\dagger w_2 = \zeta_\mu X_k^- x$, and we use these to define $z=-\sum_\mu \zeta_\mu Z_\mu X_k^- x$ and $(w_1',w_2')=g_z(w_1,w_2) \in \Omega_m^{(k;-)}$. The check that $Uw_j'=0$ for any $U \in \mathcal{U}^+_{-m+j+\ell_k^-}$ is done by writing $L_{-j}U^\dagger \in \mathcal{U}^-_{m-\ell_k^-}$ in the above basis: We thereby obtain the analog of Eq. (5.12) (but with separate terms for the $X^{(k+1;+)}$ and $X^{(k+1;-)}$ contributions). This leads to $Uw_j' =0$ for all U as in the chain case. However, from this we are only able to conclude that $w_j' \in \mathcal{U}X_k^+ x$, not that w_j' belongs to the maximal proper submodule $\mathcal{U}X_{k+1}^- x + \mathcal{U}X_{k+1}^+ x$ of $\mathcal{U}X_k^- x$ (for this, we need the last case below). We therefore have $(w_1',w_2') \in \Omega_m^{(k;+)}$.

Braid case $\Omega_m^{(k;+)} \to \Omega_m^{(k+1;-)}$: In this final case we suppose that $(w_1,w_2) \in \Omega_m^{(k;+)}$ and again $k <r-1$, to guarantee that $m>\ell_{k+1}^+$. We choose a basis of $\mathcal{U}^-_{m-\ell_k^+}$ as in the first braid case and use this to construct z so that $g_z(w_1,w_2)$ is in $\Omega^{(k+1;-)}$. Everything now works as in the previous cases. We only mention that proving $Uw_j'=0$ for all $U \in \mathcal{U}^+_{-m+j+\ell_k^+}$ here lets us conclude that the w_j' belong to the maximal proper submodule $\mathcal{U}X_{k+1}^- x + \mathcal{U}X_{k+1}^+ x$ because we have been working entirely in $\mathcal{U}X_k^+ x$. Thus, $(w_1',w_2') \in \Omega_m^{(k+1;-)}$ as required. ∎

We conclude this section with two small remarks pertaining to this proof. First, we call this result the Projection Lemma because each subsequent gauge transformation can be thought of as projecting the (w_1,w_2) onto the next-smallest subspace in the filtration. Indeed, if (w_1,w_2) is already in the next-smallest subspace, then the ζ_μ defined in the proof must already vanish, hence $z=0$ and g_z is the identity map.

The second remark addresses why the sequence of projections defined in the proof terminates. Once in the submodule corresponding to the rank k singular vector(s) $\Omega_m^{(k)}$ ($\Omega_m^{(k;\pm)}$), we were able to project further provided that $m>\ell_{k+1}$ ($m>\ell_{k+1}^\pm$). This guaranteed that the V-type basis elements of the maximal proper submodule of $\mathcal{U}^-_{m-\ell_k}$ ($\mathcal{U}^-_{m-\ell_k^\pm}$) were not scalars, and so could be written as a sum of terms with L_1 or L_2 on the right. As soon as $k=r-1$, we find that some V-type basis elements are scalars, and so cannot be written in this form. The proof then breaks down at the point of Eq. (5.12) and its analogs.

And so it should: In the chain case with $m=\ell_r$, the grade of the w_j would be ℓ_r-j, so it is completely unreasonable to expect that we can construct w_j' belonging to $\mathcal{U}X_r x$. In the braid case, we get the same conclusion if $m=\ell_r^-$. When $m=\ell_r^+$, one might hope to be able to find w_j' belonging to $\mathcal{U}X_r^- x$. However, it is possible to show (using Proposition 4.6 and Theorem 6.15 below, for example) that this is only possible in a rather trivial case: Essentially, the "data" (w_1,w_2) must be equivalent to (0,0).

VI. CONSTRUCTION WHEN THE RIGHT MODULE IS VERMA

Throughout this section we assume that $\mathcal{H}^R=\mathcal{V}_{h^R}$. In particular, this means that in the construction of Sec. IV, the submodule $\mathcal{N}$ of $\mathcal{H}^L \oplus \mathcal{U}$ is generated by (ω_0, h^R-L_0), $(\omega_1,-L_1)$ and $(\omega_2,-L_2)$ (there is no ϖ or $\bar{X}$). The corresponding exact sequence is

$$0 \to \mathcal{H}^L \to \mathcal{S} \to \mathcal{V}_{h^R} \to 0. \tag{6.1}$$

In principle, we have everything we need for our attack on the question of the existence of staggered modules $\mathcal{S}$ with exact sequence (6.1). However, the proofs which follow are necessarily rather technical, given that they apply to completely general left modules. We will therefore first briefly outline the main ideas behind them. We also suggest that the reader might like to keep in mind the simplest case in which ω_0 is the singular vector of minimal (positive) grade in $\mathcal{H}^L$. This case not only avoids the most troublesome technicalities (for example, we do not need the Projection Lemma for this case) but it also has the advantage of covering the majority of staggered modules which have thus far found physical application.[47]

Our overall plan is straightforward. The analysis of the $\mathcal{H}^R$ Verma case turns out to afford an important simplification, namely, that the admissibility of the data is completely captured by the set Ω, defined in Eq. (4.6). This allows us to identify the set of nonisomorphic staggered modules with exact sequence (6.1) as the vector space Ω/G, thereby settling the existence question when $\ell=0$ (Theorem 6.4). We then turn to the computation of the dimension of the space Ω/G. First, we use the Projection Lemma to reduce this to the dimension of an equivalent space Ω'/G', where $\Omega' \subseteq \Omega$ is significantly smaller, in general (Proposition 6.6). This allows us to separate the computation into four cases, according to the singular vector structure of $\mathcal{H}^L$ around ω_0. In each case, we reformulate the definition of Ω' so as to realize it as an intersection of kernels of certain linear functionals (Theorem 6.11). The computation of the dimension of Ω' is then just an exercise in linear algebra, albeit a rather nontrivial one. The results of this computation are given in Theorem 6.14. Finally, we discuss generalizations of the beta invariant of Eq. (3.5) which reduce the identification of a staggered module with exact sequence (6.1) to the computation of at most two numbers.

A. Admissibility

In this section, we study the question of admissibility of data (ω_1, ω_2) under the hypothesis that the right module is Verma. The result is reported in Proposition 6.2 below. First, however, we need a simple but very useful lemma. Recall that the submodule $\mathcal{N}^\circ$ may be naturally viewed as a submodule of $\mathcal{H}^L$.

Lemma 6.1: When $\mathcal{H}^R$ is Verma, $u \in \mathcal{N}^\circ$ if and only if there exist $U_1, U_2 \in \mathcal{U}$, such that

$$U_1\omega_1 + U_2\omega_2 = u \quad \textit{and} \quad U_1L_1 + U_2L_2 = 0. \tag{6.2}$$

Proof: By definition, $u \in \mathcal{N}^\circ$ if and only if there exist $U_0, U_1, U_2 \in \mathcal{U}$, such that

$$U_0\omega_0 + U_1\omega_1 + U_2\omega_2 = u \quad (\text{in } \mathcal{H}^L) \tag{6.3a}$$

and

$$U_0(L_0 - h^R) + U_1L_1 + U_2L_2 = 0 \quad (\text{in } \mathcal{U}), \tag{6.3b}$$

so one direction is trivial. What we need to show is that we may take $U_0=0$, without loss of generality. Note that by taking $u \in \mathcal{H}^L$ homogeneous, we may assume that U_0, U_1, and U_2 are homogeneous in $\mathcal{U}$.

Consider $U_1L_1 + U_2L_2$. Poincaré–Birkhoff–Witt ordering this combination will give a variety of terms, each of which must have a positive index on the rightmost mode. If Poincaré–Birkhoff–Witt-ordering U_0 produced any term which did not have a positive index on the rightmost mode, then right multiplying by $(L_0 - h^R)$ would preserve the ordering, and so this term could not be cancelled by any (ordered) term of $U_1L_1 + U_2L_2$. This contradicts (6.3b), so all the ordered terms of U_0 must have a positive index on the rightmost mode. Then, $U_0\omega_0=0$, and (6.3a) has the desired form.

But if every Poincaré–Birkhoff–Witt ordered term of U_0 has a positive index on the rightmost mode, we may write $U_0 = U_1'L_1 + U_2'L_2$ for some $U_1', U_2' \in \mathcal{U}$. Hence (for $U_0 \in \mathcal{U}_m$),

$$U_0(L_0-h^R)+U_1L_1+U_2L_2=(L_0-h^R-m)U_0+U_1L_1+U_2L_2$$
$$=(U_1+(L_0-h^R-m)U_1')L_1+(U_2+(L_0-h^R-m)U_2')L_2=0, \tag{6.4}$$

and a simple redefinition of U_1 and U_2 will put (6.3b) in the required form. This redefinition would affect (6.3a), but for the fact that

$$(L_0-h^R-m)(U_1'\omega_1+U_2'\omega_2)=0, \tag{6.5}$$

as $U_1'\omega_1+U_2'\omega_2$ is an L_0-eigenvector of eigenvalue h^R+m. ∎

Recall that Lemma 4.1 gave a necessary condition for (ω_1,ω_2) to be data of a staggered module. Theorem 4.4 and Lemma 6.1 now tell us that under the hypothesis that $\mathcal{H}^R$ is Verma, this condition is also sufficient: $\mathcal{N}^\circ=\{0\}$ if and only if

$$U_1\omega_1+U_2\omega_2=0 \quad \text{for all } U=U_1L_1=-U_2L_2\in\mathcal{U}L_1\cap\mathcal{U}L_2. \tag{6.6}$$

In the language of Sec. IV [see Eq. (4.2), in particular], this becomes:

Proposition 6.2: When $\mathcal{H}^R$ is Verma, (ω_1,ω_2) is admissible if and only if $(\omega_1,\omega_2)\in\Omega$.

Example 4 shows that this hypothesis is not superfluous. Combining this result with Proposition 3.6 now gives the following important characterization.

Proposition 6.3: The space of (nonisomorphic) staggered modules with exact sequence (6.1) may be identified with the vector space Ω/G.

Example 7: At $c=-2$ ($t=2$), one can use the algorithm detailed in Ref. 24 to fuse $\mathcal{L}_{-1/8}$ with $\mathcal{V}_{3/8}$ and $\mathcal{L}_1$ with $\mathcal{V}_0$. In both cases, a staggered module is obtained with the short exact sequence,

$$0\to\mathcal{V}_0\to\mathcal{S}\to\mathcal{V}_1\to 0. \tag{6.7}$$

The respective beta invariants turn out to be $\beta=-1$ (as in Example 3) and $\beta=\frac{1}{2}$. This exact sequence therefore admits two distinct staggered modules, hence by Proposition 6.3, there is (at least) a one-parameter family of such modules.

This example highlights in a novel way the physical importance of a good theory of staggered modules. It shows concretely how physically relevant constructions (here fusion products) can result in modules that cannot be distinguished from each other by their characters (graded dimensions) or even by the action of L_0 alone.

Finally, since $\ell=0$ implies that $\omega_1=\omega_2=0$, we thereby obtain the first piece of our classification, the case when rank $\omega_0=0$.[48] For consistency with Sec. VI C below, we will refer to this case as case (0).

Theorem 6.4: Case (0) of the classification. *Given left and right modules $\mathcal{H}^L$ and $\mathcal{H}^R$, for which the latter is Verma and $h^L=h^R$, there exists a* unique *staggered module $\mathcal{S}$ with short exact sequence (3.1).*

We remark that it should not be surprising that the precise form of $\mathcal{H}^L$ plays no role in this result. For existence when $\mathcal{H}^L$ is also Verma implies existence for general $\mathcal{H}^L$ (subject only to the nonvanishing of ω_0) by Proposition 4.5.

B. Choosing data

We have determined that the space of (nonisomorphic) staggered modules with exact sequence (6.1) is naturally realized as the quotient of Ω under the action of G, by Proposition 6.3. These spaces are a little large, in general, so it proves convenient to prune them into something a little more manageable. This will be achieved by applying the Projection Lemma (Lemma 5.1).

Let us denote by $\mathcal{M}$ the submodule of $\mathcal{H}^L$ generated by the singular vectors whose rank is one less than that of ω_0. For example, if rank $\omega_0=1$ we have $\mathcal{M}=\mathcal{H}^L$.[49] When rank $\omega_0>1$, $\mathcal{M}$ is generated by one or two singular vectors according as to whether $\mathcal{H}^L$ is of chain or braid type (this follows from $\omega_0\neq0$). We now define our "pruned" space of admissible data to be

$$\Omega' = \{(\omega_1,\omega_2) \in \Omega : \omega_1,\omega_2 \in \mathcal{M}\}. \tag{6.8}$$

The Projection Lemma with $m=\ell$ (so $r=\text{rank } \omega_0$) immediately gives the following.

Lemma 6.5: For any $(\omega_1,\omega_2)\in\Omega$, *there exists* $g_u \in G$ *such that* $g_u(\omega_1,\omega_2)=(\omega_1',\omega_2')\in\Omega'$.

The proof only requires realizing that in this application, the subspace $\Omega_m^{(r-1)}$ or $\Omega_m^{(r-1;-)}$ appearing in the Projection Lemma is precisely Ω'.

The new choice of data (ω_1',ω_2') is equivalent to the old data (ω_1,ω_2), so the underlying staggered module remains unchanged. Of course, we still have some freedom in the choice. There is a residual set of gauge transformations, namely, $G'=\{g_u \in G : u \in \mathcal{M}_\ell\} \subseteq G$, which preserves Ω'. Analogous to the case of the full G (Sec. III), we have $G' \cong \mathcal{M}_\ell/\mathbb{C}\omega_0$ (as vector spaces), hence

$$\dim G' = \dim \mathcal{M}_\ell - 1. \tag{6.9}$$

Moreover, Proposition 6.3 can now be replaced by the following.

Proposition 6.6: The space of (nonisomorphic) staggered modules with exact sequence (6.1) may be identified with the vector space Ω'/G'.

We point out that ω_0 need not be the singular vector of lowest grade in $\mathcal{M}$ (excluding of course the obvious generating ones). In the braid case when $\omega_0=Xx=X_\rho^+x$ (with $\rho=\text{rank } \omega_0$), X_ρ^-x may be a nonzero singular vector. Then, $X_\rho^-x \in \mathcal{M}$ has the same rank as ω_0, but its grade is strictly less than that of ω_0. This case is the source of the most trouble in the following analysis.

C. Characterizing admissible data

In this section we give a tractable characterization of the admissibility of pairs $(\omega_1,\omega_2) \in \mathcal{M}_{\ell-1}\oplus\mathcal{M}_{\ell-2}$. As the case rank $\omega_0=0$ (that is, $\ell=0$) has already been settled, we will assume that rank $\omega_0 \equiv \rho \geqslant 1$ for the rest of the section.

We will separate this characterization into four cases, according to the number of generating singular vectors of $\mathcal{M}$ and whether there is a nongenerating (nonzero) singular vector in $\mathcal{M}$ whose grade is less than ℓ (the troublesome cases). Explicitly, the cases are

Case (1): $\mathcal{M}$ is generated by a single singular vector and this is the only singular vector in $\mathcal{M}$ of grade less than ℓ. This applies in two situations: When $\mathcal{H}^L$ is of chain (or link) type and when $\mathcal{H}^L$ is of braid type with either $\omega_0=X_1^-x$ or $\omega_0=X_1^+x$ and $X_1^-x=0$.

Case (1′): $\mathcal{M}$ is generated by a single singular vector and there is another singular vector in $\mathcal{M}$ of grade less than ℓ. This only applies when $\mathcal{H}^L$ is of braid type with $\omega_0=X_1^+x$ and $X_1^-x\neq 0$.

Case (2): $\mathcal{M}$ is generated by two distinct singular vectors and these are the only singular vectors in $\mathcal{M}$ of grades less than ℓ. This only applies when $\mathcal{H}^L$ is of braid type with either $\omega_0=X_\rho^-x$ or $\omega_0=X_\rho^+x$ and $X_\rho^-x=0$.[50]

Case (2′): $\mathcal{M}$ is generated by two distinct singular vectors and there is another singular vector in $\mathcal{M}$ of grade less than ℓ. This only applies when $\mathcal{H}^L$ is of braid type with $\omega_0=X_\rho^+x$ and $X_\rho^-x \neq 0$.

It is easy to verify that any possibility is covered by exactly one of these cases. We illustrate them for convenience in Fig. 5.

To analyze each of these cases further, it is useful to first sharpen the conclusions of Lemma 6.1 somewhat. Specifically, we show that taking u to be a singular vector of "minimal rank" allows us to choose U_1 and U_2 in $\mathcal{U}^+$.

Lemma 6.7: If $(\omega_1,\omega_2)\in\mathcal{H}^L_{\ell-1}\oplus\mathcal{H}^L_{\ell-2}$ *is not admissible (and* $\mathcal{H}^R$ *is Verma), then* $\mathcal{N}^\circ$ *contains singular vectors of* $\mathcal{H}^L$ *of grade less than* ℓ. *For a singular vector* x' *whose rank is minimal among those in* $\mathcal{N}^\circ$, *there then exist* $U_1,U_2\in\mathcal{U}^+$ *such that*

$$U_1\omega_1 + U_2\omega_2 = x' \quad \text{and} \quad U_1L_1 + U_2L_2 = 0. \tag{6.10}$$

We have stated only one direction, but the converse is already implied by Lemma 6.1.

Proof of Lemma 6.7: Suppose that (ω_1,ω_2) is not admissible, which, in view of Theorem 4.4, means that $\mathcal{N}^\circ$ is a nonzero submodule of $\mathcal{H}^L$. Therefore $\mathcal{N}^\circ$ contains nonzero singular vectors, and it is generated by its minimal rank singular vectors. Take x' to be one such generator.

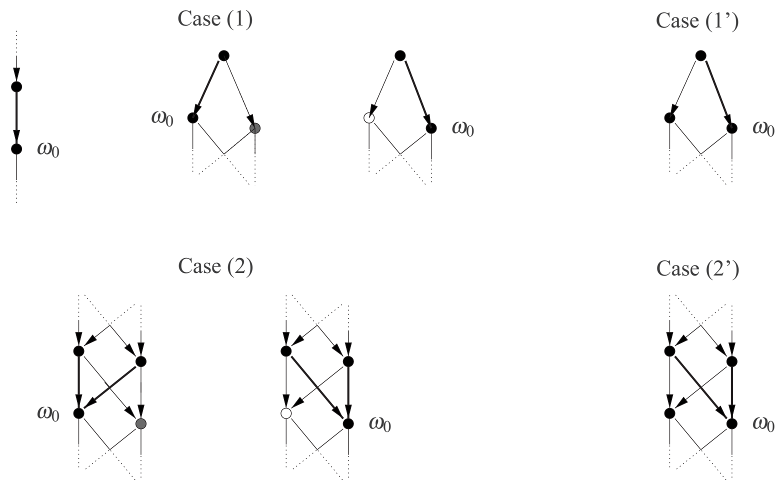

FIG. 5. An illustration of the possible structures of the left module $\mathcal{H}^L$ in cases (1), (1′), (2), and (2′). As with earlier figures, the black circle represents a singular vector of $\mathcal{H}^L$, whereas the white circle indicates a singular vector of the corresponding Verma module which has been set to zero. We use a gray circle when it does not matter if the singular vector has been set to zero or not. Note that the picture corresponding to case (1) with $t \notin \mathbb{Q}$ has been omitted—it is understood as a subcase of the chain case pictured. Similarly, the degenerate braid case $(t \in \mathbb{Q}, t<0)$ has not been explicitly portrayed—it is regarded as a subcase of case (2).

