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# On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions

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## Introduction

In this paper, we shall discuss on stochastic differential equations for sample functions of multi-dimensional diffusion processes with boundary conditions. On this subject, important works were given by Ikeda [2] and Skorohod [7]. Ikeda discussed the construction of two dimensional diffusion processes with Wentzell's boundary conditions on a unit disk using the known property of one dimensional reflecting Bessel processes. Skorohod discussed the stochastic differential equations for reflecting diffusion processes. Our main objective of the present paper is to unify these two works. We shall formulate the stochastic differential equations with boundary condition in Definition 1 and show the existence and the uniqueness of solutions in Theorem 1, which is our main result. The uniqueness obtained there is that in the sense of the probability law. It seems difficult to give a natural formulation of the pathwise uniqueness except some special cases. As a consequence, we can construct, in a purely probabilistic way, a class of diffusion processes with Wentzell's boundary conditions. In analytic way, such a problem has been discussed by Sato-Ueno [6] and Bony-Courrège-Priouret [1].

Let $\sigma(x)$ and $b(x)$ be defined on $R^n_+=\{x=(x_1, x_2, \cdots, x_n)\,|\,x_1\geq 0\}$, Borel measurable in $x$ such that

(i) $\sigma(x)=(\sigma^i_j(x))$, $i, j=1, 2, \cdots, n$, is an $n\times n$-matrix,

(ii) $b(x)=(b^i(x))$, $i=1, 2, \cdots, n$, is an $n\times 1$-matrix,

and $\tau(\tilde{x})$ and $\beta(\tilde{x})$ be defined on $\partial R^n_+=\{\tilde{x}=(x_2, x_3, \cdots, x_n)\}$, Borel measurable in $\tilde{x}$ such that

(iii) $\tau(\tilde{x})=(\tau^i_j(\tilde{x}))$, $i, j=2, 3, \cdots, n$, is an $(n-1)\times(n-1)$-matrix,

(iv) $\beta(\tilde{x})=(\beta^i(\tilde{x}))$, $i=2, 3, \cdots, n$, is an $(n-1)\times 1$-matrix.

We consider a stochastic differential equation of the following form;

$$(1)\qquad \begin{cases} dx^1_t=\sigma^1(x_t)\,dB_t+b^1(x_t)\,dt+d\varphi_t, \\ dx^i_t=\sigma^i(x_t)\,dB_t+b^i(x_t)\,dt+\tau^i(\tilde{x}_t)\,dM_t+\beta^i(\tilde{x}_t)\,d\varphi_t, \\ \qquad\qquad\qquad\qquad i=2, 3, \cdots, n, \end{cases}$$

where $B_t=(B^1_t, B^2_t, \cdots, B^n_t)$, $M_t=(M^2_t, M^3_t, \cdots, M^n_t)$, $\tilde{x}_t=(x^2_t, x^3_t, \cdots, x^n_t)$,

$$\sigma^i(x_t)\,dB_t=\sum_{j=1}^n \sigma^i_j(x_t)\,dB^j_t \quad \text{and} \quad \tau^i(\tilde{x}_t)\,dM_t=\sum_{j=2}^n \tau^i_j(\tilde{x}_t)\,dM^j_t.$$

Intuitively speaking, $\varphi_t$ is a non-decreasing process which increases only when the process $x_t=(x^1_t, x^2_t, \cdots, x^n_t)$ is on the boundary i.e., when $x^1_t=0$ and which causes the reflection of the process at the boundary. $\tau^i(\tilde{x})dM_t+\beta^i(\tilde{x}_t)d\varphi_t$ represents a random motion of the process $x_t$ on the boundary. Now we shall give a precise formulation of the equation (1). By a probability space with an increasing family of Borel fields $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, we mean a probability space $(\Omega, \mathscr{F}, P)$ with a system $\{\mathscr{F}_t\}_{t\in[0,\infty)}$ of sub-Borel fields of $\mathscr{F}$ such that it is increasing and right-continuous, i.e., $\mathscr{F}_t\subset\mathscr{F}_s$ if $t<s$ and $\mathscr{F}_{t+0}\equiv\bigcap_{\varepsilon>0}\mathscr{F}_{t+\varepsilon}=\mathscr{F}_t$ for every $t$.

