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On strictly positive solutions for some semilinear elliptic problems

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Abstract. Let B be a ball in $\mathbb{R}^N$, $N \geq 1$, let m be a possibly discontinuous and unbounded function that changes sign in B and let $0 < p < 1$. We study existence and nonexistence of strictly positive solutions for semilinear elliptic problems of the form $-\Delta u = m\,(x)\,u^p$ in B, $u = 0$ on ∂B.

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, let $p > 0$ and let $m : \Omega \to \mathbb{R}$ be a function that changes sign in Ω. Consider the problem

$$\begin{cases} -\Delta u = mu^p & \text{in } \Omega \\ 0 \not\equiv u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

When $1 < p < (N+2)\,/\,(N-2)$ (superlinear problem) it is well known that (1.1) has a solution if and only if $m > 0$ in a set of positive measure (see e.g., [3], [1] and its references). A similar result is true if $p = 1$ (linear eigenvalue problem with indefinite weight) in the sense that the aforementioned condition on m is necessary and sufficient for the existence of a unique positive principal eigenvalue with respect to the weight m (e.g., [17], [16] and its references). Furthermore, as a direct consequence of the strong maximum principle and Hopf's lemma, under suitable regularity assumptions (namely, $m \in L^r(\Omega)$, $r > N$), in both cases the solution u is strictly positive in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$, where ν denotes the outward unit normal to $\partial\Omega$. In other words, the solution automatically belongs to the interior of the positive cone of $C^{1+\theta}\left(\overline{\Omega}\right)$,

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$\theta \in (0,1)$. However, it turns out that for the sublinear problem (i.e. $0 < p < 1$) the matter of existence of strictly positive solutions in Ω for (1.1) is much more involved, even in the one-dimensional case. Our aim in the present paper is to study the existence and nonexistence of strictly positive solutions in this latter situation, when Ω is a ball in $\mathbb{R}^N$.

For $p \in (0,1)$ and $m \in C^{\theta}\left(\overline{\Omega}\right)$, it is known that (1.1) possesses a solution if and only if $m(x_0) > 0$ for some $x_0 \in \Omega$. This was established using sub- and supersolutions in Theorem 2.2 in [2] under some additional hypothesis on m, and a generalization of this result for a strongly uniformly elliptic differential operator and only asking that $m \in C\left(\overline{\Omega}\right)$ was proved by the authors in [12], Theorem 3.2 and Remark 3.3, employing iterative and fixed points methods. We note that in both works the only information available on the solution u is that $u > 0$ in the set $\{x \in \Omega : m(x) > 0\}$.

Concerning the question of existence of strictly positive solutions for (1.1) when $p \in (0,1)$, to our knowledge no necessary condition on m is known (other than the obvious one derived from the maximum principle), and the only sufficient condition we found in the literature (see [15, Theorem 4.4], or [14, Theorem 10.6]) is that the solution Ψ of the linear problem

$$\begin{cases} -\Delta\Psi = m & \text{in } \Omega \\ \Psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

satisfies $\Psi > 0$ in Ω, and even in this case the solution of (1.1) need not belong to the interior of the positive cone. Moreover, this last condition is far from being in some sense necessary as the following example shows.

Consider the problem

$$\begin{cases} -u'' = 2\left(1 - 4\cos^2 x\right) u^{1/2} & \text{in } (0,\pi) \\ u(0) = u(\pi) = 0. \end{cases} \tag{1.3}$$

A few computations yield that $u \doteq \frac{\sin^4 x}{4}$ is a solution of (1.3), but the solution of (1.2) with $2\left(1 - 4\cos^2 x\right)$ in place of m is $\Psi(x) \doteq x^2 - \pi x + 1 - \cos(2x)$ which satisfies $\Psi < 0$ in $(0,\pi)$. Let us observe however that the aforementioned condition is in fact necessary and sufficient for the existence of positive solutions for some related nonlinear problems, see [7] and [13]. Let us also mention that examples as (1.3) provide an easy way to obtain situations in which there exists a solution of (1.1) which actually vanishes in a subset of the domain. Indeed, one simply has to take any domain $\widetilde{\Omega} \supset \Omega$ and extend m (in any form) and u by zero outside Ω.