By Lemma 6.1, we can find $U_1, U_2 \in \mathcal{U}$ such that both equations in (6.10) are satisfied. But, if we Poincaré–Birkhoff–Witt order U_1 and U_2, we see that terms with negative modes on the left cannot contribute to $U_1\omega_1 + U_2\omega_2$ by the assumption that x' was of minimal rank. We therefore drop them. Furthermore, any L_0 on the left may be replaced by the appropriate eigenvalue, so we may assume that $U_1, U_2 \in \mathcal{U}^+$ in the first equation. Linear independence of Poincaré–Birkhoff–Witt monomials then allows us to likewise drop the terms with negative modes in the second equation. We may therefore write $U_1L_1 + U_2L_2 = \Sigma_n L_0^n U^{(n)} = 0$ with $U^{(n)} \in \mathcal{U}^+$. Independence and the lack of zero divisors in $\mathcal{U}$ then mean that each $U^{(n)}$ must vanish separately, so we can certainly replace each L_0 by its eigenvalue here too. This means that the $U_1, U_2 \in \mathcal{U}^+$ of the first equation also satisfy the second. Finally, we conclude from $U_1, U_2 \in \mathcal{U}^+$ in the first equation that the grade of x' must be less than ℓ. ■

Assuming that $(\omega_1, \omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, the submodule $\mathcal{N}^\circ$ is contained in $\mathcal{M}$ by Lemma 6.1. The minimal rank referred to in Lemma 6.7 is then either $\rho-1$ or ρ. In concrete terms, we need to check whether the rank $\rho-1$ singular vectors are in $\mathcal{N}^\circ$, and if this can be ruled out, we do the same for the rank ρ singular vector of grade less than ℓ if necessary [cases (1′) and (2′) only]. Below, we introduce functionals ψ, $\psi^{\pm}$, and $\psi^{\cap}$ with the aim of reducing these checks to a problem in linear algebra. We first separate our considerations according to the number of rank $\rho-1$ singular vectors in $\mathcal{H}^L$, and then analyze the further constraints stemming from the presence of a second rank ρ singular vector.

1. *Cases* (1) *and* (1′)

In these cases, $\mathcal{M}$ is generated by the normalized singular vector $X_{\rho-1}x$ of grade $\ell_{\rho-1}$. Making use of the fact that $\mathcal{M}_{\ell_{\rho-1}}$ is one dimensional, we define for each $U = U_1L_1 = -U_2L_2 \in (\mathcal{U}^+L_1 \cap \mathcal{U}^+L_2)_{\ell_{\rho-1}-\ell}$, a linear functional

$$\psi_U : \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2} \to \mathbb{C} \quad \text{by } U_1\omega_1 + U_2\omega_2 = \psi_U(\omega_1, \omega_2) X_{\rho-1}x. \tag{6.11}$$

Taking $(\omega_1, \omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, the submodule $\mathcal{N}^\circ$ contains no singular vectors of rank less than $\rho-1$. In view of Lemma 6.7, $\mathcal{N}^\circ$ contains the rank $\rho-1$ singular vector if and only if $\psi_U(\omega_1, \omega_2) \neq 0$ for some $U \in (\mathcal{U}^+L_1 \cap \mathcal{U}^+L_2)_{\ell_{\rho-1}-\ell}$. We formulate this result as follows.

Proposition 6.8: In cases (1) *and* (1′), *assuming* $(\omega_1, \omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, *we have*

$X_{\rho-1}x \notin \mathcal{N}^{\circ}$ *if and only if*

$$(\omega_1, \omega_2) \in \bigcap_{U \in (\mathcal{U}^{+}L_1 \cap \mathcal{U}^{+}L_2)_{\ell_{\rho-1}-\ell}} \operatorname{Ker} \psi_U. \tag{6.12}$$

We point out that in case (1), $X_{\rho-1}x$ is the only singular vector in $\mathcal{M}$ of grade less than ℓ, so by Lemma 6.7, the above condition completely characterizes the admissible data $(\omega_1, \omega_2) \in \Omega'$. In case $(1')$, there is another such singular vector, and so we will have to work harder to get a complete characterization (Sec. VI C 3). This proposition is of course crucial for case $(1')$ as well, since it tells us how to rule out the rank $\rho-1$ singular vectors. After that, ρ becomes the candidate for the minimal rank referred to in Lemma 6.7.

2. Cases (2) *and* (2′)

In this case there are two rank $\rho-1$ highest weight vectors in $\mathcal{H}^L$, namely, $X^{\pm}_{\rho-1}x$, and the submodule $\mathcal{M} = \mathcal{U}X^{-}_{\rho-1}x + \mathcal{U}X^{+}_{\rho-1}x$ is not a highest weight module. We have

$$\mathcal{M}_{\ell-j} = \mathcal{U}^{-}_{\ell-\ell^{-}_{\rho-1}-j}X^{-}_{\rho-1}x + \mathcal{U}^{-}_{\ell-\ell^{+}_{\rho-1}-j}X^{+}_{\rho-1}x \quad \text{for } j=1,2, \tag{6.13}$$

where the sum is direct in case (2), but not in case $(2')$. In either case, given $(\omega_1, \omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, we can write $\omega_j = \omega_j^{-} + \omega_j^{+}$ with $\omega_j^{\pm} \in \mathcal{U}X^{\pm}_{\rho-1}x$. The two conditions we will obtain below can be understood as one for each part, "$-$" and "$+$."

In analogy with the ψ_U above, we define the functionals $\psi^{\pm}_{U^{\pm}} : \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2} \to \mathbb{C}$ by the formulas,

$$U_1^{-}\omega_1 + U_2^{-}\omega_2 = \psi^{-}_{U^{-}}(\omega_1, \omega_2)X^{-}_{\rho-1}x \tag{6.14a}$$

and

$$U_1^{+}\omega_1 + U_2^{+}\omega_2 = \psi^{+}_{U^{+}}(\omega_1, \omega_2)X^{+}_{\rho-1}x \quad (\operatorname{mod} \mathcal{U}X^{-}_{\rho-1}x), \tag{6.14b}$$

where $U^{\pm} = U_1^{\pm}L_1 = -U_2^{\pm}L_2 \in (\mathcal{U}^{+}L_1 \cap \mathcal{U}^{+}L_2)_{\ell^{\pm}_{\rho-1}-\ell}$. These definitions again rely on the fact that both $\mathcal{M}_{\ell^{-}_{\rho-1}}$ and $(\mathcal{M}/\mathcal{U}X^{-}_{\rho-1}x)_{\ell^{+}_{\rho-1}}$ are one dimensional.

Assuming $(\omega_1, \omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, so there can again be no highest weight vectors of rank less than $\rho-1$ in $\mathcal{N}^{\circ}$, Lemma 6.7 tells us under which condition the singulars $X^{\pm}_{\rho-1}x$ are in $\mathcal{N}^{\circ}$. Precisely as above, $X^{-}_{\rho-1}x \in \mathcal{N}^{\circ}$ if and only if there is a U^{-} such that $\psi^{-}_{U^{-}}(\omega_1, \omega_2) \neq 0$. The case of $X^{+}_{\rho-1}x$ works out similarly, despite the slightly more involved definition of ψ^{+}. The easy direction is given by Lemma 6.7: If $X^{+}_{\rho-1}x \in \mathcal{N}^{\circ}$, then there exists $U^{+} = U_1^{+}L_1 = -U_2^{+}L_2$ such that $\psi^{+}_{U^{+}}(\omega_1, \omega_2) = 1$. To see the converse, assume that there exists U^{+} such that $\psi^{+}_{U^{+}}(\omega_1, \omega_2) \neq 0$, and without loss of generality choose U^{+} so that this value is unity. Explicitly, this means that

$$U_1^{+}\omega_1 + U_2^{+}\omega_2 = X^{+}_{\rho-1}x + u \quad \text{for some } u \in \mathcal{U}X^{-}_{\rho-1}x. \tag{6.15}$$

If $u=0$, we are done, so assume that $u = V^{-}X^{-}_{\rho-1}x \neq 0$ with $V^{-} \in \mathcal{U}^{-}$. The maximal proper submodule of $\mathcal{U}X^{-}_{\rho-1}x$ is trivial at the grade of u, so there must exist $V^{+} \in \mathcal{U}^{+}$, such that $V^{+}u = X^{-}_{\rho-1}x$. As $u = V^{-}V^{+}u$, it now follows that

$$(\mathbf{1} - V^{-}V^{+})(U_1^{+}\omega_1 + U_2^{+}\omega_2) = (\mathbf{1} - V^{-}V^{+})(X^{+}_{\rho-1}x + u) = X^{+}_{\rho-1}x. \tag{6.16}$$

Applying Lemma 6.1 to $(\mathbf{1} - V^{-}V^{+})U_j^{+}$, we conclude that $X^{+}_{\rho-1}x \in \mathcal{N}^{\circ}$.

In conclusion, $X^{\pm}_{\rho-1}x \in \mathcal{N}^{\circ}$ if and only if for some $U^{\pm}$ the value of $\psi^{\pm}_{U^{\pm}}(\omega_1, \omega_2)$ is nonzero. This gives the analog of Proposition 6.8.

Proposition 6.9: In cases (2) *and* $(2')$, *assuming* $(\omega_1, \omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, *we have* $X^{\pm}_{\rho-1}x \notin \mathcal{N}^{\circ}$ *if and only if*

$$(\omega_1,\omega_2) \in \bigcap_{U^{\pm} \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^{\pm}_{\rho-1}-\ell}} \text{Ker } \psi^{\pm}_{U^{\pm}}. \tag{6.17}$$

In case (2), the above two conditions again completely characterize when $(\omega_1,\omega_2) \in \Omega'$. As with case $(1')$, however, case $(2')$ involves an additional singular vector which leads to a further condition to check. However, we can now use Proposition 6.9 to rule out the rank $\rho-1$ singular vectors, so we may assume that the minimal rank of Lemma 6.7 is ρ. We now turn to the derivation of conditions for the additional rank ρ singular vector in cases $(1')$ and $(2')$.

3. Further conditions in cases $(1')$ *and* $(2')$

When $(\omega_1,\omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$, Propositions 6.8 and 6.9 give complete characterizations of the absence of rank $\rho-1$ singular vectors in $\mathcal{N}^{\circ}$, which suffices to settle the existence question of staggered modules in cases (1) and (2). In cases $(1')$ and $(2')$, Lemma 6.7 still leaves the possibility that $\mathcal{N}^{\circ}$ is nontrivial. We must therefore also characterize the absence of the singular vector $X^-_{\rho}x$ (which has a lower grade than $\omega_0 = X^+_{\rho}x$) in $\mathcal{N}^{\circ}$.

The derivation of this characterization is similar in flavor to the considerations of Secs. VI C 1 and VI C 2, although there are also important differences. The most immediate difference is that we must assume from the outset that the $\rho-1$ singular vectors have already been ruled out. This is necessary for the application of Lemma 6.7, and we will see that the definition of the functional $\psi^{\cap}_{U^{\cap}}$ below will only make sense when (ω_1,ω_2) satisfies this condition. We point out also another difference that will be relevant later. In Sec. VI E, we will construct invariants of staggered modules in a manner closely related to the considerations of the two previous sections. However, there will be no invariant related to what we have to do next. We will return to this point in Sec. VI E.

To decide whether $X^-_{\rho}x$ is in $\mathcal{N}^{\circ}$, we will define yet another set of functionals $\psi^{\cap}_{U^{\cap}}$. We recall that cases $(1')$ and $(2')$ both require $\mathcal{H}^L$ to be of braid type, the former corresponding to $\rho=1$ and the latter to $\rho>1$. To uniformize notation, we understand in the following that if $\rho=1$ then $\psi^+_{U^+}$ stands for ψ_U (as given in Sec. VI C 1) and $\psi^-_{U^-}$ is ignored (that is, each $\psi^-_{U^-}$ is to be regarded as the zero map). For $U^{\cap} = U^{\cap}_1 L_1 = -U^{\cap}_2 L_2 \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^-_{\rho}-\ell}$ and $(\omega_1,\omega_2) \in (\cap_{U^-} \text{Ker } \psi^-_{U^-}) \cap (\cap_{U^+} \text{Ker } \psi^+_{U^+})$, the defining formula is

$$U^{\cap}_1 \omega_1 + U^{\cap}_2 \omega_2 = \psi^{\cap}_{U^{\cap}}(\omega_1,\omega_2) X^-_{\rho}x. \tag{6.18}$$

The definition makes sense, but only because of the restriction that (ω_1,ω_2) is already annihilated by every $\psi^{\pm}_{U^{\pm}}$. This follows from the fact that the maximal proper submodule of $\mathcal{U}X^{\pm}_{\rho-1}$ is generated by the rank ρ singular vectors. For if $U^{\cap}_1\omega_1 + U^{\cap}_2\omega_2$ were not proportional to $X^-_{\rho}x$, so $U^{\cap}_1\omega_1 + U^{\cap}_2\omega_2$ would not be in the submodule $\mathcal{U}X^-_{\rho}x \subset \mathcal{M}$, there would exist a $U \in \mathcal{U}^+$, such that $\psi^{\pm}_{UU^{\cap}}(\omega_1,\omega_2)$ is equal to either $X^-_{\rho-1}x$ or $X^+_{\rho-1}x$, a contradiction.

The reason for this definition is the same as always. Assuming that both $X^-_{\rho-1}x$ and $X^+_{\rho-1}x$ are not in $\mathcal{N}^{\circ}$, so that $\psi^{\cap}_{U^{\cap}}(\omega_1,\omega_2)$ can be defined, Lemma 6.7 tells us that $\mathcal{N}^{\circ}$ is either zero or generated by $X^-_{\rho}x$. The analog of Propositions 6.8 and 6.9 is then the following.

Proposition 6.10: In the cases $(1')$ and $(2')$, *assuming that* $(\omega_1,\omega_2) \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$ *is such that* $\mathcal{N}^{\circ}$ *contains no rank* $\rho-1$ *singular vectors, we have* $\mathcal{N}^{\circ} = \{0\}$ *if and only if*

$$(\omega_1,\omega_2) \in \bigcap_{U^{\cap} \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^-_{\rho}-\ell}} \text{Ker } \psi^{\cap}_{U^{\cap}}. \tag{6.19}$$

This completes the characterization of admissibility in these cases.

4. Summary

We have defined functionals ψ_U, $\psi^{\pm}_{U^{\pm}}$, and $\psi^{\cap}_{U^{\cap}}$ whose kernels characterize when data are admissible. We note that each of these functionals is manifestly gauge invariant, so these kernels respect the gauge freedom we have in choosing the data. By combining Lemma 6.5 with Propositions 6.8–6.10, we now arrive at the complete classification of the admissible data in terms of these functionals.

Theorem 6.11: [Cases (1), (1′), (2), (2′) of the classification] *Given* $\mathcal{H}^L$ and $\mathcal{H}^R \cong \mathcal{V}_{h^R}$ *with* $\ell > 0$, *choose* $(\omega_1', \omega_2') \in \mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2}$. *Then,* $(\omega_1', \omega_2') \in \Omega'$, *so is the data of a staggered module* $\mathcal{S}$ *[with exact sequence (6.1)], if and only if the appropriate condition below is satisfied.*

Case (1): $\psi_U(\omega_1', \omega_2') = 0$ *for all* $U \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell_{\rho-1}-\ell}$.

Case (1′): *In addition to condition* (1), $\psi^{\cap}_{U^{\cap}}(\omega_1', \omega_2') = 0$ *for all* $U^{\cap} \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell_1^- - \ell}$.

Case (2): $\psi^{\pm}_{U^{\pm}}(\omega_1', \omega_2') = 0$ *for all* $U^{\pm} \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^{\pm}_{\rho-1}-\ell}$.

Case (2′): *In addition to condition* (2), $\psi^{\cap}_{U^{\cap}}(\omega_1', \omega_2') = 0$ *for all* $U^{\cap} \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell_\rho^- - \ell}$.

Here, $\rho = \mathrm{rank}\,\omega_0$, *and the relevant condition to use matches the case numbering given at the beginning of Sec. VI C. Moreover,* $(\omega_1, \omega_2) \in \mathcal{H}^L_{\ell-1} \oplus \mathcal{H}^L_{\ell-2}$ *is in* Ω, *hence is the data of a staggered module* $\mathcal{S}$, *if and only if there exist equivalent data* $(\omega_1', \omega_2') \in \Omega'$.

We remark that the single case excluded from the above theorem [$\ell = 0$, case (0)] was already settled in Theorem 6.4.

D. Counting dimensions

The results of Theorem 6.11 are very concrete descriptions of the possible data of staggered modules with $\mathcal{H}^R = \mathcal{V}_{h^R}$, even if they might seem somewhat technical. Their value is that they involve linear maps with simple definitions, and so allow reasonably straightforward computations, in each case, of the dimensions of the vector space Ω'/G' of inequivalent staggered modules.

To use Theorem 6.11 to compute the dimension of Ω'/G', we will analyze the functionals ψ_U, $\psi^-_{U^-}$, $\psi^+_{U^+}$, and $\psi^{\cap}_{U^{\cap}}$. In fact, it proves convenient to abstract one level further and consider also the induced maps,

$$\psi : (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell_{k-1}-\ell} \to (\mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2})^*, \quad U \mapsto \psi_U, \tag{6.20a}$$

$$\psi^- : (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^-_{k-1}-\ell} \to (\mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2})^*, \quad U^- \mapsto \psi^-_{U^-}, \tag{6.20b}$$

$$\psi^+ : (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^+_{k-1}-\ell} \to (\mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2})^*, \quad U^+ \mapsto \psi^+_{U^+}, \tag{6.20c}$$

$$\psi^{\cap} : (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell^-_k-\ell} \to ((\cap_{U^-} \mathrm{Ker}\, \psi^-_{U^-}) \cap (\cap_{U^+} \mathrm{Ker}\, \psi^+_{U^+}))^*, \quad U^{\cap} \mapsto \psi^{\cap}_{U^{\cap}}. \tag{6.20d}$$

All of these analyses are somewhat similar so we present instead two abstract results along these lines from which the required dimension results will be extracted on a case-by-case basis. So consider a highest weight module $\mathcal{K}$ with highest weight (h, c) and cyclic highest weight vector $\tilde{x}$. Fix a grade m. Then for $(w_1, w_2) \in \mathcal{K}_{m-1} \oplus \mathcal{K}_{m-2}$ and $U = U_1 L_1 = -U_2 L_2 \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{-m}$, we define $\Psi_U(w_1, w_2)$ by

$$U_1 w_1 + U_2 w_2 = \Psi_U(w_1, w_2)\tilde{x}. \tag{6.21}$$

This definition is clearly in the same spirit as those of ψ, $\psi^{\pm}$, and $\psi^{\cap}$. As above, Ψ_U is then the corresponding functional on $\mathcal{K}_{m-1} \oplus \mathcal{K}_{m-2}$, and Ψ alone stands for the map from $(\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{-m}$ to $(\mathcal{K}_{m-1} \oplus \mathcal{K}_{m-2})^*$ that associates with any U the functional Ψ_U.

We want to know when Ψ_U is nontrivial. This is the subject of the following result.

Lemma 6.12: The functional $\Psi_U \in (\mathcal{K}_{m-1} \oplus \mathcal{K}_{m-2})^*$ *is zero if and only if* $U = U_1 L_1 = -U_2 L_2 \in (\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{-m}$ *is such that the* $U_j^\dagger v_h$ $(j = 1, 2)$ *are in the maximal proper submodule of the Verma module* $\mathcal{V}_h$.

In particular, we will often use this result to establish the injectivity of Ψ by noting that if there is no proper singular vector in $\mathcal{V}_h$ of grade less than m, then the only U for which Ψ_U vanishes is $U=0$ (at grades $m-j$ the maximal proper submodule is trivial).

