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On Stochastic Inequalities and Comparisons of Reliability Measures for Weighted Distributions

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Inequalities, relations and stochastic orderings, as well as useful ageing notions for weighted distributions are established. Also presented are preservation and stability results and comparisons for weighted and length-biased distributions. Relations for length-biased and equilibrium distributions as examples of weighted distributions are also presented.

Key words: Life distribution; Reliability functions; Weight function; Residual life function

1 INTRODUCTION

It is well known that if data is unknowingly sampled from the weighted distribution as opposed to the parent distribution, the survival function, hazard function and mean residual life function may be under or overestimated, depending on the weight function. For size-biased or length-biased sampling, the analyst will usually give an over optimistic estimate of the survival function and mean residual life function. Blumenthal (1967), Patel and Ord (1976), Patil and Rao (1977, 1978), Gupta and Keating (1985), among others obtained relations for reliability measures of length-biased distributions and discussed some applications of weighted distributions in general and length-biased distributions in particular.

In this paper, we derive reliability inequalities for weighted distributions in general and length-biased distributions in particular. In section 2, some basic notions and definitions useful in modeling for reliability, economic and biometry applications are given. In section 3, we establish results on inequalities and relations for reliability measures under weighted models. Closure results on mixtures in the class of new worst than used in failure rate (NWUFR) is established. Section 4 deals with stability results, inequalities and comparisons for length-biased distributions. Section 5 contains a summary and conclusion.

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2 BASIC NOTIONS AND DEFINITIONS

Let $\mathcal{F}$ be the set of absolutely continuous random variables whose distribution function F satisfy

$$F(0) = 0, \quad \lim_{x\to\infty} F(x) = 1, \quad \text{Sup}\{x: F(x) < 1\} = \infty. \tag{2.1}$$

It is clear that if the mean of a random variable in $\mathcal{F}$ is finite, it is positive. The well known survival function, hazard rate, and mean residual life functions (MRLF) are given by $\bar{F}(x) = 1 - F(x)$, $\lambda_F(x) = f(x)/\bar{F}(x)$ and

$$\delta_F(x) = E(X - t|X > t) = \int_x^\infty \frac{\bar{F}(y)\,\mathrm{d}y}{\bar{F}(x)},$$

where $f(x) = \mathrm{d}F(x)/\mathrm{d}x$ is the probability density function. The corresponding functions of a random variable Y are denoted by $\bar{G}(x)$, $\lambda_G(x)$ and $\delta_G(x)$, respectively.

It is well known that $\bar{F}(x)$, $\lambda_F(x)$ and $\delta_F(x)$ are equivalent, as

$$\bar{F}(x) = \frac{\delta_F(0)}{\delta_F(x)}\exp\left(\int_0^x \frac{\mathrm{d}y}{\delta_F(y)}\right), \quad \lambda_F(x) = \frac{\{1 + \delta_F'(x)\}}{\delta_F(x)},$$

and

$$\delta_F(x) = \int_x^\infty \frac{\bar{F}(y)\,\mathrm{d}y}{\bar{F}(x)}. \tag{2.2}$$

Let X and Y be two non-negative random variables with distribution functions F and G in $\mathcal{F}$.

DEFINITION 2.1 *We say $F <_{\text{st}} G$ or $X <_{\text{st}} Y$ if $\bar{F}(x) \geq \bar{G}(x)$, for $x \geq 0$ or equivalently, for any increasing function Ψ,*

$$E\Psi(X) \leq E\Psi(Y). \tag{2.3}$$

DEFINITION 2.2 *The mean residual life is decreasing in convex order if*

$$\int_{x+t_1}^\infty \frac{\bar{F}(y)\,\mathrm{d}y}{\bar{F}(t_1)} \geq \int_{x+t_2}^\infty \frac{\bar{F}(y)\,\mathrm{d}y}{\bar{F}(t_2)}, \quad \textit{for all } x \geq 0, \ 0 \leq t_1 \leq t_2. \tag{2.4}$$

This is denoted by $X_{t_1} \leq_c X_{t_2}$, for all $0 \leq t_1 \leq t_2$.

DEFINITION 2.3 *Let $M_F(x) = \gamma_F(x)/\bar{F}(x)$, where $\gamma_F(x) = \int_x^\infty \bar{F}(y)W'(y)\,\mathrm{d}y$, $W(x) > 0$ and $W'(x) = \mathrm{d}W(x)/\mathrm{d}x$. The weighted mean residual life function $M_F(x)$ is decreasing in convex order if*

$$\int_{x+t_1}^\infty \frac{\bar{F}(y)W'(y)\,\mathrm{d}y}{\bar{F}(t_1)} \geq \int_{x+t_2}^\infty \frac{\bar{F}(y)W'(y)\,\mathrm{d}y}{\bar{F}(t_2)}, \tag{2.5}$$

for all $x \geq 0$, $0 \leq t_1 \leq t_2$, provided $W(x)\bar{F}(x) \to 0$ as $x \to \infty$.