In order to prove our results we shall mainly rely on the sub- and supersolution method. There are various reasons for which this is a natural choice. One of them is that many closely related semilinear problems can be tackled quite satisfactorily with this approach. Another one is given by the following elementary remark:

Remark 1.1. Let ψ be the solution of (1.2) with m^+ in place of m, where as usual we write $m = m^+ - m^-$ with $m^+ = \max(m,0)$ and $m^- = \max(-m,0)$.

Let $k \geq (\|\psi\|_\infty + 1)^{p/(1-p)}$. Then $k(\psi+1)$ is a supersolution of (1.1) since

$$-\Delta (k(\psi+1)) = km^+ \geq (k(\|\psi\|_\infty + 1))^p m^+ \geq m(k(\psi+1))^p \qquad \text{in } \Omega$$

and $\Phi = k$ on $\partial\Omega$. Furthermore, it is clear that this remains valid if we replace $-\Delta$ by a linear second order elliptic operator with nonnegative zero order coefficient.

It follows from the above remark that the only task is to construct some subsolution for (1.1) that is strictly positive in Ω. This will be carried out, roughly speaking, splitting the domain in parts, building subsolutions in each of them and checking that they can be joined correctly in order to have a subsolution for the whole domain. In the next section we shall accomplish this for the one-dimensional problem. In this case our results give a quite complete picture of what happens with the strictly positive solutions of (1.1). More precisely, in Theorem 2.1 we shall supply conditions that assure either the existence of strictly positive solutions or solutions in the interior of the positive cone. Moreover, under some evenness assumptions on m we shall provide further sufficient conditions for the existence of strictly positive solutions or belonging to the interior of the positive cone (see Theorem 2.3 and Proposition 2.5 respectively). Let us mention that neither of these conditions are comparable with each other (see Remarks 2.2, 2.4 and 2.6). Let us also notice that the distinction between strictly positive solutions and solutions in the interior of the positive cone is of importance since the positive solution of (1.1), if it exists, is unique (see e.g., [9], Theorem 2.1). On the other hand, in Theorem 2.7 we shall exhibit necessary conditions for the existence of strictly positive solutions, which are of similar type as the ones stated in the above theorems. We would like to point out that all these conditions are given in terms of some L^q-norms and positive principal eigenvalues, which is quite natural for this kind of semilinear problems (see e.g., [5,8,18,11]).

In Sect. 3 we shall adapt to the radial case some of the techniques developed for the one-dimensional problem, and we shall also be able to prove necessary and sufficient conditions on m (see Theorems 3.1, 3.2 and 3.3 for the sufficient conditions, and Theorem 3.4 and Remark 3.5 for the necessary ones). Let us note however that the bounds in the latter case are sometimes sharper or give more information than the ones that we find for the N-dimensional problem.

We conclude this introduction with some last few comments. There are of course many interesting questions about the strictly positive solutions of (1.1) that still remain open. It is not clear whether the ideas in the present paper are applicable to a non-radial m when $N > 1$, and it is even less clear how to attack (1.1) in the case of a general smooth bounded domain and/or a general linear second order differential operator. We are strongly convinced that similar conditions (i.e., in terms of some L^q-norm or principal eigenvalues) to the ones that appear here should still be true in this cases, but we are not able to supply a proof.

2. The one-dimensional case

For $a < b$, let $\Omega \doteq (a,b)$ and $m : \Omega \to \mathbb{R}$ with $m^- \not\equiv 0$. We consider the problem

$$\begin{cases} -u'' = mu^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Let

$$P^\circ \doteq \text{interior of the positive cone of } C^{1+\theta}\left(\overline{\Omega}\right),$$

$$\lambda_1(m,I) \doteq \text{positive principal eigenvalue for } m \text{ in } I, \text{ for any interval } I \subseteq \Omega.$$

Theorem 2.1. *Let $m \in L^2(\Omega)$ and suppose there exist $a \le x_0 < x_1 \le b$ such that $0 \not\equiv m \ge 0$ in $I \doteq (x_0, x_1)$.*

(i) *Assume that $m^- \in L^\infty(\Omega)$ and*

$$\left\| m^- \right\|_{L^\infty(\Omega)} \le \frac{2(1+p)}{\left((1-p)\max\{b-x_0, x_1-a\}\right)^2 \lambda_1(m,I)}. \tag{2.2}$$

Then there exists $u \in W^{2,2}(\Omega)$ solution of (2.1).