Proof: Write $U=U_1L_1=-U_2L_2$. By definition, $\Psi_U=0$ means $U_1w_1+U_2w_2=0$ for all $(w_1,w_2) \in \mathcal{K}_{m-1}\oplus\mathcal{K}_{m-2}$. Taking $w_1=0$ and $w_2=0$ (separately), we see that this is equivalent to $U_jw_j=0$ for all $w_j\in\mathcal{K}_{m-j}(j=1,2)$. Writing $w_j=V_j\tilde{x}$, we can further reformulate this as $U_jV_j\tilde{x}=0$ for all $V_j \in\mathcal{U}^-_{m-j}(j=1,2)$, from which we derive

$$0=\langle U_jV_j\tilde{x},\tilde{x}\rangle_{\mathcal{K}}=\langle V_j\tilde{x},U_j^\dagger\tilde{x}\rangle_{\mathcal{K}}=\langle V_jv_h,U_j^\dagger v_h\rangle_{\mathcal{V}_h}, \tag{6.22}$$

where we have distinguished the Shapovalov forms by a subscript displaying the relevant highest weight module. We therefore conclude that $\Psi_U=0$ if and only if both $U_1^\dagger v_h$ and $U_2^\dagger v_h$ belong to the maximal proper submodule of $\mathcal{V}_h$. ∎

Let us now assume that there is a nonzero *prime* singular vector $\chi\tilde{x}$ in $\mathcal{K}$ (playing the role of ω_0), where $\chi\in\mathcal{U}^-_l$ is singular (and normalized, although this is not strictly necessary). We take $m=l$. If the corresponding Verma module $\mathcal{V}_h$ has another (normalized, prime) singular vector of grade less than l, we will denote it by χ^-v_h and its grade by $l^-<l$. In this new setup, the content of Lemma 6.12 is simply described as follows. If χ^- is not defined, then $\Psi_U=0$ only if $U=0$, as follows from the remark immediately following the statement. On the other hand, if χ^- is defined, then we see that $\Psi_U=0$ if and only if $U\in(\chi^-)^\dagger(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)_{l^--l}$. This follows from factorizing each $U_j^\dagger$ as $(U_j')^\dagger\chi^-$, which leads to $U=U_1L_1=(\chi^-)^\dagger U_1'L_1$ and $U=-U_2L_2=-(\chi^-)^\dagger U_2'L_2$.

The following result will allow us to compute the dimensions of the space of inequivalent staggered modules. We mention that the first of the three cases appearing here was at the heart of Rohsiepe's analysis,[26] although he only stated it for modules of chain type.

Lemma 6.13: The subspace of $\mathcal{K}_{l-1}\oplus\mathcal{K}_{l-2}$ that is annihilated by every Ψ_U has dimension given by

$$\dim\bigcap_U \operatorname{Ker}\Psi_U=\begin{cases} p(l), & \text{if } \chi^- \text{ is not defined}\\ p(l)-p(l-l^-), & \text{if } \chi^-\tilde{x}=0\\ p(l)-p(l-l^-)+p(l-l^--1)+p(l-l^--2), & \text{if } \chi^-\tilde{x}\neq 0. \end{cases} \tag{6.23}$$

In the first two cases the result coincides with $\dim\mathcal{K}_l$ *and in the third case with* $\dim\mathcal{K}_l +\dim(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)_{l^--l}$.

Proof: Taking U_μ such that $\{\Psi_{U_\mu}\}$ is a basis for $\operatorname{Im}\Psi$, the mapping

$$(w_1,w_2)\mapsto(\Psi_{U_1}(w_1,w_2),\ldots,\Psi_{U_n}(w_1,w_2))\in\mathbb{C}^n \tag{6.24}$$

has kernel given by $\cap_U\operatorname{Ker}\Psi_U$ and rank equal to $\dim\operatorname{Im}\Psi$. In other words, each linearly independent equation $\Psi_U(w_1,w_2)=0$ reduces the dimension we want to compute by one,

$$\begin{aligned}\dim\bigcap_U\operatorname{Ker}\Psi_U&=\dim(\mathcal{K}_{l-1}\oplus\mathcal{K}_{l-2})-\dim\operatorname{Im}\Psi\\ &=\dim(\mathcal{K}_{l-1}\oplus\mathcal{K}_{l-2})-\dim(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)_{-l}+\dim\operatorname{Ker}\Psi.\end{aligned} \tag{6.25}$$

Consider therefore the case in which $\mathcal{V}_h$ has no singular vector of grade less than l (except v_h), so χ^- is not defined. Then we have $\dim\mathcal{K}_{l-j}=p(l-j)$ for $j=1,2$. But, Lemma 6.12 tells us that in this case (with $m=l$), $U\mapsto\Psi_U$ has a trivial kernel: $\dim\operatorname{Ker}\Psi=0$. Plugging these facts and the result of Lemma 4.3 into Eq. (6.25), the first formula follows.

Consider now the cases for which χ^- is defined. Regardless of whether $\chi^-\tilde{x}$ vanishes or not, Lemma 6.12 gives (with $m=l$) $\operatorname{Ker}\Psi=(\chi^-)^\dagger(\mathcal{U}^+L_1\cap\mathcal{U}^+L_2)_{l^--l}$, hence

$$\dim \operatorname{Ker} \Psi = \dim(\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{l^- - l} = p(l - l^- - 1) + p(l - l^- - 2) - p(l - l^-), \tag{6.26}$$

by Lemma 4.3. When $\chi^- \tilde{x} = 0$, the graded dimensions of $\mathcal{K}$ are $\dim \mathcal{K}_{l-j} = p(l-j) - p(l-l^- - j)$ for $j = 1, 2$. Plugging everything in and observing cancellations gives the second formula. On the other hand, if $\chi^- \tilde{x} \neq 0$ the graded dimensions are $\dim \mathcal{K}_{l-j} = p(l-j)$ and the third formula follows. ∎

With help of Lemma 6.13, we are ready to state and prove one of our main results, that giving the dimensions of the space of nonisomorphic staggered modules, Ω'/G', when the right module is Verma.

Theorem 6.14: *The dimension of the vector space Ω'/G' of isomorphism classes of staggered modules $\mathcal{S}$ with short exact sequence (6.1) is the number of rank $\rho - 1$ highest weight vectors in $\mathcal{H}^L$. Explicitly,*

$$\text{Case (0):} \quad (\ell = 0) \quad \dim \Omega'/G' = 0,$$

$$\text{Cases (1) and (1}'\text{):} \quad (\mathcal{H}^L \textit{ of chain type or } \rho = 1 \textit{ braid type}) \quad \dim \Omega'/G' = 1,$$

$$\text{Cases (2) and (2}'\text{):} \quad (\mathcal{H}^L \textit{ of } \rho > 1 \textit{ braid type}) \quad \dim \Omega'/G' = 2.$$

Proof: Case (0) being already done (Theorem 6.4), we will have to work out the cases (1), (1′), (2), and (2′) of Theorem 6.11 separately. As we already know that $\dim G' = \dim \mathcal{M}_\ell - 1$ [Equation (6.9)], it remains to be shown that $\dim \Omega' = \dim \mathcal{M}_\ell$ in cases (1) and (1′), and that $\dim \Omega' = \dim \mathcal{M}_\ell + 1$ in cases (2) and (2′).

Case (1): Let $\mathcal{K} = \mathcal{M} = \mathcal{U} X_{\rho-1} x$ and define χ by $X \equiv X_\rho = \chi X_{\rho-1}$. This χ is then normalized and prime, and l is given by $\ell - \ell_{\rho-1}$. Let Ψ be ψ, as defined in Eq. (6.20a). When $\mathcal{H}^L$ is of chain type or of braid type with $\omega_0 = X_1^- x$, Lemma 6.13 applies with χ^- undefined. Since $\Omega' = \cap_U \operatorname{Ker} \psi_U$, we read off the dimension

$$\dim \Omega' = p(\ell - \ell_{\rho-1}) = \dim \mathcal{M}_\ell. \tag{6.27}$$

The outstanding possibility, when $\mathcal{H}^L$ is of braid type with $\omega_0 = X_1^+ x$, is such that Lemma 6.13 applies with $\chi^- = X_1^-$, hence $l^- = \ell_1^-$. But for case (1), $X_1^- x = 0$, so the second formula in the lemma also gives the dimension of Ω' as

$$\dim \Omega' = p(\ell) - p(\ell_1^-) = \dim \mathcal{M}_\ell. \tag{6.28}$$

Case (1′): This can only occur in the $\rho = 1$ braid case with $\omega_0 = X_1^+ x$ and $X_1^- x \neq 0$. We set $\mathcal{K} = \mathcal{M} = \mathcal{H}^L$ and $\chi = X_1^+ = X$, $\chi^- = X_1^-$, so $l = \ell$ and $l^- = \ell_1^-$. From the third case of Lemma 6.13, we read off

$$\dim \bigcap_U \operatorname{Ker} \psi_U = \dim \mathcal{M}_\ell + \dim(\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell_1^- - \ell}. \tag{6.29}$$

But in case (1′), Ω' is a only a subset of this intersection: $\Omega' = \cap_{U^\cap} \psi^\cap_{U^\cap} \subseteq \cap_U \operatorname{Ker} \psi_U$ (and the inclusion is typically strict). Accounting for the extra conditions imposed by the $\psi^\cap_{U^\cap}$ means that the dimension of the admissible data is reduced by $\dim \operatorname{Im} \psi^\cap$, which is of course given by

$$\dim \operatorname{Im} \psi^\cap = \dim(\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{\ell_1^- - \ell} - \dim \operatorname{Ker} \psi^\cap. \tag{6.30}$$

Thus, $\dim \Omega' = \dim \mathcal{M}_\ell + \dim \operatorname{Ker} \psi^\cap$.

To show injectivity of $\psi^\cap$ and complete the computation, note first that $(\mathcal{U} X_1^- x)_{\ell-1} \oplus (\mathcal{U} X_1^- x)_{\ell-2} \subseteq \cap_U \operatorname{Ker} \psi_U$, so $\psi^\cap$ is defined on this subspace. Now we apply Lemma 6.12 to $\mathcal{K} = \mathcal{U} X_1^- x$, $m = \ell - \ell_1^-$, and $\Psi = \psi^\cap$. Since $\mathcal{V}_{h^L + \ell_1^-}$ has no singular vectors of grade less than l (except $v_{h^L + \ell_1^-}$ itself), we conclude that $\psi^\cap_{U^\cap} = 0$ implies $U^\cap = 0$.

Case (2): In the braid case with $\rho > 1$ we have

$$\mathcal{M} = \mathcal{U}X^-_{\rho-1}x + \mathcal{U}X^+_{\rho-1}x. \tag{6.31}$$

In case (2), the sum is direct at grades smaller than ℓ, so we may uniquely decompose every $w_j \in \mathcal{M}_{\ell-j}$ as

$$w_j = w_j^- + w_j^+, \quad \text{with } w_j^\pm \in (\mathcal{U}X^\pm_{\rho-1}x)_{\ell-j}. \tag{6.32}$$

We proceed by considering the "−" and "+" pieces separately.

The space whose dimension we want to compute is

$$\Omega' = (\bigcap_{U^-} \text{Ker } \psi^-_{U^-}) \cap (\bigcap_{U^+} \text{Ker } \psi^+_{U^+}). \tag{6.33}$$

We take $\mathcal{K}=\mathcal{U}X^\pm_{\rho-1}x$, $\Psi=\psi^\pm$, $X=\chi X^\pm_{\rho-1}$, $l=\ell-\ell^\pm_{\rho-1}$, and if defined, $X^-_\rho=\chi^- X^\pm_{\rho-1}$ and $l^-=\ell^-_\rho-\ell^\pm_{\rho-1}$. Then, the first or second formula of Lemma 6.13 (as appropriate) gives the dimension of $\cap_{U^\pm}$ Ker $\psi^\pm_{U^\pm}$, where the $\psi^\pm_{U^\pm}$ are restricted to the direct sum of the subspaces $(\mathcal{U}X^\pm_{\rho-1}x)_{\ell-1} \oplus (\mathcal{U}X^\pm_{\rho-1}x)_{\ell-2}$ [spanned by the $(w_1^\pm, w_2^\pm)$ of Eq. (6.32)]. The result is that this dimension coincides with that of $\mathcal{K}_l=(\mathcal{U}X^\pm_{\rho-1}x)_\ell$.

But from the definition of the $\psi^\pm_{U^\pm}$, Eq. (6.14a) and (6.14b), we quickly determine that the $\psi^\pm_{U^\pm}$ always annihilate the subspace $(\mathcal{U}X^\mp_{\rho-1}x)_{\ell-1} \oplus (\mathcal{U}X^\mp_{\rho-1}x)_{\ell-2}$. The dimension we want is therefore just the sum

$$\dim \Omega' = \dim(\mathcal{U}X^-_{\rho-1}x)_\ell + \dim(\mathcal{U}X^+_{\rho-1}x)_\ell = \dim \mathcal{M}_\ell + 1, \tag{6.34}$$

where the additional 1 derives from the fact that the decomposition (6.31) is not direct at grade ℓ because of the one-dimensional intersection spanned by ω_0.

Case (2′): As in the previous case, we use Lemma 6.13 to compute the dimension of $\cap_{U^\pm}$ Ker $\psi^\pm_{U^\pm}$, where the $\psi^\pm_{U^\pm}$ are restricted to act on pairs of descendants (of the appropriate grade) of $X^\pm_{\rho-1}x$. This time we must use the third formula, with the result that this dimension is

$$\dim(\mathcal{U}X^\pm_{\rho-1}x)_\ell + \dim(\mathcal{U}^+L_1 \cap \mathcal{U}^+L_2)_{\ell^-_\rho-\ell}.$$

The sum (6.31) is no longer direct at grades less than ℓ, but we still know that each $\psi^\pm_{U^\pm}$ annihilates pairs (w_1, w_2) whose elements w_j are in $(\mathcal{U}X^\mp_{\rho-1}x)_{\ell-j}$. Consequently, any pair whose elements are in the intersection of these subspaces, $(\mathcal{U}X^-_\rho x)_{\ell-j}$, is also annihilated. It follows then that

$$\begin{aligned}
&\dim((\cap_{U^-}\text{Ker } \psi^-_{U^-}) \cap (\cap_{U^+}\text{Ker } \psi^+_{U^+}))\\
&\quad = \dim(\mathcal{U}X^-_{\rho-1}x)_\ell + \dim(\mathcal{U}X^+_{\rho-1}x)_\ell + 2\dim(\mathcal{U}^+L_1 \cap \mathcal{U}^+L_2)_{\ell^-_\rho-\ell} - \dim(\mathcal{U}X^-_\rho)_{\ell-1} - \dim(\mathcal{U}X^-_\rho x)_{\ell-2}\\
&\quad = p(\ell-\ell^-_{\rho-1}) + p(\ell-\ell^+_{\rho-1}) + p(\ell-\ell^-_\rho-1) + p(\ell-\ell^-_\rho-2) - 2p(\ell-\ell^-_\rho)\\
&\quad = \dim \mathcal{M}_\ell + 1 + \dim(\mathcal{U}^+L_1 \cap \mathcal{U}^+L_2)_{\ell^-_\rho-\ell}.
\end{aligned} \tag{6.35}$$

Finally, we recall that $\Omega' = \cap_{U^\cap}\text{Ker } \psi^\cap_{U^\cap} \subseteq (\cap_{U^-}\text{Ker } \psi^-_{U^-}) \cap (\cap_{U^+}\text{Ker } \psi^+_{U^+})$. As in case (1′), this implies that

$$\dim \Omega' = \dim \mathcal{M}_\ell + 1 + \dim \text{Ker } \psi^\cap, \tag{6.36}$$

and the injectivity of $\psi^\cap$ follows from the same argument as before. This completes our computations. ∎

E. Invariants as coordinates

We have seen in Theorem 6.14 that the number of rank $\rho-1$ singular vectors of $\mathcal{H}^L$ coincides with the dimension of the vector space Ω'/G' (equivalently Ω/G) of nonisomorphic staggered modules with the short exact sequence (6.1). Next, we will construct coordinates on this vector space by defining invariants β or $\beta_{\pm}$ of the data defining the staggered module.

In cases (1) and (1′), recall that $\mathcal{M}$ is generated by the singular vector $X_{\rho-1}x$. We define $\chi \in \mathcal{U}^{-}$ so that $X=\chi X_{\rho-1}$ (χ is then singular, normalized, and prime). Since χ is not a scalar, we may write $\chi^{\dagger}=Y_1L_1+Y_2L_2$, although Y_1 and Y_2 are not uniquely specified. Nevertheless, every choice of Y_1 and Y_2 defines a functional $\tilde{\beta} \in (\mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2})^*$ by

$$Y_1\omega_1 + Y_2\omega_2 = \tilde{\beta}(\omega_1,\omega_2)X_{\rho-1}x. \tag{6.37}$$

Because χ is singular, this functional is invariant under the action of the gauge group G',

$$\tilde{\beta}(\omega_1 + L_1u, \omega_2 + L_2u) - \tilde{\beta}(\omega_1,\omega_2) = (Y_1L_1 + Y_2L_2)u = \chi^{\dagger}u = 0 \quad (u \in \mathcal{M}_{\ell}). \tag{6.38}$$

Moreover, it should be clear that if the data are admissible, $(\omega_1,\omega_2) \in \Omega'$, then $\tilde{\beta}$ does not depend on the choice made for Y_1 and Y_2. In this case, gauge invariance implies that we have a well-defined functional on Ω'/G'. This is our coordinate, and we denote it by β.

Similarly, in cases (2) and (2′), $\mathcal{M}$ is generated by the singular vectors $X^{-}_{\rho-1}x$ and $X^{+}_{\rho-1}x$, and we define $\chi_{\pm} \in \mathcal{U}^{-}$ so that $X=\chi_{\pm}X^{\pm}_{\rho-1}$ (making the $\chi_{\pm}$ singular, normalized, and prime). Again, the $\chi_{\pm}$ are not scalars, hence we may write $(\chi_{\pm})^{\dagger}=Y^{\pm}_1L_1+Y^{\pm}_2L_2$ (nonuniquely) and define functionals $\tilde{\beta}_{\pm} \in (\mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2})^*$ by

$$Y^{-}_1\omega_1 + Y^{-}_2\omega_2 = \tilde{\beta}_{-}(\omega_1,\omega_2)X^{-}_{\rho-1}x \tag{6.39a}$$

and

$$Y^{+}_1\omega_1 + Y^{+}_2\omega_2 = \tilde{\beta}_{+}(\omega_1,\omega_2)X^{+}_{\rho-1}x \quad (\mathrm{mod}\, \mathcal{U}X^{-}_{\rho-1}x). \tag{6.39b}$$

As above, the singularity of the $\chi_{\pm}$ implies that these functionals are invariant under G', and when $(\omega_1,\omega_2) \in \Omega'$, the definitions do not depend on the choice of $Y^{\pm}_1$ and $Y^{\pm}_2$. Thus, we obtain two coordinates on Ω'/G' in this case, and we denote the corresponding functionals by $\beta_{\pm}$.

We remark that even in the cases (1′) and (2′), we do not define an invariant related to the singular vector $X^{-}_{\rho}x$. We cannot even write down a formula analogous to Eqs. (6.37), (6.39a), and (6.39b) because $\omega_0=X^{+}_{\rho}x$ is *not* a descendant of $X^{-}_{\rho}x$. Even if one could concoct such a formula, it is difficult to imagine why the corresponding quantity should be gauge invariant. In any case, we will see in Theorem 6.15 below that the invariants we have already defined are sufficient to completely characterize a staggered module with exact sequence (6.1).

