On stochastic approximation procedures in continuous time

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Abstract

Under appropriate conditions on a partially known function $f: [0, \infty) \times R^N \to R^N$ we establish the existence of a wide class of stochastic dynamical systems with a closed loop relation so that the input $X(\cdot)$ converges to the unknown root α of the equation $f(t, \alpha) = 0$. The output $Y(\cdot)$ of the dynamical system is a nonlinear transformation of the input $X(\cdot)$ involving f, and which is corrupted by additive noise terms modelled by Ito differentials. By martingale arguments we demonstrate convergence in mean and with probability one. The procedure may be considered as a continuous-time analog of the Robbins-Monroe scheme for discrete-time processes.

Key words : Stochastic approximation, Ito integrals, martingales.

1. Introduction

Stochastic approximation procedures in continuous-time consist in generating stochastic processes $\{X(t);\ t \geq 0\}$ that converge to α as $t \to \infty$ where α is a zero of a partially known function f. The discrete-time case has been studied extensively since the pioneering paper of Robbins and Monroe[8] appeared in the early fifties. The continuous time case has been investigated by, among others, Driml and Nedoma[2] and Krasulina[5] (*see* also refs. 1 and 4). M. T. Wasan's monograph[10] has a fairly complete bibliography of the work prior to 1969.

In this paper we establish the existence of a wide class of stochastic dynamical systems with a closed loop relation between the input $X(\cdot)$ and the output $Y(\cdot)$ and such that $X(t)$ converges to the desired value as $t \to \infty$. The dynamical system is such that the output $Y(\cdot)$ is a nonlinear transformation (*via* f) of the input $X(\cdot)$ and is corrupted by a multidimensional 'white' noise term. The output is hence modelled *via* an Ito differential equation. This gives the dynamical system a reasonable degree of flexibility from the point of view of applications and also makes available the tools of martingales and Ito calculus to establish the desired convergence. The precise model is described in section 2, especially by equation (4).

An important aspect of the theory that does not seem to be adequately treated in some of the applied literature is the closed loop relation (*see* Fig. 1) between the output and the input. Whereas in discrete-time situations the input X_{n+1} at time $(n+1)$ is a function of the input X_n and output Y_n at time n, the situation in continuous time is not so simple. We clarify this point and show here that the input process $X(t)$ is adapted to the output process up to time t: $\{Y(s), s \leq t\}$. A precise statement of this is given in Theorem 1.

As regards applications, we mention that our results can be extended to be used in the problem of establishing asymptotic state estimators for stochastic dynamical systems in the realm of estimation theory. Several results in this direction have been obtained and will be published in detail elsewhere, and are also available in the first author's (P. S.) doctoral thesis.[9]

2. The problem and results

We are given partial information on a function f, namely that it maps $[0, \infty) \times R^N \rightarrow R^N$ is Borel measurable and satisfies

(1) For every $0 \leq T < \infty$ there is a $0 \leqslant K_T < \infty$ such that

$$\sup_{0 \leq t \leq T} \| f(t, x_1) - f(t, x_2) \| \leq K_T \| x_1 - x_2 \|$$

(2) There is a unique (unknown) α in R^N such that

$$f(t, \alpha) = 0 \text{ for every } t \geq 0.$$

(3) For each $\varepsilon > 0$ we can find $b_\varepsilon > 0$ such that

$$\inf_{\epsilon < \|X - \alpha\| < \epsilon^{-1}} (x - \alpha)^T f(t, x) \geqslant b_\epsilon$$

where R^N is Euclidean N-space, $\| . \|$ is the Euclidean norm, and superscript 'T' denotes transpose. [Also in the following, the matrix norm used will be $\| A \|^2 = Tr\ (AA^T)$ where Tr = trace.]