(ii) *Assume that*

$$\max\left\{ \int_{x_0}^{b} \left\| m^- \right\|_{L^1(t,b)} dt, \int_a^{x_1} \left\| m^- \right\|_{L^1(a,t)} dt \right\} < \frac{1}{\lambda_1(m,I)}. \tag{2.3}$$

Then there exists $u \in W^{2,2}(\Omega) \cap P^\circ$ solution of (2.1).

Proof. We proceed in several steps. Since $m^- \not\equiv 0$ it must occur that either $x_1 < b$ or $x_0 > a$ (or both). If $x_1 < b$ we argue as follows. By (2.2) we may pick c such that

$$\lambda_1(m,I) \le c \le \frac{2(1+p)}{\left((1-p)(b-x_0)\right)^2 \left\| m^- \right\|_\infty},$$

and due to the homogeneity of (2.1) it is enough to prove (i) for $m_c \doteq cm$. Define

$$\beta \doteq \frac{2}{1-p}, \qquad \lambda \doteq \left[\frac{c(1-p)^2 \left\| m^- \right\|_{L^\infty(\Omega)}}{2(1+p)} \right]^{\beta/2}, \tag{2.4}$$

$$\Phi_0(x) \doteq \lambda (b-x)^\beta, \qquad \text{for } x \in [x_0, b].$$

Clearly Φ_0 is decreasing and convex in $[x_0,b]$ and $\Phi_0(b) = 0 < \Phi_0(x)$ for all $x \in [x_0,b)$. Also, taking into account (2.4) some computations yield that $-\Phi_0'' \le m_c \Phi_0^p$ in (x_0,b) and $\|\Phi_0\|_\infty \le 1$.

On the other hand, let $\varphi > 0$ (not depending on c) and $\lambda_1(m_c, I)$ be satisfying

$$\begin{cases} -\varphi'' = \lambda_1(m_c,I)\, m_c \varphi & \text{in } I \\ \varphi = 0 & \text{on } \partial I \end{cases} \tag{2.5}$$

with $\|\varphi\|_\infty = 1$. From the election of c we get $-\varphi'' = \lambda_1(m_c, I)\, m_c\varphi \le m_c\varphi^p$ in I. Also, since $m_{c|I} \in L^2(I)$, by standard regularity arguments $\varphi \in W^{2,2}(I)$. Then, if $t_1, t_2 \in I$ with $t_1 < t_2$ we may integrate over (t_1, t_2) (see e.g., [4], Theorem 8.2) to obtain that

$$\varphi'(t_1) - \varphi'(t_2) = -\int_{t_1}^{t_2} \varphi'' = \lambda_1(m_c, I)\int_{t_1}^{t_2} m_c\varphi \ge 0 \tag{2.6}$$

and thus φ is concave in I.

Let $\overline{\rho} \doteq \max\{x \in I : \varphi(x) = 1\}$, and let $\overline{x} \in (\overline{\rho}, x_1)$ be the greatest abscissa of the intersection points between the graphs of $\Phi_{0|(\overline{\rho},x_1)}$ and $\varphi_{|(\overline{\rho},x_1)}$. Such a $\overline{x}$ exists because $x_1 < b$ and $\|\Phi_0\|_\infty \le \|\varphi\|_\infty$. We note that $\Phi_0'(\overline{x}) \ge \varphi'(\overline{x})$. Indeed, let γ_M be the maximum $\gamma > 0$ such that the graph of $(\Phi_0 + \gamma)_{|(\overline{\rho},x_1)}$ intersects the graph of $\varphi_{|(\overline{\rho},x_1)}$, and denote by x_M the abscissa of the point of intersection between the graphs of $(\Phi_0 + \gamma_M)_{|(\overline{\rho},x_1)}$ and $\varphi_{|(\overline{\rho},x_1)}$. Then $x_M \le \overline{x}$ and $(\Phi_0 + \gamma_M)'(x_M) = \varphi'(x_M)$. Therefore the concavity of φ and the convexity of Φ_0 say that $\Phi_0'(\overline{x}) \ge \varphi'(\overline{x})$.

In a similar way, if also $x_0 > a$ we set $\Phi_1(x) \doteq \lambda(x-a)^\beta$ for $x \in [a, x_1]$, where λ and β are given by (2.4). Reasoning as above we find that $-\Phi_1'' \le m_c\Phi_1^p$ in (a, x_1) and that there exist some $\underline{\rho} \in (x_0, \overline{\rho}]$ and $\underline{x} \in \left(x_0, \underline{\rho}\right)$ such that $\Phi_1(\underline{x}) = \varphi(\underline{x})$ and $\Phi_1'(\underline{x}) \le \varphi'(\underline{x})$.