Note that if $(\omega_1,\omega_2) \in \Omega'$, so we do indeed have a staggered module (with right module Verma), then $\omega_j=L_jy$ for $j=1,2$. Hence we may write (abusing notation in an obvious manner)

$$\beta X_{\rho-1}x = (Y_1L_1 + Y_2L_2)y = \chi^{\dagger}y \iff \beta = \langle X_{\rho-1}x, \chi^{\dagger}y\rangle_{\mathcal{U}X_{\rho-1}x}. \tag{6.40}$$

Similar formulas may be written for $\beta_{\pm}$, although for β_{+}, one should include a projection from $\mathcal{M}$ onto $\mathcal{U}X^{+}_{\rho-1}x$. It is in this form that we may compare these invariant coordinates with the beta invariant defined in Eq. (3.5).

It is the latter invariant which has been used in the literature to distinguish staggered modules with the same exact sequence, although we have already noted [Eq. (3.7)] that this beta invariant vanishes whenever $\rho=\mathrm{rank}\,\omega_0>1$. This has not been found to problematic thus far because, to the best of our knowledge, only modules with $\rho \leqslant 1$ have been found to be relevant in applications. Nevertheless, this vanishing is a conceptual problem which is solved by the invariant coordinates introduced above. Namely, when $\rho=1$ [cases (1) and (1′)], the beta invariant of Eq. (3.5) coincides with the (value of the) coordinate β because $\chi=X$ (this is why we have risked some confusion by using the same notation for the coordinates and invariants). When $\rho>1$ and the beta invariant vanishes identically, we have instead the coordinates β [cases (1) and (1′)] or $\beta_{\pm}$ [cases (2) and (2′)]. We therefore feel justified in concluding that the invariant coordinates defined here should *replace* the (in hindsight, naïve) definition of the beta invariant given in Sec. III.

There is one point that remains to be addressed. The beta invariant of Sec. III vanishes identically when $\rho > 1$, hence is useless in this case for distinguishing staggered modules with the same exact sequence. We claim that the invariant coordinates defined above are superior in this respect, so we need to establish that the invariant coordinates β or $\beta_{\pm}$ are *linearly independent* functionals on the vector space Ω'/G', that is, that they are actually coordinates. We remark that this would complete our analysis of staggered modules when the right module is Verma. Indeed, the vector space of inequivalent staggered modules with a given short exact sequence (6.1) was seen in Theorem 6.14 to have dimension 0, 1, or 2. As the number of coordinates we have constructed precisely matches the dimension of Ω'/G' in each case, they completely characterize the staggered module (again, given a short exact sequence). Practically, this means that the formulas given in Eqs. (6.37), (6.39a), and (6.39b) reduce the identification of a staggered module (6.1) to the computation of one or two numbers.

Theorem 6.15: *In cases* (1) *and* (1′), *the functional* β *is not identically zero on the one-dimensional vector space* Ω'/G', *and so parametrizes it. In cases* (2) *and* (2′), *the functionals* β_- *and* β_+ *are nonzero and linearly independent on the two-dimensional vector space* Ω'/G', *and so parametrize it.*

Proof: We first note that to show that a functional $\tilde{\beta}$ on a finite-dimensional vector space V is nonvanishing on the intersection of the kernels of a collection of functionals $\{\psi_U\}$, it is enough to prove that $\tilde{\beta}$ is linearly independent of this collection. This follows quite readily by taking a basis for the span of $\{\psi_U\}$, extending it to a basis of V^*, and then considering the action of $\tilde{\beta}$ on the dual basis (identifying V^{**} and V in the standard way). Our strategy below is therefore to prove that $\tilde{\beta}$ and its variants are linearly independent of the ψ_U (and its variants), so β is nonzero.

Case (1): Assume that $\tilde{\beta}$ is a linear combination of the ψ_U : $\tilde{\beta}=\Sigma_U b_U \psi_U = \psi_B \in (\mathcal{M}_{\ell-1} \oplus \mathcal{M}_{\ell-2})^*$ for some $B=\Sigma_U b_U U = B_1 L_1 = -B_2 L_2$. Then, from the definitions (6.11) and (6.37), we get

$$Y_1 w_1 + Y_2 w_2 = B_1 w_1 + B_2 w_2 \quad \text{for all } w_1 \in \mathcal{M}_{\ell-1} \text{ and } w_2 \in \mathcal{M}_{\ell-2}, \tag{6.41}$$

where $Y_1 L_1 + Y_2 L_2 = \chi^\dagger$ is such that $X = \chi X_{\rho-1}$ (so χ is nonzero and singular). Setting $w_2=0$, we find that $Y_1 - B_1$ must annihilate $\mathcal{M}_{\ell-1}$. However, this implies that

$$\langle (Y_1 - B_1)^\dagger X_{\rho-1} x, \mathcal{M}_{\ell-1} \rangle_{\mathcal{M}} = \langle X_{\rho-1} x, (Y_1 - B_1)\mathcal{M}_{\ell-1} \rangle_{\mathcal{M}} = 0, \tag{6.42}$$

hence that $(Y_1 - B_1)^\dagger X_{\rho-1} x$ is a grade $\ell-1$ descendant of a (noncyclic) singular vector of $\mathcal{M}$. But, in case (1), $\mathcal{M}$ has no nontrivial singular vectors of grade less than ℓ (except of course for $X_{\rho-1}x$ itself). Thus, $(Y_1 - B_1)^\dagger X_{\rho-1} x = 0$.

When the Verma module corresponding to $\mathcal{M}$ has no singular vectors of grade less than ℓ, we may conclude that $Y_1 = B_1$, and repeating this argument for $w_1 = 0$, that $Y_2 = B_2$. Then, we obtain a contradiction,

$$\chi^\dagger = Y_1 L_1 + Y_2 L_2 = B_1 L_1 + B_2 L_2 = 0. \tag{6.43}$$

However, case (1) also includes the possibility that $\mathcal{M} = \mathcal{H}^L$ is of braid type with $\rho = 1$, $\chi = X = X_1^+$, and $X_1^- x = 0$. Then, we can only conclude that $(Y_1 - B_1)^\dagger = V_1 \chi^-$ for some $V_1 \in \mathcal{U}^-$, where $\chi^- = X_1^-$ is singular. Similarly, taking $w_2 = 0$ now leads to $(Y_2 - B_2)^\dagger = V_2 \chi^-$ for some $V_2 \in \mathcal{U}^-$, and we arrive at

$$\chi^\dagger = Y_1 L_1 + Y_2 L_2 = B_1 L_1 + B_2 L_2 + (\chi^-)^\dagger (V_1^\dagger L_1 + V_2^\dagger L_2) = (\chi^-)^\dagger (V_1^\dagger L_1 + V_2^\dagger L_2). \tag{6.44}$$

This is again a contradiction because it implies that $\chi x = X_1^+ x$ is a descendant of $\chi^- x = X_1^- x$. It therefore follows that in case (1), $\tilde{\beta}$ is linearly independent of the (ψ_U), so $\beta \in (\Omega'/G')^*$ is nonvanishing.

Case (1′): In this case, $\Omega' \subseteq \cap_U \mathrm{Ker}\, \psi_U$, so we again need $\tilde{\beta}$ to be linearly independent of the ψ_U. If this was not the case, we would use the argument which settles case (1) to derive the contradiction of Eq. (6.44) [the sole difference arises because $\chi^- x \neq 0 (\chi^- = X_1^-)$, so Eq. (6.42) would give $(Y_j-B_j)^\dagger x = V_j\chi^- x$ for some $V_j \in \mathcal{U}^-$, recovering $(Y_j-B_j)^\dagger = V_j\chi^-$]. Therefore, $\tilde{\beta}$ does not vanish identically on $\cap_U \mathrm{Ker}\ \psi_U$. However, we still have to rule out the possibility that $\tilde{\beta}$ might vanish on the (typically proper) subset $\Omega' = \cap_{U^\cap} \mathrm{Ker}\ \psi^\cap_{U^\cap}$. To do this, note that there exists a pair $(w_1, w_2) \in \cap_U \mathrm{Ker}\ \psi_U$ for which $\tilde{\beta}(w_1,w_2) \neq 0$. We will use this pair to construct a pair $(w_1', w_2') \in \cap_{U^\cap} \mathrm{Ker}\ \psi^\cap_{U^\cap}$ which has the same (nonzero) value as (w_1,w_2) under $\tilde{\beta}$, thereby establishing that $\tilde{\beta} \neq 0$ on Ω'.

The key observation is that any $(w_1^\cap, w_2^\cap) \in (\mathcal{U}\chi^- x)_{\ell-1} \oplus (\mathcal{U}\chi^- x)_{\ell-2}$ is annihilated by $\tilde{\beta}$ and every ψ_U, but not, in general, by the $\psi^\cap_{U^\cap}$. We may therefore "shift" our pair (w_1,w_2) by any such $(w_1^\cap, w_2^\cap)$ without affecting membership in $\cap_U \mathrm{Ker}\ \psi_U$ or changing its value under $\tilde{\beta}$. Take then a basis $\{\psi^\cap_{U^\cap_\mu}\}$ of Im $\psi^\cap$, and notice that as the restriction to $(\mathcal{U}\chi^- x)_{\ell-1} \oplus (\mathcal{U}\chi^- x)_{\ell-2}$ of $\psi^\cap_{U^\cap}$ is zero only for $U^\cap = 0$ (Lemma 6.12), this remains a basis for the restrictions. Extend arbitrarily to a basis of $((\mathcal{U}\chi^- x)_{\ell-1} \oplus (\mathcal{U}\chi^- x)_{\ell-2})^*$. Let the corresponding dual basis of $(\mathcal{U}\chi^- x)_{\ell-1} \oplus (\mathcal{U}\chi^- x)_{\ell-2}$ be denoted by $\{(w_1^{(\mu)}, w_2^{(\mu)})\}$, so, in particular, $\psi^\cap_{U^\cap_\mu}(w_1^{(\nu)}, w_2^{(\nu)}) = \delta_{\mu,\nu}$. Choosing now

$$w_j^\cap = \sum_\mu \psi^\cap_{U^\cap_\mu}(w_1,w_2) w_j^{(\mu)}, \tag{6.45}$$

we quickly compute that $(w_1', w_2') = (w_1 - w_1^\cap, w_2 - w_2^\cap)$ is annihilated by every $\psi^\cap_{U^\cap}$. Since $(w_1', w_2') \in \cap_U \mathrm{Ker}\ \psi_U$ and $\tilde{\beta}(w_1', w_2') = \tilde{\beta}(w_1, w_2) \neq 0$, this proves that $\tilde{\beta} \neq 0$ on Ω'.

Cases (2) *and* (2′): In these cases, we once again use the decomposition

$$\mathcal{M}_{\ell-j} = (\mathcal{U}X^-_{\rho-1}x)_{\ell-j} + (\mathcal{U}X^+_{\rho-1}x)_{\ell-j}, \tag{6.46}$$

where the sum is direct in case (2) but not in case (2′). We therefore write $w_j = w_j^- + w_j^+$ with $w_j^\pm \in \mathcal{U}X^\pm_{\rho-1}x (j=1,2)$. The nonuniqueness of this decomposition in case (2′) leads to no difficulties in what follows.

We start by observing that the restrictions of our functionals to the "wrong" subspaces are trivial,

$$\psi^\pm_{U^\pm} = \tilde{\beta}_\pm = 0 \quad \text{on } (\mathcal{U}X^\mp_{\rho-1}x)_{\ell-1} \oplus (\mathcal{U}X^\mp_{\rho-1}x)_{\ell-2}. \tag{6.47}$$

In particular, in case (2′), all these functionals vanish on the intersection $(\mathcal{U}X^-_\rho x)_{\ell-1} \oplus (\mathcal{U}X^-_\rho x)_{\ell-2}$ (which is why nonuniqueness leads to no difficulties). It follows from this that if the $\tilde{\beta}_\pm$ are nonzero on Ω', their linear independence, and hence that of the $\beta_\pm$, follows for free.

However, proving that the functionals $\beta_\pm$ are nonzero reduces to demonstrating (separately for "−" and "+") that the corresponding $\tilde{\beta}_\pm$ are linearly independent of the $\psi^\pm_{U^\pm}$ and, furthermore [in case (2′) only], to checking that the $\tilde{\beta}_\pm$ do not vanish identically on $\cap_{U^\cap} \mathrm{Ker}\ \psi^\cap_{U^\cap}$. After splitting the (w_1,w_2) according to Eq. (6.46), the arguments establishing these results are identical to those presented in cases (1) and (1′), so we do not repeat them here. ∎

We close this section with a couple of examples illustrating the formalism constructed above. The first illustrates a simple case in which there are two invariant coordinates $\beta_\pm$.

Example 8: By Theorem 6.14, there is a two-dimensional space of nonisomorphic staggered modules $\mathcal{S}$ with $c=0$ $\left(t=\frac{3}{2}\right)$ and short exact sequence,

$$0 \to \mathcal{V}_0 \to \mathcal{S} \to \mathcal{V}_5 \to 0, \tag{6.48}$$

because $\mathcal{H}^L = \mathcal{V}_0$ is of braid type and its grade $\ell=5$ singular vector ω_0 has rank 2 *(this is a case* (2) *example). The dimensionality of Ω'/G' can also be demonstrated directly as follows.*

The normalized rank 1 *singular vectors generating the submodule $\mathcal{M}$ of $\mathcal{H}^L$ are*

$$L_{-1}x \quad \textit{and} \quad \left(L_{-1}^2 - \tfrac{2}{3}L_{-2}\right)x. \tag{6.49}$$

This example is rather special because the only states of $\mathcal{H}^L$ not in $\mathcal{M}$ are x and its (nonzero) *multiples (the irreducible highest weight module* $\mathcal{L}_0$ *is one dimensional). It follows that* $\Omega' = \Omega \cap \mathcal{M} = \Omega$. *Since* $\ell = 5$, *we should check the constraint on the possible data* $(\omega_1, \omega_2) \in \mathcal{M}_4 \oplus \mathcal{M}_3$ *coming from the nontrivial element of* $(\mathcal{U}^+ L_1 \cap \mathcal{U}^+ L_2)_{-5}$ *given in Eq. (4.8),*

$$(L_1^2 L_2 + 6L_2^2 - L_1 L_3 + 2L_4)\omega_1 = (L_1^3 + 6L_1 L_2 + 12L_3)\omega_2. \tag{6.50}$$

However, both sides must be proportional to $x \notin \mathcal{M}$, *hence must vanish for all* ω_1 and ω_2. *There is therefore no constraint upon the data.*

Since $\dim \mathcal{M}_4 = 5$ *and* $\dim \mathcal{M}_3 = 3$, *the space of admissible data has dimension* 8. *As the space of gauge transformations* $G' = G$ *has dimension* $\dim \mathcal{M}_5 - 1 = 6$, *we conclude that the space of inequivalent staggered modules with exact sequence (6.48) is two dimensional, as expected. Finally, as* ω_0 *may be represented in the forms*

$$\omega_0 = \left(L_{-1}^4 - \tfrac{20}{3}L_{-2}L_{-1}^2 + 4L_{-2}^2 + 4L_{-3}L_{-1} - 4L_{-4}\right)L_{-1}x \tag{6.51a}$$

$$= (L_{-1}^3 - 6L_{-2}L_{-1} + 6L_{-3})\left(L_{-1}^2 - \tfrac{2}{3}L_{-2}\right)x, \tag{6.51b}$$

it follows from Eq. (6.39a) and (6.39b) and Theorem 6.15 that this space is parametrized by two invariants,

$$\beta_- L_{-1}x = \left(L_1^4 - \tfrac{20}{3}L_1^2 L_2 + 4L_2^2 + 4L_1 L_3 - 4L_4\right)y \tag{6.52a}$$

and

$$\beta_+ \left(L_{-1}^2 - \tfrac{2}{3}L_{-2}\right)x = (L_1^3 - 6L_1 L_2 + 6L_3)y \quad (\mathrm{mod}\ \mathbb{C}L_{-1}^2 x). \tag{6.52b}$$

Any choice of values for these beta invariants corresponds to a distinct staggered module.

This example is admittedly special because $\mathcal{M}$ coincides with $\mathcal{H}^L$ at all positive grades. One consequence is that both β_- and β_+ are defined for all $(\omega_1, \omega_2) \in \Omega$ and are invariant under the full group of gauge transformations G. In general, however, this is not true. Practically, the beta invariants may be viewed as numbers to be computed in order to identify representations. It is therefore somewhat inconvenient that they are, in general, only defined for data $(\omega_1, \omega_2) \in \Omega'$, hence for only certain choices of y, and are consequently only invariant under the restricted set of gauge transformations $G' \subseteq G$ preserving Ω'.

While the Projection Lemma, Lemma 5.1, guarantees that we can always choose (equivalent) data in Ω', it is sometimes desirable to define the invariants so that one can easily compute them for general data $(\omega_1, \omega_2) \in \Omega$, and hence for general choices of y. In the following example, we illustrate how to combine the content of the Projection Lemma with the above definitions of the beta invariants to deduce a generally valid formula.

Example 9: We consider the one-dimensional space of $c = -2$ $(t = 2)$ *staggered modules with short exact sequence,*

$$0 \to \mathcal{V}_0 \to \mathcal{S} \to \mathcal{V}_3 \to 0, \tag{6.53}$$

$\mathcal{H}^L$ *is of chain type with singular vectors* $L_{-1}x$ and $\omega_0 = (L_{-1}^2 - 2L_{-2})L_{-1}x$ *at grades* 1 *and* 3, *respectively.* ω_0 *is therefore composite, of rank* 2, *and* $\mathcal{M}$ *is generated by* $L_{-1}x$. *We note first of all, supposing that* y *is chosen such that* $(\omega_1, \omega_2) \in \Omega'$, *that the invariant* β *of Eq. (6.37) may be defined by*

$$\beta L_{-1}x = (L_1^2 - 2L_2)y. \tag{6.54}$$

Our aim is to derive a similar formula that can be used with any choice of y *(assuming only that it is correctly normalized).*

To do this, we recall that in the proof of the Projection Lemma, we constructed projections onto appropriate submodules of $\mathcal{H}^L$ *which take data to equivalent data. This was achieved by considering an orthonormal basis of the complement of the submodule (as a vector space) at the* *right grade. In the case at hand, we only need one projection to get from Ω to Ω', the submodule we want to project onto is $\mathcal{M}$, and the grade of our basis is $\ell=3$. Since* $\dim \mathcal{M}_3=2$ *and* $\dim \mathcal{H}_3^L=3$, *we may take $Z=(i/2)L_{-3}$ to define our orthonormal basis $\{Zx\}$ (recall that the Shapovalov form is assumed bilinear, not sesquilinear). On the other hand, the vectors L_{-1}^3x and $L_{-2}L_{-1}x$ span $\mathcal{M}$ at grade 3 (they are the $V_\lambda L_{-1}x$ in the notation of Sec. V).*

Given data in Ω, the key step in the proof of the Projection Lemma was to find equivalent data in Ω' using a carefully chosen gauge transformation g_z. In the case at hand, one can check that the choice amounts to $z=-ZZ^\dagger y$. In terms of gauge transforming y, this corresponds to applying the operator $\mathbf{1}-ZZ^\dagger$ to obtain the new y (for which the corresponding data are in Ω'). This immediately yields an improved version of the definition (6.54) of β,

$$\beta L_{-1}x = (L_1^2 - 2L_2)(1 - ZZ^\dagger)y = (L_1^2 - 2L_2)\left(1 + \tfrac{1}{4}L_{-3}L_3\right)y, \tag{6.55}$$

which may be used for any (admissible) choice of y.