DEFINITION 2.4 *The specific age factor of a system at time t, specific with respect to a positive parameter s, is defined as* $A_F(t,s) = \bar{F}(t)\bar{F}(s)/\bar{F}(t,s)$, $t, s \geq 0$, *and the specific interval-average hazard rate is*

$$H_F(t,s) = t^{-1}\int_s^{s+t} \lambda_F(x)\,\mathrm{d}x, \quad t > 0, \ s \geq 0. \tag{2.6}$$

DEFINITION 2.5 *The random variable X is said to be:*

(i) *Increasing in likelihood ratio (ILR) if and only if* $f(x+t)/f(x)$ *is decreasing in* $x \geq 0$ *for all* $t \geq 0$;
(ii) *Decreasing in mean residual life (DMRL) if and only if* $\int_x^\infty \bar{F}(t)\,\mathrm{d}t/\bar{F}(x)$ *is decreasing in* $x \geq 0$;
(iii) *Increasing failure rate (IFR) if and only if* $\bar{F}(x+t)/\bar{F}(x)$ *is decreasing in* $x \geq 0$, *for every* $t \geq 0$.

The next definition is due to Loh (1984).

DEFINITION 2.6 *An absolutely continuous distribution function for which* $\lim_{x\to 0} F(x)/x$ *exist is new better than used in average failure rate (NBAFR) if*

$$\bar{\lambda}_F(0) = \lim_{x\to 0^+} x^{-1}\int_0^x \lambda_F(y)\,\mathrm{d}y \leq x^{-1}\int_0^x \lambda_F(y)\,\mathrm{d}y \quad \textit{for all } x \geq 0. \tag{2.7}$$

The inequality is reversed for new worst than used in average failure rate (NWAFR). Similarly, we say F is new better than used in failure rate (NBUFR) if $\bar{\lambda}_F(0) \leq \lambda_F(x)$ for all $x > 0$. The inequality is reversed if F is new worst than used in failure rate (NWUFR).

The next definition is due to Bhattacharjee (1986).

DEFINITION 2.7 *Let X be in* $\mathcal{F}$. *The distribution function* $F(x)$ *of X is said to be finitely and positively smooth if there exists a number* $\alpha \in (0,\infty)$ *such that*

$$\lim_{t\to\infty} \bar{F}_t(x) = \mathrm{e}^{-\alpha x} \quad \textit{for all } x \geq 0,$$

where

$$\bar{F}_t(x) = P(X > x+t \mid X > t) = \frac{\bar{F}(x+t)}{\bar{F}(x)}, \tag{2.8}$$

is the survival function of a random variable X_t *(lifetime of a device at age t,* $t \geq 0$*) and* α *the asymptotic decay coefficient of X.*

The following definition is due to Alzaid (1994).

DEFINITION 2.8 *Let X be in* $\mathcal{F}$. *If its distribution function* $F(x)$ *is finitely and positively smooth with asymptotic decay coefficient* α, *X is called*

(i) *Used better than age (UBA) if*

$$\bar{F}_t(x) \geq e^{-\alpha x} \quad \textit{for all } t, x \geq 0; \tag{2.9}$$

(ii) *Harmonic used better than aged in expectation (HUBAE) if*

$$\int_x^\infty \bar{F}(t)\,dt \geq e^{-\alpha x} \int_0^\infty \bar{F}(y)\,dy \quad \textit{for all } x \geq 0, \quad \textit{provided } \mu = \int_0^\infty \bar{F}(y)\,dy < \infty. \tag{2.10}$$

The inequalities in (2.9) *and* (2.10) *are reversed for used worse than aged (UWA) and harmonic used worse than aged in expectation (HUWAE).*

3 RELATIONS AND STOCHASTIC ORDERINGS FOR RELIABILITY MEASURES

Let X be a nonnegative random variable in $\mathcal{F}$, with distribution function F and probability density function (pdf) f. The weighted distribution of X or the pdf of the weighted random variable X_w is given by

$$f_w(x) = \frac{W(x)f(x)}{\delta^*}, \tag{3.1}$$

where δ^* is a normalizing constant. Patel and Rao (1977) referred to (3.1) as a weighted distribution with weight function $W(x) \geq 0$. The distribution given in Eq. (3.1) arises naturally when one subsamples from the original distribution with large sample size say N, given a chance proportional to $W(x)$ to observation x. In such a case, as $N \to \infty$, the subsample of fixed size n may be considered as a random sample on a random variable X_w with probability density function

$$f_w(x) = \frac{W(x)f(x)}{E(W(X))}, \tag{3.2}$$

where $0 < E(W(x)) < \infty$. In studies in reliability, biometry, renewal theory, survival analyses and wildlife populations the weighted distribution is often applicable for certain sampling plans. See Patel and Rao (1977) for a survey and their applications. In this section, we obtain inequalities and stability results on the survival function, hazard rate and mean residual life function for weighted distributions.