**Definition 1.** By a solution of the equation (1)*), we mean a probability space with an increasing family of Borel fields $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$

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*) We call it also "a solution corresponding to $[\sigma, b, \tau, \beta]$".

and a family of stochastic processes $\mathfrak{X}=\{x_t=(x_t^1, x_t^2, \cdots, x_t^n),\ B_t=(B_t^1, B_t^2, \cdots, B_t^n),\ M_t=(M_t^2, \cdots, M_t^n),\ \varphi_t\}$ defined on it such that

(i) with probability one, they are all continuous in $t$ such that

$$B_0=0,\ M_0=0 \quad \text{and} \quad \varphi_0=0,$$

(ii) they are all adapted to $\mathscr{F}_t$, i.e., for each $t$, they are $\mathscr{F}_t$-measurable,

(iii) with probability one, $x_t \in R_+^n$ (i.e., $x_t^1 \geqq 0$) for all $t$ and $\varphi_t$ is non-decreasing; furthermore, $\varphi_t$ increases only when $x_t^1=0$, i.e., if $x_t^1>0$ for some $t$ then there exists $\varepsilon>0$ such that

$$\varphi_{t+\varepsilon}-\varphi_{(t-\varepsilon)\vee 0}=0,$$

(iv) $(B_t, M_t)$ is a system of $\mathscr{F}_t$-martingales such that

$$\langle B^i, B^j\rangle_t=\delta_{ij}t, \quad \langle B^i, M^j\rangle_t=0, \quad \langle M^i, M^j\rangle_t=\delta_{ij}\varphi_t{}^{*)},$$

and

(v) $\mathfrak{X}=\{x_t, B_t, M_t, \varphi_t\}$ satisfies

$$(1)' \quad \begin{cases} x_t^1-x_0^1=\displaystyle\int_0^t \sigma^1(x_s)\,dB_s+\int_0^t b^1(x_s)\,ds+\varphi_t, \\ x_t^i-x_0^i=\displaystyle\int_0^t \sigma^i(x_s)\,dB_s+\int_0^t b^i(x_s)\,ds+\int_0^t \tau^i(\tilde{x}_s)\,dM_s+\int_0^t \beta^i(\tilde{x}_s)\,d\varphi_s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=2, 3, \cdots, n, \end{cases}$$

where $\tilde{x}_t=(x_t^2, x_t^3, \cdots, x_t^n)$ and the integrals by $dB$ and by $dM$ are understood in the sense of stochastic integrals, cf. [4].

**Remark 1.** As is well known (cf. e.g. [4]), $B_t$ is an $n$-dimensional Brownian motion such that $B_t-B_s$ is independent of $\mathscr{F}_s$, $t>s$.

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*) For a system $\{X_t, Y_t\}$ of $\mathscr{F}_t$-martingales, $\langle X, Y\rangle$ is the continuous adapted process of bounded variation such that $X_t Y_t-\langle X, Y\rangle_t$ is a martingale, cf. [4].

Now we shall define the uniqueness of solutions. It is defined, as usual, in the sense of probability law.

**Definition 2.** We shall say that the uniqueness holds for (1) if, for any two solutions $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ and $\mathfrak{X}'=(x'_t, B'_t, M'_t, \varphi'_t)$ (which may be defined on different probability spaces) such that $x_0=x$ and $x'_0=x$ *a.s.* for some $x\in R^n_+$, the probability law of the processes $x_t$ and $x'_t$ on the space $\{W^+, \mathscr{B}(W^+)\}$ coincides, where $W^+$ is the Fréchet space of all $R^n_+$-valued continuous functions on $[0, \infty)$ with the compact uniform topology and $\mathscr{B}(W^+)$ is the topological Borel field on $W^+$.

