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Research Article

On Step-Like Contrast Structure of Singularly Perturbed Systems

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The existence of a step-like contrast structure for a class of high-dimensional singularly perturbed system is shown by a smooth connection method based on the existence of a first integral for an associated system. In the framework of this paper, we not only give the conditions under which there exists an internal transition layer but also determine where an internal transition time is. Meanwhile, the uniformly valid asymptotic expansion of a solution with a step-like contrast structure is presented.

1. Introduction

The problem of contrast structures is a singularly perturbed problem whose solutions with both internal transition layers and boundary layers. In recent years, the study of contrast structures is one of the hot research topics in the study of singular perturbation theory. In western society, most works on internal layer solutions concentrate on singularly perturbed parabolic systems by geometric method (see [1] and the references therein). In Russia, the works on singularly perturbed ordinary equations are concerned by boundary function method [2–5]. One of the basic difficulties for such a problem is unknown of where an internal transition layer is in advance.

Butuzov and Vasil'eva initiated the concept of contrast structures in 1987 [6] and studied the following boundary value problem of a second-order semilinear equation with a step-like contrast structure, which is called a monolayer solution in [1]

$$\begin{gathered} \mu^2 y'' = F(y,t), \quad 0 \le t \le 1, \\ y(0,\mu) = y^0, \qquad y(1,\mu) = y^1, \end{gathered} \tag{1.1}$$

where $\mu > 0$ is a small parameter and F has a desired smooth scalar function on its arguments.

Suppose that the reduced equation $F(y,t)=0$ has two isolated solutions $\overline{y}=\varphi_i(t)$ $(i=1,2)$ on $0\le t\le 1$, which satisfy the following condition:

$$\varphi_1(t)<\varphi_2(t), \qquad F_y(\varphi_i(t),t)>0, \quad i=1,2. \tag{1.2}$$

The condition (1.2) indicates that there exist two saddle equilibria $M_i(\varphi_i(\bar{t}),0)$ $(i=1,2)$ in the phase plane $(\tilde{y},\tilde{z})$ of the associated equations given by

$$\frac{d\tilde{y}}{d\tau}=\tilde{z}, \qquad \frac{d\tilde{z}}{d\tau}=F\left(\tilde{y},\bar{t}\right), \quad 0<\bar{t}<1, \tag{1.3}$$

where $\bar{t}$ is fixed and $-\infty<\tau<+\infty$.

It is shown in [6] that the existence of an internal transition layer for the problem (1.1) is closely related to the existence of a heteroclinic orbit connecting M_1 and M_2. The principal value t_0 of an internal transition time t^* is determined by an equation as follows:

$$\int_{\varphi_1(t_0)}^{\varphi_2(t_0)} F(y,t_0)\,dy=0. \tag{1.4}$$

In [7], Vasil'eva further studied the existence of step-like contrast structures for a class of singularly perturbed equations given by

$$\begin{aligned} \mu\frac{du}{dt}&=f(u,v,t),\\ \mu\frac{dv}{dt}&=g(u,v,t), \quad 0\le t\le 1, \end{aligned} \tag{1.5}$$

where f and g are scalar functions. For (1.5), we may impose either a first class of boundary condition or a second class of boundary condition.

Suppose that there exist two solutions $\{\varphi_i(t),\psi_i(t)\}$ $(i=1,2)$ of the reduced equations $f(\overline{u},\overline{v},t)=0$; $g(\overline{u},\overline{v},t)=0$, and $M_i(\varphi_i(\bar{t}),\psi_i(\bar{t}))$ $(i=1,2)$ are two saddle equilibria in the phase plane $(\tilde{u},\tilde{v})$ of the associated equations given by

$$\begin{aligned} \frac{d\tilde{u}}{d\tau}&=f\left(\tilde{u},\tilde{v},\bar{t}\right);\\ \frac{d\tilde{v}}{d\tau}&=f\left(\tilde{u},\tilde{v},\bar{t}\right), \end{aligned} \tag{1.6}$$

where $\bar{t}$ is fixed with $0<\bar{t}<1$. This indicates that the eigenvalues $\lambda_{ik}(\bar{t})$ $(i,k=1,2)$ of the Jacobian matrix

$$A\left(\bar{t}\right)=\begin{pmatrix} f_{\tilde{u}} & f_{\tilde{v}}\\ g_{\tilde{u}} & g_{\tilde{v}}\end{pmatrix}_{\tilde{u}=\varphi_i(\bar{t}),\,\tilde{v}=\psi_i(\bar{t})} \tag{1.7}$$

satisfy the condition as follows:

$$\text{either } \lambda_{i1}\left(\bar{t}\right) > 0, \quad \lambda_{i2}\left(\bar{t}\right) < 0, \quad \text{or} \quad \lambda_{i1}\left(\bar{t}\right) < 0, \quad \lambda_{i2}\left(\bar{t}\right) > 0. \tag{1.8}$$

If (1.6) is a Hamilton equation, that is, $g_{\tilde{u}} = -f_{\tilde{v}}$, it implies that $g d\tilde{u} - f d\tilde{v} = dH(\tilde{u}, \tilde{v}, \bar{t})$. Then, the equation to determine t_0 is given by

$$H(\varphi_1(t_0), \psi_1(t_0), t_0) = H(\varphi_2(t_0), \psi_2(t_0), t_0). \tag{1.9}$$

Geometrically, (1.9) is also a condition for the existence of a heteroclinic orbit connecting $M_1(\varphi_1(\bar{t}), \psi_1(\bar{t}))$ and $M_2(\varphi_2(\bar{t}), \psi_2(\bar{t}))$.

Unfortunately, for a high dimensional singularly perturbed system, we cannot always find such an equation like (1.9) to determine t_0 at which there exists a heteroclinic orbit. This is one difficulty to further study the problem on step-like contrast structures. On the other hand, we know that the existence of a spike-like or a step-like contrast structure of high dimension is closely related to the existence of a homoclinic or heteroclinic orbit in its corresponding phase space, respectively. However, the existence of a homoclinic or heteroclinic orbit in high dimension space and how to construct such an orbit are themselves open in general in the qualitative analysis (geometric method) theory [8–10]. To explore these high dimensional contrast structure problems, we just start from some particular class of singularly perturbed system and are trying to develop some approach to construct a desired heteroclinic orbit by using a first integral method for such a class of the system and determine its internal transition time t_0.