We also obtain closure results on mixtures of weighted distributions under NWUFR class of life distributions. The following results given by Eqs. (3.3)–(3.6) connecting the weighted distribution to the unweighted distribution can be easily obtained.

The weighted survival function of the weighted distribution function F_W is given by

$$\begin{aligned}\bar{F}_W(x) &= -\int_x^{\infty} \frac{W(t)\bar{F}'(t)\,\mathrm{d}t}{E[W(X)]} \\ &= \frac{W(x)\bar{F}(x)}{E[W(X)]} + \frac{1}{E[W(X)]}\int_x^{\infty} \bar{F}(t)W'(t)\,\mathrm{d}t \\ &= \frac{\bar{F}(x)\{W(x)+M_F(x)\}}{E[W(X)]},\end{aligned}$$

where

$$M_F(x) = \int_x^{\infty} \frac{\bar{F}(t)\bar{W}'(t)\,\mathrm{d}t}{\bar{F}(x)}, \quad \text{assuming } W(x)\bar{F}(x) \to 0 \text{ as } x \to \infty. \tag{3.3}$$

Note that if $W'(x) \geq 0$, then $M_F(x) \geq 0$ for all $x \geq 0$. The hazard function of the weighted distribution F_W is given by

$$\lambda_{F_W}(x) = \frac{W(x)f(x)}{\bar{F}(x)\{W(x)+M_F(x)\}} = \frac{W(x)\lambda_F(x)}{W(x)+M_F(x)} \tag{3.4}$$

and the mean residual life function of the weighted distribution function F_W is

$$\begin{aligned}\delta_{F_W}(x) &= \int_x^{\infty} \frac{\bar{F}_W(t)\,\mathrm{d}t}{\bar{F}_W(x)} \\ &= \int_x^{\infty} \frac{\bar{F}(t)\{W(t)+M_F(t)\}\,\mathrm{d}t}{\bar{F}(x)\{W(x)+M_F(x)\}}.\end{aligned} \tag{3.5}$$

The hazard rate $\lambda_F(x)$ in terms of $\lambda_{F_W}(x)$ is given by

$$\lambda_F(x) = \frac{\lambda_{F_W}(x)/W(x)}{\int_x^{\infty}(\lambda_{F_W}(t)/W(t))\exp(-\int_x^t \lambda_{F_W}(y)\,\mathrm{d}y)\,\mathrm{d}t} \tag{3.6}$$

The weighted specific age factor of a system at time t, specific with respect to a positive time parameter s, is given by

$$A_{F_W}(t,s) = A_F(t,s)\left[\frac{(W(t)+M_F(t))(W(s)+M_F(s))EW(t+s)}{W(t+s)+M_F(t+s))EW(t)EW(s)}\right] \tag{3.7}$$

and the weighted specific interval-average hazard rate is

$$H_{F_W}(t,s) = t^{-1}\int_s^{s+t} \frac{W(x)\lambda_F(x)}{\{W(x)+M_F(x)\}}\mathrm{d}x, \quad t>0, \ s \geq 0. \tag{3.8}$$

Remark 3.1

(i) Note that $H_{F_W}(t_2,s) \geq H_{F_W}(t_1,s)$ for all $s \geq 0$ and $t_2 \geq t_1$, if $\lambda_{F_W}(s) \leq \lambda_{F_W}(s+t)$ for all $s \geq 0, t > 0$.
(ii) Clearly, if $W(x)$ is increasing in x, then $\lambda_{F_W}(x) \leq \lambda_F(x)$ for all $x \geq 0$.
(iii) If F is IFR(DFR), and $W(x)$ is increasing and concave (decreasing and convex), then F_W is IFR(DFR).
(iv) If F is DMRL(ILR), then F_W is DMRL (ILR).

(v) If $f_W(x)$ is continous and twice differentiable then $\lambda_{F_W}(x)$ is increasing in $x \geq 0$, provided $W''(x)W(x) < (W'(x))^2$ and $f''(x)f'(x) < (f'(x))^2$, where $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$ and $f''(x) = \mathrm{d}^2f(x)/\mathrm{d}x^2$.