**Proposition 1.** *Suppose that, for the equation* (1), *the uniqueness holds and that, for every Borel probability measure* $\mu$ *on* $R^n_+$, *a solution of* (1) *exists such that* $P[x_0\in dx]=\mu(dx)$. *Then, if* $P_x$ *is the probability law of the process* $x_t$ *such that* $x_0=x$ *a.s. which is unique by the first assumption,* $x \rightsquigarrow P_x(B)$ *is universally measurable for every* $B\in\mathscr{B}(W^+)$ *and* $\{P_x, x\in R^n_+\}$ *has the strong Markov property. In particular, for any solution of* (1) *such that* $P[x_0\in dx]=\mu(dx)$, *the probability law* $Q$ *of the process* $x_t$ *is uniquely determined and is given by* $Q(B)=\int P_x(B)\,\mu(dx)$, $B\in\mathscr{B}(W^+)$.

This can be proved in exactly the same way as Proposition 2 and Corollary 2 of [8] and hence the proof is omitted.

Now we shall discuss the existence and uniqueness of solutions of (1). The result is summarized in the following

**Theorem 1.** *Suppose* $\sigma$, $b$, $\tau$ *and* $\beta$ *are all bounded and Lipschitz continuous. Further, suppose a constant* $c>0$ *exists such that*

$$(2)\qquad |\sigma^1(x)|\equiv\Big(\sum_{j=1}^{n}\sigma^1_j(x)^2\Big)^{1/2}\geqq c.$$

*Then, for any probability law* $\mu$ *on* $R^n_+$, *a solution* $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ *of* (1) *exists such that* $P(x_0\in dx)=\mu(dx)$. *Furthermore, the uniqueness* *of solutions (cf. Definition 2) holds. Thus, $x_t$ defines a diffusion process on $R^n_+$ by Proposition 1.*

**Proof.**

(i) First, we shall consider the following special case; $\sigma^1_1(x)\equiv 1$, $\sigma^1_j(x)\equiv 0$, $j=2, 3, \cdots, n$ and $b^1(x)\equiv 0$. Then, the first equation of (1) is of the form

$$dx^1_t = dB^1_t + d\varphi_t. \tag{3}$$

By Skorohod [7] (cf. also McKean [5]), $\varphi_t$ and $x^1_t$ are uniquely determined if $x^1_0$ and $B^1_t$ are given;

$$\begin{aligned} x^1_t &= B^1_t + x^1_0, & t\leq \sigma_0 \equiv \inf\{t;\ B^1_t + x^1_0 = 0\} \\ &= B^1_t + x^1_0 - \min_{\sigma_0\leq s\leq t}[B^1_s + x^1_0], & t>\sigma_0, \end{aligned} \tag{4}$$

$$\begin{aligned} \varphi_t &= 0, & t\leq \sigma_0 \\ &= -\min_{\sigma_0\leq s\leq t}[B^1_s + x^1_0], & t>\sigma_0. \end{aligned} \tag{5}$$

We shall show that there exists an $(n-1)$-dimensional Brownian motion $\hat{B}_t$ independent of $B_t$ such that $M_t = \hat{B}_{\varphi_t}$. This implies, in particular, that the joint distribution of $(x_0, B_t, M_t)$ is uniquely determined by the distribution of $x_0$. For this, we note first that, since $\varphi_t$ is the local time of the one-dimensional reflecting Brownian motion $x^1_t$, $\lim_{t\to\infty}\varphi_t = \infty$ *a.s.*. By a general theory ([4]), $\hat{B}_t \equiv M_{\varphi_t^{-1}}$ is an $(n-1)$-dimensional Brownian motion and hence, it is sufficient to prove that $B_t$ and $\hat{B}_t$ are independent. Let $P(\,|\mathfrak{B})$ be the regular conditional distribution given $\mathfrak{B}=\{B_t;\ t\in[0, \infty)\}$. Now we have