2. Problem Formulation

We consider a class of semilinear singularly perturbed system as follows:

$$\begin{aligned}
\mu^2 y_1'' &= f_1(y_1, y_2, \ldots, y_n, t); \\
\mu^2 y_2'' &= f_2(y_1, y_2, \ldots, y_n, t); \\
&\vdots \\
\mu^2 y_n'' &= f_n(y_1, y_2, \ldots, y_n, t),
\end{aligned} \tag{2.1}$$

with a first class of boundary condition given by

$$\begin{aligned}
y_k(0, \mu) &= y_k^0, \quad k = 1, 2, \ldots, n; \\
y_j'(0, \mu) &= z_j^0 \quad j = 1, 2, \ldots, n-1; \\
y_n'(1, \mu) &= z_n^1,
\end{aligned} \tag{2.2}$$

where $\mu > 0$ is a small parameter.

The class of system (2.1) in question has a strong application background in engineering. For example, in the study of smart materials of variated current of liquid [11, 12], its math model is a kind of such a system like (2.1), where the small parameter μ indicates a particle. The given boundary condition (2.2) corresponds the stability condition $[H_3]$ listed later to ensure that there exists a solution for the problem in question.

For our convenience, the system (2.1) can also be written in the following equivalent form,

$$\begin{aligned}
\mu y_1' &= z_1;\\
\mu y_2' &= z_2;\\
&\vdots\\
\mu y_n' &= z_n;\\
\mu z_1' &= f_1(y_1, y_2, \dots, y_n, t);\\
\mu z_2' &= f_2(y_1, y_2, \dots, y_n, t);\\
&\vdots\\
\mu z_n' &= f_n(y_1, y_2, \dots, y_n, t).
\end{aligned} \tag{2.3}$$

Then, the corresponding boundary condition (2.2) is now written as

$$y_k(0,\mu) = y_k^0, \quad z_j(0,\mu) = \mu z_j^0, \quad z_n(1,\mu) = \mu z_n^1; \quad k = 1, 2, \dots, n, \ j = 1, 2, \dots, n-1. \tag{2.4}$$

The following assumptions are fundamental in theory for the problem in question.

$[H_1]$ Suppose that the functions f_i $(i = 1, 2, \dots, n)$ are sufficiently smooth on the domain $D = \{(y_1, y_2, \dots, y_n, t) \mid |y_i| \le l_i,\ 0 \le t \le 1,\ i = 1, 2, \dots, n\}$, where $l_i > 0$ are real numbers.

$[H_2]$ Suppose that the reduced system of (2.1) given by

$$\begin{aligned}
f_1(\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n, t) &= 0;\\
f_2(\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n, t) &= 0;\\
&\vdots\\
f_n(\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n, t) &= 0
\end{aligned} \tag{2.5}$$

has two isolated solutions on D:

$$\left\{\overline{y}_1 = a_1^1(t), \overline{y}_2 = a_2^1(t), \dots, \overline{y}_n = a_n^1(t)\right\}, \qquad \left\{\overline{y}_1 = a_1^2(t), \overline{y}_2 = a_2^2(t), \dots, \overline{y}_n = a_n^2(t)\right\}. \tag{2.6}$$

$[H_3]$ Suppose that the characteristic equation of the system (2.3) given by

$$\left|\begin{matrix} -\lambda & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\lambda & 0 & \cdots & 1 \\ f_{1y_1} & \cdots & f_{1y_n} & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ f_{ny_1} & \cdots & f_{ny_n} & 0 & \cdots & -\lambda \end{matrix}\right|_{y_1=a_1^i(t), y_2=a_2^i(t), \dots, y_n=a_n^i(t)} = 0; \quad (i = 1, 2) \tag{2.7}$$

has $2n$ real valued solutions $\overline{\lambda}_k(t)$, $k = 1, 2, \dots, 2n$, where

$$\begin{gathered} \operatorname{Re} \overline{\lambda}_k(t) < 0, \quad k = 1, 2, \dots, 2n-1; \\ \operatorname{Re} \overline{\lambda}_{2n}(t) > 0. \end{gathered} \tag{2.8}$$

Remark 2.1. $[H_3]$ is called as a stability condition. For a more general stability condition given by

$$\begin{gathered} \operatorname{Re} \overline{\lambda}_1(t) < 0, \dots, \operatorname{Re} \overline{\lambda}_k(t) < 0; \\ \operatorname{Re} \overline{\lambda}_{k+1}(t) > 0, \dots, \operatorname{Re} \overline{\lambda}_{2n}(t) > 0, \quad 1 < k < 2n, \end{gathered} \tag{2.9}$$

it will be studied in the other paper because of more complicated dynamic performance presented.

Under the assumption of $[H_3]$, there may exist a solution $y(t, \mu)$ with only two boundary layers that occurred at $t = 0$ and $t = 1$, for which the detailed discussion has been given by [13, Theorem 4.2], or it may consults [5, Theorem 2.4, page 49]. We are only interested in a solution $y(t, \mu)$ with a step-like contrast structure in this paper. That is, there exists $t^* \in (0, 1)$ such that the following limit holds:

$$\lim_{\mu \to 0} y(t, \mu) = \begin{cases} a^1(t), & 0 < t < t^*, \\ a^2(t), & t^* < t < 1. \end{cases} \tag{2.10}$$

We regard the solution $y(t, \mu)$ defined above with such a step-like contrast structure as being smoothly connected by two pure boundary solutions: $y^{(-)}(t, \mu)$, $0 \le t < t^*$ and $y^{(+)}(t, \mu)$, $t^* < t \le 1$. That is,

$$y^{(-)}(t^*, \mu) = y^{(+)}(t^*, \mu); \qquad z^{(-)}(t^*, \mu) = z^{(+)}(t^*, \mu). \tag{2.11}$$

The assumption $[H_3]$ ensures that the corresponding associated system given by

$$\begin{aligned} \frac{d\tilde{y}_k}{d\tau} &= \tilde{z}_k, \quad 0 < \bar{t} < 1, \\ \frac{d\tilde{z}_k}{d\tau} &= f_k\left(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n, \bar{t}\right), \quad k = 1, 2, \dots, n, \end{aligned} \tag{2.12}$$

has two equilibria $M_i(a_1^i(t), a_2^i(t), \dots, a_n^i(t), 0, \dots, 0)$ $(i = 1, 2)$, where $\bar{t}$ is fixed. They are both hyperbolic saddle points. From [13, Theorem 4.2] (or [5, Theorem 2.4]), it yields that there exists a stable manifold $W^s(M_i)$ of $2n - 1$ dimensions and an unstable manifold $W^u(M_i)$ of one-dimension in a neighborhood of M_i. To get a heteroclinic orbit connecting M_1 and M_2 in the corresponding phase space, we need some more assumptions as follows.

$[H_4]$ Suppose that the associated system (2.12) has a first integral

$$\Phi\left(\tilde{y}_1, \dots, \tilde{y}_n, \tilde{z}_1, \dots, \tilde{z}_n, \bar{t}\right) = C, \tag{2.13}$$

where C is an arbitrary constant and Φ is a smooth function on its arguments.