It should be clear that the same strategy will recover formulas for the beta invariants of general staggered modules [with exact sequence (6.1)] which are valid for every y corresponding to admissible data. All that will change is that the orthonormal basis may consist of several elements Z_μ, and that one might need several consecutive projections. Indeed, in the chain case we let $Z_\mu^{(k)}$ denote the basis elements chosen at the kth step of the projections of Sec. V and define

$$\mathbf{1} - \mathbb{P} = \left(\mathbf{1} - \sum_\mu Z_\mu^{(\rho-1)}(Z_\mu^{(\rho-1)})^\dagger\right) \cdots \left(\mathbf{1} - \sum_\mu Z_\mu^{(2)}(Z_\mu^{(2)})^\dagger\right)\left(\mathbf{1} - \sum_\mu Z_\mu^{(1)}(Z_\mu^{(1)})^\dagger\right). \tag{6.56}$$

The formula defining the invariant now becomes

$$\beta X_{\rho-1}x = \chi^\dagger(\mathbf{1} - \mathbb{P})y. \tag{6.57}$$

In the braid case, projecting from rank $k-1$ to k required two steps and we will denote the corresponding orthonormal bases by $Z_\mu^{(k-1;+)}$ and $Z_\mu^{(k;-)}$. Now,

$$\begin{aligned}\mathbf{1} - \mathbb{P} = &\left(\mathbf{1} - \sum_\mu Z_\mu^{(\rho-1;-)}(Z_\mu^{(\rho-1;-)})^\dagger\right)\left(\mathbf{1} - \sum_\mu Z_\mu^{(\rho-2;+)}(Z_\mu^{(\rho-2;+)})^\dagger\right)\left(\mathbf{1} - \sum_\mu Z_\mu^{(\rho-2;-)}(Z_\mu^{(\rho-2;-)})^\dagger\right)\\ &\cdots\left(\mathbf{1} - \sum_\mu Z_\mu^{(2;-)}(Z_\mu^{(2;-)})^\dagger\right)\left(\mathbf{1} - \sum_\mu Z_\mu^{(1;+)}(Z_\mu^{(1;+)})^\dagger\right)\left(\mathbf{1} - \sum_\mu Z_\mu^{(1;-)}(Z_\mu^{(1;-)})^\dagger\right),\end{aligned} \tag{6.58}$$

and the invariants are defined by

$$\beta_- X_{\rho-1}^- x = (\chi_-)^\dagger(\mathbf{1} - \mathbb{P})y \quad \text{and} \quad \beta_+ X_{\rho-1}^+ x = (\chi_+)^\dagger(\mathbf{1} - \mathbb{P})y \quad (\operatorname{mod} \mathcal{U}X_{\rho-1}^- x). \tag{6.59}$$

As a final simplification in such formulas, we can even remove the inconvenient quotient in the definition of β_+. Specifically, we can choose one more basis $\{W_\mu\}$, this time for $\mathcal{U}_{\ell_{\rho-1}^+ - \ell_{\rho-1}^-}$, such that the corresponding basis $\{W_\mu v_{h^L + \ell_{\rho-1}^-}\}$ of $\mathcal{V}_{h^L + \ell_{\rho-1}^-}$ is orthonormal. Then for any $u \in \mathcal{M}_{\ell_{\rho-1}^+}$, the vector $\Sigma_\mu W_\mu W_\mu^\dagger u$ is in $\mathcal{U}X_{\rho-1}^- x$, and $u - \Sigma_\mu W_\mu W_\mu^\dagger u$ is proportional to $X_{\rho-1}^+ x$ by virtue of orthonormality. A completely explicit formula for β_+ is thus

$$\beta_+ X_{\rho-1}^+ x = \left(\mathbf{1} - \sum_\mu W_\mu W_\mu^\dagger\right)(\chi_+)^\dagger(\mathbf{1} - \mathbb{P})y. \tag{6.60}$$

We make one final remark about this way of defining invariants. The explicit forms of the projections $\mathbf{1} - \mathbb{P} \in \mathcal{U}$ will, in general, depend on the choices of orthonormal bases used. However, the values taken by the beta invariants of course do not.

VII. GENERAL RIGHT MODULES

In view of the general construction of Theorem 4.4, the existence question in the case of arbitrary right modules $\mathcal{H}^R$ seems at first to be more involved than the $\mathcal{H}^R$ Verma case elucidated in Sec. VI. The vectors $(\varpi,-\bar{X})$ that are added to the list of generators of $\mathcal{N}$ [Eq. (4.10)] contain both ϖ, which is only determined by the data in a rather indirect fashion, and $\bar{X}$, for which no simple, explicit general formula is known. However, Proposition 4.6 suggests an alternate strategy. Indeed, given left and right modules $\mathcal{H}^L$ and $\mathcal{H}^R$, we can first use Theorem 6.14 to determine the space of isomorphism classes of staggered modules $\check{\mathcal{S}}$ with exact sequence,

$$0 \to \mathcal{H}^L \xrightarrow{\check{\iota}} \check{\mathcal{S}} \xrightarrow{\check{\pi}} \mathcal{V}_{h^R} \to 0, \tag{7.1}$$

and then for each isomorphism class, decide whether the right module can be replaced by $\mathcal{H}^R$, obtaining

$$0 \to \mathcal{H}^L \xrightarrow{\iota} \mathcal{S} \xrightarrow{\pi} \mathcal{H}^R \to 0. \tag{7.2}$$

In practice, the isomorphism classes are determined by the beta invariants of $\check{\mathcal{S}}$, so our task in this section is to analyze, in terms of these coordinates, when such a replacement is permitted. Throughout this section, we will assume that the right module $\mathcal{H}^R$ of $\mathcal{S}$ is not a Verma module.

Note that the definitions of Ω, G, Ω', G', and $\mathcal{M}$ depend only on the left module $\mathcal{H}^L$ which is unchanged in the replacement proposed above. We will therefore continue to use these notations in this section without comment. Similarly, the important definitions [Eqs. (6.37), (6.39a), and (6.39b)] of β and $\beta_{\pm}$ make perfect sense for $\mathcal{S}$. Indeed, since the data of $\mathcal{S}$ and $\check{\mathcal{S}}$ coincide by Proposition 4.6, it follows that their beta invariants coincide too. We therefore obtain, as an immediate consequence of Theorems 6.14 and 6.15, a uniqueness result covering every case except that which was already treated in Corollary 3.5.

Corollary 7.1: There exists at most one staggered module $\mathcal{S}$ (up to isomorphism) for any given choice of left and right modules and beta invariants β or *$\beta_{\pm}$ of Sec. VI E (as appropriate).*

A. Singular vectors of staggered modules

It was observed in Corollary 4.7 that $\mathcal{S}$ may be realized as a quotient of $\check{\mathcal{S}}$. We give in this section a sharpening of this result. First, however, we recall from Proposition 3.3 that when $\bar{X} \in \mathcal{U}_{\bar{\ell}}^-$ is defined, it is necessary for the existence of $\mathcal{S}$ that $\bar{X}\omega_0=0$ in $\mathcal{H}^L$. A similar statement holds if both $\bar{X}^- \in \mathcal{U}_{\bar{\ell}^-}^-$ and $\bar{X}^+ \in \mathcal{U}_{\bar{\ell}^+}^-$ are defined. We therefore assume in what follows that $\mathcal{H}^L$ satisfies this requirement.

Proposition 7.2: When $\bar{X}$ is defined, a staggered module $\mathcal{S}$ exists if and only if $\check{\mathcal{S}}$ has a singular vector $\bar{y}$ at grade $\ell+\bar{\ell}$. Then, $\mathcal{S}=\check{\mathcal{S}}/\mathcal{U}\bar{y}$. When $\bar{X}^-$ and *$\bar{X}^+$ are defined, $\mathcal{S}$ exists if and only if $\check{\mathcal{S}}$ has singular vectors $\bar{y}^-$* and *$\bar{y}^+$ at grades $\ell+\bar{\ell}^-$* and *$\ell+\bar{\ell}^+$, respectively. Then, $\mathcal{S}=\check{\mathcal{S}}/(\mathcal{U}\bar{y}^-+\mathcal{U}\bar{y}^+)$.*

We remark immediately that by Proposition 3.3, the left module $\mathcal{H}^L$ does not have a (nonzero) singular vector at grade $\ell+\bar{\ell}$, so the singular vectors $\bar{y}$ or $\bar{y}^{\pm}$ in $\check{\mathcal{S}}$ are not annihilated by the projection onto $\mathcal{V}_{h^R}$. Indeed, we may assume the normalizations,

$$\check{\pi}(\bar{y}) = \bar{X}v_{h^R} \quad \text{or} \quad \check{\pi}(\bar{y}^{\pm}) = \bar{X}^{\pm}v_{h^R}. \tag{7.3}$$

The singular vectors therefore have the form $\bar{X}y-\varpi$ or $\bar{X}^{\pm}y-\varpi^{\pm}$, where $\varpi,\varpi^{\pm} \in \mathcal{H}^L$. The uniqueness of such singular vectors follows again from Proposition 3.3. We mention that it is in considering situations such as these that the terminology employed by Rohsiepe in Ref. 26 becomes inconvenient. In particular, we see once again that for $\check{\mathcal{S}}$, Rohsiepe's lower module, which he defines as the subspace of L_0 eigenvectors, is not a highest weight module (it contains $\bar{y}$).

Proof: We first assume that $\mathcal{S}$ exists. Denote by $\check{\mathcal{N}}$ and $\mathcal{N}$ the submodules of $\mathcal{H}^L \oplus \mathcal{U}$ in the constructions (Theorem 4.4) of $\check{\mathcal{S}}$ and $\mathcal{S}$, respectively. As we have seen in the proof of Proposition 4.6, $\check{\mathcal{N}} \subseteq \mathcal{N}$. We will show that $L_n(\varpi,-\bar{X}) \in \check{\mathcal{N}}$ for all $n>0$ and $(L_0-h^R-\bar{\ell})(\varpi,-\bar{X}) \in \check{\mathcal{N}}$, thereby establishing that $(\varpi,-\bar{X})$ becomes singular in the quotient $(\mathcal{H}^L\oplus\mathcal{U})/\check{\mathcal{N}}=\check{\mathcal{S}}$ (we will only detail this direction in the $\bar{X}$ case, that of $\bar{X}^{\pm}$ being identical).

We first write $L_n\bar{X}=U_0(L_0-h^R)+U_1L_1+U_2L_2$, as usual because $\bar{X}$ is singular. Then in $\mathcal{H}^L\oplus\mathcal{U}$, the definition of $\check{\mathcal{N}}$ gives

$$\begin{aligned} L_n(\varpi,-\bar{X}) &= L_n(\varpi,0) - U_0(0,L_0-h^R) - U_1(0,L_1) - U_2(0,L_2) \\ &= L_n(\varpi,0) - U_0(\omega_0,0) - U_1(\omega_1,0) - U_2(\omega_2,0) \quad (\mathrm{mod}\ \check{\mathcal{N}}), \end{aligned} \tag{7.4}$$

and each of the four terms on the right hand side is obviously in $\iota^L(\mathcal{H}^L)=\mathcal{H}^L\oplus\{0\}$. Now, $(\varpi,-\bar{X})$ is one of the generators of $\mathcal{N}$, so the sum of the four terms is in $\mathcal{N}$ (since $\check{\mathcal{N}}\subseteq\mathcal{N}$). But the existence of $\mathcal{S}$ implies that $\mathcal{N}\cap\iota^L(\mathcal{H}^L)=\mathcal{N}^{\circ}=\{0\}$ by Theorem 4.4, hence that the sum of these four terms is zero. We conclude that $L_n(\varpi,-\bar{X})\in\check{\mathcal{N}}$ as required.

The argument for $(L_0-h^R-\bar{\ell})$ is similar. In fact, we have $(L_0-h^R-\bar{\ell})\bar{X}=\bar{X}(L_0-h^R)$, so

$$(L_0-h^R-\bar{\ell})(\varpi,-\bar{X}) = -\bar{X}(0,L_0-h^R) = -\bar{X}(\omega_0,0) \quad (\mathrm{mod}\ \check{\mathcal{N}}). \tag{7.5}$$

But we are assuming that $\bar{X}\omega_0=0$ (Proposition 3.3), hence we find that $(L_0-h^R-\bar{\ell})(\varpi,-\bar{X})\in\check{\mathcal{N}}$, as required. This completes the proof that the class of $(\varpi,-\bar{X})$ modulo $\check{\mathcal{N}}$ is a singular vector of $\check{\mathcal{S}}$.

The other direction requires us to show that the existence of the singular vector $\bar{y}\in\check{\mathcal{S}}$ implies that the quotient by the submodule generated by $\bar{y}$ is the desired staggered module $\mathcal{S}$. The strategy here is rather similar to that used to prove Theorem 4.4. First observe from Eq. (7.3) that $\check{\pi}(\mathcal{U}\bar{y})=\mathcal{J}$, where $\mathcal{H}^R=\mathcal{V}_{h^R}/\mathcal{J}$. We denote the projections from $\check{\mathcal{S}}$ to $\check{\mathcal{S}}/\mathcal{U}\bar{y}$ and from $\mathcal{V}_{h^R}$ to $\mathcal{H}^R$ by $\bar{\pi}$ and $\pi_{\mathcal{J}}$, respectively. We then define module homomorphisms ι and π so as to make the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{H}^L & \xrightarrow{\check{\iota}} & \check{\mathcal{S}} & \xrightarrow{\check{\pi}} & \mathcal{V}_{h^R} & \rightarrow & 0 \\ & & \| & & \downarrow\bar{\pi} & & \downarrow\pi_{\mathcal{J}} & & \\ 0 & \rightarrow & \mathcal{H}^L & \xrightarrow{\iota} & \dfrac{\check{\mathcal{S}}}{\mathcal{U}\bar{y}} & \xrightarrow{\pi} & \mathcal{H}^R & \rightarrow & 0 \end{array} \tag{7.6}$$

Our task is now to show that the bottom row is exact. For injectivity of $\iota=\bar{\pi}\circ\check{\iota}$, we must show that $\mathrm{Ker}\ \bar{\pi}\cap\mathrm{Im}\ \check{\iota}=\mathcal{U}\bar{y}\cap\check{\iota}(\mathcal{H}^L)=\{0\}$. But, if $U\bar{y}\in\check{\iota}(\mathcal{H}^L)$ for some $U\in\mathcal{U}$, then we can assume that $U\in\mathcal{U}^-$ by the singularity of $\bar{y}$. Exactness of the top row now gives $0=\check{\pi}(U\bar{y})=U\bar{X}v_{h^R}\in\mathcal{V}_{h^R}$, hence $U=0$ as Verma modules are free as $\mathcal{U}^-$-modules. This proves that ι is injective. The projection π is well defined by $\pi\circ\bar{\pi}=\pi_{\mathcal{J}}\circ\check{\pi}$ because $\mathrm{Ker}\ \bar{\pi}$ is annihilated by the right hand side by construction. Its surjectivity follows from that of $\pi_{\mathcal{J}}$ and $\check{\pi}$.

Exactness then follows from that of the top row, whence $\pi\circ\iota=\pi\circ\bar{\pi}\circ\check{\iota}=\pi_{\mathcal{J}}\circ\check{\pi}\circ\check{\iota}=0$, and the following argument: If $\pi\circ\bar{\pi}(u)=0$ for some $u\in\check{\mathcal{S}}$, then $\pi_{\mathcal{J}}\circ\check{\pi}(u)=0$, hence $\check{\pi}(u)=U\bar{X}v_{h^R}$ for some $U\in\mathcal{U}$. We therefore conclude that

$$u = U\bar{y} \quad (\mathrm{mod}\ \check{\iota}(\mathcal{H}^L)), \quad \text{so } \bar{\pi}(u)=0 \quad (\mathrm{mod}\ \bar{\pi}\check{\iota}(\mathcal{H}^L)=\iota(\mathcal{H}^L)). \tag{7.7}$$

As $\bar{\pi}$ is surjective, we are done.

We have therefore constructed an exact sequence with the left and right modules of $\mathcal{S}$. The data of $\check{\mathcal{S}}/\mathcal{U}\bar{y}$ are obtained by acting on $y=\bar{\pi}(\check{y})$ (which is indeed mapped to $x^R\in\mathcal{H}^R$ under π) and coincide with that of $\mathcal{S}$ (and $\check{\mathcal{S}}$). By Proposition 3.6, $\mathcal{S}\cong\check{\mathcal{S}}/\mathcal{U}\bar{y}$ as required.

The argument for the $\bar{X}^{\pm}$ case is similar, although slightly more complicated. First we construct an exact sequence for $\check{\mathcal{S}}/\mathcal{U}\bar{y}^-$ as above (with injection ι' and surjection π'), obtaining a commutative diagram very similar to (7.6). The arguments for this step are exactly the same as those above. Then, we define ι and π so as to make the following augmented diagram commute:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{H}^L & \xrightarrow{\check{\iota}} & \check{\mathcal{S}} & \xrightarrow{\check{\pi}} & \mathcal{V}_{h^R} & \rightarrow & 0 \\ & & \| & & \downarrow \bar{\pi}^- & & \downarrow \pi_{\mathcal{J}^-} & & \\ 0 & \rightarrow & \mathcal{H}^L & \xrightarrow{\iota'} & \dfrac{\check{\mathcal{S}}}{\mathcal{U}\bar{y}^-} & \xrightarrow{\pi'} & \dfrac{\mathcal{V}_{h^R}}{\mathcal{J}^-} & \rightarrow & 0 \\ & & \| & & \downarrow \bar{\pi}^+ & & \downarrow \pi_{\mathcal{J}^+} & & \\ 0 & \rightarrow & \mathcal{H}^L & \xrightarrow{\iota} & \dfrac{\check{\mathcal{S}}}{\mathcal{U}\bar{y}^- + \mathcal{U}\bar{y}^+} & \xrightarrow{\pi} & \mathcal{H}^R & \rightarrow & 0 \end{array} \tag{7.8}$$

Here, $\bar{\pi}^{\pm}$ corresponds to quotienting by the submodule generated by $\bar{y}^{\pm}$ and $\pi_{\mathcal{J}^{\pm}}$ corresponds to quotienting by the submodule $\mathcal{J}^{\pm}$ generated by $\bar{X}^- v_{h^R}$ or $\bar{X}^+ x'$ as appropriate, where x' is the highest weight vector of $\mathcal{V}_{h^R}/\mathcal{J}^-$. The arguments demonstrating exactness and the isomorphism of $\mathcal{S}$ and $\check{\mathcal{S}}/(\mathcal{U}\bar{y}^- + \mathcal{U}\bar{y}^+)$ are also identical to those above, except as regards the proof that ι is injective.

Note then that ι will be injective if the only $U \in \mathcal{U}^-$ for which $U\bar{y}^+ \in \iota'(\mathcal{H}^L)$ are such that $U\bar{y}^+ \in \mathcal{U}\bar{y}^-$. But applying π' and using the exactness of the middle row of (7.8) gives $U\bar{X}^+ x' = 0$. The module generated by the highest weight vector x' is not Verma, so we can only conclude that $U = V^- \chi_+^- + V^+ \chi_+^+$ for some $V^{\pm} \in \mathcal{U}^-$, where the $\chi_+^{\pm} \bar{X}^+ v_{h^R}$ denote the (normalized) singular vectors in $\mathcal{V}_{h^R}$ whose rank is one higher than that of $\bar{X}^+ v_{h^R}$ (in particular, the $\chi_+^{\pm}$ are singular). Thus,

$$U\bar{y}^+ = V^- \chi_+^- \bar{y}^+ + V^+ \chi_+^+ \bar{y}^+. \tag{7.9}$$

We use the fact that $\chi_+^{\pm} \bar{X}^+ = \chi_-^{\pm} \bar{X}^-$ for some singular $\chi_-^{\pm} \in \mathcal{U}^-$ (which follows from the Feigin–Fuchs classification of singular vectors in Verma modules). It is easily verified that $\chi_+^{\pm} \bar{y}^+ - \chi_-^{\pm} \bar{y}^- \in \check{\mathcal{S}}$ are singular vectors and furthermore that they are in $\operatorname{Ker} \check{\pi} = \operatorname{Im} \check{\iota}$. By Proposition 3.3, we then have $\chi_+^{\pm} \bar{y}^+ = \chi_-^{\pm} \bar{y}^-$. Substituting into Eq. (7.9), we therefore see that the vector $U\bar{y}^+ \in \check{\mathcal{S}}$ must be in the submodule generated by $\bar{y}^-$, establishing the injectivity of ι. ∎

This result validates the practical technique proposed in Ref. 21 to find constraints on the beta invariant of a staggered module $\mathcal{S}$ by searching for singular vectors in the corresponding $\check{\mathcal{S}}$. The power of Proposition 7.2, when combined with the classification of Theorem 6.14, is evidenced by the following examples.