In addition to the *a priori* information given above, the only additional information about f is available in the following form. A stochastic dynamical system involving f exists, or can be designed, such that for any input $\{x(w, t), t \geq 0\}$, the output $\{y(w, t), \geq 0\}$ satisfies

(4) $dy(w, t) = f(t, x(w, t))\, dt + G(w, t)\, db(w, t)$

where y and x are R^N valued, b is R^M valued and G is $N \times M$ matrix valued, and are all measurable stochastic processes defined on a probability space $(\mathcal{W}, \mathcal{B}, P)$ and are adapted to an increasing family $\{\mathcal{F}_t, t \geqslant 0\}$ of sub-sigma algebras of $\mathcal{B}$. Moreover, $\{b(w, t)\}$ is a Wiener process which is also non-anticipating with respect to the family $\{\mathcal{F}_t\}$. $G(w, t)$ satisfies

(5) $\sup_{0 \leq t \leq T} E \| G(w, t) \|^2 < \infty$ for each $T < \infty$

and equation (4) is to be interpreted in the Ito sense.[3,7] *The problem now is to find* α. We solve this problem by producing an input process $\{x(w,t),\ t \geqslant 0\}$ which is adapted to the family of σ-algebras $\{\mathcal{G}_t,\ t \geqslant 0\}$ generated by the *corresponding* output process $y(w,t)$ ($\mathcal{G}_t = \sigma\{y(w,s),\ 0 \leqslant s \leqslant t\}$), and is such that $x(w,t) \to \alpha$ almost surely as $t \to \infty$. In other words, we will be able to close the loop in Fig. 1 in such a way that the required convergence holds.

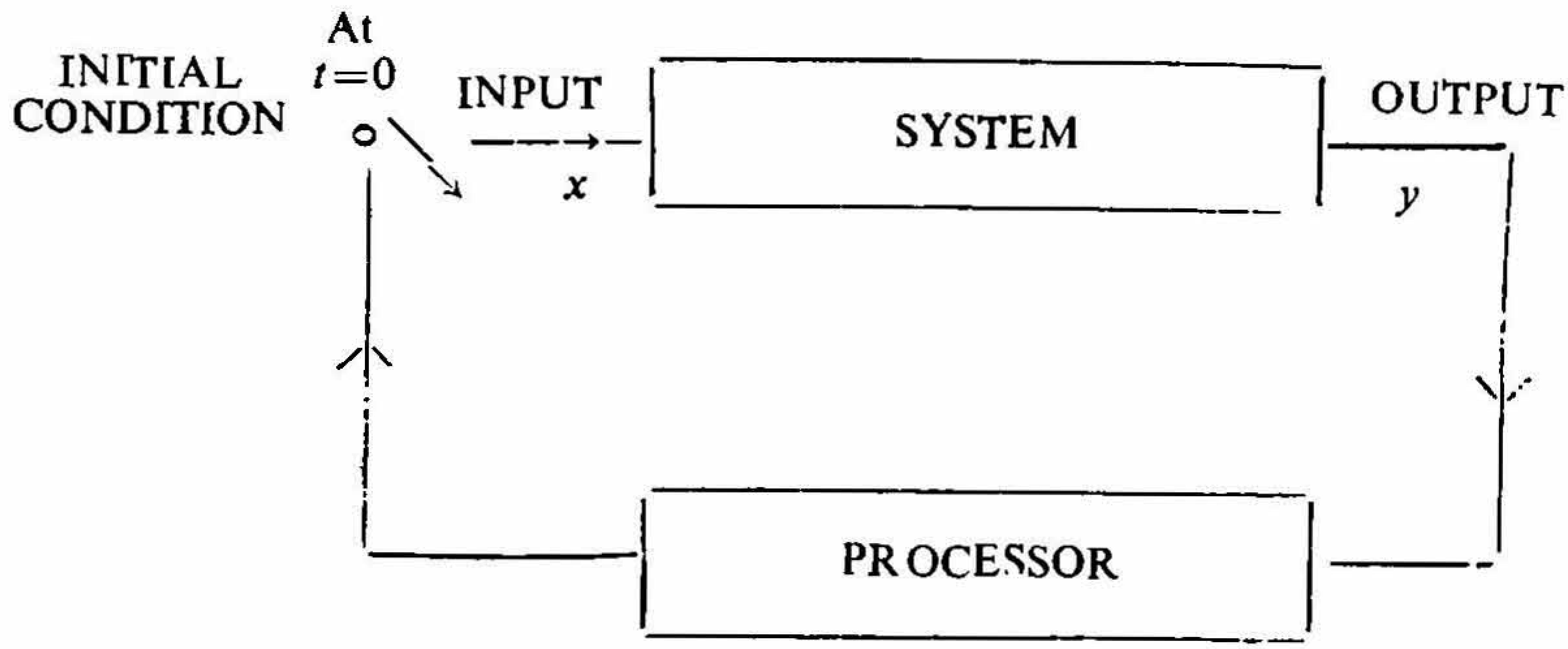

FIG. 1

Remark: Only separable versions of all processes will be considered.