Then, the first integral passing through M_i $(i = 1, 2)$ can be represented by

$$\Phi\left(\tilde{y}_1, \dots, \tilde{y}_n, \tilde{z}_1, \dots, \tilde{z}_n, \bar{t}\right) = \Phi\left(M_i, \bar{t}\right). \tag{2.14}$$

$[H_5]$ Suppose that (2.14) is solvable with respect to $\tilde{z}_n$, which is denoted by

$$\tilde{z}_n = h\left(\tilde{y}_1, \dots, \tilde{y}_n, \tilde{z}_1, \dots, \tilde{z}_{n-1}, \bar{t}, M_i\right), \quad (i = 1, 2). \tag{2.15}$$

Let $\tilde{z}_n^{(-)} = h^{(-)}(\tilde{y}_1^{(-)}, \dots, \tilde{y}_n^{(-)}, \tilde{z}_1^{(-)}, \dots, \tilde{z}_{n-1}^{(-)}, \bar{t}, M_1)$ and $\tilde{z}_n^{(+)} = h^{(+)}(\tilde{y}_1^{(+)}, \dots, \tilde{y}_n^{(+)}, \tilde{z}_1^{(+)}, \dots, \tilde{z}_{n-1}^{(+)}, \bar{t}, M_2)$ be the parametric expressions of orbit passing through the hyperbolic saddle points M_1 and M_2, respectively.

Corresponding to the given boundary condition (2.2), we consider the following initial value relation at $\tau = 0$

$$\tilde{y}_k^{(-)}(0) = \tilde{y}_k^{(+)}(0), \quad k = 1, \dots, n; \qquad \tilde{z}_j^{(-)}(0) = \tilde{z}_j^{(+)}(0), \quad j = 1, \dots, n-1. \tag{2.16}$$

Let

$$H\left(\bar{t}\right) = h^{(-)} - h^{(+)} = 0, \tag{2.17}$$

where

$$\begin{aligned} h^{(-)} &= h^{(-)}\left(\tilde{y}_1^{(-)}(0), \dots, \tilde{y}_n^{(-)}(0), \tilde{z}_1^{(-)}(0), \dots, \tilde{z}_{n-1}^{(-)}(0), \bar{t}, M_1\right); \\ h^{(+)} &= h^{(+)}\left(\tilde{y}_1^{(+)}(0), \dots, \tilde{y}_n^{(+)}(0), \tilde{z}_1^{(+)}(0), \dots, \tilde{z}_{n-1}^{(+)}(0), \bar{t}, M_2\right). \end{aligned} \tag{2.18}$$

$[H_6]$ Suppose that (2.17) is solvable with respect to $\bar{t}$ and it yields a solution $\bar{t} = t_0$. That is, $H(t_0) = 0$ and $H'(t_0) \neq 0$.

Remark 2.2. It is easy to see from (2.14) and (2.17) that the necessary condition of the existence of a heteroclinic orbit connecting M_1 and M_2 can also be expressed as "the equation

$$\Phi\left(M_1, \bar{t}\right) = \Phi\left(M_2, \bar{t}\right) \tag{2.19}$$

is solvable with respect to $\bar{t} = t_0$."

3. Construction of Asymptotic Solution

We seek an asymptotic solution of the problem (2.1)-(2.2) of the form

$$\begin{aligned} y_k(t,\mu) &= \begin{cases} \displaystyle\sum_{l=0}^{\infty} \mu^l \left[\overline{y}_{kl}^{(-)}(t) + \Pi_l y_k(\tau_0) + Q_l^{(-)} y_k(\tau)\right], & 0 \le t \le t^*; \\ \displaystyle\sum_{l=0}^{\infty} \mu^l \left[\overline{y}_{kl}^{(+)}(t) + R_l y_k(\tau_1) + Q_l^{(+)} y_k(\tau)\right], & t^* \le t \le 1, \end{cases} \\ z_k(t,\mu) &= \begin{cases} \displaystyle\sum_{l=0}^{\infty} \mu^l \left[\overline{z}_{kl}^{(-)}(t) + \Pi_l z_k(\tau_0) + Q_l^{(-)} z_k(\tau)\right], & 0 \le t \le t^*; \\ \displaystyle\sum_{l=0}^{\infty} \mu^l \left[\overline{z}_{kl}^{(+)}(t) + R_l z_k(\tau_1) + Q_l^{(+)} z_k(\tau)\right], & t^* \le t \le 1, \end{cases} \end{aligned} \tag{3.1}$$

where $\tau_0 = t/\mu$, $\tau = (t - t^*)/\mu$, $\tau_1 = (t-1)/\mu$; and for $k = 1, 2, \ldots, n$, $\overline{y}_{kl}^{(\mp)}(t)$ $(0 < t < 1)$ are coefficients of regular terms; $\Pi_l y_k(\tau_0)$ $(\tau_0 \geq 0)$ are coefficients of boundary layer terms at $t = 0$; $R_l y_k(\tau_1)$ $(\tau_1 \leq 0)$ are coefficients of boundary layer terms at $t = 1$; and $Q_l^{(\mp)} y_k(\tau)$ $(-\infty < \tau < +\infty)$ are left and right coefficients of internal transition terms at $t = t^*$. Meanwhile, similar definitions are for $\overline{z}_{kl}^{(\mp)}(t)$, $\Pi_l z_k(\tau_0)$, $R_l z_k(\tau_1)$, and $Q_l^{(\mp)} z_k(\tau)$.

The position of a transition time $t^* \in (0,1)$ is unknown in advance. It needs being determined during the construction of an asymptotic solution. Suppose that t^* has also an asymptotic expression of the form

$$t^* = t_0 + \mu t_1 + \mu^2 t_2 + \cdots, \tag{3.2}$$

where t_l $(l = 0, 1, 2, \ldots)$ are temporarily unknown at the moment and will be determined later.

Meanwhile, let

$$y_1(t^*) = y_{10}^* + \mu y_{11}^* + \mu^2 y_{12}^* + \cdots, \tag{3.3}$$

where y_{1l}^* $(l = 0, 1, 2, \ldots)$ are all constants, independent of μ, and $y_1(t^*)$ takes value between $a_1^1(t^*)$ and $a_1^2(t^*)$. For example, $y_1(t^*) = (1/2)(a_1^1(t^*) + a_1^2(t^*))$.

