On Stokes' problem for the flow of a third-grade fluid induced by a variable shear stress

S. Asghar, Muhammad R. Mohyuddin, P. Donald Ariel, and T. Hayat

Abstract: The flow of an incompressible third-grade fluid over an infinite wall is considered. The flow is due to a variable shear stress. Both the series and the numerical solutions of the nonlinear partial-differential equation resulting from the momentum equation are obtained. Effects of non-Newtonian parameters on the flow phenomena are analyzed. It is found that with an increase in second-grade parameter and third-grade parameter, the velocity decreases and thus, the boundary-layer thickness increases.

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Résumé : Nous étudions l'écoulement d'un fluide incompressible de troisième catégorie au dessus d'une surface semi-infinie. Le flot est causé par un effort de déchirement variable. Nous obtenons les solutions en série et numériques de l'équation différentielle non linéaire aux dérivépartielles résultant de l'équation des moments. Nous analysons les effets des paramètres non newtoniens sur le flot. Nous trouvons qu'une augmentation des paramètres de seconde et de troisième catégories résulte en une diminution de la vitesse et une augmentation de l'épaisseur de la couche frontière.

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1. Introduction

The fluid motion induced because of the motion of a flat plate, also named the Stokes' problem [1], occurs in many applied problems [2]. Interesting solutions of this problem for a Newtonian fluid have been obtained by Zierep [3]. Soundalgekar [4], Puri [5], Bandelli et al. [6], Tigoiu [7], Rajagopal and Na [8], Fetecau and Zierep [9], Fetecau and Fetecau [10] have studied the problems for various non-Newtonian fluid models. Recently, Asghar et al. [11] studied the flow of a third-grade fluid on a porous plate executing oscillations in its own plane with superimposed injection or suction.

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There have been several studies on flows of non-Newtonian fluids due to their technological significance and also in view of the interesting mathematical challenges because of the nonlinearities in the constitutive equations. Owing to the complexity of fluids, many models of non-Newtonian fluids are proposed. Amongst those there is one subclass, namely, viscoelastic fluids that have been studied extensively. Related studies are given in refs. 12–22.

Although the second-grade fluid model is able to predict the normal stress differences that are characteristics of non-Newtonian fluids, it does not take into account the shear thinning and thickening phenomena that many fluids show. The third-grade model represents a further, although inconclusive, attempt toward a more comprehensive description of the behaviour of viscoelastic fluids. With this in view, the model in the present work is a third-grade fluid and its flow over an infinite flat plate is considered. The governing equation is a nonlinear third-order partial-differential equation. The flow is induced by a variable shear stress dependent on time, which, incidentally, also makes the boundary conditions nonlinear.

2. Basic equations

The stress in a third-grade fluid is given by [23]

$$\mathsf{T} = -p\boldsymbol{i} + \mu\mathsf{A}_1 + \alpha_1\mathsf{A}_2 + \alpha_2\mathsf{A}_1^2 + \beta_1\mathsf{A}_3 + \beta_2\left(\mathsf{A}_2\mathsf{A}_1 + \mathsf{A}_1\mathsf{A}_2\right) + \beta_3\left(\mathrm{tr}\,\mathsf{A}_2\right)\mathsf{A}_1 \tag{2.1}$$

in which T is the stress tensor; p is the scalar pressure; μ is the coefficient of viscosity, and α_1, α_2, β_1, β_2, and β_3 are the material moduli. The kinematic tensors A_1, A_2, and A_3 are defined by

$$\mathsf{A}_1 = (\mathrm{grad}\,\boldsymbol{V}) + (\mathrm{grad}\,\boldsymbol{V})^\top \tag{2.2}$$

$$\mathsf{A}_n = \frac{\mathrm{d}\mathsf{A}_{n-1}}{\mathrm{d}t} + \mathsf{A}_{n-1}\,(\mathrm{grad}\,\boldsymbol{V}) + (\mathrm{grad}\,\boldsymbol{V})^\top\,\mathsf{A}_{n-1}, \qquad n = 2, 3 \tag{2.3}$$

where $\boldsymbol{V}$ denotes the velocity field and $\mathrm{d}/\mathrm{d}t$ the material time derivative. A detailed thermodynamic analysis of the model, represented by (2.1) is given by Fosdick and Rajagopal [23]. They show that if all the motions of the fluid are to be compatible with thermodynamics in the sense that these motions meet the Clausius–Duhem inequality and if it is further assumed that the specific Helmholtz free energy is a minimum when the fluid is locally at rest, then

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \qquad \beta_1 = \beta_2 = 0, \qquad \beta_3 \geq 0 \tag{2.4}$$

Therefore, the constitutive relation for a thermodynamically compatible fluid of third-grade becomes

$$\mathsf{T} = -p\boldsymbol{i} + \mu\mathsf{A}_1 + \alpha_1\mathsf{A}_2 + \alpha_2\mathsf{A}_1^2 + \beta_3\left(tr\mathsf{A}_2\right)\mathsf{A}_1 \tag{2.5}$$

If we put $\beta_1 = \beta_2 = \beta_3 = 0$ in (2.1), we obtain the model for second grade fluid.

3. Statement of the problem

We consider a third-grade fluid over an infinite plate placed along the x-axis, and choose the y-axis to be perpendicular to the plate. The plate is under a variable shear stress with magnitude $c\tau(t)$ where c is a constant having the dimension ρU_0 (ρ is the density and U_0 is some reference velocity). The basic governing equations are the conservation of mass and linear momentum. These are

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\boldsymbol{V}) = 0 \tag{3.1}$$

$$\rho\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \mathrm{div}\mathsf{T} + \rho\boldsymbol{b} \tag{3.2}$$

where $\boldsymbol{b}$ is the body force. For an incompressible flow, (3.1) reduces to

$$\operatorname{div}\boldsymbol{V} = 0 \tag{3.3}$$

For the problem under consideration the velocity field is assumed to be

$$\boldsymbol{V} = u\,(y, t)\,\boldsymbol{i} \tag{3.4}$$

where u and $\boldsymbol{i}$ are the velocity and unit vector in the x-coordinate direction, respectively.