$$E((M^i_t - M^i_s)F_1(\omega)F_2(\omega)) = 0, \qquad i=2, 3, \cdots, n, \quad t>s \tag{6}$$

where $F_1(\omega)$ is $\mathfrak{M}_s = \{M_u;\ u\in[0, s]\}$-measurable and $F_2(\omega)$ is $\mathfrak{B}$-measurable. For, by noting that $F_2(\omega)$ has an expression (cf. [3], or [4])

$$F_2(\omega)=c+\int_0^\infty \Phi_s(\omega)\,dB_s \qquad a.s.,$$

where $\Phi_s(\omega)=(\Phi_s^1, \cdots, \Phi_s^n)$ is a measurable process adapted to $\mathfrak{B}_s = \{B_t;\ t\in[0, s]\}$ and also that

$$E\left\{(M_t^i-M_s^i)\int_s^\infty \Phi_u(\omega)\,dB_u F_1(\omega)\right\}=0$$

because of $<M, B>=0$,
we have

$$E\{(M_t^i-M_s^i)F_1(\omega)F_2(\omega)\}=E\left\{(M_t^i-M_s^i)F_1(\omega)\left(c+\int_0^s \Phi_u(\omega)\,dB_u\right)\right\}=0$$

since $F_1(\omega)\left(c+\int_0^s \Phi_u(\omega)\,dB_u\right)$ is $\mathscr{F}_s$-measurable. (6) implies that

$$E\{(M_t^i-M_s^i)F_1(\omega)\,|\,\mathfrak{B}\}=0 \qquad a.s..$$

Similarly, we can prove that

$$E[\{(M_t^i-M_s^i)(M_t^j-M_s^j)-\delta_{ij}(\varphi_t-\varphi_s)\}F_1(\omega)\,|\,\mathfrak{B}]=0 \qquad a.s..$$

Thus, $\{M_t, \mathfrak{M}_t, P(\,|\mathfrak{B})\}$ is a system of martingales such that $<M^i, M^j> =\delta_{ij}\varphi_t$. This implies that $\{\hat{B}_t, P(\,|\mathfrak{B})\}$ is $(n-1)$-dimensional Brownian motion *a.s.* and hence $B_t$ and $\hat{B}_t$ are independent.

Now, we shall show that the pathwise uniqueness of solutions for (1) holds; for any two solutions $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ and $\mathfrak{X}'=(x_t', B_t', M_t', \varphi_t')$ on the same probability space, $x_0=x_0'$, $B_t\equiv B_t'$ and $M_t\equiv M_t'$ imply that $x_t\equiv x_t'$. We have remarked above that $x_0=x_0'$ and $B_t\equiv B_t'$ imply $\varphi_t=\varphi_t'$ and $x_t^1=x_t'^1$. Then, by denoting $\tilde{x}(t)=(x^2(t), \cdots, x^n(t))$ and $\tilde{x}'(t)=(x'^2(t), \cdots, x'^n(t))$,

$$x^i(t)=x^i(0)+\int_0^t \sigma^i(x^1(s), \tilde{x}(s))\,dB_s+\int_0^t b^i(x^1(s), \tilde{x}(s))\,ds$$

$$+\int_0^t \tau^i(\tilde{x}(s))\,dM_s+\int_0^t \beta^i(\tilde{x}(s))\,d\varphi_s$$

and

$$x'^i(t)=x^i(0)+\int_0^t \sigma^i(x^1(s),\ \tilde{x}'(s))\,dB_s+\int_0^t b^i(x^1(s),\ \tilde{x}'(s))\,ds$$

$$+\int_0^t \tau^i(\tilde{x}'(s))\,dM_s+\int_0^t \beta^i(\tilde{x}'(s))\,d\varphi_s \qquad i=2,\cdots,n.$$