(vi) $M_F(x)$ is non-increasing if and only if $\bar{F}^2(x)W'(x) \geq \gamma_F(x)f(x)$, where $\gamma_F(x) = \int_x^{\infty} \bar{F}(y)W'(y)\,\mathrm{d}y$. This follows from the fact that

$$M_F'(x) = \frac{\{f(x)\gamma_F(x) - \bar{F}^2(x)W'(x)\}}{\bar{F}^2(x)}. \tag{3.9}$$

A measure of deviation of $M_F(x)$ from constant is given by

$$\begin{aligned}\beta_{F_W} &= \int_0^{\infty} \{\bar{F}^2(x)W'(x) - f(x)\gamma_F(x)\}\,\mathrm{d}x \\ &= \int_0^{\infty} (2W(x)\bar{F}(x) - \gamma_F(x))\,\mathrm{d}F(x), \text{ assuming } W(x)\bar{F}^2(x) \to 0 \text{ as } x \to \infty. \end{aligned} \tag{3.10}$$

In the following theorem, we prove consistency of test based on empirical estimates of $\bar{F}$ and γ_F.

THEOREM 3.1 *$\beta_{F_W} > 0$ and any test based on the empirical estimates of $\bar{F}$ and γ_F in Eq.* (3.10) *is consistent.*

Proof Let the weight function $W(x)$ be positive and continous. Set $g(x) = 2W(x)\bar{F}(x) - \gamma_F(x)$. Then $g(x) \geq 0$ and continous, since F is continous and has decreasing mean residual life. In fact $g(x) > 0$ for at least one x, that is $g(x_1) > 0$ for some x_1. Therefore $g(x_2) = 2W(x_2)\bar{F}(x_2) - \gamma_F(x_2) \geq g(x_1) > 0$, where $x_2 = \inf\{x | x \leq x_1, \bar{F}(x) = \bar{F}(x_1)\}$. Consequently, $\beta_{F_W} > 0$, and the test based on empirical estimates is consistent. ∎

An empirical estimate of β_F is given by $\hat{\beta}_F = \int_0^{\infty}(2W(x)\bar{F}_n(x) - \hat{\gamma}F_n(x))\,\mathrm{d}F_n(x)$, where F_n is the empirical distribution function and $\hat{\gamma}_{f_n}(x)$ an empirical estimate of $\gamma_F(x)$.

THEOREM 3.2 *Suppose* $\int_a^b \mathrm{d}F_W(x)/\bar{F}_W(x) \leq \int_{a+t}^{b+t} \mathrm{d}F_W(x)/\bar{F}_W(x)$ *for* $t > 0$, *then* $M^*_{FW}(x+t) \leq M^*_F(x)$ *for all* $x \in [a, b)$ *such that* $x + t < b$, *where* $M^*_{FW}(x) = \int_x^b \bar{F}_W(t)\,\mathrm{d}t/\bar{F}_W(x), \bar{F}_W(x) = \bar{F}(x)\{W(x) + M_F(x)\}/E[W(X)]$ *and* $M_F(x) = \int_x^{\infty} \bar{F}(t)$ $W'(t)\,\mathrm{d}t/\bar{F}(x)$.

Proof Note that

$$\begin{aligned} M^*_{F_W}(x) &= \frac{1}{\bar{F}_W(x)} \int_x^b \bar{F}_W(y)\,\mathrm{d}y \\ &= \int_x^{\infty} \frac{\bar{F}_W(y)}{\bar{F}_W(x)}\,\mathrm{d}y \\ &\geq \int_x^{b-t} \frac{\bar{F}_W(y)\,\mathrm{d}y}{\bar{F}_W(x)} \\ &\geq \int_{x+t}^{b} \frac{\bar{F}_W(s)}{\bar{F}_W(x+t)}\,\mathrm{d}s = M^*_{F_W}(x+t). \end{aligned}$$

∎

Let $\bar{H}(x) = \int_\Theta \bar{F}_\theta(x)\,\mathrm{d}K(\theta)$ be a mixture of F_θ, where K is the mixing distribution and $\{F_\theta : \theta \in \Theta\}$ is a family of distributions. If each $F_\theta(x)$ has a density $f_\theta(x)$, then the probability density function of H is given by

$$h(x) = \int_\Theta f_\theta(x)\,\mathrm{d}K(\theta) \tag{3.11}$$

THEOREM 3.3 *Suppose each $\bar{F}_\theta$ in NWUFR, $\theta \in \Theta$ and $\bar{H}$ is an arbitrary mixture of F_θ, then $\bar{H}_W$, the weighted survival function is NWUFR.*

Proof Let $\lambda_{H_W}(t)$ and $\lambda_{F_\theta}(t)$ be the hazard functions corresponding to the distribution function H_W and F_θ, respectively. We show that

$$\bar{\lambda}_{H_W}(0) \geq \frac{1}{t}\int_0^t \lambda_{H_W}(x)\,\mathrm{d}x \quad \text{for all } t > 0.$$