The boundary conditions on the flow are

$$\left[\mu \frac{\partial u}{\partial y} + \alpha_1 \frac{\partial^2 u}{\partial y \partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^3\right]_{y=0} = c\tau\,(t)\,, \qquad t > 0 \tag{3.5}$$

$$u(y, t) \to 0 \quad \text{as} \quad y \to \infty \tag{3.6}$$

Using (3.4), (3.3) is identically satisfied and (3.2) in the absence of body forces yields

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3 \left[\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2\right] \tag{3.7}$$

Introducing the nondimensional parameters

$$\overline{\alpha}_1 = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \qquad \varepsilon = \frac{6\beta_3 U_0^4}{\rho \nu^3}, \qquad \overline{u} = \frac{u}{U_0}, \qquad \overline{t} = \frac{U_0^2 t}{\nu}, \qquad \eta = \frac{U_0}{\nu} y \tag{3.8}$$

in (3.7) and the boundary conditions (3.5) and (3.6), and omitting the bars for simplicity we get

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u}{\partial \eta^2 \partial t} + \varepsilon \left[\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial u}{\partial \eta}\right)^2\right] \tag{3.9}$$

$$\left[\frac{\partial u}{\partial \eta} + \alpha_1 \frac{\partial^2 u}{\partial \eta \partial t} + \frac{1}{3}\varepsilon \left(\frac{\partial u}{\partial \eta}\right)^3\right]_{\eta=0} = \tau\,(t)\,, \quad t > 0 \tag{3.10}$$

$$u(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty \tag{3.11}$$

In this paper, we consider two cases (*i*) $\tau\,(t) = \mathrm{e}^{\lambda t}$ (λ is real and positive constant) and (*ii*) $\tau\,(t) = \mathrm{e}^{i\omega t}$ (ω is the imposed frequency). In the former case, since λ is positive, it is prudent to obtain a numerical solution besides an analytical solution in the form of a perturbation series in terms of ε. On the other hand, in the latter case, since the solution is essentially bounded a perturbation solution should, therefore, give acceptable results.

4. Solution for case 1: $\tau\,(t) = \mathrm{e}^{\lambda t}$, λ is purely real (acceleration)

4.1. Numerical solution

In problems of this type, usually no initial condition is given at $t = 0$. For example, for a second grade fluid ($\varepsilon = 0$) Hayat et al. [19] and Rajagopal [24] derived the analytical solutions for a number of unsteady unidirectional flow problems, without using any initial condition. The initial condition(s), if needed to be derived, can be obtained from the solution.

Because of the nonlinearity introduced on account of the third-grade-fluid parameter, it is not feasible, in general, to obtain a closed-form analytical solution and a numerical solution naturally enters into the consideration. For the latter, it appears that an initial condition must be prescribed at $t = 0$.

However, as Ariel [22] recently demonstrated in an analogous situation, the initial condition can be deduced if appropriate transformations are used.

We choose

$$u(\eta, t) = e^{\lambda t} f(\eta, t) \tag{4.1}$$

so that the differential equation for f takes the form

$$\frac{\partial f}{\partial t} + \lambda f = \frac{\partial^2 f}{\partial \eta^2} + \alpha_1 \left(\frac{\partial^3 f}{\partial t \partial \eta^2} + \lambda \frac{\partial^2 f}{\partial \eta^2} \right) + \varepsilon\, e^{2\lambda t} \left(\frac{\partial f}{\partial \eta} \right)^2 \frac{\partial^2 f}{\partial \eta^2} \tag{4.2}$$

and the boundary conditions become

$$(1 + \alpha_1 \lambda) \frac{\partial f(0, t)}{\partial \eta} + \frac{1}{3} \varepsilon\, e^{2\lambda t} \left[\frac{\partial f(0, t)}{\partial \eta} \right]^3 = 1, \qquad f(\infty, t) = 0 \tag{4.3}$$

Next we introduce the transformation

$$\xi = e^{2\lambda t} \tag{4.4}$$

which leads us the boundary-value problem

$$2\lambda \xi \frac{\partial f}{\partial \xi} + \lambda f = (1 + \alpha_1 \lambda) \frac{\partial^2 f}{\partial \eta^2} + 2\alpha_1 \lambda \xi \frac{\partial^3 f}{\partial \xi \partial \eta^2} + \varepsilon \xi \left(\frac{\partial f}{\partial \eta} \right)^2 \frac{\partial^2 f}{\partial \eta^2} \tag{4.5}$$

$$(1 + \alpha_1 \lambda) \frac{\partial f(0, \xi)}{\partial \eta} + \frac{1}{3} \varepsilon \xi \left[\frac{\partial f(0, \xi)}{\partial \eta} \right]^3 = 1, \qquad f(\infty, \xi) = 0 \tag{4.6}$$

Equation (4.5) only has the boundary conditions at $\eta = 0$ and $\eta = \infty$, but not the initial condition at $\xi = 0$. But if we make the reasonable assumption that f is regular at $\xi = 0$, we do not need the initial condition to get the integration started at $\xi = 0$. Equation (4.5) can thus be integrated in the entire domain $0 \leq \xi < \infty \cap 0 \leq \eta \leq \infty$. When one reaches $\xi = 1$, the initial condition is recovered for the problem. Now either the original (3.9) can be integrated in the usual manner, or the integration can be carried further beyond $\xi = 1$ by (4.5). We have chosen the latter approach in the present work.

Equation (4.6) can be integrated with respect to ξ from ξ_i to ξ_{i+1}, integrating the terms that can be readily integrated and replacing the remaining integrals by their numerical equivalent using the trapezoidal rule. The resulting equation can then be solved for f at $\xi = \xi_{i+1}$ by treating the equation as a boundary-value problem in the η-domain. The details of the integration scheme have been furnished in Ariel [22] and are omitted here, except that in the present work, the situation is slightly complicated on account of the boundary condition at $\eta = 0$. Now we have

$$(1 + \alpha_1 \lambda) \frac{\partial f(0, \xi)}{\partial \eta} + \frac{1}{3} \varepsilon \xi \left[\frac{\partial f(0, \xi)}{\partial \eta} \right]^3 = 1 \tag{4.7}$$

which is cubic in $\partial f(0, \xi) / \partial \eta$, that must be solved for each value of ξ. Also at $\xi = 0$, the solution for f is

$$f(\eta, 0) = -\frac{1}{\sqrt{\lambda}\sqrt{1 + \lambda \alpha_1}} \exp\left(-\sqrt{\frac{\lambda}{1 + \lambda \alpha_1}}\, \eta \right) \tag{4.8}$$

In Table 1, the values of $u(0)$, the velocity of the fluid at the plate, are presented for $\lambda = 0.5$, $\alpha = 0.2$, 0.5, 1, and 2, and for various values of t. Also in Table 1, the corresponding values of $u(0)$ obtained by the perturbation technique are given. The latter is described in the next section.