Hence, if we set $z(t)=x(t)-\tilde{x}'(t)$, then for $i=2,\cdots,n$,

$$z^i(t)-\int_0^t [b^i(x^1(s),\ \tilde{x}(s))-b^i(x^1(s),\ \tilde{x}'(s))]\,ds$$

$$-\int_0^t [\beta^i(\tilde{x}(s))-\beta^i(\tilde{x}'(s))]\,d\varphi_s$$

$$=\int_0^t [\sigma^i(x^1(s),\ \tilde{x}(s))-\sigma^i(x^1(s),\ \tilde{x}'(s))]\,dB_s$$

$$+\int_0^t [\tau^i(\tilde{x}(s))-\tau^i(\tilde{x}'(s))]\,dM_s$$

is $\mathscr{F}_t$-martingale and hence, for any bounded $\mathscr{F}_t$-stopping time $\sigma$,

$$E\Big\{|z^i(\sigma)-\int_0^\sigma [b^i(x^1(s),\ \tilde{x}(s))-b^i(x^1(s),\ \tilde{x}'(s))]\,ds$$

$$-\int_0^\sigma [\beta^i(\tilde{x}(s))-\beta^i(\tilde{x}'(s))]\,d\varphi_s|^2\Big\}$$

$$=E\Big\{\int_0^\sigma \sum_{j=1}^n [\sigma^i_j(x^1(s).\ \tilde{x}(s))-\sigma^i_j(x^1(s),\ \tilde{x}'(s))]^2 ds$$

$$+\int_0^\sigma \sum_{j=2}^n [\tau^i_j(\tilde{x}(s))-\tau^i_j(\tilde{x}'(s))]^2 d\varphi_s\Big\}$$

$$\leqq K_1 E\Big\{\int_0^\sigma |z(s)|^2 dA_s\Big\}$$

where $A_t=t+\varphi_t$ and $K_1>0$ is a constant. Also, we have

$$\Big\{\int_0^\sigma [\beta^i(\tilde{x}(s))-\beta^i(\tilde{x}'(s))]\,d\varphi_s\Big\}^2$$

$$\leqq \int_0^\sigma [\beta^i(\tilde{x}(s)) - \beta^i(\tilde{x}'(s))]^2 d\varphi_s \cdot \varphi_\sigma$$

$$\leqq K_2 \int_0^\sigma |z(s)|^2 d\varphi_s \cdot \varphi_\sigma,$$

and similarly,

$$\left\{\int_0^\sigma [b^i(x^1(s), \tilde{x}(s)) - b^i(x^1(s), \tilde{x}'(s))] ds\right\}^2$$

$$\leqq K_3 \int_0^\sigma |z(s)|^2 ds \cdot \sigma.$$

Hence, there exists a constant $K_4 > 0$ such that

(7) $$E\{|z(\sigma)|^2\} \leqq K_4 E\left\{\int_0^\sigma |z(s)|^2 dA_s \cdot (1 + A_\sigma)\right\}.$$

Let $T > 0$ be fixed and $\sigma = A_t^{-1}$, $t \in [0, T]$, where $A_t^{-1}$ is the inverse function of $t \rightsquigarrow A_t$. Then $\sigma$ is an $\mathscr{F}_t$-stopping time such that $\sigma \leqslant t \leqslant T$ and $\varphi_\sigma \leqslant t \leqslant T$. Therefore, by (7), there exists a constant $K = K(T) > 0$ such that, for every $t \in [0, T]$,

(8) $$E\{|z(A_t^{-1})|^2\} \leqq KE\left\{\int_0^{A_t^{-1}} |z(s)|^2 dA_s\right\}$$

$$= KE\left\{\int_0^t |z(A_s^{-1})|^2 ds\right\}$$

$$= K\int_0^t E\{|z(A_s^{-1})|^2\} ds.$$

This implies that $z(t) = 0$ *a.s.* $t \in [0, A_T^{-1}]$ and, since $T$ is arbitrary, $z(t) \equiv 0$, i.e., $\tilde{x}(t) \equiv \tilde{x}'(t)$ or $x(t) \equiv x'(t)$. Thus, the pathwise uniqueness holds.