Theorem 1 below constructs a candidate for such a solution, while Theorems 2 and 3 prove the desired convergence.

Theorem 1: Let $g : [0,\infty) \to [0,\infty)$ satisfy

$$\int_0^\infty g(t)\,dt = \infty \tag{6}$$

$$\int_0^T g^2(t)\,dt < \infty \text{ for each } T < \infty, \tag{7}$$

and

$$\int_0^\infty g^2(t)\,E\,\|G(w,t)\|^2\,dt < \infty. \tag{8}$$

Let $x(w,0)$ be a random variable such that

$$E\,\|x(w,0)\|^2 < \infty. \tag{9}$$

Then there exists a unique almost sure sample continuous process $\{x(w,t);\ t \geqslant 0\}$ adapted to $\{\mathcal{F}_t\}$, which satisfies the Ito equation

$$x(w,t) = x(w,0) - \int_0^t g(u)\,f\big(u, x(w,u)\big)\,du - \int_0^t g(u)\,G(w,u)\,db(w,u). \tag{10}$$

Moreover with this $x(\cdot,\cdot)$ in (4), x is recoverable from y in the sense that

$$x(w,t) = x(w,0) - \int_0^t g(u)\,dy(w,u) \tag{11}$$

where the integral on the right is meant in the mean-square sense.

Remark : Notice that (4) and (11) provide a complete description of the closed loop relation between the input and output thus providing a mathematical basis for the claim implied by Fig. 1. Also, the definition of the mean-square integral yields the right side of (11) and hence $x(w,t)$ to be adapted to $\mathcal{F}_t$.

Theorem 2 : Under the conditions of Theorem 1, the process $\{x(w,t)\,;\ t \geq 0\}$ converges to α with probability one as $t \to \infty$.

Theorem 3: If in addition to the conditions of Theorem 1 we have the stronger conditions

$$\int_0^\infty g^2(t)\ (E\,\|\,G(w,t)\,\|^4)^{1/2}\ dt < \infty, \tag{8A}$$

and

$$E\,\|\,x(w,0)\,\|^4 < \infty, \tag{9A}$$

then $E(\|\,x(w,t) - \infty\,\|^r) \to 0$ as $t \to \infty$ for every $0 < r < 4$.

3. Proofs

(*a*) *Proof of Theorem* 1: Existence and uniqueness of a solution to (10) are proved by the usual iteration argument[3] and we shall only sketch the same here. Without loss of generality we restrict t to $[0, T]$, $T < \infty$, Define

$$x_0(w,t) = x(w,0), \text{ and for } n \geq 0 \tag{12}$$

$$x_{n+1}(w,t) = x(w,0) - \int_0^t g(u)\ f(u, x_n(w,u))\ du - \int_0^t g(u)\ G(w,u)\ db(w,u).$$

Observe that the Ito stochastic integral on the right side in (12) is well defined because of (8) whereas the Lebesgue integral exists in view of (1), (2) on f, (7) on g, (9) on x_0, and the easily proved relation that

$$a_n = \sup_{0 \leq t \leq T} \|\,x_n(w,t)\,\|^2 < \infty$$

with probability one implies the same for a_{n+1}.

Now by the Schwartz inequality (1) and (7) we obtain an integral inequality for $\|\,x_n(w,t) - x_{n-1}(w,t)\,\|^2$ from (12), iterating which leads to

$$b_n(w) = \sup_{0 \leq t \leq T} \|\,x_{n+1}(w,t) - x_n(w,t)\,\|^2 \leq A(w)\, C^n{}_T / n! \tag{13}$$

where $C_T < \infty$ and

$$A(w) = 2K_T{}^2 \left(\int_0^T g(u)\,\|\,x(w,0) - \alpha\,\|\, du \right)^2$$

$$+ 2 \sup_{0 \leq t \leq T} \left\| \int_0^t g(u)\, G(w,u)\, db(w,u) \right\|^2. \tag{14}$$

That the first term on the right of (14) has finite expectation can be seen by an application of the Schwartz inequality, (7) and (9). For the second term we need (8) and a special property of the Ito integral as given in ref. 3, (p. 20, Theorem 1), to establish its finite expectation. Thus $E(A(w)) < \infty$ and since A is non-negative

$$A(w) < \infty \text{ with probability one.} \tag{15}$$

Then from (15) and (13) we have

$$\sum_{n=0}^{\infty} b_n(w) < \infty \text{ with probability one.} \tag{16}$$

It is now standard that with probability one $x_n(w, .)$ is a Cauchy sequence in $C[0, T]$, the space of continuous R^N-valued functions on $[0, T]$, and that the limit $x(w, .)$ will be almost surely in $C[0, T]$, and will be a solution of (10). Also, from (12), Gronwall's lemma and the above convergence, we have,

$$\sup_{0 \leq t \leq T} E \| x(w, t) \|^2 < \infty. \tag{17}$$

We omit the proof of uniqueness, which is again standard, and proceed to establish the important relation (11).