Table 1. Illustrating the variation of $u(0)$, the velocity at the plate with α_1, the viscoelastic fluid parameter and ε, the second-grade fluid parameter for $\lambda = 0.5$ using (*i*) the exact numerical solution and (*ii*) the perturbation solution.

$u(0)$											
$\lambda = 0.5$											
		$t = 0$		$t = 1$		$t = 2$		$t = 3$		$t = 5$	
α_1	ε	Exact	Perturbation	Exact	Perturbation	Exact	Perturbation	Exact	Perturbation	Exact	Perturbation
0.2	0.1	−1.339476	−1.342384	−2.186549	−2.197751	−3.529042	−3.570781	−5.598185	−5.853466	−13.222440	−60.142478
	0.2	−1.3314858	−1.3367778	−2.1579308	−2.1773578	−3.4432068	−3.5370308	−5.3843428	−6.4045018	−12.355852	−213.784337
	0.5	−1.311407	−1.322416	−2.095671	−2.146119	−3.282987	−3.800568	−5.033027	−12.501669	−11.154726	−1334.267421
	1	−1.285830	−1.306674	−2.027334	−2.193869	−3.128698	−5.455769	−4.727634	−37.477161	−10.240992	−5400.264261
0.5	0.1	−1.257607	−1.261516	−2.055104	−2.070731	−3.322842	−3.377852	−5.285384	−5.465808	−12.620225	−25.166529
	0.2	−1.250940	−1.258241	−2.030566	−2.057429	−3.247671	−3.335021	−5.097938	−5.478324	−11.854309	−66.928162
	0.5	−1.233809	−1.249130	−1.976187	−2.026241	−3.107124	−3.312736	−4.790754	−6.809757	−10.776368	−384.2423041
	1	−1.211545	−1.236331	−1.916194	−2.003322	−2.972230	−3.629622	−4.522355	−13.341755	−9.942527	−1553.196693
1	0.1	−1.149875	−1.153137	−1.883191	−1.896866	−3.056709	−3.108968	−4.885331	−5.055345	−11.798322	−14.513506
	0.2	−1.145326	−1.151598	−1.865411	−1.890251	−2.997434	−3.082781	−4.727786	−4.980096	−11.147837	−21.551781
	0.5	−1.133065	−1.147127	−1.823182	−1.872203	−2.880334	−3.026095	−4.466168	−5.020847	−10.224618	−82.218010
	1	−1.116092	−1.140168	−1.773710	−1.848109	−2.765489	−3.004536	−4.238184	−5.977082	−9.498322	−315.166407
2	0.1	−0.997836	−0.999508	−1.639238	−1.646527	−2.678170	−2.708566	−4.324706	−4.439598	−10.638175	−11.632051
	0.2	−0.9957281	−0.9990191	−1.6303941	−1.6443631	−2.6443891	−2.6992271	−4.2169871	−4.4021041	−10.107024	−11.763908
	0.5	−0.9897228	−0.9975668	−1.6069828	−1.6380588	−2.5669318	−2.6734758	−4.0154818	−4.3172068	−9.3598608	−16.253280
	1	−0.980624	−0.995196	−1.575504	−1.628170	−2.480478	−2.638103	−3.830112	−4.267656	−8.773695	−37.381574

4.2. Perturbation solution

We perturb the velocity u in ε as follows [8]:

$$u\,(\eta, t; \varepsilon) = u_0\,(\eta, t) + \varepsilon u_1\,(\eta, t) + \varepsilon^2 u_2\,(\eta, t) + \cdots \tag{4.9}$$

For $\varepsilon = 0$, (4.9) gives an exact solution for the reduced problem corresponding to a second-grade fluid. Using (4.9) in (3.9) and conditions (3.10) and (3.11), we obtain the following systems:

Zeroth-order system

$$\frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u_0}{\partial \eta^2 \partial t} \tag{4.10}$$

$$\left[\frac{\partial u_0}{\partial \eta} + \alpha_1 \frac{\partial^2 u_0}{\partial \eta \partial t}\right]_{\eta=0} = \mathrm{e}^{\lambda t}, \qquad t > 0 \tag{4.11}$$

$$u_0\,(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty \tag{4.12}$$

First-order system

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u_1}{\partial \eta^2 \partial t} + \frac{\partial^2 u_0}{\partial \eta^2}\left(\frac{\partial u_0}{\partial \eta}\right)^2 \tag{4.13}$$

$$\left[\frac{\partial u_1}{\partial \eta} + \alpha_1 \frac{\partial^2 u_1}{\partial \eta \partial t} + \frac{1}{3}\left(\frac{\partial u_0}{\partial \eta}\right)^3\right]_{\eta=0} = 0 \tag{4.14}$$

$$u_1\,(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty \tag{4.15}$$

Second-order system

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u_2}{\partial \eta^2 \partial t} + \frac{\partial^2 u_1}{\partial \eta^2}\left(\frac{\partial u_0}{\partial \eta}\right)^2 + 2\frac{\partial u_0}{\partial \eta}\frac{\partial u_1}{\partial \eta}\frac{\partial^2 u_0}{\partial \eta^2} \tag{4.16}$$

$$\left[\frac{\partial u_2}{\partial \eta} + \alpha_1 \frac{\partial^2 u_2}{\partial \eta \partial t} + \left(\frac{\partial u_0}{\partial \eta}\right)^2 \frac{\partial u_1}{\partial \eta}\right]_{\eta=0} = 0 \tag{4.17}$$

$$u_2\,(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty \tag{4.18}$$

On defining the transformations

$$u_0\,(\eta, t) = f_0\,(\eta)\,\mathrm{e}^{\lambda t}, \qquad u_1\,(\eta, t) = f_1\,(\eta)\, e^{3\lambda t}, \qquad u_2\,(\eta, t) = f_2\,(\eta)\,\mathrm{e}^{5\lambda t} \tag{4.19}$$