Note that

$$\begin{aligned}
\log \bar{H}_W(t) &= \log\left(\int_\Theta \bar{F}_W(x)\,\mathrm{d}K(\theta)\right)\\
&\geq \log\left(\int_\Theta \bar{F}(x)\,\mathrm{d}K(\theta)\right)\\
&\geq \int_\Theta \log \bar{F}(x)\,\mathrm{d}K(\theta)\\
&= \int_\Theta\left(\int_0^t -\lambda_F(x)\,\mathrm{d}x\right)\mathrm{d}K(\theta)\\
&\geq \int_\Theta\left(\int_0^t -\lambda_{F_W}(x)\,\mathrm{d}x\right)\mathrm{d}K(\theta).
\end{aligned}$$

The first inequality is due to the fact that $\bar{F}_W(x) \geq \bar{F}(x)$ for all $x > 0$. The second inequality follows from Jensen's inequality and the last inequality follows from the fact that $\lambda_F(x) \geq \lambda_{F_W}(x)$ for all $x > 0$. It follows therefore that

$$\begin{aligned}
\int_0^t \lambda_{H_W}(x)\,\mathrm{d}x &\leq \int_\Theta\left(\int_0^t \lambda_{F_W}(x)\,\mathrm{d}x\right)\mathrm{d}K(\theta)\\
&\leq t\int_\Theta \bar{\lambda}_{F_W}(0)\,\mathrm{d}K(\theta)
\end{aligned}$$

and by the dominated convergence theorem

$$\int_0^t \lambda_{H_W}(x)\,\mathrm{d}x \leq t\bar{\lambda}_{H_W}(0)$$

for all $t > 0$. Consequently, H_W is NWUFR. ■

The next result, whose proof is straightforward and omitted deals with the closure of the HUBAE class of life distributions for weighted and in particular length-biased distributions.

THEOREM 3.4 *Let $F_{W_i}(x)$ be distribution functions of weighted random variables $X_{W_i}, i = 1, 2, \ldots$, in $\mathcal{F}$ that are HUBAE with equal asymptotic decay coefficient α, then the distribution function $F_W(x) = \sum_{i=1}^{\infty} p_i F_{W_i}(x)$, $0 \leq p_i \leq 1$, $\sum_{i=1}^{\infty} p_i = 1$, is HUBAE with asymptotic decay coefficient α provided that $F_W(x)$ is finitely and positively smooth.* ■

We now discuss problem of distance between H_F and H_G where $\mu_F = \mu_G = \int_0^\infty \bar{G}(t)\,dt$. Let $H_F^*(x) = \int_0^x W(t)\,dF(t)$ and $H_G^*(x) = \int_0^x W(t)\,dG(t)$, $x \geq 0$, be bounded nondecreasing functions with $H_F^*(0) = H_G^*(0) = 0$. The Lévy distance between H_F^* and H_G^* denoted by $L(H_F^*, H_G^*)$ is the infimum of the numbers $\delta > 0$ satisfying

$$H_F^*(x-\delta) - \delta \leq H_G^*(x) \leq H_F^*(x+\delta) + \delta, \tag{3.12}$$

for all $x \geq 0$, where $W(x)$ is continuous, nonnegative, and nondecreasing on $[0, \infty)$. We consider the Lévy distance for the distributions functions of the minimum and maximum order statistic from H_F and H_G.

Let $X_W = (X_1, X_2, \ldots, X_n)$ and $Y_W = (Y_1, Y_2, \ldots, Y_n)$ be two random vectors with independent components following the distributions $\{H_{F_i}\}$ and $\{H_{G_i}\}$, respectively. Let $H_{F_{(1)}}$, $H_{F_{(n)}}$ and $H_{G_{(1)}}$, $H_{G_{(n)}}$ denote the distribution functions of the minimum and maximum of the order statistics from H_F and H_G, respectively.

THEOREM 3.5 $L\left(H_{F_{(1)}}, H_{G_{(1)}}\right) \leq \sum_{i=1}^n L\left(H_{F_i}^*, H_{G_i}^*\right)$, *and* $L\left(H_{F_{(n)}}, H_{G_{(n)}}\right) \leq \sum_{i=1}^n L\left(H_{F_i}^*, H_{Gi}^*\right)$.