Let

$$\{t_i^{(n)}: \; i = 0, 1, 2, \ldots, m_n\}$$

be a sequence of partitions of $[0, t]$. Define

$$g_n(s) = g(t_i^{(n)}) \text{ for } t_i^{(n)} \leq s < t_{i+1}^{(n)} \tag{18}$$

$$I_n(w) = \sum_{i=0}^{m_n - 1} g(t_i^{(n)}) [y(w, t_{i+1}^{(n)}) - y(w, t_i^{(n)})].$$

From the definition of $y(w, s)$ in (4) we may re-write I_n as

$$I_n(w) = \int_0^t g_n(s) f(s, x(w, s)) \, ds + \int_0^t g_n(s) G(w, s) \, db(w, s), \tag{19}$$

where the linearity of the integrals involved and the step nature of g_n have been used. Setting $I(w) = x(w, 0) - x(w, t)$, from (10) and (19) we obtain after using (1), the Schwartz inequality (17), (5) and Ito's formula

$$E \| I_n - I \|^2 \leq \text{constant} \int_0^t | g_n(s) - g(s) |^2 \, ds. \tag{20}$$

By (7) g is in $L^2[0, t]$ for each $t < \infty$ and hence there always exist sequence of partitions $\{t_i^{(n)}\}$ such that the g_n defined by (18) converge to g in $L_2[0, t]$. This together with (20) implies that $I(w)$ is the mean-square limit of $I_n(w)$. The mean-square limit of the I_n is, by definition of the mean-square integral, the integral on the right of (11) and from the definition of $I(w)$, (11) follows.

(b) *Proof of Theorem* 2: The following is a key step in the proof.

Proposition 1: Under conditions (1) through (9) we have

(i) For $0 \leq s \leq t < \infty$

$$E(\| x(w,t) - \alpha \|^2 \mid \mathcal{F}_s) \leq \| x(w,s) - \alpha \|^2 + \int_s^t g^2(u) E(\|G(w,u)\|^2 \mid \mathcal{F}_s)\, du \tag{21}$$

with probability one.

(ii) There is a constant C independent of s, t such that

$$0 \leq E \int_s^t g(u)\, (x(w,u) - \alpha)^T f(u, x(w,u))\, du \leq C < \infty. \tag{22}$$

(iii) If in addition (8 A) and (9 A) are satisfied then

$$\sup_{t \geq 0} E(\| x(w,t) - \alpha \|^4) < \infty. \tag{23}$$

We defer the proof of this proposition and proceed to prove the convergence results. Define

$$a(w,t) = \int_t^\infty g^2(u) E(\|G(w,u)\|^2 \mid \mathcal{F}_t)\, du \tag{24}$$

$$z(w,t) = a(w,t) + \| x(w,t) - \alpha \|^2$$

The integral is well defined due to (8) and the assumed measurability conditions on G. Since $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leqslant t$ we have

$$E(a(w,t) \mid \mathcal{F}_s) \leq a(w,s) \tag{25}$$

with probability one for $s \leq t$.

Combining (21), (24) and (25) we obtain for $s \leq t$

$$E(z(w,t) \mid \mathcal{F}_s) \leq z(w,s) \tag{26}$$

with probability one.

(25) and (26) together with the non-negativity of a and z from (24) yield the fact that the families

$$\{z(w,t);\ \mathcal{F}_t;\ t \geq 0\}$$

and

$$\{a(w,t);\ \mathcal{F}_t;\ t \geq 0\}$$

are both non-negative supermartingales, thus ensuring their almost sure convergence as $t \to \infty$.[6] Call these limits $Z(w)$ and $a(w)$ respectively. By (8) $E(a(w,t))$ decreases to zero as $t \to \infty$, forcing $a(w)$ to vanish almost surely and we thus have

$$\| x(w,t) - \alpha \|^2 \to Z(w) \tag{27}$$

almost surely as $t \to \infty$.