We next define $\underline{u}$ by $\underline{u} \doteq \Phi_1$ in $[a, \underline{x}]$, $\underline{u} \doteq \varphi$ in $[\underline{x}, \overline{x}]$ and $\underline{u} \doteq \Phi_0$ in $[\overline{x}, b]$ (if $x_1 = b$ we simply set $\underline{u} \doteq \Phi_1$ in $[a, \underline{x}]$ and $\underline{u} \doteq \varphi$ in $[\underline{x}, b]$; and if $x_0 = a$ we proceed analogously). Then $\underline{u}$ is well defined, $\underline{u} \in C\left(\overline{\Omega}\right) \cap H_0^1(\Omega)$, $\underline{u} > 0$ in Ω (and $\underline{u} = 0$ on $\partial\Omega$). Furthermore, an integration by parts shows that $\underline{u}$ satisfies $-\underline{u}'' \le m_c\underline{u}^p$ in the weak sense in Ω because $\Phi_1'(\underline{x}) \le \varphi'(\underline{x})$ and $\Phi_0'(\overline{x}) \ge \varphi'(\overline{x})$. Therefore, taking into account Remark 1.1 we get a bounded solution $\underline{u} \le u \in H_0^1(\Omega)$ of (2.1) with m_c in place of m, and hence also $u \in W^{2,2}(\Omega)$.

Let us prove (ii). Without loss of generality we assume that $a < x_0 < x_1 < b$ (in fact, one can can see that the cases in which this does not happen are as in (i) easier to treat). It follows that $m^- \not\equiv 0$ in $(x_{1,}b)$ and (a, x_0). For $0 < \varepsilon < (b - x_0)^{-1}$ let

$$m_\varepsilon \doteq \frac{1 - \varepsilon(b - x_0)}{\int_{x_0}^b \|m^-\|_{L^1(t,b)}\,dt} m \tag{2.7}$$

and define

$$\Phi_\varepsilon(x) \doteq \int_x^b \left\|m_\varepsilon^-\right\|_{L^1(t,b)} dt + \varepsilon(b - x), \qquad x \in [x_0, b]. \tag{2.8}$$

It holds that Φ_ε is decreasing, Φ_ε' is increasing (in particular, Φ_ε is convex) and $0 < \Phi_\varepsilon(x) \le 1$ for all $x \in [x_0, b)$. Moreover, it is clear that for a.e. such x, $-\Phi_\varepsilon''(x) = -m_\varepsilon^-(x) \le m_\varepsilon(x)\,\Phi_\varepsilon^p(x)$ and also $\Phi_\varepsilon(b) = 0$, $\Phi_\varepsilon'(b) = -\varepsilon$.

On the other side, let $\varphi > 0$ and $\lambda_1(m_\varepsilon, I)$ be given by (2.5) with m_ε in place of m_c. Making ε smaller if necessary, by (2.3) we can choose ε such that $\lambda_1(m_\varepsilon, I) \le 1$. Indeed, the strict inequality is immediate from (2.3) for $\varepsilon = 0$ and thus also for $\varepsilon > 0$ sufficiently small. Since $\|\varphi\|_\infty = 1$ it follows that

$-\varphi'' \leq m_\varepsilon \varphi^p$ in I. Moreover, if $\rho \in I$ satisfies $\varphi(\rho) = 1$, arguing as in (i) we find some $\overline{x} \in (\rho, x_1)$ such that $\Phi_\varepsilon(\overline{x}) = \varphi(\overline{x})$ and $\Phi'_\varepsilon(\overline{x}) \geq \varphi'(\overline{x})$.

Define now u_ε by $u_\varepsilon \doteq 0$ in $[a, x_0]$, $u_\varepsilon \doteq \varphi$ in $[x_0, \overline{x}]$ and $u_\varepsilon \doteq \Phi_\varepsilon$ in $[\overline{x}, b]$. Then u_ε is a weak subsolution for (2.1) with m_ε in place of m, and hence recalling Remark 1.1 we obtain a nonnegative solution v of (2.1) (due to the homogeneity) which satisfies $v > 0$ in (x_0, b) and $v'(b) < 0$.