Example 10: We are finally ready to demonstrate the claims made in Examples 3 and 4 concerning the allowed values of β. In the former case, the staggered module $\mathcal{S}$ had $c=-2$ ($t=2$), $\mathcal{H}^L = \mathcal{V}_0/\mathcal{V}_3$, and $\mathcal{H}^R = \mathcal{V}_1/\mathcal{V}_6$. By Theorem 6.14, there is a one-dimensional space of staggered modules $\check{\mathcal{S}}$ with the same left module but $\mathcal{H}^R = \mathcal{V}_1$, parametrized by β (Theorem 6.15). We search in $\check{\mathcal{S}}$ for a singular vector at grade 6, finding one for every $\beta \in \mathbb{C}$,

$$\begin{aligned} \bar{y} = {} & (L_{-1}^3 - 8L_{-2}L_{-1} + 12L_{-3})(L_{-1}^2 - 2L_{-2})y - \Big(-\tfrac{16}{3}(\beta+1)L_{-2}^2 L_{-1}^2 + \tfrac{4}{3}(14\beta+5)L_{-3}L_{-2}L_{-1} - 6\beta L_{-3}^2 \\ & - 6(\beta-2)L_{-4}L_{-1}^2 + 8\beta L_{-4}L_{-2} - \tfrac{2}{3}(5\beta+2)L_{-5}L_{-1} + 4\beta L_{-6}\Big)x. \end{aligned} \tag{7.10}$$

Here, we have used $(L_{-1}^2 - 2L_{-2})L_{-1}x = 0$ to eliminate terms of the form $\mathcal{U}^- L_{-1}^3 x$.[51] *It now follows from Proposition 7.2 that there also exists a one-dimensional space of staggered modules $\mathcal{S}$ (likewise parametrized by β) with the desired left and right modules.*

The case of Example 4 is different. The staggered module $\mathcal{S}$ had $c=0$ $\left(t=\tfrac{3}{2}\right)$, $\mathcal{H}^L = \mathcal{V}_0/\mathcal{V}_2$, and

$\mathcal{H}^R = \mathcal{V}_1/\mathcal{V}_5$. *Searching for a grade* 5 *singular vector in the* $\check{\mathcal{S}}$ *(with unknown* β*), we find that a singular vector exists* if and only if $\beta = -\frac{1}{2}$, *in which case it has the form*

$$\bar{y} = \left(L_{-1}^4 - \tfrac{20}{3}L_{-2}L_{-1}^2 + 4L_{-2}^2 + 4L_{-3}L_{-1} - 4L_{-4}\right)y - \left(-\tfrac{32}{9}L_{-3}L_{-2} + \tfrac{16}{3}L_{-4}L_{-1} + 2L_{-5}\right)x. \tag{7.11}$$

Here, we have used $(L_{-1}^2 - \frac{2}{3}L_{-2})x = 0$ *to eliminate terms of the form* $\mathcal{U}^- L_{-1}^2 x$. *Proposition 7.2 now states that there is a unique staggered module* $\mathcal{S}$ *with the desired left and right modules, and that it has beta invariant* $\beta = -\frac{1}{2}$.

While searching for singular vectors gives a useful general technique to determine how many staggered modules correspond to a given exact sequence, it is clear that this method is computationally intensive. For instance, even the relatively simple module discussed in Example 6 requires searching for singular vectors at grade 14, hence determining the form of ϖ (when it exists) within a space of dimension $\dim \mathcal{H}_{14}^L = p(14) - p(10) = 93$. Clearly, it would be very helpful to have stronger existence results, and it is these that we turn to now.

B. Submodules and the Projection Lemma

The previous section reduces the existence question for $\mathcal{S}$ to a question about singular vectors $\bar{y}$ (or $\bar{y}^{\pm}$) in $\check{\mathcal{S}}$. We will first develop the idea of this section in the case in which there is only one $\bar{X}$, briefly noting afterward the slight changes needed in the $\bar{X}^{\pm}$ case. Recall that these singular vectors $\bar{y}$ necessarily take the form $\bar{X}y - \varpi$, where $\varpi \in \mathcal{H}_{\ell+\bar{\ell}}^L$. In searching for these singular vectors, we are naturally led to consider the set of elements obtained from $\bar{X}y$ through translating by an element of $\mathcal{H}_{\ell+\bar{\ell}}^L$. This translation is strongly reminiscent of gauge transforming data, and it is this similarity that we shall exploit in this section.

To make matters more transparent, let us consider instead of $\check{\mathcal{S}}$, a staggered module $\mathcal{T}$ that differs only in that its left module is also Verma. This does not change the dimension of the space of isomorphism classes, by Theorem 6.14, and we have the usual definitions of x, y, ω_0, ω_1, ω_2, and β (or $\beta_{\pm}$). However, this slight change in viewpoint necessitates a reinterpretation of the results of the previous section because a Verma left module obviously conflicts with the conclusion of Proposition 3.3 when the right module is not Verma (upon setting a singular vector of $\mathcal{H}^R$ to zero). Instead of searching for singular vectors of the form $\bar{X}y - \varpi$, we will therefore instead consider the submodules of $\mathcal{T}$ generated by the $\bar{X}y - u$, where u ranges over $(\mathcal{V}_{h^L})_{\ell+\bar{\ell}} \subset \mathcal{T}$.

More precisely, let us consider the submodules $\bar{\mathcal{T}}(u) \subset \mathcal{T}$ which are generated by x and $\bar{X}y - u$. Because we have insisted that the left module is Verma (and this is why we are insisting upon this in the first place), these are all staggered modules with exact sequence,

$$0 \to \mathcal{V}_{h^L} \to \bar{\mathcal{T}}(u) \to \mathcal{V}_{h^R+\bar{\ell}} \to 0. \tag{7.12}$$

Indeed, putting $\bar{y} = \bar{X}y - u$, we define in the usual way $\bar{\omega}_0 = (L_0 - h^R - \bar{\ell})\bar{y} = \bar{X}\omega_0$, $\bar{\omega}_1 = L_1\bar{y}$, $\bar{\omega}_2 = L_2\bar{y}$, and thence $\bar{\beta}$ (or $\bar{\beta}_{\pm}$) by Eq. (6.37) [or (6.39a) and (6.39b)]. Varying $u \in (\mathcal{V}_{h^L})_{\ell+\bar{\ell}}$ then really does amount to performing gauge transformations on any given representative, $\bar{\mathcal{T}}(0)$ say. In particular, all the $\bar{\mathcal{T}}(u)$ are isomorphic.

Apply now the Projection Lemma, Lemma 5.1, to the staggered module $\bar{\mathcal{T}}(0)$. This tells us that we can always make a gauge transformation so that the transformed data $(\bar{\omega}_1', \bar{\omega}_2')$ belong to the submodule $\bar{\mathcal{M}}$ of $\mathcal{V}_{h^L}$ generated by the singular vectors of rank $\rho + \bar{\rho} - 1$, where ρ is the rank of $\omega_0 = Xx$ in $\mathcal{V}_{h^L}$ and $\bar{\rho}$ is the rank of $\bar{X}v_{h^R}$ in $\mathcal{V}_{h^R}$ (so $\rho + \bar{\rho}$ is the rank of $\bar{\omega}_0 \in \mathcal{V}_{h^L}$). In other words, there exists $\varpi \in (\mathcal{V}_{h^L})_{\ell+\bar{\ell}}$, such that $\mathfrak{vir}^+(\bar{X}y - \varpi) \subseteq \bar{\mathcal{M}}$. The submodule $\mathcal{U}(\bar{X}y - \varpi) \subset \mathcal{T}$ is then a staggered module with left module $\bar{\mathcal{M}}$ (or even some submodule thereof), right module $\mathcal{V}_{h^R+\bar{\ell}}$, and beta invariant $\bar{\beta}$ (or $\bar{\beta}_{\pm}$).

Consider now the quotient of $\mathcal{T}$ by $\bar{\mathcal{M}} \subseteq \mathcal{V}_{h^L}$. If we assume that $\omega_0 \notin \bar{\mathcal{M}}$ (which is equivalent to assuming that $\bar{\rho} > 1$), then Proposition 4.5 tells us that this is a staggered module $\check{\mathcal{S}}$ with exact sequence

$$0 \to \mathcal{V}_{h^L} / \bar{\mathcal{M}} \to \check{\mathcal{S}} \to \mathcal{V}_{h^R} \to 0. \tag{7.13}$$

Moreover, its beta invariant is obviously the same as that of $\mathcal{T}$, namely, β. It should now be evident that $\bar{y} = \bar{X}y - \varpi$ is a singular vector of $\check{\mathcal{S}}$, so by Proposition 7.2, we may construct a module $\mathcal{S} = \check{\mathcal{S}} / \mathcal{U}\bar{y}$ for each beta invariant β whose exact sequence is

$$0 \to \mathcal{V}_{h^L} / \bar{\mathcal{M}} \to \mathcal{S} \to \mathcal{V}_{h^R} / \mathcal{U}\bar{X}v_{h^R} \to 0. \tag{7.14}$$

We can even reduce the left module of $\mathcal{S}$ further by quotienting by any submodule not containing ω_0.

It remains only to remark upon the differences in the $\bar{X}^{\pm}$ case. We may apply the above formalism to consider separately the submodules $\bar{\mathcal{T}}^{\pm}(u) \subset \mathcal{T}$ which are generated by x and $\bar{X}^{\pm}y - u$. Applying the Projection Lemma to each, we conclude that there exist $\varpi^{\pm}$, such that $\mathfrak{vir}^{+}(\bar{X}^{\pm}y - \varpi^{\pm}) \subseteq \bar{\mathcal{M}}$ for the submodule $\bar{\mathcal{M}} \subseteq \mathcal{V}_{h^L}$ generated by the rank $\rho + \bar{\rho} - 1$ singular vectors (we emphasize that this is the same submodule for both "$-$" and "$+$"). The vectors $\bar{y}^{\pm} = \bar{X}^{\pm}y - \varpi^{\pm}$ are therefore both singular in the quotient $\check{\mathcal{S}} = \mathcal{T} / \bar{\mathcal{M}}$, so an appeal to Proposition 7.2 then settles this case. Putting this all together, we have proven the following result.

Proposition 7.3: Let ρ and $\bar{\rho}$ denote the ranks of the singular vectors $\omega_0 = Xx \in \mathcal{V}_{h^L}$ and $\bar{X}v_{h^R} \in \mathcal{V}_{h^R}$. If there are no (nonzero) singular vectors in $\mathcal{H}^L$ of rank $\rho + \bar{\rho} - 1$, then the dimension of the space of staggered modules $\mathcal{S}$ with exact sequence (7.2) matches the dimension of the space of staggered modules $\check{\mathcal{S}}$ with exact sequence (7.1).

Example 11: This result allows us to understand why the exact sequence (3.19) of Example 3 admits a one-parameter family of staggered modules. In Example 10, we proved that this was indeed the case, but now we see it as a direct consequence of Proposition 7.3, and hence as a corollary of the Projection Lemma. To whit, the left module is $\mathcal{V}_0 / \mathcal{V}_3$ and the right module is $\mathcal{V}_1 / \mathcal{V}_6$ (see Fig. 3 in Sec. III). The ranks of $\omega_0 = L_{-1}x$ and $\bar{X}v_{h^R}$ are 1 and 2, respectively, so that of $\bar{\omega}_0$ is $\rho + \bar{\rho} = 3$. But there is no (nonvanishing) rank 2 singular vector of $\mathcal{H}^L$ (it would have dimension 3), hence the proposition applies.

We note that the proposition does not apply to the exact sequence (3.20) considered in Example 4. In this case, the left module is $\mathcal{V}_0 / \mathcal{V}_2$ and the right module is $\mathcal{V}_1 / \mathcal{V}_5$, so we find that $\rho + \bar{\rho} = 2$. But there is a nonvanishing rank 1 singular vector in $\mathcal{H}^L$, namely, ω_0. This failure to meet the hypotheses should be expected as we have already shown (Example 10) that the dimension of the space of $\mathcal{S}$ differs from that of the corresponding $\check{\mathcal{S}}$. We will therefore have to work harder to get an intuitive understanding of why this is so (beyond a brute force computation of singular vectors).

Example 12: In the study of the so-called $LM(1,q)$ logarithmic conformal field theories,[24] one encounters staggered modules $\mathcal{S}_s$ with $c = 13 - 6\,(q + q^{-1})$ $(t = q)$ and exact sequence,

$$0 \to \mathcal{Q}_{1,s} \to \mathcal{S}_s \to \mathcal{Q}_{1,s+2(q-\sigma)} \to 0, \tag{7.15}$$

where $\mathcal{Q}_{1,s} = \mathcal{V}_{h_{1,s}} / \mathcal{V}_{h_{1,s}+s}$. Here, s is a positive integer not divisible by q, and $0 < \sigma < q$ is the remainder obtained upon dividing s by q. The left and right modules are of chain type, the former being irreducible if $s < q$ and reducible with singular vectors of ranks 0 and 1 if $s > q$. The right module is always reducible with singular vectors of ranks 0 and 1.

We then have $\rho = 0$ when $s < q$ and $\rho = 1$ otherwise, $\ell = (q - \sigma)(s - \sigma)/q$, $\bar{\rho} = 2$, and $\bar{\ell} = s + 2(q - \sigma)$. Since the left module has no singular vectors of rank $\rho + 1$, it follows from Proposition 7.3 and Theorem 6.14 that the exact sequence (7.15) describes a one-parameter family of staggered modules. Identifying the staggered modules appearing in the $LM(1,q)$ models therefore requires

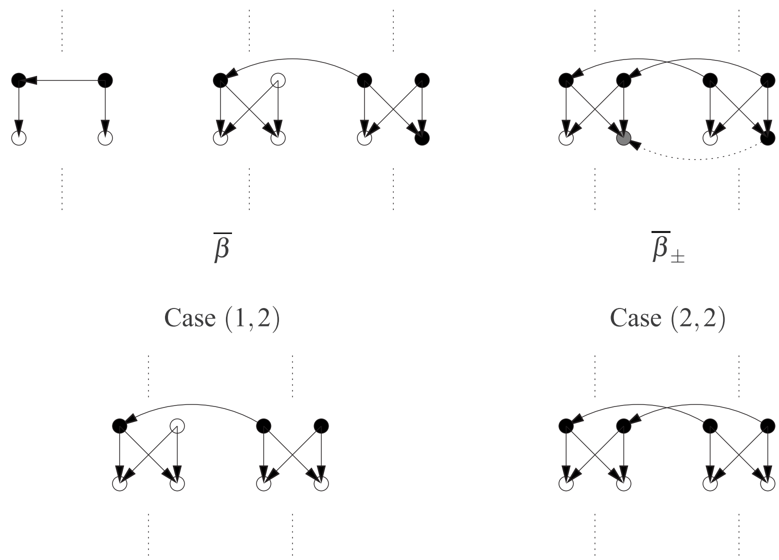

FIG. 6. The critical rank configurations for which Propositions 7.3 and 3.3 are not sufficient to settle the existence question. Pictured are the singular vectors of $\mathcal{H}^L$ of ranks $\rho+\bar{\rho}-1$ and $\rho+\bar{\rho}$, and their counterparts of ranks $\bar{\rho}-1$ and $\bar{\rho}$ in $\mathcal{H}^R$. Black indicates that the singular vector is present, white that it has been set to zero, and gray that either possibility is admissible. The (curved) horizontal arrows indicate the nondiagonal action of L_0. It is understood that in certain circumstances, some of the singular vectors pictured may not actually be present in the braid cases (for example, when $\bar{\rho}=1$, $\mathcal{H}^R$ has only one singular vector of rank $\bar{\rho}-1=0$). We also indicate for each configuration the beta invariants of Eq. (7.16) whose vanishing is equivalent to the existence of the associated staggered module.

computing the corresponding beta invariants. Unfortunately, this has only been done for certain small s.

Proposition 7.3 states that if $\mathcal{H}^L$ has no (nonzero) singular vectors of rank greater than or equal to $\rho+\bar{\rho}-1$, then the existence question for staggered modules $\mathcal{S}$ is equivalent to the same question for the corresponding $\check{\mathcal{S}}$. Moreover, Proposition 3.3 tells us that the left module $\mathcal{H}^L$ of $\mathcal{S}$ cannot have a singular vector of rank $\rho+\bar{\rho}$ or $\rho+\bar{\rho}+1$, according as to whether $\mathcal{H}^L$ is of chain or braid type, respectively. We have therefore solved the existence question for staggered modules in all but a finite number of outstanding *critical rank* cases. It is these cases that we now turn to.

C. Existence at the critical ranks

If $\mathcal{H}^L$ has nonzero singular vectors at the critical rank $\rho+\bar{\rho}-1$, we can still follow the strategy of Sec. VII B to try to construct $\mathcal{S}$, but we cannot, in general, quotient away the full submodule $\bar{\mathcal{M}}$ without ending up with a left module smaller than $\mathcal{H}^L$. We will therefore have to perform a more detailed analysis to determine when we can quotient by a smaller submodule.

For convenience, we will separate the outstanding cases according to the configurations of the singular vectors of $\mathcal{H}^L$ and $\mathcal{H}^R$ at the critical ranks. We let $\mathbf{g}\in\{0,1,2\}$ denote the number of rank $\rho+\bar{\rho}-1$ singular vectors of $\mathcal{H}^L$ and $\mathbf{n}\in\{0,1,2\}$ denote the (minimal) number of singular vectors needed to generate $\mathcal{J}$, where $\mathcal{H}^R=\mathcal{V}_{h^R}/\mathcal{J}$. The critical rank cases correspond to neither $\mathbf{g}$ nor $\mathbf{n}$ vanishing, so we have four singular vector configurations which we illustrate in Fig. 6. There, $\mathbf{g}$ represents the number of black circles in the top row for $\mathcal{H}^L$ and $\mathbf{n}$ represents the number of white circles in the bottom row for $\mathcal{H}^R$. We label the critical rank cases by this pair of integers $(\mathbf{g},\mathbf{n})$.

Let us first consider the case (1,1) with modules of chain type for simplicity. Recall that the data of the module $\bar{\mathcal{T}}(\varpi)$ were denoted by $(\bar{\omega}_1,\bar{\omega}_2)$, where ϖ were chosen so that $\bar{\omega}_j=L_j\bar{y}$

$=L_j(\bar{X}y-\varpi)\in\bar{\mathcal{M}}$. Instead of quotienting $\mathcal{T}$ by $\bar{\mathcal{M}}$, we would now like to quotient by the smaller submodule $\mathcal{U}\bar{\omega}_0=\mathcal{U}\bar{X}Xx\subset\bar{\mathcal{M}}$. It is clear that $\bar{y}$ will become singular in the quotient if and only if $\bar{\omega}_j=0$ for $j=1,2$. Of course, we have the freedom of gauge transformations in choosing ϖ, so the question should be whether $(\bar{\omega}_1,\bar{\omega}_2)$ is equivalent to (0,0). From this, we conclude that $\bar{y}=\bar{X}y-\varpi$ will be singular in $\mathcal{T}/\mathcal{U}\bar{\omega}_0$ (for some choice of ϖ) if and only if the beta invariant $\bar{\beta}$ of $\bar{\mathcal{T}}(\varpi)$ vanishes. We remark that this is equivalent to the vanishing of $\bar{\beta}$ for any $\bar{\mathcal{T}}(u)$, $u\in(\mathcal{V}_{h^L})_{\ell+\bar{\ell}}$, by gauge invariance.