To finish the proof of Theorem 2 we have to show that Z vanishes almost surely.

Suppose

$$P\{w : Z(w) > 0\} > 0.$$

Then by Egoroff's theorem, there exist $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, $t_0 < \infty$ such that $P(A) > \varepsilon_0$, where

$$A = \{w : \varepsilon_1 < \| x(w, t) - \alpha \|^2 < \varepsilon_1^{-1} \quad \text{for } t \geq t_0\}.$$

But by (3), for $w \varepsilon A$ and $u \geq t_0$, we must have

$$(x(w, u) - \alpha)^T f(u, x(w, u)) \geq \beta_{\varepsilon_1} > 0$$

and again by (3)

$(x(w, t) - \alpha)^T f(t, x(w, t)) \geq 0$ for all w and t, making

$$E \int_{t_0}^{t} g(u)(x(w, u) - \alpha)^T f(u, x(w, u))\, du \geq \varepsilon_0 \beta_{\varepsilon_1} \int_{t_0}^{t} g(u)\, du$$

$\to \infty$ as $t \to \infty$ by (6). This obviously contradicts (22). Hence $Z(w) = 0$ almost surely.

(*c*) *Proof of Theorem* 3: By Proposition 1 (iii), we know that under (8 A) and (9 A), (23) holds making the family

$$\{\| x(w, t) - \alpha \|^r ; \quad t \geq 0\}$$

uniformly integrable with respect to expectation, for each $0 < r < 4$.[6] This with the almost sure convergence of $\| x(w, t) - \alpha \|$ to zero from Theorem 2 [(8 A) $\Rightarrow$ (8), (9 A) $\Rightarrow$ (9)] yields the desired conclusion.

(*d*) *Proof of Proposition* 1: For proving (i) and (ii) directly via Ito's lemma we would require $E(\| x(w, t) - \alpha \|^4) < \infty$ for each t, and in order to prove (iii) *via* Ito's lemma and an integral inequality proved in the appendix we would need

$$E(\| x(w, t) - \alpha \|^6) < \infty$$

for each t. Both of these are *a priori* unverifiable and so we consider a sequence of truncated processes, apply the preceding ideas to each such process and then take limits to complete the proof.

Define a sequence of stopping times with respect to $\{\mathcal{F}_t\}$ by

$$T_N(w) = \inf\{t : x(w, t) > N\}, \quad N = 1, 2.... \tag{28}$$

and a related sequence of stopped processes by

$$x_N(w, t) = x(w, t \wedge T_N(w)), \quad N = 1, 2, ... \tag{29}$$

where $a \wedge b =$ minimum of a and b.

By appealing to a standard result on stopped Ito integrals (ref 7, p. 24) we obtain the important relation

$$\begin{aligned} x_N(w, t) = x_N(w, 0) &- \int_0^t I_{E_N}(u)\, g(u)\, f(u, x_N(w, u))\, du \\ &- \int_0^t I_{E_N}(u)\, G(w, u)\, db(w, u) \end{aligned} \tag{30}$$

where I_{E_N} denotes the characteristic function of the set E_N and

$$E_N = \{u : \; T_N(w) \geq u\}. \tag{31}$$

Also it is obvious from (28) and (29) that

$$x_N(w, 0) = x(w, 0) \text{ for every } N. \tag{32}$$

The process $x(w, t)$ is finite valued and has continuous samplep aths, with probability one, and hence is bounded for t in [0, T] almost surely for each $T < \infty$. Thus

$$T_N(w) \to \infty \text{ almost surely as } N \to \infty \tag{33}$$

and using (29)

$$x_N(w, t) \to x(w, t) \text{ with probability one, as } N \to \infty. \tag{34}$$

Applying Ito's differentiation rule to

$$\beta_N(w, t) = \| x_N(w, t) - \alpha \|^2, \tag{35}$$

we get for $0 \leq s \leq t$,

$$\begin{aligned} \beta_N(w, t) = \beta_N(w, s) &- 2 \int_s^t I_{E_N}(u) \; g(u) \; (x_N(w, u) - \alpha)^T \cdot f(u, x_N(w, u)) \, du \\ &- 2 \int_s^t I_{EN}(u) \; g(u) \, (x_N(w, u) - \alpha)^T \cdot G(w, u) db(w, u) \\ &+ \int_s^t I_{E_N}(u) \, g^2(u) \; \| G(w, u) \|^2 \, du. \end{aligned} \tag{36}$$