Reasoning similarly we may construct some $w \geq 0$ solution for (2.1) such that $w > 0$ in (a, x_1) and $w'(a) > 0$. If we set $\underline{u} \doteq \max(v, w)$ then $\underline{u}$ is a strictly positive subsolution of (2.1) satisfying $\underline{u}'(b) < 0 < \underline{u}'(a)$ and so recalling once again Remark 1.1 the theorem follows. □

Remark 2.2. (i) Observe that Theorem 2.1 (i) says in particular that for any m that changes sign in Ω (with $m^- \in L^\infty(\Omega)$), (2.1) has a solution if p is close enough to 1.

(ii) Let us mention that the conditions (2.2) and (2.3) are not comparable. Indeed, it is clear that if (2.1) is almost linear (i.e. $p \approx 1$) then (2.2) is weaker than (2.3). Conversely, fixing everything except m^-, if the L^1-norm of m^- is small enough and the L^∞-norm of m^- is sufficiently large then (2.3) is fulfilled but (2.2) is not.

Under some evenness assumptions, further sufficient conditions on m can be stated. For $b > 0$ and $x_0 \in (0, b)$ we fix for the rest of the section

$$\Omega \doteq (-b, b), \qquad \Omega_0 \doteq (-x_0, x_0).$$

Theorem 2.3. (i) *Let $m \in L^2(\Omega)$ with $m^- \in L^\infty(\Omega)$. Suppose there exists $x_0 > 0$ such that in Ω_0 m is even and $0 \not\equiv m \geq 0$. Let*

$$M_1 \doteq \frac{(1+p)\|m\|_{L^1(0,x_0)}}{(1-p)(b-x_0)}, \quad M_2 \doteq \frac{2^p(1+p)}{((1-p)(b-x_0))^{1+p} x_0^{1-p} \lambda_1(m, \Omega_0)}$$

and assume that

$$\left\|m^-\right\|_{L^\infty(\Omega)} \leq \max(M_1, M_2).$$

Then there exists $u \in W^{2,2}(\Omega)$ solution of (2.1).

(ii) *Let $m \in L^2(\Omega)$ and suppose there exists $\underline{m} \leq m$ and satisfying* (i)*. Then the same conclusion of* (i) *holds.*

Proof. We start arguing as in the first part of Theorem 2.1. Suppose first $\|m^-\|_{L^\infty(\Omega)} \leq M_1$. For $x \in [x_0, b]$, let Φ be the "polynomial" given by (2.4) with β as there and λ also as there but with $c = 1$. We set $\alpha \doteq \Phi(x_0)$. After some computations one can check that $-\Phi'' \leq m\Phi^p$ in (x_0, b) and also that $\|m^-\|_\infty \leq M_1$ implies

$$\Phi'(x_0) \geq -\alpha^p \|m\|_{L^1(0,x_0)}. \tag{2.9}$$

We shall next construct a solution for

$$\begin{cases} -w'' = mw^p & \text{in } \Omega_0 \\ w = \alpha & \text{on } \partial\Omega_0. \end{cases} \tag{2.10}$$

Let $\Psi > 0$ be the solution of $-\Psi'' = m$ in Ω_0, $\Psi = 0$ on $\partial\Omega_0$. As in Remark 1.1 it can be verified that $K(\Psi + 1)$ is a supersolution of (2.10) for every $K \geq \max\left(\alpha, (\|\Psi\|_\infty + 1)^{p/(1-p)}\right)$.

On the other hand, let $0 < \varphi \in W^{2,2}(\Omega)$ and $\lambda_1(m, \Omega_0)$ be given by (2.5) with m and Ω_0 in place of m_c and I respectively. Recalling that $\|\varphi\|_\infty = 1$ one can also see that $\alpha + \lambda_1(m, \Omega_0)^{-1/(1-p)} \varphi$ is a subsolution of (2.10). It follows that there exists some $\alpha < w \in W^{2,2}(\Omega_0)$ solution of (2.10). We notice next that w is even. Indeed, since m is even in Ω_0, $w(-x)$ is also a solution, but the solution of (2.10) is unique because the positive solution of $-v'' = m(v + \alpha)^p$ in Ω_0, $v = 0$ on $\partial\Omega_0$ is unique (see e.g., [9], Theorem 2.1). In particular, $w'(0) = 0$. Let $v \doteq w - \alpha$. Integrating over $(0, x_0)$ we find that

$$w'(x_0) = v'(x_0) = \int_0^{x_0} v'' = -\int_0^{x_0} m(v + \alpha)^p \leq -\alpha^p \|m\|_{L^1(0,x_0)}. \tag{2.11}$$

Thus, taking into account (2.9) and (2.11), defining $\underline{u} \doteq \Phi$ in $[x_0, b]$ and $\underline{u} \doteq w$ in $[0, x_0]$ and extending $\underline{u}$ as an even function we obtain a strictly positive weak subsolution for (2.1) (although m is not necessarily even in Ω, this is correct since the definition of Φ is based on the L^∞ norm of m^-) and so from Remark 1.1 we get the solution.