In general, **g** and **n** may be greater than 1 and there are a few possibilities among the submodules of $\bar{\mathcal{M}}$ that we might want to quotient out. We will analyze whether the submodules $\mathcal{U}(\bar{X}^{\varepsilon}y-\varpi^{\varepsilon})\subset\mathcal{T}$ contain the singular vectors $X^{\varepsilon'}_{\rho+\bar{\rho}-1}x\in\mathcal{V}_{h^L}$, where $\varepsilon'\in\{-,+\}$ parametrizes the nonvanishing singular vectors $X^{\varepsilon'}_{\rho+\bar{\rho}-1}x\neq0$ of $\mathcal{H}^L$. Analogous to the argument above, we find that to each generating singular vector of $\mathcal{J}$ and each rank $\rho+\bar{\rho}-1$ singular vector of $\mathcal{H}^L$, there is a corresponding beta invariant which must vanish. Specifically, given vectors $\bar{y}^{\varepsilon}=\bar{X}^{\varepsilon}y-\varpi^{\varepsilon}$ such that $\bar{\omega}^{\varepsilon}_j=L_j\bar{y}^{\varepsilon}\in\bar{\mathcal{M}}$ and elements $\bar{\chi}^{\varepsilon}_{\varepsilon'}\in\mathcal{U}^-$ (singular and prime), such that $X^{\varepsilon}_{\rho+\bar{\rho}}=\bar{\chi}^{\varepsilon}_{\varepsilon'}X^{\varepsilon'}_{\rho+\bar{\rho}-1}$ and $X^{\varepsilon'}_{\rho+\bar{\rho}-1}x\neq0$ in $\mathcal{H}^L$, we define $\mathbf{gn}\in\{0,1,2,4\}$ beta invariants by

$$(\bar{\chi}^{\varepsilon}_{-})^{\dagger}\bar{y}^{\varepsilon}=\bar{\beta}^{\varepsilon}_{-}X^{-}_{\rho+\bar{\rho}-1}x\quad\text{and}\quad(\bar{\chi}^{\varepsilon}_{+})^{\dagger}\bar{y}^{\varepsilon}=\bar{\beta}^{\varepsilon}_{+}X^{+}_{\rho+\bar{\rho}-1}x\quad(\operatorname{mod}\mathcal{U}X^{-}_{\rho+\bar{\rho}-1}x).\tag{7.16}$$

These are the beta invariants of the $\bar{\mathcal{T}}^{\varepsilon}(\varpi^{\varepsilon})$, and we may quotient $\mathcal{T}$ to get a staggered module $\check{\mathcal{S}}$ with left module $\mathcal{H}^L$ and singular vectors $\bar{y}^{\varepsilon}$ if and only if all of the $\bar{\beta}^{\varepsilon}_{\varepsilon'}$ vanish (the easy proof of this is sketched below). We have indicated which beta invariants are relevant to each critical rank case in Fig. 6 for convenience. We further remark that we will suppress the indices ε and ε' in cases where they take a single value [as in case (1,1) above].

Theorem 7.4: *Given $\mathcal{H}^L$, $\mathcal{H}^R=\mathcal{V}_{h^R}/\mathcal{J}$, and $(\omega_1,\omega_2)\in\Omega$ such that $\mathcal{H}^L$ contains nonzero singular vectors of rank $\rho+\bar{\rho}-1$, a staggered module $\mathcal{S}$ with these left and right modules and data exists if and only if $\bar{\beta}^{\varepsilon}_{\varepsilon'}=0$ for all $\varepsilon\in\{-,+\}$ such that $\bar{X}^{\varepsilon}v_{h^R}\in\mathcal{J}$ and all $\varepsilon'\in\{-,+\}$ such that $X^{\varepsilon'}_{\rho+\bar{\rho}-1}x\neq0$.*

Proof: In view of Proposition 7.2 and the above, all that needs to be proven is that the vanishing of the appropriate invariants $\bar{\beta}$ occurs precisely when the $\bar{y}$ become nonvanishing singular vectors in the quotient $\check{\mathcal{S}}=\mathcal{T}/\mathcal{K}$ (recall that $\mathcal{K}$ is then a submodule of $\bar{\mathcal{M}}$). To lighten the notation, we will omit superscript indices ε. It is understood that what follows must be repeated separately for the **n** values that ε takes.

It is clear that $\bar{y}$ will be singular if and only if both $\bar{\omega}_1$ and $\bar{\omega}_2$ belong to $\mathcal{K}$. When $\mathcal{K}$ is generated by singular vectors of grades $\ell+\bar{\ell}$ or greater, for example, when $\mathcal{V}_{h^L}$ is of chain type, this requires that the data vanish (this direction is always easy, in fact). Now, the data can be *chosen* to vanish using a gauge transformation if and only if all beta invariants $\bar{\beta}$ or $\bar{\beta}_{\pm}$ vanish because vanishing data is admissible (Proposition 6.2), gauge transformations connect any two equivalent pieces of data (Proposition 3.6) and beta invariants completely determine the isomorphism class (Theorem 6.15). The proof is then complete for such $\mathcal{K}$.

However, $\mathcal{K}\subset\bar{\mathcal{M}}$ may be generated by singular vectors of lower grade than $\ell+\bar{\ell}$. To deal with this possibility, note that

$$\bar{\omega}_j\in\mathcal{U}X^{\pm}_{\rho+\bar{\rho}-1}x\Rightarrow\bar{\beta}_{\mp}=0.\tag{7.17}$$

Indeed, this is just the analog of (a part of) Eq. (6.47) in the present situation, and it immediately implies that if $\bar{y}$ becomes singular in the quotient $\mathcal{T}/\mathcal{K}$, then the invariants $\bar{\beta}_{\varepsilon'}$ vanish. Roughly speaking, the converse is also true: Split the data as $\bar{\omega}_j=\bar{\omega}^{+}_j+\bar{\omega}^{-}_j$, where $\bar{\omega}^{\pm}_j\in\mathcal{U}X^{\pm}_{\rho+\bar{\rho}-1}x$. From the arguments in Secs. VI D and VI E, we can infer that the admissible $(\bar{\omega}^{\pm}_1,\bar{\omega}^{\pm}_2)$ modulo the gauge transformations g_u, $u\in(\mathcal{U}X^{\pm}_{\rho+\bar{\rho}-1})_{\ell+\bar{\ell}}$, form a one-dimensional vector space parametrized by $\bar{\beta}_{\pm}$.

Then, $\bar{\beta}_{\pm}=0$ implies that we can choose $\bar{\omega}_j^{\pm}=0$ by a gauge transformation. It follows that the "full data" $\bar{\omega}_j$ can be chosen to belong to $\mathcal{U}X^{\mp}_{\rho+\bar{\rho}-1}x$. When $\mathcal{K}$ is generated by $X^{\mp}_{\rho+\bar{\rho}-1}x$, the vanishing of $\bar{\beta}_{\pm}$ therefore implies that $\bar{y}$ becomes singular in the quotient $\mathcal{T}/\mathcal{K}$. When $\mathcal{K}$ is generated by singular vectors of grade $\rho+\bar{\rho}$, the vanishing of both the $\bar{\beta}_{\pm}$ implies the same. This completes the proof. ∎

Determining when the beta invariants $\bar{\beta}^{\varepsilon}_{\varepsilon'}$ of the staggered modules $\bar{\mathcal{T}}^{\varepsilon}(u)$ vanish is an explicit condition which can be checked in particular examples (see Example 13 below). To get more insight into this, we revisit the definitions of these beta invariants using the forms given in Eqs. (6.57) and (6.59). This allows us to set $u=0$ and write

$$\bar{\beta}^{\varepsilon}_{\varepsilon'}X^{\varepsilon'}_{\rho+\bar{\rho}-1}x=(\bar{\chi}^{\varepsilon}_{\varepsilon'})^{\dagger}(\mathbf{1}-\bar{\mathbb{P}})\bar{y}^{\varepsilon}=(\bar{\chi}^{\varepsilon}_{\varepsilon'})^{\dagger}(\mathbf{1}-\bar{\mathbb{P}})\bar{X}^{\varepsilon}y, \tag{7.18}$$

modulo $\mathcal{U}X^{-}_{\rho+\bar{\rho}-1}x$ if $\varepsilon'=+$, where $\mathbf{1}-\bar{\mathbb{P}}$ denotes the net effect of the Projection Lemma (as in Sec. VI E). Now, $\bar{X}^{\varepsilon}$ is singular, and both $(\bar{\chi}^{\varepsilon}_{\varepsilon'})^{\dagger}$ and $\bar{\mathbb{P}}$ have positive modes on the right of each of their terms [see Eqs. (6.56) and (6.58)]. We may therefore write

$$(\bar{\chi}^{\varepsilon}_{\varepsilon'})^{\dagger}(\mathbf{1}-\bar{\mathbb{P}})\bar{X}^{\varepsilon}=U_0^{(\varepsilon,\varepsilon')}(L_0-h^R)+U_1^{(\varepsilon,\varepsilon')}L_1+U_2^{(\varepsilon,\varepsilon')}L_2 \tag{7.19}$$

for some $U_0^{(\varepsilon,\varepsilon')},U_1^{(\varepsilon,\varepsilon')},U_2^{(\varepsilon,\varepsilon')}\in\mathcal{U}$, hence

$$\bar{\beta}^{\varepsilon}_{\varepsilon'}X^{\varepsilon'}_{\rho+\bar{\rho}-1}x=U_0^{(\varepsilon,\varepsilon')}\omega_0+U_1^{(\varepsilon,\varepsilon')}\omega_1+U_2^{(\varepsilon,\varepsilon')}\omega_2. \tag{7.20}$$

This expresses the $\bar{\beta}^{\varepsilon}_{\varepsilon'}$ as *affine-linear* functionals of the data (ω_1,ω_2) of $\mathcal{T}$ (and thus also of $\check{\mathcal{S}}$). Finally, applying a gauge transformation g_u to (ω_1,ω_2) results in the left hand side of Eq. (7.20) changing by

$$(\bar{\chi}^{\varepsilon}_{\varepsilon'})^{\dagger}(\mathbf{1}-\bar{\mathbb{P}})\bar{X}^{\varepsilon}u-U_0^{(\varepsilon,\varepsilon')}(L_0-h^R)u=0, \tag{7.21}$$

since u has conformal dimension h^R and $(\mathbf{1}-\bar{\mathbb{P}})\bar{X}^{\varepsilon}u\in\bar{\mathcal{M}}_{\ell+\bar{\ell}}$. This gauge invariance then lets us conclude that the $\bar{\beta}^{\varepsilon}_{\varepsilon'}$ are affine functions on the space Ω/G of isomorphism classes of staggered modules $\check{\mathcal{S}}$ with exact sequence (7.1). Assuming that $\ell>0$, so that the beta invariants β or $\beta_{\pm}$ of $\check{\mathcal{S}}$ are defined, we can therefore consider the $\bar{\beta}^{\varepsilon}_{\varepsilon'}$ as affine functions of β or $\beta_{\pm}$.

Example 13: We consider the existence of a $c=-2$ ($t=2$) staggered module $\mathcal{S}$ with exact sequence,

$$0\to\mathcal{V}_0/\mathcal{V}_3\to\mathcal{S}\to\mathcal{V}_1/\mathcal{V}_3\to 0. \tag{7.22}$$

We therefore have $X=L_{-1}$, $\rho=1$, $\bar{X}=\bar{\chi}=L_{-1}^2-2L_{-2}$, and $\bar{\rho}=1$. Since $\omega_0=L_{-1}x$ has rank $\rho+\bar{\rho}-1=1$, this is a critical rank example.

By Theorem 6.14, there is a one-dimensional space of staggered modules $\mathcal{T}$ with left module $\mathcal{V}_0$ and right module $\mathcal{V}_1$, parametrized by β. We must determine the beta invariant $\bar{\beta}$ of the submodule $\bar{\mathcal{T}}(0)$ generated by x and $\bar{X}y$. Referring to the calculation of Example 9, we have

$$\begin{aligned}\bar{\beta}\omega_0&=\bar{X}^{\dagger}(\mathbf{1}-\bar{\mathbb{P}})\bar{X}y=(L_1^2-2L_2)\left(\mathbf{1}+\tfrac{1}{4}L_{-3}L_3\right)(L_{-1}^2-2L_{-2})y\\&=(8L_{-1}L_0L_1-15L_{-1}L_1+4(2L_0+1)(L_0-1))y=(-15\beta+12)\omega_0.\end{aligned} \tag{7.23}$$

The conclusion is then that $\mathcal{S}$ exists by Theorem 7.4 if and only if $\bar{\beta}=0$, hence $\beta=\frac{4}{5}$. This value is of course reproduced by searching for an explicit singular vector of the form $\bar{y}=\bar{X}y-\varpi$ with $\varpi\in\mathcal{V}_0/\mathcal{U}\bar{X}\omega_0=\mathcal{H}^L$ (as in Sec. VII A).

Consider a case (1,1) staggered module $\mathcal{S}$ of chain type (or $\rho=1$ braid type). If $\ell>0$ (so $\rho>0$), then there is a single invariant β to consider. By Theorem 7.4, $\mathcal{S}$ exists if and only if a single invariant $\bar{\beta}$ vanishes. We have shown that the latter invariant is an affine function of the former, so there are three possibilities.

- $\bar{\beta}$ is constant and zero, so $\mathcal{S}$ exists for all β.
- $\bar{\beta}$ is constant and nonzero, so $\mathcal{S}$ does not exist for any β.
- $\bar{\beta}$ is not constant, so $\mathcal{S}$ exists for a unique β.

In the absence of any information to the contrary, we should expect that the last possibility is overwhelmingly more likely to occur. Moreover, indeed, this is what we observe. For instance, the staggered modules of Examples 4, 6, and 13 all admit only a single value of β. We can now finally understand this as the generic consequence of imposing one (linear, inhomogeneous) relation, $\bar{\beta}=0$, on one unknown, β.

More generally, we can use Eq. (7.20) to decompose the beta invariants of the $\bar{\mathcal{T}}^{\varepsilon}(0)$ as

$$\bar{\beta}^{\varepsilon}_{\varepsilon'}(\omega_1,\omega_2)=\gamma^{\varepsilon}_{\varepsilon'}(\omega_1,\omega_2)+\bar{\beta}^{\varepsilon}_{\varepsilon'}(0,0). \tag{7.24}$$

This defines *linear* functionals $\gamma^{\varepsilon}_{\varepsilon'}$ on the space of data of the $\mathcal{T}$ (and $\check{\mathcal{S}}$). Let $\mathbf{b}\in\{0,1,2\}$ denote the number of beta invariants needed to describe the $\mathcal{T}$. Assuming that the $\gamma^{\varepsilon}_{\varepsilon'}$ are all linearly independent, we therefore obtain $\mathbf{gn}$ linear relations in $\mathbf{b}$ unknowns.[52] Analyzing these numbers in each case then leads to simple expectations for the dimension of the space of staggered modules $\mathcal{S}$.

More specifically, when the left and right modules are of chain type, $\mathbf{b}$ is 0 or 1, depending on whether $\rho=0$ or not. In the braid case, $\mathbf{b}$ is 0, 1, or 2, depending on whether $\rho=0$, $\rho=1$, or $\rho>1$ (this is a direct restatement of Theorem 6.14). We should therefore *expect* that the staggered modules $\mathcal{S}$ corresponding to the critical rank configurations of Fig. 6 will exist in case (1,1) provided that $\rho>0$ and cases (1,2) and (2,1) provided that $\rho>1$. We should not expect the $\mathcal{S}$ to exist otherwise. Moreover, we expect that when $\mathcal{S}$ exists, it is unique, except in case (1,1) with braid type and $\rho>1$, in which case we expect a one-parameter family of staggered modules.

Example 14: It is easy to investigate examples of critical rank staggered modules using the singular vector result of Proposition 7.2. For example, we know from Example 10 that a $c=0$ $\left(t=\frac{3}{2}\right)$ *staggered module with* $\mathcal{H}^L=\mathcal{V}_0/\mathcal{V}_2$ *and* $\mathcal{H}^R=\mathcal{V}_1/\mathcal{V}_5$ *is unique, admitting only* $\beta=-\frac{1}{2}$*. Similarly, replacing the right module by* $\mathcal{V}_1/\mathcal{V}_7$ *leads to a unique staggered module with* $\beta=\frac{1}{3}$*. These are both case* (1,1) *examples, but we may deduce from their (expected) uniqueness that the case* (1,2) *staggered module* $\mathcal{S}$ *corresponding to replacing the right module by* $\mathcal{V}_1/(\mathcal{V}_5+\mathcal{V}_7)$ does not exist: *The associated* $\check{\mathcal{S}}$ *would have to have singular vectors at grades* 5 *and* 7*, requiring both* $\beta=-\frac{1}{2}$ and $\beta=\frac{1}{3}$.

For case (2,1) examples, we take $\mathcal{H}^L=\mathcal{V}_0/(\mathcal{V}_5+\mathcal{V}_7)$ *and* $\mathcal{H}^R=\mathcal{V}_h/\mathcal{V}_{h'}$ *for* $h=1,2$ and $h'=5,7$ *(and* $c=0$*). In all these cases,* $\rho=1$*, so we do not expect that such staggered modules exist. Moreover, one can explicitly check in each case that the appropriate singular vector does not exist, confirming our expectations. It is more interesting to consider the* $\rho=2$ *examples with* $\mathcal{H}^L=\mathcal{V}_0/(\mathcal{V}_{12}+\mathcal{V}_{15})$ *and* $\mathcal{H}^R=\mathcal{V}_5/\mathcal{V}_h$ *for* $h=12$ *and* 15*. The singular vectors turn out to exist if and only if*

$$\beta_-=-\frac{11\,200}{51},\quad \beta_+=\frac{1680}{17}\quad \textit{and}\quad \beta_-=-\frac{5600}{57},\quad \beta_+=\frac{3360}{19}, \tag{7.25}$$

respectively, in line with expectations. Finally, if we replace the right module by $\mathcal{V}_5/(\mathcal{V}_{12}+\mathcal{V}_{15})$ *to get a case* (2,2) *example, we see from the different* $\beta_\pm$ *above that this staggered module cannot exist, again as expected.*

Our last example illustrates case (1,1) *with* $\rho>1$*. We search for a* $c=0$ *staggered module* $\mathcal{S}$ *with* $\mathcal{H}^L=\mathcal{V}_0/\mathcal{V}_7$ *and* $\mathcal{H}^R=\mathcal{V}_5/\mathcal{V}_{12}$*, hence* $\rho=2$*. The corresponding* $\check{\mathcal{S}}$ *turns out to have a singular vector at grade* 12 *provided that*

$$189\beta_- + 80\beta_+ = -3360. \tag{7.26}$$

It follows that there exists a one-parameter family of such staggered modules $\mathcal{S}$, just as we expect.

The above examples completely support our naive expectations concerning the dimensions of the spaces of critical rank staggered modules. However, things are never quite as simple as one might like.

Example 15: Let $c=1$ $(t=1)$ and $\mathcal{H}^L=\mathcal{H}^R=\mathcal{V}_{1/4}/\mathcal{V}_{9/4}$. These are chain-type modules with $\ell=0$, so the corresponding staggered module $\mathcal{S}$ would be a case (1,1) critical rank example with $\rho=0$. With no β but one *$\bar{\beta}$, we should not expect that such an $\mathcal{S}$ exists. Nevertheless, it is easy to check that the vector*

$$(L_{-1}^2 - L_{-2})y - \tfrac{4}{3}L_{-2}x \in \check{\mathcal{S}} \tag{7.27}$$

is singular. By Proposition 7.2, a staggered module with these left and right modules does therefore exist, contrary to our expectations.