Now, the existence of solutions is shown in the following way; let $\{B(t), \hat{B}(t), x(0)\}$ be given on a probability space $(\Omega, \mathscr{F}, P)$, where $B(t)$ $(B(0) = 0)$ is an $n$-dimensional Brownian motion, $\hat{B}(t)$ $(\hat{B}(0) = 0)$ is an $(n-1)$-dimensional Brownian motion and $x(0)$ is an $R_+^n$-valued random variable such that they are mutually independent. Let $x^1(t)$ and $\varphi(t)$ be given by (4) and (5) as the unique solution of (3). Let $M_t=\hat{B}_{\varphi_t}$, $\mathscr{F}'_t$ be the Borel field generated by $\{x(0), B(s), M(s'); s\in[0, t], s'\in[0, t]\}$ and $\mathscr{F}_t=\bigcap_{\varepsilon>0}\mathscr{F}'_{t+\varepsilon}$. It is easy to see that $(B_t, M_t; \mathscr{F}_t)$ is a system of martingales which satisfies the condition (iv) of Def. 1. Let $\tilde{x}_0(t)=(x^2(0), \cdots, x^n(0))$ and define $\tilde{x}_k(t)$, $k=1, 2, \cdots$ inductively by

$$x_k^i(t)=x^i(0)+\int_0^t\sigma^i(x^1(s), \tilde{x}_{k-1}(s))\,dB_s+\int_0^t b^i(x^1(s), \tilde{x}_{k-1}(s))\,ds$$

$$+\int_0^t\tau^i(\tilde{x}_{k-1}(s))\,dM_s+\int_0^t\beta^i(\tilde{x}_{k-1}(s))\,d\varphi_s, \qquad i=2, 3, \cdots, n.$$

Then, by the same estimate as (8), we have

$$(9) \qquad E\{|\tilde{x}_k(A_t^{-1})-\tilde{x}_{k-1}(A_t^{-1})|^2\}\leq K\int_0^t E\{|\tilde{x}_{k-1}(A_s^{-1})-\tilde{x}_{k-2}(A_s^{-1})|^2\}\,ds$$

and hence, by a usual argument, $\tilde{x}(t)=\lim\tilde{x}_k(t)$ exists *a.s.*, the convergence being uniform in $t$ on each compact set. Clearly, $\mathfrak{X}=(x(t)=(x^1(t), \tilde{x}(t)), B(t), M(t), \varphi(t))$ is a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$. Also it is clear, by the way of construction, that there exists $F(x, w_1, w_2)$; $(x, w_1, w_2)\in \boldsymbol{R}_+^n\times W_1\times W_2 \rightsquigarrow F\in W_3\times W_4$, $W_1$, $W_2$, $W_3$ and $W_4$ being the space of all continuous functions $t\in[0, \infty)\rightsquigarrow w(t)\in R^n$ (resp. $R^{n-1}$, resp. $R_+^n$, resp. $R_+^1$) such that it is $\mathscr{B}(R^n)\times\mathscr{B}_t(W_1)\times\mathscr{B}_t(W_2)/\mathscr{B}_t(W_3)\times\mathscr{B}_t(W_4)$-measurable for every $t$ and

$$(10) \qquad (x_\cdot, \varphi_\cdot)=F(x_0, B_\cdot, M_\cdot) \qquad a.s..$$

By the above arguments, every solution must be given in this way and hence, the uniqueness in the sense of Def. 2 of solutions is obvious.