the zeroth-, first-, and second-order systems become

$$(1 + \lambda\alpha_1)\, f_0''\,(\eta) - \lambda f_0 = 0 \tag{4.20}$$

$$(1 + \lambda\alpha_1)\, f_0'\,(0) = 1 \tag{4.21}$$

$$f_0\,(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{4.22}$$

$$(1 + 3\lambda\alpha_1)\, f_1''\,(\eta) - 3\lambda f_1 = -\left(f_0'\right)^2 f_0'' \tag{4.23}$$

$$(1 + 3\lambda\alpha_1)\, f_1'(0) + \frac{1}{3}\left[f_0'(0)\right]^3 = 0 \tag{4.24}$$

$$f_1(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{4.25}$$

$$(1+5\lambda\alpha_1)\, f_2''(\eta) - 5\lambda f_2 = -\left(f_0'\right)^2 f_1'' - 2 f_0' f_0'' f_1' \tag{4.26}$$

$$(1+5\lambda\alpha_1)\, f_2'(0) + \left[f_0'(0)\right]^2 f_1'(0) = 0 \tag{4.27}$$

$$f_2(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{4.28}$$

where a prime indicates the derivative with respect to η.

The solutions of the zeroth-, first-, and second-order systems are given, respectively, by

$$f_0(\eta) = -A_0 c_0\, \mathrm{e}^{-c_0\eta} \tag{4.29}$$

$$f_1(\eta) = A_1\left(c_1\, \mathrm{e}^{-c_1\eta} - 3c_0\, \mathrm{e}^{-3c_0\eta}\right) \tag{4.30}$$

$$f_2(\eta) = A_2\left(c_2\, \mathrm{e}^{-c_2\eta} - 5c_0\, \mathrm{e}^{-5c_0\eta}\right) + B_2\left[c_2\, \mathrm{e}^{-c_2\eta} - (2c_0+c_1)\, \mathrm{e}^{-(2c_0+c_1)\eta}\right] \tag{4.31}$$

where

$$A_0 = \frac{1}{\lambda}, \qquad c_0 = \sqrt{\frac{\lambda}{1+\lambda\alpha_1}}, \qquad A_1 = -\frac{A_0^3 c_0^4}{18\,(1+4\lambda\alpha_1)}, \qquad c_1 = \sqrt{\frac{3\lambda}{1+3\lambda\alpha_1}}$$

$$A_2 = \frac{-9A_0^2 A_1 c_0^4}{20\,(1+6\lambda\alpha_1)}, \qquad B_2 = \frac{A_0^2 A_1 c_0^4 c_1^2}{(2c_0+c_1)^2\,(1+5\lambda\alpha_1) - 5\lambda}, \qquad c_2 = \sqrt{\frac{5\lambda}{1+5\lambda\alpha_1}}$$

The expression for *skin friction* is given as

$$\tau_1 = \frac{\overline{\tau}_1}{\rho U_0^2} = \left[\mathrm{e}^{\lambda t} f_0'(0) + \varepsilon\, \mathrm{e}^{3\lambda t} f_1'(0) + \varepsilon^2\, \mathrm{e}^{5\lambda t} f_2'(0) \cdots\right] \tag{4.32}$$

From (4.29)–(4.31) we easily obtain

$$f_0'(\eta) = A_0 c_0^2\, \mathrm{e}^{-c_0\eta} \tag{4.33}$$

$$f_1'(\eta) = A_1\left(-c_1^2\, \mathrm{e}^{-c_1\eta} + 9c_0^2\, \mathrm{e}^{-3c_0\eta}\right) \tag{4.34}$$

$$f_2'(\eta) = A_2\left(-c_2^2\, \mathrm{e}^{-c_2\eta} + 25c_0^2\, \mathrm{e}^{-5c_0\eta}\right) + B_2\left[-c_2^2\, \mathrm{e}^{-c_2\eta} + (2c_0+c_1)^2\, \mathrm{e}^{-(2c_0+c_1)\eta}\right] \tag{4.35}$$

4.3. Results and discussion

From Table 1, we observe that there is very good agreement between the numerical solution and the perturbation solution for $t = 0$ and small values of t ($t < 1$). For the values of t greater than 3, there is sufficient discrepancy in the results so that the perturbation solution can no longer be accepted — the results from the numerical solution only should be used.

Figure 1 is plotted for the velocity field u against η for ($\alpha_1 = 0, 1, 2$; $t = 1$; $\lambda = 0.5$, and $\varepsilon = 0.1$; 0.5). It can be seen that with an increase in the viscoelastic parameter α_1 the velocity increases near the boundary but then decreases away from the boundary thus causing the boundary-layer thickness to increase. Also it is found that when α_1 is fixed i.e., ($\alpha_1 = 0$) and the third-grade parameter is increased from $\varepsilon = 0.1$ to $\varepsilon = 0.5$ the velocity is again increased near the plate and then decreased away from the boundary, though the effect of the third-grade fluid parameter is not as pronounced as that of the viscoelastic fluid parameter. The same behaviour is observed when $\alpha_1 = 1$ and $\alpha_1 = 2$. In Fig. 2 the velocity field u is plotted against η for ($\alpha_1 = 0, 1, 2$; $t = 2$; $\lambda = 0.5$; and $\varepsilon = 0.1$ and 0.5) and in Fig. 3 for ($\alpha_1 = 0, 1, 2$; $t = 5$; $\lambda = 0.5$, and $\varepsilon = 0.1$ and 0.5). Similar observations for the velocity field and the boundary-layer thickness are seen in these figures as in Fig. 1 except that the difference between the velocity profiles for $\varepsilon = 0.1$ and $\varepsilon = 0.5$ become prominent as we increase t from 2 to 5.

Fig. 1. Variation of velocity profile u with η for $t = 1$.