Then, we will determine the asymptotic solution (3.1) step by step using "a smooth connection method" based on the boundary function method [13] or [5]. The smooth connection condition (2.11) can be written as

$$\begin{aligned} &Q_0^{(-)}y_k(0) + a_k^1(t_0) = Q_0^{(+)}y_k(0) + a_k^2(t_0), \qquad Q_0^{(-)}z_k(0) = Q_0^{(+)}z_k(0); \\ &Q_l^{(-)}y_k(0) + \left[\overline{y}_{1l}'^{(-)}(t_0)t_l + \xi_{1l}^{(-)}\right] = Q_0^{(-)}y_k(0) + \left[\overline{y}_{1l}'^{(+)}(t_0)t_l + \xi_{1l}^{(+)}\right]; \\ &Q_l^{(-)}z_k(0) + \left[\overline{z}_{1l}'^{(-)}(t_0)t_l + \eta_{1l}^{(-)}\right] = Q_0^{(-)}z_k(0) + \left[\overline{z}_{1l}'^{(+)}(t_0)t_l + \eta_{1l}^{(+)}\right], \end{aligned} \tag{3.4}$$

where $k = 1,2,\ldots,n$, $l = 1,2,\ldots$; $\xi_{1l}^{(\mp)} = \xi_{1l}^{(\mp)}(t_0,\ldots,t_{l-1})$, and $\xi_{1l}^{(\mp)} = \xi_{1l}^{(\mp)}(t_0,\ldots,t_{l-1})$ are all the known functions depending only on $t_0,\ldots,t_{l-1}$.

Substituting (3.1) into (2.1)-(2.2) and equating separately the terms depending on t, τ_0, τ_1, and τ by the boundary function method, we can obtain the equations to determine $\{\overline{y}_{kl}^{(\mp)}(t), \overline{z}_{kl}^{(\mp)}(t)\}$; $\{\Pi_l y_k(\tau_0), \Pi_l z_k(\tau_0)\}$, $\{R_l y_k(\tau_1), R_l z_k(\tau_1)\}$, and $\{Q_l^{(\mp)}y_k(\tau), Q_l^{(\mp)}z_k(\tau)\}$, respectively. The equations to determine the zero-order coefficients of regular terms $\{\overline{y}_{k0}^{(\mp)}(t), \overline{z}_{k0}^{(\mp)}(t)\}$ $(k = 1,2,\ldots,n)$ are given by

$$\begin{aligned} &\overline{z}_{10}^{(\mp)}(t) = \overline{z}_{20}^{(\mp)}(t) = \cdots = \overline{z}_{n0}^{(\mp)}(t) = 0; \\ &f_1\left(\overline{y}_{10}^{(\mp)}, \overline{y}_{20}^{(\mp)}, \ldots, \overline{y}_{n0}^{(\mp)}, t\right) = 0; \\ &f_2\left(\overline{y}_{10}^{(\mp)}, \overline{y}_{20}^{(\mp)}, \ldots, \overline{y}_{n0}^{(\mp)}, t\right) = 0; \\ &\qquad\vdots \\ &f_n\left(\overline{y}_{10}^{(\mp)}, \overline{y}_{20}^{(\mp)}, \ldots, \overline{y}_{n0}^{(\mp)}, t\right) = 0. \end{aligned} \tag{3.5}$$

It is clear to see that (3.6) coincides with the reduced system (2.11). Therefore, by $[H_2]$, (3.6) has the solution

$$\left\{\overline{y}_{10}^{(\mp)}, \overline{y}_{20}^{(\mp)}, \ldots, \overline{y}_{n0}^{(\mp)}\right\} = \left\{a_1^i(t), a_2^i(t), \ldots, a_n^i(t)\right\}, \quad i = 1,2. \tag{3.6}$$

The equations to determine $\{\overline{y}_{kl}^{(\mp)}(t), \overline{z}_{kl}^{(\mp)}(t)\}$ $(k = 1,2,\ldots,n;\ l = 1,2,\ldots)$ are given by

$$\begin{aligned} &\overline{y}_{1l-1}' = \overline{z}_{1l}, \overline{y}_{2l-1}' = \overline{z}_{2l}, \ldots, \overline{y}_{nl-1}' = \overline{z}_{nl}; \\ &\overline{z}_{1l-1}' = \overline{f}_{1y_1}(t)\overline{y}_{1l} + \overline{f}_{1y_2}(t)\overline{y}_{2l} + \cdots + \overline{f}_{1y_n}(t)\overline{y}_{nl} + \overline{h}_{1l}(t); \\ &\overline{z}_{2l-1}' = \overline{f}_{2y_1}(t)\overline{y}_{1l} + \overline{f}_{2y_2}(t)\overline{y}_{2l} + \cdots + \overline{f}_{2y_n}(t)\overline{y}_{nl} + \overline{h}_{2l}(t); \\ &\qquad\vdots \\ &\overline{z}_{nl-1}' = \overline{f}_{ny_1}(t)\overline{y}_{1l} + \overline{f}_{ny_2}(t)\overline{y}_{2l} + \cdots + \overline{f}_{ny_n}(t)\overline{y}_{nl} + \overline{h}_{nl}(t). \end{aligned} \tag{3.7}$$

Here the superscript $(\mp)$ is omitted for the variables $\overline{y}_{kl}^{(\mp)}$ and $\overline{z}_{kl}^{(\mp)}$ in (3.8) for simplicity in notation. To understand $\overline{y}_{kl}$ and $\overline{z}_{kl}$, we agree that they take $(-)$ when $0 \leq t \leq t_0$; while they take $(+)$ when $t_0 \leq t \leq 1$. The terms $\overline{h}_{kl}(t)$ $(k = 1,2,\ldots,n;\ l = 1,2,\ldots)$ are expressed in terms of $\overline{y}_{km}$ and $\overline{z}_{km}$ $(k = 1,2,\ldots,n;\ m = 0,1,\ldots,l-1)$. Also $\overline{f}_{(\cdot)}(t)$ are known functions that take value at $(a_1^i(t), a_2^i(t),\ldots,a_n^i(t))$, where $i = 1$ when $0 \leq t \leq t_0$ and $i = 2$ when $t_0 \leq t \leq 1$.

Since (3.8) is an algebraic linear system, the solution $\{\overline{y}_{kl}^{(\mp)}(t), \overline{z}_{kl}^{(\mp)}(t)\}$ $(k = 1,2,\ldots,n;$ $l = 1,2,\ldots)$ is uniquely solvable by $[H_3]$.