(ii) Now we consider the general case. We shall reduce it to the above special case by the following three transformations;

a) *a transformation of the Brownian motion.*

Let $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ be a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ corresponding to $[\sigma, b, \tau, \beta]$. Let $p(x)$, $x\in R_+^n$ is a measurable $n\times n$-orthogonal matix and set

$$\tilde{B}_t=\int_0^t p(x_s)\,dB_s, \left(\text{i.e., } \tilde{B}_t^i=\sum_{j=1}^n \int_0^t p_j^i(x_s)\,dB_s^j\right).$$

Then $\tilde{B}_t$ is an $n$-dimensional Brownian motion and it is easy to see that $\tilde{\mathfrak{X}}=(x_t, \tilde{B}_t, M_t, \varphi_t)$ is a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ corresponding to $[\tilde{\sigma}=\sigma p^{-1}, b, \tau, \beta]$.

b) *a time change.*

Let $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ be a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ corresponding to $[\sigma, b, \tau, \beta]$. Let $c(x)$, $x\in R_+^n$ be a measurable function such that $c_1\leqq c(x)\leqq c_2$ for some constants $c_2>c_1>0$. Let $A(t)=\int_0^t c(x_s)\,ds$, $\tilde{x}_t=x_{A_t^{-1}}$, $\tilde{B}_t=\int_0^t \sqrt{c}(\tilde{x}_s)\,dB_{A_s^{-1}}$, $\tilde{M}_t=M_{A_t^{-1}}$, $\tilde{\varphi}_t=\varphi_{A_t^{-1}}$ and $\tilde{\mathscr{F}}_t=\mathscr{F}_{A_t^{-1}}$. Then, $(\tilde{x}_t, \tilde{B}_t, \tilde{M}_t, \tilde{\varphi}_t)$ is a solution on $(\Omega, \mathscr{F}, P; \tilde{\mathscr{F}}_t)$ corresponding to $[\sqrt{c}^{-1}\sigma, c^{-1}b, \tau, \beta]$. This can be proved easily if we note the following general fact: if $(Y_t, \mathscr{F}_t)$ is a system of martingales such that $<Y^i, Y^j>_t =\psi^{ij}(t)$ and $A_t$ is a strictly increasing continuous process adapted to $\mathscr{F}_t$ such that $A_\infty=\infty$ *a.s.*, then $(\tilde{Y}_t=Y_{A_t^{-1}}, \mathscr{F}_{A_t^{-1}})$ is a system of martingales such that $<\tilde{Y}^i, \tilde{Y}^j>_t=\psi^{ij}(A_t^{-1})$. This fact is a direct consequence of Doob's optional sampling theorem.

c) *a transformation of the drift.*

Let $(x_t, B_t, M_t, \varphi_t)$ be a solution on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ corresponding to $[\sigma, b, \tau, \beta]$. Let $d(x)=(d^1(x), d^2(x), \cdots, d^n(x))$ be defined on $x\in R_+^n$, bounded and measurable. Let $\tilde{P}(d\omega)$ be the probability measure on $(\Omega, \mathscr{F})$ such that, for each $t$, $\tilde{P}(B)=\int_B \exp\left[\int_0^t d(x_s)\,dB_s -1/2\int_0^t |d|^2(x_s)\,ds\right]P(d\omega)$ for every $B\in\mathscr{F}_t$. Then, $\left(x_t, \tilde{B}_t=B_t -\int_0^t d(x_s)\,ds, M_t, \varphi_t\right)$ is a solution on $(\Omega, \mathscr{F}, \tilde{P}; \mathscr{F}_t)$ which corresponds to $[\sigma, \tilde{b}=b+\sigma d, \tau, \beta]$. This is well known and is called usually Girsanov's theorem.

Now suppose the coefficients $[\sigma, b, \tau, \beta]$ satisfy the condition of the theorem. Then it is easy to see that there exists an orthogonal matrix $p(x)$, Lipschitz continuous in $x$ such that $\tilde{\sigma}=\sigma\cdot p^{-1}$ has the form

$$\sigma\cdot p^{-1}(x)=\begin{pmatrix} a(x), 0, \cdots, 0 \\ *** \end{pmatrix}.$$