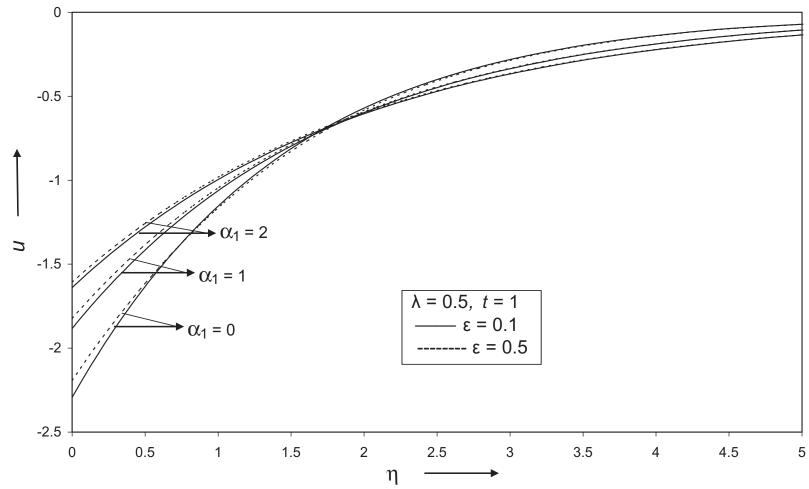
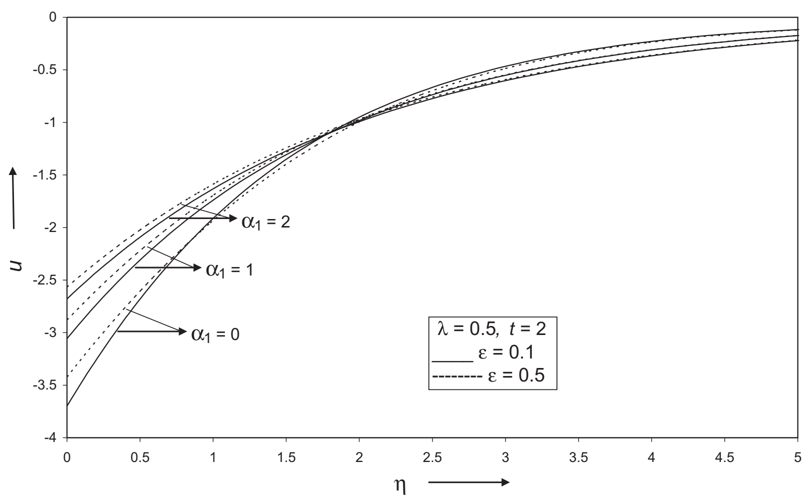


Fig. 2. Variation of velocity profile u with η for $t = 2$.



5. Solution for case 2: $\tau(t) = e^{\lambda t}$, λ is purely imaginary (oscillations)

5.1. Perturbation solution

We now discuss the case when the shear stress at the plate has an oscillating nature. For that we put $\lambda = i\omega$ in $\tau(t)$ and get, after some lengthy calculations, the following velocity field up to order ε^2:

$$u(\eta, t; \varepsilon) = [f_{0R}(\eta)\cos\omega t - f_{0I}(\eta)\sin\omega t] + \varepsilon[f_{1R}(\eta)\cos 3\omega t - f_{1I}(\eta)\sin 3\omega t] + \varepsilon^2[f_{2R}(\eta)\cos 5\omega t - f_{2I}(\eta)\sin 5\omega t]\ldots \quad (5.1)$$

where the expressions for the functions f_{0R}, f_{0I}, f_{1R}, f_{1I}, f_{2R}, and f_{2I} are given in Appendix A.

5.2. Results and discussion

In Fig. 4, u the velocity is plotted against η for a second-grade fluid ($\alpha_1 = 0, 1, 2$; $t = \pi/2$; $\omega = 0.5$; and $\varepsilon = 0$). It can be seen that with an increase in the viscoelastic fluid parameter α_1 the

Fig. 3. Variation of velocity profile u with η for $t = 5$.

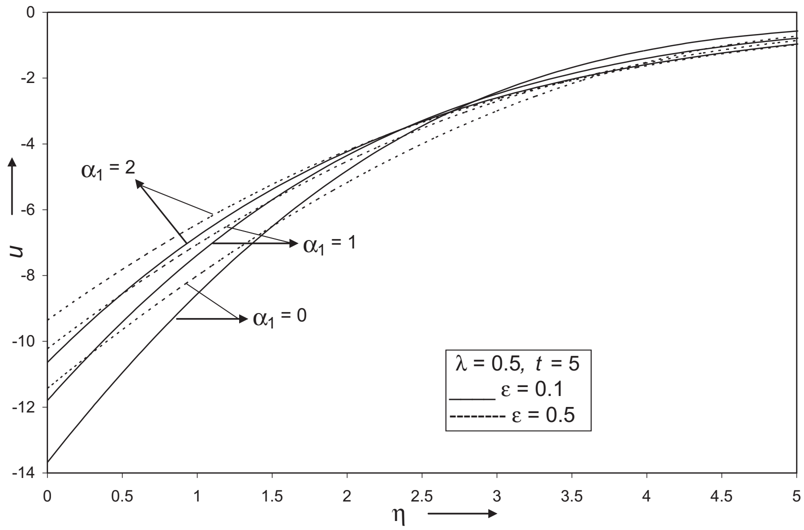
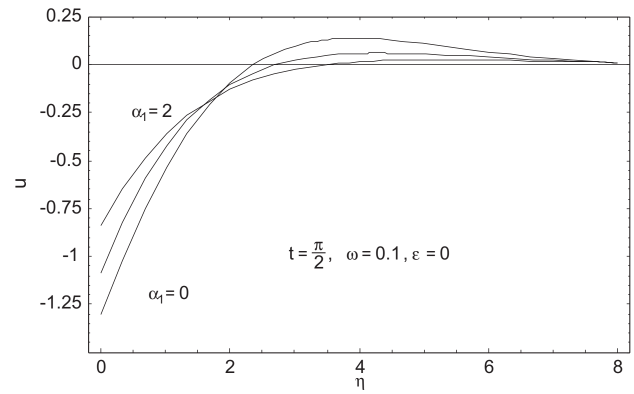


Fig. 4. Variation of velocity profile u with η for $t = \pi/2$; $\varepsilon = 0$; $\omega = 0.1$; and $\alpha_1 = 0$, 1, and 2.



velocity decreases and thus the boundary-layer thickness increases. Similar effects are seen in Fig. 5 in which $t = 2\pi$ and $\omega = 0.5$ are taken instead of $t = \pi/2$ and $\omega = 0.1$. In Fig. 6, the velocity u is plotted against η for a third-grade fluid ($\alpha_1 = 0$, 1, 2; $t = 2\pi$; $\omega = 0.5$; and $\varepsilon = 0.1$). Figure 6 shows that with an increase in the third-grade parameter the velocity decreases and thus, the boundary-layer thickness further increases. In Fig. 7, the velocity u is plotted against η for $\alpha_1 = 0.5$, $t = 2\pi$, $\varepsilon = 0.5$, and for various values of the oscillating frequency ($\omega = 0.1$, 0.3, 0.7). It is clear from Fig. 7 that the amplitude of the velocity decreases with an increase in the frequencies. Figure 8 is plotted for the stress τ_{xy} at any point in the fluid against η for various values of α_1.