Next, we give the equations and their conditions for determining the zero-order coefficient of an internal transition layer $\{Q_0^{(-)}y_k(\tau), Q_0^{(-)}z_k(\tau)\}$ as follows:

$$\begin{gathered}\frac{d}{d\tau}Q_0^{(-)}y_k = Q_0^{(-)}z_k, \quad -\infty < \tau \leq 0;\\ \frac{d}{d\tau}Q_0^{(-)}z_k = f_k\left(a_1^1(t_0) + Q_0^{(-)}y_1,\ldots,a_n^1(t_0) + Q_0^{(-)}y_n, t_0\right),\\ Q_0^{(-)}y_1(0) = y_{10}^* - a_1^1(t_0);\\ Q_0^{(-)}y_k(-\infty) = 0, Q_0^{(-)}z_k(-\infty) = 0, \quad k = 1,2,\ldots,n.\end{gathered} \tag{3.8}$$

We rewrite (3.9) in a different form by making the change of variables

$$\tilde{y}_k^{(-)} = a_k^1(t_0) + Q_0^{(-)}y_k, \quad \tilde{z}_k^{(-)} = Q_0^{(-)}z_k, \quad k = 1,2,\ldots,n. \tag{3.9}$$

Then, (3.9) is further written in these new variables as

$$\begin{gathered}\frac{d\tilde{y}_k^{(-)}}{d\tau} = \tilde{z}_k^{(-)}, \quad -\infty < \tau \leq 0;\\ \frac{d\tilde{z}_k^{(-)}}{d\tau} = f_k\left(\tilde{y}_1^{(-)}, \tilde{y}_2^{(-)},\ldots,\tilde{y}_n^{(-)}, t_0\right),\end{gathered} \tag{3.10}$$

$$\begin{gathered}\tilde{y}_0^{(-)}(0) = y_{10}^*;\\ \tilde{y}_k^{(-)}(-\infty) = a_k^1(t_0), \quad \tilde{z}_k^{(-)}(-\infty) = 0, \quad k = 1,2,\ldots,n.\end{gathered} \tag{3.11}$$

From $[H_3]$, it yields that the equilibrium $(a_1^1(t_0),\ldots,a_n^1(t_0),0,\ldots,0)$ of the autonomous system (3.11) is a hyperbolic saddle point. Therefore, there exists an unstable one-dimensional manifold $W^u(M_1)$. For the existence of a solution of (3.11) satisfying (3.12), we need the following assumption.

$[H_7]$ Suppose that the hyperplane $\tilde{y}_1^{(-)}(0) = y_{10}^*$ intersects the manifold $W^u(M_1)$ in the phase space $(\tilde{y}_1^{(-)}(t_0), \tilde{y}_2^{(-)}(t_0),\ldots,\tilde{y}_n^{(-)}(t_0)) \times (\tilde{z}_1^{(-)}(t_0), \tilde{z}_2^{(-)}(t_0),\ldots,\tilde{z}_n^{(-)}(t_0))$, where $t_0 \in (0,1)$ is a parameter.

Then, $\{\tilde{y}_k^{(-)}(0), \tilde{z}_k^{(-)}(0)\}$ $(k = 1, 2, \ldots, n)$ are known values after $\{\tilde{y}_k^{(-)}(\tau), \tilde{z}_k^{(-)}(\tau)\}$ being solved by $[H_7]$. We can get the equations and the corresponding boundary conditions to determine $\{Q_0^{(+)} y_k(\tau), Q_0^{(+)} z_k(\tau)\}$ as follows:

$$\frac{d}{d\tau} Q_0^{(+)} y_k = Q_0^{(+)} z_k, \quad 0 \le \tau < +\infty;$$
$$\frac{d}{d\tau} Q_0^{(+)} z_k = f_k\Big(a_1^2(t_0) + Q_0^{(+)} y_1, \ldots, a_n^2(t_0) + Q_0^{(+)} y_n, t_0\Big), \tag{3.12}$$

$$Q_0^{(+)} y_k(0) = y_k^{(-)}(0) - a_k^2(t_0), \quad k = 1, 2, \ldots, n;$$
$$Q_0^{(+)} z_j(0) = \tilde{z}_j^{(-)}(0), \quad j = 1, 2, \ldots, n-1; \tag{3.13}$$
$$Q_0^{(+)} y_k(+\infty) = 0, \quad Q_0^{(+)} z_k(+\infty) = 0, \quad k = 1, 2, \ldots, n.$$

Introducing a similar transformation as doing for (3.9), we can get

$$\frac{d\tilde{y}_k^{(+)}}{d\tau} = \tilde{z}_k^{(+)}, \quad 0 \le \tau < +\infty;$$
$$\frac{d\tilde{z}_k^{(+)}}{d\tau} = f_k\Big(\tilde{y}_1^{(+)}, \tilde{y}_2^{(+)}, \ldots, \tilde{y}_n^{(+)}, t_0\Big), \tag{3.14}$$

$$\tilde{y}_k^{(+)}(0) = \tilde{y}_k^{(-)}(0), \quad k = 1, 2, \ldots, n;$$
$$\tilde{z}_j^{(+)}(0) = \tilde{z}_j^{(-)}(0), \quad j = 1, 2, \ldots, n-1; \tag{3.15}$$
$$\tilde{y}_k^{(+)}(+\infty) = a_k^2(t_0), \quad \tilde{z}_k^{(+)}(+\infty) = 0, \quad k = 1, 2, \ldots, n.$$

To ensure that the existence of a solution of (3.15)-(3.16), we need the following assumption.

$[H_8]$ Suppose that the hypercurve $\{\tilde{y}_1^{(+)}(0) = \tilde{y}_1^{(-)}(0), \ldots, \tilde{y}_n^{(+)}(0) = \tilde{y}_n^{(-)}(0), \tilde{z}_1^{(+)}(0) = \tilde{z}_1^{(-)}(0), \ldots, \tilde{z}_n^{(+)}(0) = \tilde{z}_n^{(-)}(0)\}$ intersects the manifold $W^s(M_2)$ in the phase space $(\tilde{y}_1^{(+)}(t_0), \tilde{y}_2^{(+)}(t_0), \ldots, \tilde{y}_n^{(+)}(t_0)) \times (\tilde{z}_1^{(+)}(t_0), \tilde{z}_2^{(+)}(t_0), \ldots, \tilde{z}_n^{(+)}(t_0))$, where $t_0 \in (0, 1)$ is a parameter.

Here it should be emphasized that under the conditions of $[H_7]$ and $[H_8]$, the solutions $\{Q_0^{(\mp)} y_k(\tau), Q_0^{(\mp)} z_k(\tau)\}$ $(k = 1, 2, \ldots, n)$ not only exist but also decay exponentially [13], or [5].

If the parameter t_0 is determined, $\{Q_0^{(\mp)} y_k(\tau), Q_0^{(\mp)} z_k(\tau)\}$ $(k = 1, 2, \ldots, n)$ are completely known. To determine t_0, it is closely related to the existence of a heteroclinic orbit connecting M_1 and M_2 in the phase space.