Notice that for each N, $x_N(w, t)$ is uniformly bounded by N and so using (8) the Ito integral on the right side of (36) is a zero mean martingale. Also the assumptions on f and g imply that the integrand of the second term on the right of (36) is non-negative. Taking expectations conditioned on $\mathcal{F}_s$ in (36) then yields,

$$E(\beta_N(w, t) \,|\, \mathcal{F}_s) \leq \beta_N(w, s) + \int_s^t g^2(u) \, E(\| G(w, u) \|^2 \,|\, \mathcal{F}_s) \, du. \tag{37}$$

Also by the non-negativity of β_N, we get from (36) and (37)

$$\begin{aligned} 0 &\leq 2E \int_s^t I_{E_N}(u) \, g(u) \, (x_N(w, u) - \alpha)^T \cdot f(u, x_N(w, u)) \, du \\ &\leq E \, \beta_N(w, 0) + E \int_0^\infty g^2(u) \, \| G(w, u) \|^2 \, du \end{aligned} \tag{38}$$

Now letting $N \to \infty$, we use (34), (35) and Fatou's lemma to get part (i) of Proposition 1 from (37). For part (ii), using (29), (31) (33) and the non-negativity of the integrand, we apply the monotone convergence theorem to (38), noticing that the right side of (38) is finite independent of s, t, N by (8), (9) and (32). Now for part (iii) we apply Ito's formula to

$$\alpha_N(w, t) = \beta_N^2(w, t) \tag{39}$$

and obtain from (30) setting $s = 0$

$$\alpha_N(w,t) = \alpha_N(w,0) - 4\int_0^t I_{EN}(u)\, g(u)\, \beta_N(w,u)$$
$$\cdot (x_N(w,u) - \alpha)^T f(u, x_N(w,u))\, du - 4\int_0^t I_{E_N}(u)\, g(u)$$
$$\cdot \beta_N(w,u)(x_N(w,u) - \alpha)^T G(w,u)\, db(w,u)$$
$$+ 2\int_0^t I_{E_N}(u)\, g^2(u)\, \{\beta_N(w,u)\, \| G(w,u) \|^2$$
$$+ 2\, \| (x(w,u) - \alpha)^T G(w,u) \|^2\}\, du \tag{40}$$

Arguing as for β_N we get by taking expectations

$$E(\alpha_N(w,t)) \leq E(\alpha_N(w,0)) + 6\int_0^t g^2(u)\, E(\beta_N(w,u)\, \| G(w,u) \|^2)\, du \tag{41}$$

Using the Schwartz inequality on the integral in (41) we obtain

$$m_N(t) \leq m_N(0) + \int_0^t \phi(m_N(u))\; dH(u) \tag{42}$$

where

$$m_N(t) = E(\alpha_N(w,t)),\quad m_N(0) = E\, \| x(w,0) - \alpha \|^4$$
$$\phi(x) = x^{1/2}$$
$$H(t) = 6\int_0^t g^2(u)\; (E(\| G(w,u) \|^4))^{1/2}\, du \tag{43}$$

Now by (8 A), (9 A) and choosing $\beta > E\,(\| x(w,0) - \alpha \|^4)$ we get from Corollaries 1 and 2 of Lemma 1 and Lemma 2 of the Appendix applied to (42)

$$m_N(t) \leq \bar{m}(t,\beta) \leq \sup_t\ \bar{m}(t,\beta) < \infty \tag{44}$$

Taking limits and using Fatou's lemma we have (iii) of Proposition 1.

4. Concluding remarks

1. The noise term in the observations, in differential form $G(\cdot,\cdot)\, db$, is fairly general to allow for wide variety in the underlying dynamic systems, and at the same time is in a form enabling the use of the powerful tools of martingale and Ito calculus. In particular, though arising from ‘ white noise ’, the noise terms could still admit correlations among the components due to the presence of the $G(\cdot,\cdot)$ matrix. Also the noise is in general non-Gaussian due to the random elements in G. Such models are widely used in estimation and control theory.