The *skin friction* at the plate $\eta = 0$ can be obtained by finding the real part in the following equation:

$$\tau_2 = \frac{\overline{\tau}_2}{\rho U_0^2} = \begin{bmatrix} \mathrm{e}^{i\omega t}\{f'_{0R}(0) + if'_{0I}(0)\} + \varepsilon \mathrm{e}^{3i\omega t}\{f'_{1R}(0) + if'_{1I}(0)\} \\ +\varepsilon^2\,\mathrm{e}^{5i\omega t}\{f'_{2R}(0) + if'_{2I}(0)\} + \cdots \end{bmatrix} \tag{5.2}$$

where $f'_{0R}(0)$, $f'_{1R}(0)$, $f'_{2R}(0)$, $f'_{0I}(0)$, $f'_{1I}(0)$, $f'_{2I}(0)$ are given in Appendix B.

Fig. 5. Variation of velocity profile u with η for $t = 2\pi$; $\varepsilon = 0$; $\omega = 0.5$; and $\alpha_1 = 0$, 1, and 2.

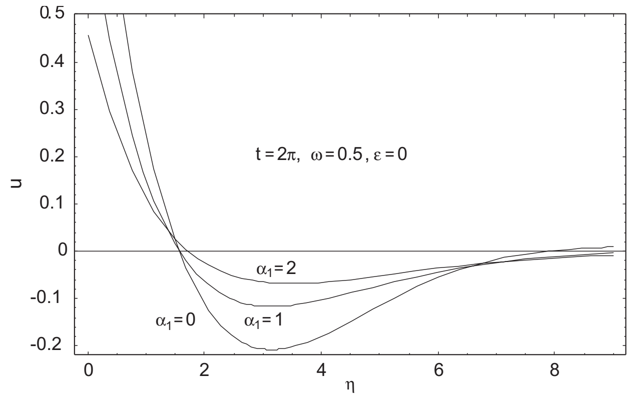


Fig. 6. Variation of velocity profile u with η for $t = 2\pi$; $\varepsilon = 0.1$; $\omega = 0.5$; and $\alpha_1 = 0$, 1, and 2.

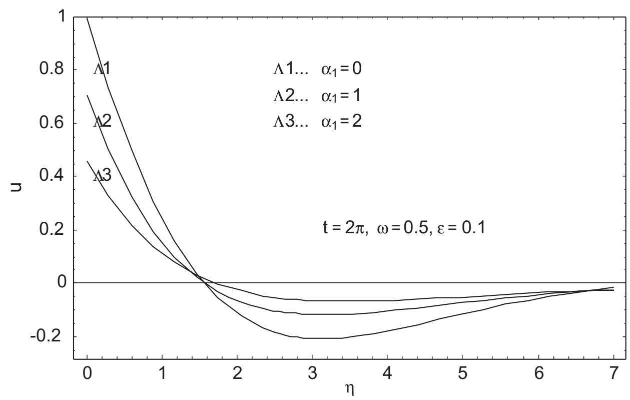
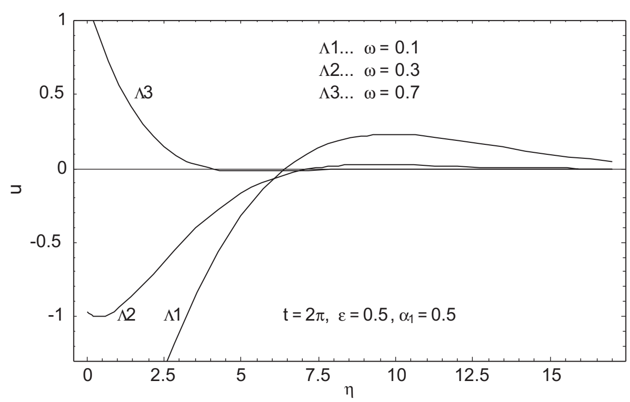


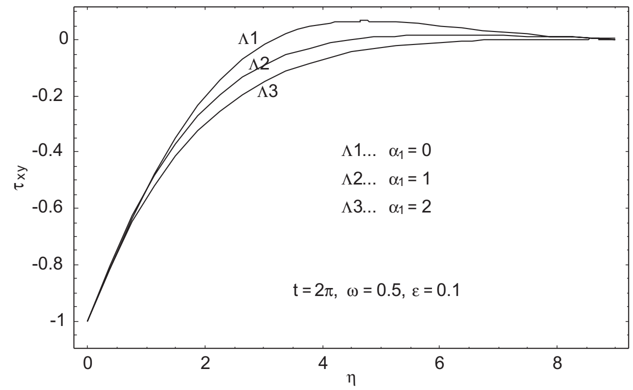
Fig. 7. Variation of velocity profile u with η for $t = 2\pi$; $\varepsilon = 0.5$; $\alpha_1 = 0.5$; and $\omega = 0.1$, 0.3, and 0.7.



6. Concluding remarks

In this work, the flow of a third-grade fluid on a plate is studied. The flow induced here is due to the variable shear stress of the plate. The analytical solution of the governing nonlinear partial-differential equation is given using a perturbation series. The numerical solution is also obtained. It should be noted

Fig. 8. Variation of shear stress τ_{xy} with η for $t = 2\pi$; $\varepsilon = 0.1$; $\omega = 0.5$; and $\alpha_1 = 0$, 1, and 2.



that the results of several unattempted problems can be obtained as the limiting cases of our solution. Specifically, the results for viscous and second-grade fluid flows due to a variable shear stress (which are not yet in the literature to the best of our knowledge) can be recovered by taking $\alpha_1 = \varepsilon = 0$ and $\varepsilon = 0$, respectively. Our investigation shows that the perturbation technique is adequate for the case when the variable shear stress has as oscillatory character, however, if the shear stress grows exponentially with time then the perturbation solution can be accepted only for small values of time. For moderate to large values of time, the numerical solution must be used.