By the given initial values (3.14) or (3.16), we have already obtained

$$\tilde{y}_k^{(+)}(0) = \tilde{y}_k^{(-)}(0), \quad k = 1, 2, \ldots, n,$$
$$\tilde{z}_j^{(+)}(0) = \tilde{z}_j^{(-)}(0), \quad j = 1, 2, \ldots, n-1. \tag{3.16}$$

If we show $\tilde{z}_n^{(+)}(0) = \tilde{z}_n^{(-)}(0)$, the smooth connection condition (2.11) for the zero-order is satisfied. By $[H_4]$ and $[H_5]$, we have

$$\tilde{z}_n^{(\mp)} = h^{(\mp)}\left(\tilde{y}_1^{(\mp)}, \ldots, \tilde{y}_n^{(\mp)}, \tilde{z}_1^{(\mp)}, \ldots, \tilde{z}_{n-1}^{(\mp)}, t_0, M_{1,2}\right). \tag{3.17}$$

Since $\tilde{y}_k^{(\mp)}$ $(k = 1,2,\ldots,n)$ and $\tilde{z}_j^{(\mp)}$ $(j = 1,2,\ldots,n-1)$ only depend on y_{10}^*, while y_{10}^* only depends on t_0, the necessary condition for existence of a heteroclinic orbit connecting M_1 and M_2 at $\tau = 0$ is given by

$$h^{(-)}\left(y_{10}^*(t_0), t_0, M_1\right) = h^{(+)}\left(y_{10}^*(t_0), t_0, M_2\right), \tag{3.18}$$

or

$$\Phi(M_1, t_0) = \Phi(M_2, t_0). \tag{3.19}$$

However (3.19) or (3.20) is the one to determine t_0. Then, by $[H_6]$, there exists an $t_0 = t_0^*$ from (3.19) or (3.20). We can see that the process of determining t_0 is the one of a smooth connection. Therefore, all the zero-order terms $\{Q_0^{(\mp)} y_k(\tau), Q_0^{(\mp)} z_k(\tau)\}$ have now been completely determined by the smooth connection for the zero-order coefficients of the asymptotic solution.

For the high-order terms $\{Q_l^{(\mp)} y_k(\tau), Q_l^{(\mp)} z_k(\tau)\}$ $(l = 1,2,\ldots)$, we have the equations and their boundary conditions as follows:

$$\begin{gathered}
\frac{d}{d\tau} Q_l^{(-)} y_k = Q_l^{(-)} z_k, \quad -\infty < \tau \leq 0; \\
\frac{d}{d\tau} Q_l^{(-)} z_k = \tilde{f}_{ky_1}^{(-)}(\tau) Q_l^{(-)} y_1 + \tilde{f}_{ky_2}^{(-)}(\tau) Q_l^{(-)} y_2 + \cdots \tilde{f}_{ky_n}^{(-)}(\tau) Q_l^{(-)} y_n + \tilde{G}_{lk}^{(-)}(\tau), \\
Q_l^{(-)} y_1(0) = y_{1l}^* - \left[\overline{y}_{1l}(t_0) t_l + \xi_{1l}(t_0, \ldots, t_{l-1})\right]; \\
Q_l^{(-)} y_k(-\infty) = 0, \quad Q_l^{(-)} z_k(-\infty) = 0, \quad k = 1,2,\ldots,n, \\
\frac{d}{d\tau} Q_l^{(+)} y_k = Q_l^{(+)} z_k, \quad 0 \leq \tau < +\infty; \\
\frac{d}{d\tau} Q_l^{(+)} z_k = \tilde{f}_{ky_1}^{(+)}(\tau) Q_l^{(+)} y_1 + \tilde{f}_{ky_2}^{(+)}(\tau) Q_l^{(+)} y_2 + \cdots \tilde{f}_{ky_n}^{(+)}(\tau) Q_l^{(-)} y_n + \tilde{G}_{lk}^{(+)}(\tau), \\
Q_l^{(+)} y_k(0) = y_{kl}^*(0) - \left[\overline{y}'_{kl}(t_0) t_l + \xi_{kl}(t_0, \ldots, t_{l-1})\right], \quad k = 1,2,\ldots,n; \\
Q_l^{(+)} z_j(0) = \tilde{z}_{jl}^* - \left[\overline{z}'_{kl}(t_0) t_l + \eta_{kl}(t_0, \ldots, t_{l-1})\right], \quad j = 1,2,\ldots,n-1; \\
Q_l^{(+)} y_k(+\infty) = 0, \quad Q_l^{(+)} z_k(+\infty) = 0, \quad k = 1,2,\ldots,n,
\end{gathered} \tag{3.20}$$

where $\tilde{f}_{(\cdot)}^{(\mp)}(\tau)$ represent known functions that take value at $(a_1^i(t_0) + Q_0^{(\mp)} y_1(\tau), \ldots, a_n^i(t_0) + Q_0^{(\mp)} y_n(\tau))$; $\tilde{G}_{lk}^{(\mp)}(\tau)$ are the known functions that only depend on those asymptotic terms whose subscript is $0, 1, \ldots, l-1$; and ξ_{kl} and η_{kl} are all the known functions. Since (3.21) are all linear boundary value problems, it is not difficult to prove the existence of solution and the exponential decaying of solution without imposing any extra condition.

As for boundary functions $\Pi y(\tau_0)$ and $Ry(\tau_1)$, it is easy to obtain their constructions by using the normal boundary function method. So we would not discuss the details on them here [13] or [5]. However, it is worth mentioning that the coefficients t_l $(l = 1, 2, \ldots)$ in (2.3) will be determined by an equation as follows:

$$H'(t_0)t_l = \xi(t_0, \ldots, t_{l-1}). \tag{3.21}$$

Then, t_l can be solved from (3.22) by $[H_6]$. Then, we have so far constructed the asymptotic expansion of a solution with an internal transition layer for the problem (2.1)-(2.2) and the asymptotic expansion of an internal transition time t^*.

4. Existence of Step-Like Solution and Its Limit Theorem

We mentioned in Section 2that the solution with a step-like contrast structure can be regarded as a smooth connection by two solutions of pure boundary value problem from left and right, respectively. To this end, we establish the following two associated problems.

For the left associated problem,

$$\begin{gathered}
\mu\left(y_k^{(-)}\right)' = z_k^{(-)}; \\
\mu\left(z_k^{(-)}\right)' = f_k\left(y_1^{(-)}, \ldots, y_n^{(-)}, t\right),
\end{gathered} \tag{4.1}$$

$$\begin{gathered}
y_k^{(-)}(0, \mu) = y_k^0, \quad k = 1, 2, \ldots, n; \\
z_j^{(-)}(0, \mu) = \mu z_j^0, \quad j = 1, 2, \ldots, n-1; \\
y_1^{(-)}(t^*, \mu) = y_1(t^*),
\end{gathered} \tag{4.2}$$

where $0 \le t \le t^* < 1$, t^* is a parameter, such a solution $\{y_k^{(-)}(t, \mu), z_k^{(-)}(t, \mu)\}$ of (4.1) and (4.2) exists by $[H_1]$–$[H_3]$ [14, 15]. Then, we have $\{y_k^{(-)}(t^*, \mu), z_k^{(-)}(t^*, \mu)\}$, $k = 1, 2, \ldots, n$.