Example 16: We can readily generalize the realization of Example 15 for other $\ell=0$ examples. Let t be arbitrary but let $h=h_{r,s}$, $r,s\in\mathbb{Z}_+$, vary with t as in Eq. (2.12). Then, $\bar{X}\in\mathcal{U}_{rs}^-$ also varies with t, although it need not remain prime (that is, $\bar{\chi}=\bar{X}$ for generic t only). We may therefore methodically investigate the existence of staggered modules with $\mathcal{H}^L=\mathcal{H}^R=\mathcal{V}_h/\mathcal{V}_{h+rs}$ by computing

$$\bar{\beta}' x = \bar{X}^\dagger \bar{X} y \in \mathcal{T} \tag{7.28}$$

for small r and s (because $\ell=0$, there is no $\bar{\mathbb{P}}$). Clearly $\bar{\beta}'$ need not coincide with the true invariant $\bar{\beta}$ if $\bar{X}$ is composite. Some results are (note that swapping r and s amounts to inverting t):

(r,s)	$\bar{\beta}'$	$\bar{\beta}=0$
(1, 1)	2	–
(2, 1)	$4(t^2-1)$	$t=\pm 1$
(3, 1)	$24(t^2-1)(4t^2-1)$	$t=\pm 1$
(4, 1)	$288(t^2-1)(4t^2-1)(9t^2-1)$	$t=\pm 1, \pm\frac{1}{2}$
(5, 1)	$5760(t^2-1)(4t^2-1)(9t^2-1)(16t^2-1)$	$t=\pm 1$
(6, 1)	$172\,800(t^2-1)(4t^2-1)(9t^2-1)(16t^2-1)(25t^2-1)$	$t=\pm 1, \pm\frac{1}{2}, \frac{1}{3}$
(2, 2)	$-8t^{-4}(t^2-1)^2(t^2-4)(4t^2-1)$	$t=\pm\frac{1}{2}, \pm 2$
(3, 2)	$-192t^{-6}(t^2-1)^3(t^2-4)(4t^2-1)^2(9t^2-1)$	$t=\pm\frac{1}{3}, \pm 2$

Here, we list those t for which $\bar{\beta}'$ vanishes and for which this vanishing implies the vanishing of $\bar{\beta}$ (which requires $\bar{X}$ to be prime), hence the existence of a staggered module with $\mathcal{H}^L=\mathcal{H}^R=\mathcal{V}_h/\mathcal{V}_{h+rs}$. This sequence of examples makes it clear that given r and s, staggered modules of this kind can certainly exist.

In the $\ell=0$ case discussed above, the invariants $\bar{\beta}$ are evidently constants. As we have seen, their vanishing is nevertheless a subtle question. However, continuing the analysis of Example 16 leads to a clear pattern for the existence question in this case, and, in fact, this question was already solved explicitly (for chain-type modules) by Rohsiepe in Ref. 26. His argument extends to any staggered module for which $\beta=0$ or $(\beta^-,\beta^+)=(0,0)$ and $\bar{X}$ is prime ($\bar{\rho}=1$), and we outline it below. Note that this is always a critical rank case.

Proposition 7.5: Suppose that $\check{\mathcal{S}}$ is a staggered module with left module $\mathcal{H}^L$, right module $\mathcal{V}_{h^R}$, and all beta invariants vanishing (if any are defined). Suppose further that the prime singular vector $\bar{X}v_{h^R}$ of smallest *grade $\bar{\ell}$ is such that $\bar{X}\omega_0=0$. Then there exists a singular vector in $\check{\mathcal{S}}$ at grade $\ell+\bar{\ell}$ if and only if $h^R=h_{r,s}$ with $t=q/p\in\mathbb{Q}$ (where $\gcd\{p,q\}=1$), $p \mid r, q \mid s$ and $|p|s\neq|q|r$.*

Proof: We will prove the existence of the singular vector by demonstrating the vanishing of $\bar{\beta}$

or $\bar{\beta}_{\pm}$. We immediately remark that the assumption of $\bar{\ell}$ being the smallest grade of a prime singular means that $\bar{\ell}=rs$ for a pair $(r,s)\in\mathbb{Z}_+\times\mathbb{Z}_+$ that satisfies $h_{r,s}=h^R$ with minimal product rs.

Since any invariants of $\check{\mathcal{S}}$ vanish, we may choose $y\in\check{\mathcal{S}}$ such that $\omega_1=\omega_2=0$, by Propositions 3.6 and 6.2 and Theorem 6.15. Writing $L_j\bar{X}=V_0(L_0-h^R)+V_1L_1+V_2L_2$, we notice that with this choice, $L_j\bar{X}y=V_0\omega_0\in\bar{\mathcal{M}}$, so we need no projections to define the $\bar{\beta}_{\pm}$. Now, one of these invariants is given (perhaps modulo $\mathcal{U}X_\rho-x$) by

$$\bar{\beta}\omega_0=\bar{X}^\dagger\bar{X}y=U_0(L_0-h^R)y, \tag{7.29}$$

where we have written $\bar{X}^\dagger\bar{X}=U_0(L_0-h^R)+U_1L_1+U_2L_2$ as usual. But by Poincaré–Birkhoff–Witt ordering appropriately, we may choose $U_0=f(L_0)$ for some polynomial f, since $\bar{X}^\dagger\bar{X}\in\mathcal{U}_0$. We therefore obtain

$$\bar{\beta}\omega_0=f(L_0)(L_0-h^R)y=f(h^R)\omega_0. \tag{7.30}$$

The vanishing of $\bar{\beta}$ is therefore equivalent to h^R being a zero of f, hence a double zero of $f(h)(h-h^R)$.

Consider now the highest weight vector $v_h\in\mathcal{V}_h$. We have

$$\bar{X}^\dagger\bar{X}v_h=U_0(L_0-h^R)v_h=f(h)(h-h^R)v_h. \tag{7.31}$$

By extending $\{\bar{X}v_h\}$ to a basis of $(\mathcal{V}_h)_{\bar{\ell}}$, it is possible to show that $\bar{\beta}=0$ if and only if the Kac determinant [Eq. (2.11)] of $\mathcal{V}_h$ at grade $\bar{\ell}=rs$ possesses a double zero at $h=h^R$ (this is an innocent generalization of the statement of Ref. 26, Lemma 6.2, its proof needs no changes). No (r',s') with $h_{r',s'}=h^R$ has $r's'<\bar{\ell}=rs$, so the double zero can only occur if there is another such pair $(r',s')\neq(r,s)$ with $r's'=rs$. Such a second distinct pair is easily verified to have the form $(r',s')=(|t|^{-1}s,|t|r)$, and integrality and distinctness yield the conditions given in the statement of the proposition.

These conditions are equivalent to the vanishing of *this* $\bar{\beta}$. But, they also imply that $\mathcal{H}^R$ and $\mathcal{H}^L$ are of chain type. Hence this is the only $\bar{\beta}$ and its vanishing is actually sufficient for the existence of the singular vector. This completes the proof. ∎

The restriction that $\bar{X}v_{h^R}$ have minimal (positive) grade is not serious, but Rohsiepe's argument requires some refining in this case. Essentially, if $\mathcal{V}_{h^R}$ is of braid type with $\bar{X}=X_1^+$, we generalize (Ref. 26, Lemma 6.2) to conclude that $\bar{\beta}=0$ is equivalent to the Kac determinant of $\mathcal{V}_h$ at grade $\bar{\ell}=\bar{\ell}_1^+$ having a zero at $h=\mathcal{H}^R$ of order $p(\bar{\ell}_1^+-\bar{\ell}_1^-)+2$ (or greater). However, coupling the explicit form of the Kac determinant formula with the conclusion of Proposition 7.5 for $\bar{X}=X_1^-$, we can deduce that the order of this zero is precisely $p(\bar{\ell}_1^+-\bar{\ell}_1^-)+1$. Thus, $\bar{\beta}$ cannot vanish.

This solves the existence question for staggered modules $\mathcal{S}$ with no nonvanishing beta invariants, $\bar{X}\omega_0=0$ and $\mathcal{H}^R=\mathcal{V}_h/\mathcal{U}\bar{X}v_h$, where $\bar{X}$ is prime: They exist if and only if $h=h_{\lambda|p|,\mu|q|}$ for some $\lambda,\mu\in\mathbb{Z}_+$, where $t=q/p\in\mathbb{Q}$ and $\lambda\neq\mu$. In particular, the left and right modules must be of chain type. One can also deduce existence for general $\bar{X}$, assuming existence when $\bar{X}$ is prime, by inductively applying Proposition 7.5 to certain submodules of (quotient modules of) the corresponding $\mathcal{T}$. However, deducing general nonexistence from nonexistence when $\bar{X}$ is prime requires far more intricate extensions of Rohsiepe's argument. Such arguments could complete the analysis in some further special cases, but the details are not in the spirit of what we have achieved here, so we will not elaborate any further upon them.

As mentioned before, the existence of these $\ell=0$ critical rank staggered modules is certainly not in line with our naive expectations based on counting constraints and unknowns. However, viewed in the light of Example 16, we can conclude that these counterexamples to our expectations are, in fact, quite rare—given $\bar{\ell}$, then in all the continuum of values of t there are only finitely many for which such staggered modules exist.

Of course, we should contrast this with the critical rank cases not covered by Proposition 7.5. In these cases, while we have not been able to rule out counterexamples to our expectations, we know of none! We would like to offer a speculative argument suggesting why this is so. Recall that the analysis of the cases covered by Proposition 7.5 is simplified by not requiring the $\bar{\mathbb{P}}$ when defining the $\bar{\beta}$. Structurally, we only need consider two singular vectors, ω_0 (which may as well be x in the analysis) and $\bar{X}v_{h^R}$ (which is prime), in our calculations. The key observation which we exploited in Example 16 was that such a configuration of two singular vectors can be continuously deformed for all t. The result was (modulo issues of $\bar{X}$ remaining prime) an expression for $\bar{\beta}$ as a *polynomial* in t and t^{-1}. Given this, it is no surprise that this polynomial will vanish for some values of t. In other words, because each $\bar{\beta}$ corresponds to a configuration of only two singular vectors, we should expect that our naive counting arguments will fail from time to time.

In contrast, the more general critical rank cases require the consideration of at least three singular vectors. Such configurations cannot be deformed smoothly—varying t without at least one of singular vectors disappearing is impossible. There is therefore very little to be gained from trying to express the $\bar{\beta}$ as polynomials in t and t^{-1} because the result will not correspond to a meaningful beta invariant for almost all t. For this reason, we suspect that counterexamples to our naive expectations of this more general type must be *significantly rarer* than those guaranteed by Proposition 7.5. Indeed, one might even be tempted to conjecture that there are, in fact, no counterexamples beyond those which we have described above. Evidently, more work is necessary to further understand this important situation.

VIII. SUMMARY OF RESULTS

In the preceding sections, we have answered our main question—that of the characterization and classification of staggered modules—in an expository yet detailed manner. We fully expect that the formalism developed throughout the course of this study will be invaluable when faced with further questions concerning these kinds of indecomposable modules and their generalizations. Moreover, we have tried throughout to illustrate with examples how such questions arise in concrete practical studies *and can be answered*.

The details should nevertheless not prevent us from presenting the reasonably simple answer that we have obtained to the original question. The results may be presented in purely structural terms, as one would hope, and we are finally in a position to summarise what we have shown.

Theorem 8.1: *Given two highest weight modules $\mathcal{H}^L$ and $\mathcal{H}^R$ of central charge c and highest weights h^L and h^R, respectively, the space $\mathbb{S}$ of isomorphism classes of staggered modules $\mathcal{S}$ with exact sequence,*

$$0 \to \mathcal{H}^L \xrightarrow{\iota} \mathcal{S} \xrightarrow{\pi} \mathcal{H}^R \to 0,$$

is described as follows. Let

- *$\ell = h^R - h^L$ be the grade of a singular vector $\omega_0 \in \mathcal{H}^L$;*
- *ρ be the rank of ω_0, if $\omega_0 \neq 0$;*
- *$\mathbf{n} \in \{0,1,2\}$ be the number of generating singular vectors $\bar{X}^{\varepsilon} v_{h^R}$ of $\mathcal{J}$, where $\mathcal{H}^R = \mathcal{V}_{h^R}/\mathcal{J}$;*
- *$\bar{\rho}$ be the rank of the $\bar{X}^{\varepsilon} v_{h^R}$, if $\mathbf{n} > 0$;*
- *$\mathbf{b} \in \{0,1,2\}$ be the number of (nonzero) rank $\rho - 1$ singular vectors in $\mathcal{H}^L$;*
- *$\mathbf{g} \in \{0,1,2\}$ be the number of (nonzero) rank $\rho + \bar{\rho} - 1$ singular vectors in $\mathcal{H}^L$, if $\mathbf{n} > 0$.*
  Then
- *there exists no such $\mathcal{S}$ unless $\omega_0 \neq 0$ (requiring ℓ to be a non-negative integer);*
- *there exists no such $\mathcal{S}$ unless each $\bar{X}^{\varepsilon} \omega_0 = 0$.*

Assuming these necessary conditions are met, we have the following.

- *When* $\mathbf{g}=0$, $\mathbb{S}$ *is a vector space* Ω/G *of dimension* $\mathbf{b}$. *When nontrivial, this vector space is parametrized by the beta invariants of Sec. VI E.*
- *In general,* $\mathbb{S}$ *is an affine subspace of* Ω/G *characterized by the vanishing of the* $\mathbf{gn}$ *auxiliary beta invariants of Sec. VII C.*

Theorem 8.1 gives a complete description of the space $\mathbb{S}$, hence a complete classification of staggered modules, when $\mathbf{g}=0$. In the few remaining cases in which $\mathbf{g}>0$ (the critical rank cases of Sec. VII C), our classification is not complete. For these cases, pictured in Fig. 6, we can, however, say that if $\mathbf{b}=0$ (or all the beta invariants of Sec. VI E vanish), then the nature of $\mathbb{S}$ is determined by Proposition 7.5 and its simple consequences. Otherwise, we expect (based on some speculative reasoning and an extensive study of examples) that the dimension of $\mathbb{S}$ is given by

$$\dim \mathbb{S} = \mathbf{b} - \mathbf{gn}, \tag{8.1}$$

where negative dimensions indicate that $\mathbb{S}$ is empty. We hope to report on the validity of this expectation in the future.

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[28] For applications to field theory, one would normally extend *antilinearly*, hence the appellation "adjoint." However, this distinction is largely irrelevant to the theory we are developing here.
[29] In applications to field theory, where the adjoint (2.5) is extended antilinearly to all of $\mathfrak{vir}$, this would define a Hermitian form. Physicists often refer to this form as the Shapovalov form as well.
[30] Admittedly, $X \in \mathcal{U}_m^-$ being singular only makes sense when h (and c) is specified. What it concretely means is that for $n=1,2$ there are $X_0^{(n)}, X_1^{(n)}, X_2^{(n)} \in \mathcal{U}$, such that $L_n X = X_0^{(n)}(L_0-h) + X_1^{(n)} L_1 + X_2^{(n)} L_2$ [compare with Eq. (2.9)]. The value of h should nevertheless always be clear from the context, so we trust that this terminology will not lead to any confusion.
[31] This is, in fact, the origin of the term "singular" in this context—it refers to the fact that the matrix representing the Shapovalov form at grade n has determinant zero.

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Field Theory, and Random Systems, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1989).

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[38] Apart from the obvious direct sums $\mathcal{H}^L \oplus \mathcal{H}^R$, reducible Verma modules form a simple class of examples of this type.

[39] We remark that when $\ell > 0$, one can extend the definition of the Shapovalov form to $\mathcal{H}^L \times \mathcal{S}$ by noting that for $u = Ux \in \mathcal{H}^L$, $\langle u, y\rangle = \langle Ux, y\rangle = \langle x, U^{\dagger} y\rangle$ and $U^{\dagger} y \in \mathcal{H}^L$. With this extension, we can write $\beta = \langle \omega_0, y\rangle$. One can also define an extended scalar product when $\ell = 0$, but in this case $\langle x, x\rangle$ necessarily vanishes, $\langle x, x\rangle = \langle x, (L_0 - h^R) y\rangle - \langle (L_0 - h^L) x, y\rangle = 0 \quad (\ell = 0)$. We must instead take $\langle x, y\rangle = 1$. These extensions are important in applications to logarithmic conformal field theory in which they give specializations of the so-called two-point correlation functions.[6,53] However, we will have no need of them here. We only mention that the nondiagonalizability of L_0 on $\mathcal{S}$ is not in conflict with its self-adjointness because such extensions of the Shapovalov form are necessarily indefinite.[54]

[40] For these parameters, we follow here and in later examples the established notation of the Schramm–Loewner evolution literature, where the curve and its growth process are often denoted simply by $\mathrm{SLE}_{\kappa}(\rho)$. Roughly speaking, κ determines the universality class (the central charge and fractal dimension of the curve), whereas ρ is related to the choice of boundary conditions. We are also using ρ to denote the rank of a singular vector (as in Sec. II). We trust that this will not lead to any confusion as it is clear that singular vector ranks are completely unrelated to SLE parameters.

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[42] We include a seemingly arbitrary "−" sign in the equation which follows (and in similar later equations) because it turns out to be convenient in the long run to be consistent with expressions such as that found in Eq. (2.9).

[43] The precise way in which one does this parallels that discussed in the context of Verma modules. One starts with the trivial one-dimensional representation of $\mathfrak{vir}^{\leq 0}$ and the induced $\mathfrak{vir}$-module is naturally identified as a graded vector space with $\mathcal{U}^{+}$.

[44] Here we lighten the notation by omitting possible superscripts "±." We also note that if $\check{\mathcal{H}}^R$ were Verma, then the inclusion $\check{\mathcal{N}} \subseteq \mathcal{N}$ would follow immediately. In the proof we may therefore exclude this trivial case and assume that both $\check{\bar{X}}$ and $\bar{X}$ are nonzero.

[45] Note that in the braid case the highest weight submodules generated by the singular vectors are not nested, which is why the definition of $\Omega_m^{(k;-)}$ requires w_j to be in the sum of two highest weight submodules instead.

[46] As usual, we can always find an orthogonal basis $\{Z_{\mu}\}$, such that $|\langle Z_{\mu}, Z_{\mu}\rangle| = 1$. Since every complex scalar is a square, it is trivial to redefine the Z_{μ} so as to obtain an orthonormal basis. We mention that if we had chosen the Shapovalov form to be sesquilinear rather than bilinear, then this would not be possible.

[47] Actually, the physically relevant modules we have in mind here do not always have right module Verma. However, Proposition 4.6 suggests that the relevant modules with non-Verma right modules should be recovered from this case as quotients. We will turn to this in Sec. VII.

[48] Although formulated differently and obtained by slightly different means, the result in this case has already appeared in Ref. 26. In fact, the result is obtained there (and could have been obtained here) without the lengthy preparation that our more general results require.

[49] The case rank $\omega_0 = 0$ (that is, $\ell = 0$) has already been analyzed in Theorem 6.4, but would be consistent with $\mathcal{M} = \{0\}$.

[50] We mention that this also covers the possibility that ω_0 is the singular vector of *maximal* grade in a braid-type Verma module with $t < 0$ (Sec. II).

[51] This is nothing but the requirement $\bar{X}\omega_0 = 0$ of Proposition 3.3. Combined with the observation that the gauge freedom here is trivial, $\mathcal{H}_1^L = \mathbb{C}\omega_0$, this also explains why Eq. (7.10) is valid independent of the choice of y.

[52] Unfortunately, demonstrating this linear independence (in particular, the nonvanishing) seems to require a significantly more delicate analysis than that presented in the proof of Theorem 6.15. We hope to return to this issue in a future publication.

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