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Appendix A. Expressions for the functions of the perturbation solution (5.1)

$$f_{0R} = \frac{-\mathrm{e}^{-R_1\eta}}{R_2^2 + I_2^2}\left(R_2 \cos I_1\eta - I_2 \sin I_1\eta\right), \qquad f_{0I} = \frac{\mathrm{e}^{-R_1\eta}}{R_2^2 + I_2^2}\left(R_2 \sin I_1\eta + I_2 \cos I_1\eta\right)$$

$$f_{1R} = \frac{\mathrm{e}^{-3R_1\eta}}{R_4^2 + I_4^2}\left[R_4\left(R_1 \cos 3I_1\eta + I_1 \sin 3I_1\eta\right) - I_4\left(I_1 \cos 3I_1\eta - R_1 \sin 3I_1\eta\right)\right] - \frac{\mathrm{e}^{-R_3\eta}}{3\left(R_4^2 + I_4^2\right)}\left[R_4\left(R_3 \cos I_3\eta + I_3 \sin 3I_3\eta\right) - I_4\left(I_3 \cos I_3\eta - R_3 \sin I_3\eta\right)\right]$$

$$f_{1I} = \frac{\mathrm{e}^{-3R_1\eta}}{R_4^2 + I_4^2}\left[R_4\left(I_1 \cos 3I_1\eta - R_1 \sin 3I_1\eta\right) + I_4\left(R_1 \cos 3I_1\eta + I_1 \sin 3I_1\eta\right)\right] - \frac{\mathrm{e}^{-R_3\eta}}{3\left(R_4^2 + I_4^2\right)}\left[R_4\left(I_3 \cos I_3\eta - R_3 \sin I_3\eta\right) + I_4\left(R_3 \cos I_3\eta + I_3 \sin 3I_3\eta\right)\right]$$

$$f_{2R} = R_9 + R_{13} + R_{14}, \qquad f_{2I} = I_9 + I_{13} + I_{14}$$

$$R_1 = \frac{1}{\sqrt{2\left(1 + \omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2 + \omega^2} + \omega^2\alpha_1}$$

$$I_1 = \frac{1}{\sqrt{2\left(1 + \omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2 + \omega^2} - \omega^2\alpha_1}$$

$$R_2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2 + \omega^2} + \omega^2}, \quad I_2 = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2 + \omega^2} - \omega^2}$$

$$R_3 = \frac{1}{\sqrt{2\left(1 + 9\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(9\omega^2\alpha_1\right)^2 + 9\omega^2} + 9\omega^2\alpha_1}$$

$$I_3 = \frac{1}{\sqrt{2\left(1 + 9\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(9\omega^2\alpha_1\right)^2 + 9\omega^2} - 9\omega^2\alpha_1}$$

$$R_4 = 2\omega^2\alpha_1\left(2\omega^2\alpha_1^2 - 3\right), \qquad I_4 = \omega\left(1 - 9\omega^2\alpha_1^2\right)$$

$$R_{14} = \left(a_7 \cos I_5\eta - b_7 \sin I_5\eta\right)\mathrm{e}^{-R_5\eta}, \qquad I_{14} = \left(b_7 \cos I_5\eta + a_7 \sin I_5\eta\right)\mathrm{e}^{-R_5\eta}$$

$$R_{13} = R_{10} - \left(a_{18}R_{12} - b_{18}I_{12}\right), \qquad I_{13} = I_{10} - \left(b_{18}R_{12} + a_{18}I_{12}\right)$$

$$R_{12} = R_{11}\cos I_5\eta + I_{11}\sin I_5\eta, \qquad I_{12} = I_{11}\cos I_5\eta - R_{11}\sin I_5\eta$$

$$R_{11} = \frac{e^{-R_5\eta}}{R_5^2 + I_5^2}\left[(R_3 + 2R_1)R_5 + I_5(I_3 + 2I_1)\right]$$

$$I_{11} = \frac{e^{-R_5\eta}}{R_5^2 + I_5^2}\left[(I_3 + 2I_1)R_5 - I_5(R_3 + 2R_1)\right]$$

$$R_{10} = e^{-(2R_1+R_3)\eta}\left[a_{18}\cos(2I_1 + I_3)\eta + b_{18}\sin(2I_1 + I_3)\eta\right]$$

$$I_{10} = e^{-(2R_1+R_3)\eta}\left[b_{18}\cos(2I_1 + I_3)\eta - a_{18}\sin(2I_1 + I_3)\eta\right]$$

$$a_{18} = \frac{a_{15}a_{17} + b_{15}b_{17}}{a_{17}^2 + b_{17}^2}, \qquad b_{18} = \frac{b_{15}b_{17} - a_{15}a_{17}}{a_{17}^2 + b_{17}^2}, \qquad a_{17} = 3(a_{16} - 5\omega\alpha_1 b_{16})$$

$$a_{16} = (2R_1 + R_3)^2 - (2I_1 + I_3)^2 - \left(R_5^2 + I_5^2\right)^2$$

$$b_{16} = (2R_1 + R_3)(2I_1 + I_3) - 2R_5I_5, \qquad b_{17} = 3(b_{16} + 5\omega\alpha_1 a_{16})$$

$$a_{15} = (2R_1 + R_3)(a_{13}a_{14} - b_{13}b_{14}) - (2I_1 + I_3)(b_{13}a_{14} + a_{13}a_{14})$$

$$b_{15} = (2I_1 + I_3)(a_{13}a_{14} - b_{13}b_{14}) + (2R_1 + R_3)(b_{13}a_{14} + a_{13}a_{14})$$

$$a_{14} = \left(R_6^2 - I_6^2\right)\left(R_1^2 - I_1^2\right) - 4R_1I_1R_6I_6$$

$$b_{14} = 2R_1I_1\left(R_6^2 - I_6^2\right) + 2R_6I_6\left(R_1^2 - I_1^2\right)$$