2. The results of this paper could be generalized in a straightforward manner to some situations when $G(w,t)$ depends on $x(w,t)$, e.g., $G(w,t) = \tilde{G}(w,t) + H(w,t,$

$x(w, t))$ where $\tilde{G}$ satisfies the same conditions as the earlier G and H is globally bounded and uniformly Lipschitz in the third argument. This can be seen to widen considerably the class of dynamical systems to which the results are applicable.

3. The truncated process need not be considered at all and the proofs could be simplified a great deal if the noise factor $G(w, t)$ is also either uniformly bounded or non-random and bounded, for compact t-intervals. Also for (8) ((8 A)) to hold it suffices that $g(\cdot)$ belongs to $L_2(0, \infty)$ and $G(w, t)$ has finite second (fourth) moments that are bounded in t. For example $g(t) = 1/(t+1)$ will satisfy (6), (7), (8) ((8 A)) if G has bounded second (fourth) moments.

4. The models proposed in refs. 1,2,4, 5 for continuous time stochastic approximation do not consider Ito differential systems and are hence not suitable for the application to estimation theory which we have in mind (*see* Introduction).

5. If the noise term were absent (*e.g.*, $G = 0$) we would be in the purely deterministic case and our result would reduce to the following stability result in the theory of ordinary differential equations :

Theorem 4: Let $\{x(t),\ t \geq 0\}$ be a solution to

$$x(t) = x(0) - \int_0^t g(u) f(u, x(u))\, du,$$

where f obeys (1) to (3) and $g \geq 0$ (6) and (7). Then $x(t) \to \alpha$ as $t \to \infty$.

Thus our main result should be thought of as a stochastic version of the above result.

Appendix

The following lemmata were used in the proof of Proposition 1. Since no convenient reference seems to be available, we supply the proofs,

Lemma 1: Let H be a non-atomic Lebesgue-Stieltjes measure on the Borel sets of $[0, \infty)$, ϕ a Borel measurable map from $(0, \infty)$ to $(0, \infty)$ and β a finite positive number. Assume

$$w(z) = \int_\beta^z (\phi(x))^{-1}\, dx < \infty \text{ for each } z < \infty, \tag{45}$$

$$H[0, \infty) \leq \int_0^\infty (\phi(x))^{-1}\, dx = \lambda \leqslant \infty$$

$$\text{and } H[0, t] < \lambda \text{ for each } t < \infty. \tag{46}$$

Then the nonlinear integral equation

$$m(t) = \beta + \int_0^t \phi(m(u))\, H(du),\quad t \geq 0 \tag{47}$$

has a finite valued, continuous, non-decreasing solution on $[0, \infty)$.

Proof: The function $w(.)$ clearly maps $[\beta, \infty)$ onto $[0, \lambda)$, is continuous [from (45)] and is strictly increasing (since ϕ is finite valued). All this implies that w is an invertible map, *i.e.*,

$w^{-1}: [0, \lambda) \rightarrow [\beta, \infty)$ exists. Now set

$$\bar{m}(t, \beta) = w^{-1}(H[0, t)) \tag{48}$$

By the definition of w and the first part of (46), $\bar{m}$ is well defined for all $t \geq 0$. Also by (45) and the second part of (46) $\bar{m}$ is finite-valued for all finite t. Since w is strictly increasing and $H[0, t)$ is non-decreasing from (48) we have $\bar{m}$ to be non-decreasing in t. Hence $\bar{m}$ can have atmost dis-continuities of the first kind, and since w is continuous, by (48), $H[0, t)$ will have the same at the corresponding points But H is atomless and so $H[0, t)$ is continuous. Hence we have $\bar{m}(t, \beta)$ to be continuous, finite-valued and non-decreasing on $[0, \infty)$. We will now show $\bar{m}$ satisfies (47) to complete the proof.

From the properties of $\bar{m}$ proved above and (48) it is clear that $\bar{m}$ maps the measure space $([0, t), \mathcal{B}[0, t), H)$ onto $([\beta, \lambda_t'), \mathcal{B}[\beta, \lambda_t'), \bar{w})$, where $\mathcal{B}[a, b)$ is the σ-algebra of Borel sets in $[a, b)$, $\bar{w}$ is the Lebesgue-Stieltjes measure generated by $w(\cdot)$ and $\lambda_t' = w^{-1}(H[0, t))$. Then by the change of variables formula for integration with respect to Lebesgue-Stieltjes measures [*see e.g.*, ref. 6]) we have

$$\int_0^t \phi(\bar{m}(u, \beta))\, H(du) = \int_\beta^{\bar{m}(t, \beta)} \phi(y)\, \bar{w}(dy)$$

$$= \int_\beta^{\bar{m}(t, \beta)} \phi(y) \frac{1}{\phi(y)}\, dy = \bar{m}(t, \beta) - \beta, \tag{49}$$

thus showing that $\bar{m}$ solves (47).