Since $a(x)^2=\sum_{j=1}^{n}\sigma_j^1(x)^2$, there exist positive constants $c_1$ and $c_2$ such that $c_1\leqq a(x)\leqq c_2$. Let $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ be a solution of (1) on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$. If we perform on it the transformation of the Brownian motion determined by $p$, the time change determined by $c(x)=a(x)^2$ and the transformation of the drift determined by $d(x)=(-[a(x)]^{-2}b^1(x), 0, \cdots, 0)$ successively, then we get a solution $(\tilde{x}_t, \tilde{B}_t, \tilde{M}_t, \tilde{\varphi}_t)$ on $(\Omega, \mathscr{F}, \tilde{P}; \widetilde{\mathscr{F}}_t)$ which corresponds to $[\tilde{\sigma}, \tilde{b}, \tau, \beta]$ where $[\tilde{\sigma}, \tilde{b}, \tau, \beta]$ are bounded and Lipschitz continuous such that $\tilde{\sigma}_1^1(x)\equiv 1$, $\tilde{\sigma}_j^1(x)\equiv 0$, $j=2, 3, \cdots, n$ and $\tilde{b}^1(x)\equiv 0$, that is, they satisfy the condition of the case (i). Then, as we saw in (i), the joint distribution of the process $(\tilde{x}_t, \tilde{B}_t, \tilde{M}_t, \tilde{\varphi}_t)$ is uniquely determined by giving the distribution of $\tilde{x}_0$. Since $x_t$ can be obtained from $\{\tilde{x}_t, \tilde{B}_t\}$ by the transformation of the drift determined by $-d(x)$ and then by the time change determined by $c(x)^{-1}$, the probability law of the process $x_t$ is uniquely determined by giving the distribution of $x_0$. Thus, the uniqueness in the sense of Def. 2 of solutions holds. The existence of solutions is also clear; as is shown in (i), a solution $(\tilde{x}_t, \tilde{B}_t, \tilde{M}_t, \tilde{\varphi}_t)$ corresponding to $[\tilde{\sigma}, \tilde{b}, \tau, \beta]$ exists. If we perform on it the transformation of the drift determined by $-d(x)$, the time change determined by $c(x)^{-1}$ and the transformation of the Brownian motion determined by $p^{-1}$ successively, we get a solution $(x_t, B_t, M_t, \varphi_t)$ which corresponds to $[\sigma, b, \tau, \beta]$. The theorem is completely proved.

**Remark 2.** Our result can be used to construct diffusion processes with boundary conditions on a manifold with boundary since the construction can be localized and therefore reduced to the case of the half-space.

**Remark 3.** By a formula on stochastic integrals, cf. [4], we have, for $f\in C_0^2(R_+^n)^{*)}$,

$$f(x_t)-f(x_0)=a \text{ martingale}+\int_0^t Af(x_s)\,ds+\int_0^t Lf(\tilde{x}_s)\,d\varphi_s,$$

where

$$Af(x)=\sum_{i,j=1}^n a^{ij}(x)\partial^2 f/\partial x^i\partial x^j+\sum_{i=1}^n b^i(x)\partial f/\partial x^i$$

and

$$Lf(\tilde{x})=\sum_{i,j=2}^n \alpha^{ij}(\tilde{x})\partial^2 f/\partial x^i\partial x^j+\sum_{i=2}^n \beta^i(\tilde{x})\partial f/\partial x^i+\partial f/\partial x^1$$

with

$$2a^{ij}(x)=\sum_{k=1}^n \sigma_k^i\sigma_k^j \quad\text{and}\quad 2\alpha^{ij}(\tilde{x})=\sum_{k=2}^n \tau_k^i\tau_k^j.$$

Thus we see that the infinitesimal generator of the semigroup of the diffusion process $x_t$ constructed above is an extention of the differential operator $A$ with the domain $\mathscr{D}(A)=\{f\in C_0^2(R_+^n);\ Lf|_{\partial R_+^n}=0\}$.

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*) $C_0^2(R_+^n)$=the space of all twice continuously differentiable functions on $R_+^n$ with compact support.