$$a_{13} = a_2\left(R_3^2 - I_3^2\right) - 2b_2R_3I_3, \qquad b_{13} = b_2\left(R_3^2 - I_3^2\right) + 2a_2R_3I_3,$$

$$R_9 = R_7 - (R_8\cos I_5\eta + I_8\sin I_5\eta), \qquad I_9 = I_7 - (I_8\cos I_5\eta - R_8\sin I_5\eta)$$

$$R_8 = \frac{5e^{-R_5\eta}}{R_5^2 + I_5^2}\left[(R_1R_5 + I_1I_5)a_{12} - b_{12}(I_1R_5 - R_1I_5)\right]$$

$$I_8 = \frac{5e^{-R_5\eta}}{R_5^2 + I_5^2}\left[(I_1R_5 - R_1I_5)a_{12} + b_{12}(R_1R_5 + I_1I_5)\right]$$

$$R_7 = e^{-5R_1\eta}(a_{12}\cos 5I_1\eta + b_{12}\sin 5I_1\eta), \qquad a_{12} = \frac{15(a_9a_{11} + b_9b_{11})}{a_{11}^2 + b_{11}^2}$$

$$I_7 = e^{-5R_1\eta}(b_{12}\cos 5I_1\eta - a_{12}\sin 5I_1\eta), \qquad b_{12} = \frac{15(b_9a_{11} - a_9b_{11})}{a_{11}^2 + b_{11}^2}$$

$$a_{11} = a_{10} - 5\omega\alpha_1 b_{10}, \qquad b_{11} = b_{10} + 5\omega\alpha_1 a_{10}$$

$$a_{10} = 25\left(R_1^2 - I_1^2\right) - \left(R_5^2 - I_5^2\right), \qquad b_{10} = 50R_1I_1 - 2R_5I_5$$

$$a_9 = \left(R_6^2 - I_6^2\right)(a_2a_8 - b_2b_8) - 2R_6I_6(b_2a_8 + a_2b_8)$$

$$b_9 = 2R_6I_6(a_2a_8 - b_2b_8) + \left(R_6^2 - I_6^2\right)(b_2a_8 + a_2b_8)$$

$$a_8 = R_1\left\{\left(R_1^2 - I_1^2\right)^2 - 4R_1^2I_1^2\right\} - 4\left(R_1^2 - I_1^2\right)R_1I_1^2, \qquad d = a_2 + ib_2$$

$$b_8 = I_1\left\{\left(R_1^2 - I_1^2\right)^2 - 4R_1^2I_1^2\right\} + 4\left(R_1^2 - I_1^2\right)R_1^2I_1, \qquad c = R_6 + iI_6$$

$$a_3 = \left(R_3^2 - I_3^2\right) - 9\left(R_1^2 - I_1^2\right), \qquad b_3 = 2R_3I_3 + 2R_1I_1, \qquad \sqrt{\theta} = R_5 + iI_5$$

$$a_4 = \left(R_6^2 - I_6^2\right)\left(R_1^2 - I_1^2\right) - 4R_1I_1R_6I_6, \qquad b = R_3 + iI_3$$

$$b_4 = 2R_1I_1\left(R_6^2 - I_6^2\right) + 2R_6I_6\left(R_1^2 - I_1^2\right), \qquad a = R_1 + iI_1$$

$$a_5 = R_5 - 5\omega\alpha_1 I_5,\ b_5 = I_5 + 5\omega\alpha_1 R_5, \qquad a_7 = \frac{a_5a_6 + b_5b_6}{3\left(a_5^2 + b_5^2\right)}, \qquad b_7 = \frac{a_5b_6 - a_6b_5}{3\left(a_5^2 + b_5^2\right)}$$

$$a_6 = a_4\left(a_2a_3 - b_2b_3\right) - b_4\left(b_3a_2 + a_3b_2\right), \qquad a_2 = \frac{-30\omega^2\alpha_1}{\left(30\omega^2\alpha_1\right)^2 + \left(6\omega - 24\omega^3\alpha_1^2\right)^2}$$

$$b_6 = b_4\left(a_2a_3 - b_2b_3\right) + a_4\left(b_3a_2 + a_3b_2\right), \qquad b_2 = \frac{24\omega^3\alpha_1^2 - 6\omega}{\left(30\omega^2\alpha_1\right)^2 + \left(6\omega - 24\omega^3\alpha_1^2\right)^2}$$

$$R_6 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_1^2 + b_1^2} + a_1}, \qquad I_6 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_1^2 + b_1^2} - a_1}$$

$$R_5 = \frac{1}{\sqrt{2\left(1 + 25\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(25\omega^2\alpha_1\right)^2 + 25\omega^2} + 25\omega^2\alpha_1}$$

$$I_5 = \frac{1}{\sqrt{2\left(1 + 25\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(25\omega^2\alpha_1\right)^2 + 25\omega^2} - 25\omega^2\alpha_1}$$

where f_{0R}, f_{0I}, f_{1R}, f_{1I}, and f_{2R}, f_{2I} are the real and imaginary parts of f_0, f_1, and f_2, respectively.

Appendix B. Expressions for the functions of the skin friction equation (5.2)

$$f_{0R}'(0) = \frac{R_1R_2 + I_1I_1}{R_2^2 + I_2^2}, \qquad f_{0I}'(0) = \frac{I_1R_2 - R_1I_2}{R_2^2 + I_2^2}$$

$$f_{1R}'(0) = \frac{1}{R_4^2 + I_4^2}\left[6R_1I_1I_4 + 3R_4\left(I_1^2 - R_1^2\right) + \frac{1}{3}R_4\left(R_3^2 - I_3^2\right) - \frac{2}{3}R_3I_1I_4\right]$$

$$f_{1I}'(0) = \frac{1}{R_4^2 + I_4^2}\left[-6R_1I_1I_4 + 3I_4\left(I_1^2 - R_1^2\right) + \frac{1}{3}I_4\left(R_3^2 - I_3^2\right) + \frac{2}{3}R_3I_3R_4\right]$$

$$f_{2R}'(0) = R_9'(0) + R_{13}'(0) + R_{14}'(0), \qquad f_{2I}'(0) = I_9'(0) + I_{13}'(0) + I_{14}'(0)$$