Proof It is clear that

$$\prod_{i=1}^n H_{F_i} - \prod_{i=1}^n H_{G_i} \leq \sum_{\{i|H_{F_i}-H_{G_i}\geq 0\}} \{H_{F_i} - H_{G_i}\} \quad \text{for } 0 \leq H_{F_i},\ H_{G_i} \leq 1,\ \ 1 \leq i \leq n.$$

Now let δ_i decrease to $L\left(H_{F_i}^*, H_{G_i}^*\right)$, $1 \leq i \leq n$ and set $\delta = \sum_{i=1}^n \delta_i$, then $H_{F_{(1)}} - H_{G_{(1)}}(x-\delta) - \delta \leq \sum_{\{i|H_{F_i}-H_{G_i}\geq 0\}} \left\{H_{F_i}(x) - H_{G_i}(x-\delta)\right\} - \delta \leq 0$, and $H_{G_{(1)}}(x+\delta) - H_{F_{(1)}}(x) - \delta \leq 0$ for all $x \geq 0$, so that $L\left(H_{F_{(1)}}, H_{G_{(1)}}\right) \leq \sum_{i=1}^n \delta_i$. Consequently, $L\left(H_{F_{(1)}}, H_{G_{(1)}}\right) \leq \sum_{i=1}^n L\left(H_{F_i}^*, H_{G_i}^*\right)$. Similarly, $L\left(H_{F_{(n)}}, H_{G_{(n)}}\right) \leq \sum_{i=1}^n L\left(H_{F_i}^*, H_{G_i}^*\right)$. ■

4 INEQUALITIES FOR LENGTH BIASED DISTRIBUTIONS

In this section, inequalities and stability results for the length-biased distribution functions and residual life function are established.

Let X be a random variable in $\mathcal{F}$, and X_t the lifetime of a device at age $t, t \geq 0$. The survival function of X_t is as $t \to \infty, X_t$ has the survival function or equilibrium survival function given by

$$\bar{F}_e(x) = \frac{1}{\mu_F} \int_x^\infty \bar{F}(y)\,\mathrm{d}y, \quad x \geq 0, \tag{4.1}$$

and the length-biased equilibrium survival function

$$\begin{aligned}\bar{G}_e(x) &= \frac{1}{\mu_G} \int_x^\infty \bar{G}(y)\,\mathrm{d}y \\ &= \frac{1}{\mu_F^2 + \sigma_F^2} \int_x^\infty \bar{F}(y)\{y + \delta_F(y)\}\,\mathrm{d}y,\end{aligned}$$

where

$$\delta_F(y) = \int_y^\infty \frac{\bar{F}(t)\,\mathrm{d}t}{\bar{F}(y)}, \quad y \geq 0, \text{ and } \sigma_F^2 \text{ is the variance of } F. \tag{4.2}$$

In general, when $W(x) = x$, the moments of the length-biased distribution and those of the original distribution are related by

$$E_{F_W}(X^r) = \frac{E_F(X^{r+1})}{\mu_F}, \tag{4.3}$$

where E_{F_W} denotes expectation with respect to the weighted distribution. It is well known that if $r > 0$ and $E_F(X^r) < \infty$, then $E_F(X^r) = \int_0^\infty x^{r-1}\bar{F}(x)\,\mathrm{d}x$ and $\lim_{x\to\infty} x^r\bar{F}(x) = 0$ and if $r < 0$ and $E_F(X^r) < \infty$, then $E_F(X^r) = |r| \int_0^\infty x^{r-1}\bar{F}(x)\,\mathrm{d}x$ and $\lim_{x\to 0^+} x^r F(x) = 0$.

Several stochastic equivalence of ordered random variables and generalizations have been considered by various authors. See for example Barcelli and Makowski (1989), and references therein.

THEOREM 4.1 *Let $H_F(x) = 1/\mu_F \int_0^x t\,\mathrm{d}F(t)$ and $H_G(x) = 1/\mu_G \int_0^x t\,\mathrm{d}G(t)$. Suppose that X stochastically dominates Y in the second order and $E_{H_F}(X^r) = E_{H_G}(X^r) < \infty$ for some $r \geq 1$, and $\mu_F = \mu_G = \mu$, then $X =^d Y$, where $=^d$ denotes equality in distribution.*

Proof If $r \geq 1$, then

$$\begin{aligned}E_{H_F}(X^r) - E_{H_G}(X^r) &= \frac{E_F(X^{r+1})}{\mu_F} - \frac{E_G(X^{r+1})}{\mu_G} \\ &= \frac{1}{\mu}[E_F(X^{r+1}) - E_G(X^{r+1})] \\ &= \frac{r+1}{\mu}\left\{\int_0^\infty x^r(\bar{F}(x) - \bar{G}(x))\,\mathrm{d}x\right\}.\end{aligned}$$

Now

$$E_{H_F}(X^r) - E_{H_G}(X^r) = 0 \text{ implies}$$
$$\frac{r+1}{\mu}\left\{\int_0^\infty x^r\big(\bar{F}(x) - \bar{G}(x)\big)\,\mathrm{d}x\right\} = 0.$$