For the right associated problem,

$$\begin{gathered}
\mu\left(y_k^{(+)}\right)' = z_k^{(+)}; \\
\mu\left(z_k^{(+)}\right)' = f_k\left(y_1^{(+)}, \ldots, y_n^{(+)}, t\right),
\end{gathered} \tag{4.3}$$

$$\begin{gathered}
y_k^{(+)}(t^*, \mu) = y_k^{(-)}(t^*), \quad k = 1, 2, \ldots, n; \\
z_j^{(+)}(t^*, \mu) = z_j^{(-)}(t^*), \quad j = 1, 2, \ldots, n-1; \\
z_n^{(+)}(1, \mu) = \mu z_n^1,
\end{gathered} \tag{4.4}$$

where $0 < t^* \le t \le 1$, t^* is still a parameter, the similar reason is for the existence of $\{y_k^{(+)}(t,\mu), z_k^{(+)}(t,\mu)\}$ of (4.3) and (4.4) [14, 15].

Then, we write the asymptotic expansion of $\{y_k^{(\mp)}(t,\mu), z_k^{(\mp)}(t,\mu)\}$ as follows:

$$y_k(t,\mu) = \begin{cases} y_k^{(-)}(t,\mu) = a_k^1(t) + \Pi_0 y_k(\tau_0) + Q_0^{(-)} y_k(\tau) + O(\mu), & 0 \le t \le t^*; \\ y_k^{(+)}(t,\mu) = a_k^2(t) + R_0 y_k(\tau_1) + Q_0^{(+)} y_k(\tau) + O(\mu), & t^* \le t \le 1, \end{cases} \tag{4.5}$$

$$z_k(t,\mu) = \begin{cases} z_k^{(-)}(t,\mu) = \Pi_0 z_k(\tau_0) + Q_0^{(-)} z_k(\tau) + O(\mu), & 0 \le t \le t^*; \\ z_k^{(+)}(t,\mu) = R_0 z_k(\tau_1) + Q_0^{(+)} z_k(\tau) + O(\mu), & t^* \le t \le 1, \end{cases} \tag{4.6}$$

where $\tau_0 = t/\mu$, $\tau_1 = (t-1)/\mu$ and $\tau = (t-t^*)/\mu$.

We proceed to show that there exists an t^* indeed in the neighborhood of t_0 such that the solution $\{y_k^{(-)}(t,\mu), z_k^{(-)}(t,\mu)\}$ of the left associated problem (4.1) and (4.2) and the solution $\{y_k^{(+)}(t,\mu), z_k^{(+)}(t,\mu)\}$ of the right associated problem (4.3) and (4.4) smoothly connect at t^*, from which we obtain the desired step-like solution.

From the asymptotic expansion of (4.5) and (4.6), we know that $\{a_1^1(t), a_2^1(t), \ldots, a_n^1(t)\}$ and $\{a_1^2(t), a_2^2(t), \ldots, a_n^2(t)\}$ are the solutions of the reduced system (2.5). In the neighborhood of t_0, the boundary functions $\{\Pi_0 y_k(\tau_0), \Pi_0 z_k(\tau_0)\}$ and $\{R_0 y_k(\tau_1), R_0 z_k(\tau_1)\}$ are both exponentially small. Thus, they can be omitted in the neighborhood of t_0.

We are now concerned with the equations and the boundary conditions for which $\{Q_0^{(-)} y_k(\tau), Q_0^{(-)} z_k(\tau)\}$ satisfy. They can be obtained easily from (3.9) just with the replacement of t_0 by t^*,

$$\begin{gathered} \frac{d}{d\tau} Q_0^{(-)} y_k = Q_0^{(-)} z_k, \quad -\infty < \tau \le 0; \\ \frac{d}{d\tau} Q_0^{(-)} z_k = f_k\left(a_1^1(t^*) + Q_0^{(-)} y_1, \ldots, a_n^1(t^*) + Q_0^{(-)} y_n, t^*\right), \\ Q_0^{(-)} y_1(0) = y_{10}^* - a_1^1(t^*); \\ Q_0^{(-)} y_k(-\infty) = 0, \quad Q_0^{(-)} z_k(-\infty) = 0, \quad k = 1, 2, \ldots, n. \end{gathered} \tag{4.7}$$

After the change of variables given by

$$\tilde{y}_k^{(-)} = a_k^1(t^*) + Q_0^{(-)} y_k(\tau); \quad \tilde{z}_k^{(-)} = Q_0^{(-)} z_k(\tau); \quad k = 1, 2, \ldots, n, \tag{4.8}$$

then (4.7) can be written as

$$\frac{d\tilde{y}_k^{(-)}}{d\tau} = \tilde{z}_k^{(-)}, \quad -\infty < \tau \le 0;$$
$$\frac{d\tilde{z}_k^{(-)}}{d\tau} = f_k\left(\tilde{y}_1^{(-)}, \dots, \tilde{y}_n^{(-)}, t^*\right), \tag{4.9}$$

$$\tilde{y}_1^{(-)} = y_1(t^*);$$
$$\tilde{y}_k^{(-)}(-\infty) = a_k^1(t^*), \quad \tilde{z}_k^{(-)}(-\infty) = 0, \quad k = 1, 2, \dots, n. \tag{4.10}$$

It is similar to get the equations and the boundary conditions for which $\{Q_0^{(+)}y_k(\tau), Q_0^{(+)}z_k(\tau)\}$ satisfy

$$\frac{d}{d\tau}Q_0^{(+)}y_k = Q_0^{(+)}z_k, \quad 0 \le \tau < +\infty;$$
$$\frac{d}{d\tau}Q_0^{(+)}z_k = f_k\left(a_1^2(t^*) + Q_0^{(+)}y_1, \dots, a_n^2(t^*) + Q_0^{(+)}y_n, t^*\right),$$
$$Q_0^{(+)}y_k(0) = y_k^{(-)}(0) - a_k^2(t^*), \quad k = 1, 2, \dots, n; \tag{4.11}$$
$$Q_0^{(+)}z_j(0) = \tilde{z}_j^{(-)}(0), \quad j = 1, 2, \dots, n-1;$$
$$Q_0^{(+)}y_k(+\infty) = 0, \quad Q_0^{(+)}z_k(+\infty) = 0.$$