Corollary 1: If (46) is replaced by

$$\int_\bullet^\infty (\phi(x))^{-1}\, dx = \infty, \tag{46 A}$$

Lemma 1 holds.

Proof: We have only to note that in this case any Lebesgue-Stieltjes measure satisfies (46).

We also have the following obvious.

Corollary 2: If $H[0, \infty) < \infty$ in Lemma 1 or Corollary 1, then

$$\sup_t \bar{m}(t, \beta) \leq C < \infty, \tag{50}$$

where C is dependent only on β, ϕ and H. Actually $C = w^{-1}(H[0, \infty))$. (The result follows from (48) and the definition of w).

Lemma 2. Let H, ϕ and β be as in Lemma 1. Assume further that ϕ is non-decreasing. If $m(\cdot)$ is any non-negative locally bounded Borel measurable map on $[0, \infty)$ such that

$$m(t) \leq m(0) + \int_0^t \phi(m(u))\, H(du), \; t \geq 0 \tag{51}$$

and if $m(0) < \beta$, then $m(t) < \bar{m}(t, \beta)$ for all finite $t \geq 0$, where $\bar{m}(t, \beta)$ is a solution of (47), given by (48).

Proof: Let $T = \inf\{t \geq 0 : m(t) \geq \bar{m}(t, \beta)\}$. If $T = \infty$, there is nothing to prove. Suppose $T < \infty$. Then since $m(u) < \bar{m}(u, \alpha)$ for $0 \leq u < T$, ϕ is non-decreasing and H is atomless, we must have

$$m(0) + \int_0^T \phi(m(u))\, H(du) \leq \beta - a + \int_0^T \phi(\bar{m}(u, \beta))\, H(du)$$

where (52)

$$a = \beta - m(0) > 0.$$

Now as $h \downarrow 0$, $[T, T+h]$ decreases to $[T]$, a set of H-measure zero, since H is atomless. Then, since m is locally bounded and ϕ is finite valued and non-decreasing, yielding $\phi(m(u))$ to be integrable, there exists an $\varepsilon > 0$ such that for all $0 \leqslant h < \varepsilon$.

$$\int_T^{T+h} \phi(m(u))\, H(du) < a/2 \tag{53}$$

From (51), (52), (53) we conclude

$$m(T+h) < \beta + \int_0^T \phi(\bar{m}(u, \beta))\, H(du) - a/2. \tag{54}$$

By Lemma 1

$$\beta + \int_0^T \phi(\bar{m}(u, \beta))\, H(du) = \bar{m}(T, \beta) \leqslant \bar{m}(T+h, \beta) \tag{55}$$

Combining (54) and (55) and using $a > 0$ we have

$$m(T+h) < \bar{m}(T+h, \beta) \text{ for all } 0 \leqslant h < \varepsilon$$

thus violating the definition of T, which contradiction forces T to be ∞.

The following corollary is included as a point of interest, though we do not need it.

Corollary 3. Under the hypotheses of Lemma 2, $\bar{m}(t, \beta)$ (defined by (48)) is the unique solution to (47).

Proof: Let $m(\cdot)$ be another solution to (47). Clearly $m(0) = \beta$, and by Lemma 2

$$m(t) < \bar{m}(t, \beta + \varepsilon) \text{ for all } t \text{ and each } \varepsilon > 0.$$

But $\bar{m}(t, \beta + \varepsilon)$ converges to $\bar{m}(t, \beta)$ uniformly on compact t-intervals as $\varepsilon \downarrow 0$. Hence

$$m(t) \leq \bar{m}(t, \beta) \text{ for all } t. \tag{56}$$

e solution $m(\cdot)$ of (47) being necessarily continuous, the proof of Lemma 2 could adapted to show that $\bar{m}(t) > \bar{m}(t, \beta - \varepsilon)$ for all t and $0 < \varepsilon < \beta$. Again taking nits as $\varepsilon \downarrow 0$ we get $\bar{m}(t) \geq \bar{m}(t, \beta)$, for all t, combining which with (56) we obtain e corollary.

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