That is,

$$\frac{r+1}{\mu}\int_0^\infty\left(\int_0^x rt^{r-1}\,\mathrm{d}t\right)\big(\bar{F}(x) - \bar{G}(x)\big)\,\mathrm{d}x = 0, \tag{4.4}$$

and

$$\frac{r(r+1)}{\mu}\int_0^\infty t^{r-1}\left(\int_t^\infty(\bar{F}(x) - \bar{G}(x))\,\mathrm{d}x\right)\mathrm{d}t = 0,$$

by Fubini's Theorem. (4.5)

Using the fact that X stochastically dominates Y in the second order and (4.5), we have

$$\int_t^\infty(\bar{F}(x) - \bar{G}(x))\,\mathrm{d}x = 0, \quad \text{and } \bar{F}(x) = \bar{G}(x) \text{ for } x > 0. \qquad \blacksquare$$

Next, we obtain inequalities for the length-biased residual life and equilibrium distribution functions. Let the distribution function F possess moment of order J, that is $\mu_j = E(X^j)$, $j = 1, 2, \ldots, J$. Let $\{S_j(x)\}, j = 0, 1, 2, \ldots, J$, be a sequence of decreasing functions given by

$$S_j(x) = \begin{cases} \bar{F}(x), & j = 0, \\ \displaystyle\int_0^\infty \bar{F}(x+t)\frac{t^{j-1}}{(j-1)!}, & j = 1, 2, \ldots, J. \end{cases} \tag{4.6}$$

We let $S_{-1}(x) = f(x)$ be the pdf of F if it exists. Then $S_j(0) = \mu_j/j!$, $S_j'(x) = -S_{j-1}(x), j = 0, 1, 2, \ldots, J$. The ratio $S_{j-1}(x)/S_j(x)$ is a hazard function of a distribution function with survival function $S_j(x)/S_j(0)$. The following is a modified version of the Lemma given by Barlow *et al.* (1963).

LEMMA (Barlow *et al.* (1963)) *If F has decreasing mean residual life (DMRL), then*

$$S_k(x) \le S_k(0)\,\mathrm{e}^{-x/\mu}, \quad k = 1, 2, \ldots, \quad \textit{and}$$
$$S_k(x) \ge \mu S_{k-1}(0)\,\mathrm{e}^{-x/\mu} - \mu S_{k-1}(0) + S_k(0), \qquad k = 2, 3, \ldots. \tag{4.7}$$

The inequalities are reversed if F has increasing mean residual life (IMRL).

THEOREM 4.2 *If $\bar{G}_e$ has increasing hazard rate (IHR), then for $\delta \ge 0$,*

$$\int_0^\infty |\bar{G}_e(x) - x\,\mathrm{e}^{-x/\mu}|\,\mathrm{d}x \le 2\delta, \quad \textit{where } \delta = \mu^2 - \frac{\mu^2}{2\mu}.$$

Proof Let $A = \{x|\bar{G}_e(x) \le x \exp\{-x/\mu\}\}$. Then for $x > 0$, we have

$$
\begin{aligned}
&\int_0^\infty |\bar{G}_e(x) - x\exp\{-x/\mu\}|\,\mathrm{d}x \\
&= \int_A \{x\exp\{-x/\mu\} - \bar{G}_e(x)\}\,\mathrm{d}x - \int_{A^c} \{x\exp\{-x/\mu\} - \bar{G}_e(x)\}\,\mathrm{d}x \\
&\le 2\int_A \{x\exp\{-x/\mu\} - \bar{G}_e(x)\}\,\mathrm{d}x \\
&\le 2\int_0^\infty \{x\exp\{-x/\mu\} - \bar{F}_e(x)\}\,\mathrm{d}x \\
&= 2\int_0^\infty \left\{x\exp\{-x/\mu\} - \left(\frac{1}{\mu_F}\int_x^\infty \bar{F}(y)\,\mathrm{d}y\right)\right\}\mathrm{d}x \\
&= 2\int_0^\infty \left\{x\exp\{-x/\mu\} - \left(\frac{1}{\mu}\int_x^\infty \bar{F}(y)\,\mathrm{d}y\right)\right\}\mathrm{d}x \\
&= 2\int_0^\infty \{x\exp\{-x/\mu\} - S_1(x)/\mu\}\,\mathrm{d}x \\
&= 2(\mu^2 - S_2(0)/\mu) \\
&= 2\left(\mu^2 - \frac{\mu_2}{2\mu}\right) = 2\delta.
\end{aligned}
$$

■

PROPOSITION 4.1 *Let X be in $\mathcal{F}$. If X_ℓ is HUBAE, then the length biased random variable X_ℓ or its distribution function G is HUBAE.*