After the transformation given by

$$\tilde{y}_k^{(+)} = a_k^2(t^*) + Q_0^{(+)}y_k(\tau); \quad \tilde{z}_k^{(+)} = Q_0^{(+)}z_k(\tau); \quad k = 1, 2, \dots, n, \tag{4.12}$$

then (4.11) can be written as

$$\frac{d\tilde{y}_k^{(+)}}{d\tau} = \tilde{z}_k^{(+)}, \quad 0 \le \tau < +\infty;$$
$$\frac{d\tilde{z}_k^{(+)}}{d\tau} = f_k(\tilde{y}_1, \dots, \tilde{y}_n, t^*), \tag{4.13}$$

$$\tilde{y}_k^{(+)}(0) = y_k^{(-)}(0), \quad k = 1, 2, \dots, n;$$
$$\tilde{z}_k^{(+)}(0) = \tilde{z}_j^{(-)}(0), \quad j = 1, 2, \dots, n-1; \tag{4.14}$$
$$\tilde{y}_k^{(+)}(+\infty) = 0, \quad \tilde{z}_k^{(+)}(+\infty) = 0.$$

$[H_3]$ and $[H_4]$ imply that there exists a first integral

$$\Phi\left(\tilde{y}_1^{(-)}, \dots, \tilde{y}_n^{(-)}, \tilde{z}_1^{(-)}, \dots, \tilde{z}_n^{(-)}, t^*\right) = \Phi(M_1, t^*) \tag{4.15}$$

of the system (4.9) that approaches $M_1(a_1^1(t^*), a_2^1(t^*), \dots, a_n^1(t^*), 0, \dots, 0)$ as $\tau \to -\infty$; and there exists a first integral

$$\Phi\left(\tilde{y}_1^{(+)}, \dots, \tilde{y}_n^{(+)}, \tilde{z}_1^{(+)}, \dots, \tilde{z}_n^{(+)}, t^*\right) = \Phi(M_2, t^*) \tag{4.16}$$

of the system (4.13) that approaches $M_2(a_1^2(t^*), a_2^2(t^*), \dots, a_n^2(t^*), 0, \dots, 0)$ as $\tau \to +\infty$.

In views of $[H_5]$, from (4.15) and (4.16) we have

$$\tilde{z}_n^{(\mp)} = h^{(\mp)}\left(\tilde{y}_1^{(\mp)}, \dots, \tilde{y}_n^{(\mp)}, \tilde{z}_1^{(\mp)}, \dots, \tilde{z}_{n-1}^{(\mp)}, t^*, M_{1,2}\right). \tag{4.17}$$

Then, we know from (4.2) and (4.4) that $y_k^{(-)}(t^*, \mu) = y_k^{(+)}(t^*, \mu)$, $k = 1, 2, \dots, n$; and $z_j^{(-)}(t, \mu) = z_j^{(+)}(t, \mu)$, $j = 1, 2, \dots, n-1$ for the solutions $\{y_k^{(\mp)}(t, \mu), z_k^{(\mp)}(t, \mu)\}$ of the left and right associated problems. For a smooth connection of the solutions, the remaining is to prove

$$z_n^{(-)}(t^*, \mu) = z_n^{(+)}(t^*, \mu). \tag{4.18}$$

Let

$$\Delta(t^*) = z_n^{(-)}(t^*, \mu) - z_n^{(+)}(t^*, \mu). \tag{4.19}$$

Substituting (4.6) into (4.19), we have

$$\begin{aligned} \Delta(t^*) &= Q_0^{(-)} z_n(0) - Q_0^{(+)} z_n(0) + O(\mu) = h^{(-)} - h^{(+)} + O(\mu) \\ &= H(t^*) + O(\mu) = H(t_0) + H'(t_0)(t^* - t_0) + O(t^* - t_0)^2 + O(\mu), \end{aligned} \tag{4.20}$$

where $O(\mu)$ can be regarded as $C\mu$ for simplicity.

If we take $t^* = \mathrm{t}_0 \pm K\mu$ in (4.20), we have

$$\Delta(t^*) = \pm H'(t_0) K\mu + O(\mu). \tag{4.21}$$

Since the sign of $H'(t_0)$ is fixed, (4.21) has an opposite sign when K is sufficiently large, for example, $K > C$, and μ is sufficiently small. That is,

$$\left(H'(t_0)K\mu + O(\mu)\right)\left(-H'(t_0)K\mu + O(\mu)\right) < 0, \quad \text{as } K > C,\ \mu \ll 1. \tag{4.22}$$

Then, there exists $\hat{t} \in [t_0 - K\mu, t_0 + K\mu]$ such that $\Delta(\hat{t}) = 0$ by applying the intermediate value theorem to (4.21). This implies in turn that (4.18) holds.

Therefore, we have shown that there exists a step-like contrast structure for the problem (2.1)-(2.2). We summarize it as the following main theorem of this paper.

Theorem 4.1. *Suppose that [H_1]–[H_8] hold. Then, there exists an* $\mu_0 > 0$ *such that there exists a step-like contrast structure solution* $y_k(t,\mu)$ $(k = 1,2,\ldots,n)$ *of the problem* (2.1)-(2.2) *when* $0 < \mu < \mu_0$. *Moreover, the following asymptotic expansion holds*

$$y_k(t,\mu) = \begin{cases} a_k^1(t) + \Pi_0 y_k(\tau_0) + Q_0^{(-)} y_k(\tau) + O(\mu), & 0 \le t \le \hat{t}; \\ a_k^2(t) + R_0 y_k(\tau_1) + Q_0^{(+)} y_k(\tau) + O(\mu), & \hat{t} \le t \le 1. \end{cases} \tag{4.23}$$

Remark 4.2. Only existence of solution with a step-like contrast structures is guarantied under the conditions of [H_1]–[H_8]. There may exists a spike-like contrast structure, or the combination of them [16] for the problem (2.1)-(2.2). They need further study.

5. Conclusive Remarks

The existence of solution with step-like contrast structures for a class of high-dimensional singular perturbation problem investigated in this paper shows that how to get a heteroclinic orbit connecting saddle equilibria M_1 and M_2 in the corresponding phase space is a key to find a step-like internal layer solution. Using only one first integral of the associated system, this demands only bit information on solution, is our first try to construct a desired heteroclinic orbit in high-dimensional phase space. It needs surely further study for this interesting connection between the existence of a heteroclinic orbit of high-dimension in qualitative theory and the existence of a step-like contrast structure (internal layer solution) in a high-dimensional singular perturbation boundary value problem of ordinary differential equations. The particular boundary condition we adopt in this paper is just for the corresponding stability condition, which ensures the existence of solution of the problem in this paper. For the other type of boundary condition, we need some different stability condition to ensure the existence of solution of the problem in question, which we also need to study separately. Finally, if we want to construct a higher-order asymptotic expansion, it is similar with obvious modifications in which only more complicated techniques involved.

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