Proof The results follows from the fact that

$$\bar{G}(x) = \frac{\bar{F}(x)\{x + \delta_F(x)\}}{\mu_F} \ge \bar{F}(x) \quad \text{for all } x \ge 0,$$

where

$$\delta_F(x) = \frac{1}{\bar{F}(x)}\int_x^\infty \bar{F}(t)\,\mathrm{d}t.$$

Consequently,

$$
\begin{aligned}
\int_x^\infty \bar{G}(t)\,\mathrm{d}t &\ge \int_x^\infty \bar{F}(t)\,\mathrm{d}t \\
&\ge \mathrm{e}^{-\alpha x}\int_0^\infty \bar{F}(t)\,\mathrm{d}t, \quad \text{for all } x \ge 0.
\end{aligned}
$$

■

The next theorem shows that if the length-biased random variable is UWA, then the original random variable is also UWA.

THEOREM 4.3 *If* $\bar{G}(x+t) \le \bar{G}(t)\ \mathrm{e}^{\alpha x}$ *for all* $x, t \ge 0$, *then* $\bar{F}(x+t) \le \bar{F}(t)\,\mathrm{e}^{-\alpha x}$ *for all* $x, t \ge 0$, *where* $\bar{G}(t) = 1/\mu_F \int_t^{\infty} x\,\mathrm{d}F(x)$.

Proof $\bar{G}(x+t) \le \bar{G}(t)\,\mathrm{e}^{-\alpha x}$, for all $x, t \ge 0$ is equivalent to

$$\bar{F}(x+t)\{x+t+\delta_F(x+t)\}/\mu_F \le \bar{F}(t)\{t+\delta_F(t)\}\mathrm{e}^{-\alpha x}/\mu_F, \quad \text{for all } x, t \ge 0,$$

$$\Longleftrightarrow \frac{\bar{F}(x+t)}{\bar{F}(t)} \le \frac{t+\delta_F(t)}{x+t+\delta_F(x+t)}\mathrm{e}^{\alpha x}, \quad \text{for all } x, t \ge 0,$$

$$\le \mathrm{e}^{-\alpha x}, \quad \text{for all } x \ge 0. \qquad \blacksquare$$

The last inequality follows from the fact that $(y + \delta_F(y))$ is an increasing function of y. This is due to the fact that for $0 \le y_1 < y_2$, we have

$$\begin{aligned} y_2 + \delta_F(y_2) - (y_1 + \delta_F(y_1)) = \frac{1}{\bar{F}(y_1)\bar{F}(y_2)} &\{(y_2 - y_1)\bar{F}(y_2)\bar{F}(y_1) \\ &+ (\bar{F}(y_1) - \bar{F}(y_2)) \int_{y_2}^{\infty} \bar{F}(x)\,\mathrm{d}x \\ &- \bar{F}(y_2) \int_{y_1}^{y_2} \bar{F}(x)\,\mathrm{d}x \ge 0 \end{aligned}$$

by noting that,

$$\bar{F}(y_1)\bar{F}(y_2)(y_2 - y_1) \ge \bar{F}(y_2) \int_{y_1}^{y_2} \bar{F}(x)\,\mathrm{d}x. \qquad \blacksquare$$

THEOREM 4.4 *Let* G *be in* $\mathcal{F}$ *with finite mean* μ_G. *If* G *is HUBAE with asymptotic decay coefficient* α, *then* $\mu_G \ge 1/\alpha$.

Proof Using the fact that G is HUBAE we have

$$\begin{aligned} \mu_G &= \int_0^{\infty} \bar{G}(t)\,\mathrm{d}t \\ &= \int_0^{x} \bar{G}(t)\,\mathrm{d}t + \int_x^{\infty} \bar{G}(t)\,\mathrm{d}t \\ &\ge \int_0^{x} \bar{G}(t)\,\mathrm{d}t/(1 - \mathrm{e}^{-\alpha x}) \quad \text{for all } x > 0, \\ &\ge \int_0^{x} \bar{F}(t)\,\mathrm{d}t/(1 - \mathrm{e}^{-\alpha x}) \quad \text{for all } x > 0. \end{aligned}$$

Now

$$\mu_G \geq \lim_{x\to 0^+} \frac{\bar{G}(x)}{\alpha\, \mathrm{e}^{-\alpha x}} \geq \lim_{x\to 0^+} \frac{\bar{F}(x)}{\alpha\, \mathrm{e}^{-\alpha x}} = \frac{1}{\alpha}. \qquad \blacksquare$$

5 CONCLUSION

In this article, we have obtained useful inequalities and established relations for weighted distributions function in general, and in particular length-biased distribution functions, and their unweighted counterparts. Results on mixtures of NWUFR and HUBAE classes are established for weighted distributions. Finally, some basic results on Levy distance and reliability measures for weighted and length-biased models are presented.

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