Suppose now that $\|m^-\|_\infty \leq M_2$. Then it holds that

$$\Phi'(x_0) \geq -x_0^{-1} \lambda_1(m, \Omega_0)^{-1/(1-p)}. \tag{2.12}$$

We claim that $w'(x_0) \leq -x_0^{-1} \lambda_1(m, \Omega_0)^{-1/(1-p)}$. Indeed, arguing as in (2.6) we see that φ is concave and it is also even (by uniqueness, because the positive principal eigenvalue is simple and $\|\varphi\|_\infty = 1$). It follows that φ is nonincreasing in $(0, x_0)$ (because φ is C^1) and so $\varphi(0) = 1$ since $\|\varphi\|_\infty = 1$. Hence $\varphi'(x_0) \leq -1/x_0$. Furthermore, recalling that $w - \alpha \geq \lambda_1(m, \Omega_0)^{-1/(1-p)} \varphi$ (by the construction of w) and $w(x_0) = \alpha$, we deduce that $w'(x_0) \leq \lambda_1(m, \Omega_0)^{-1/(1-p)} \varphi'(x_0)$ and thus the claim is proved. Therefore the existence of a solution for (2.1) can be derived as in the final part of the above paragraph taking into account the claim and (2.12).

To end the proof we note that if some $\underline{m}$ satisfies (i) then there exists a solution $\underline{u}$ of (2.1) with $\underline{m}$ in place of m. If $\underline{m} \leq m$, then $\underline{u}$ is a subsolution of (2.1) and so Remark 1.1 proves (ii). □

Remark 2.4. (i) Let us note that M_1 and M_2 neither are comparable. Indeed, fixing m and x_0, one can see that when $b >> 0$ it holds that $M_1 > M_2$, but either if $p \approx 1$ or if $b - x_0 \approx 0$ then $M_2 > M_1$.

(ii) It is also easy to check that in general the conditions in Theorems 2.1 and 2.3 are not comparable.

We next provide further sufficient conditions for the existence of solutions in P°. We observe that here m is assumed to be even in the whole Ω.

Proposition 2.5. (i) *Let $m \in L^2(\Omega)$ be an even function. Suppose there exists $x_0 > 0$ such that $0 \not\equiv m \geq 0$ in Ω_0, and assume either*

$$\left\|m^-\right\|_{L^1(x_0,b)} < \|m\|_{L^1(0,x_0)} \qquad or \tag{2.13a}$$

$$\left\|m^-\right\|_{L^1(x_0,b)}^{1-p} \left[\int_{x_0}^b \left\|m^-\right\|_{L^1(t,b)} dt\right]^p < \frac{1}{x_0^{1-p}\lambda_1(m,\Omega_0)}. \tag{2.13b}$$

Then there exists $u \in W^{2,2}(\Omega) \cap P^\circ$ solution of (2.1)

(ii) *Let $m \in L^2(\Omega)$, and suppose there exists $\underline{m} \leq m$ and satisfying* (i). *Then the same conclusion of* (i) *holds.*

Proof. The proof is similar to the proof of Theorems 2.1 and 2.3 and hence we omit the details. Let us first assume (2.13a). For $\varepsilon > 0$ sufficiently small and $x \in [x_0, b]$, let m_ε and Φ_ε be given by (2.7) and (2.8) respectively, and let $\alpha_\varepsilon \doteq \Phi_\varepsilon(x_0)$. Then $-\Phi_\varepsilon'' \leq m_\varepsilon \Phi_\varepsilon^p$ in (x_0, b) and also $\Phi_\varepsilon(b) = 0$, $\Phi_\varepsilon'(b) = -\varepsilon$. Making ε smaller if necessary, by (2.13a) we can take ε such that $\Phi_\varepsilon'(x_0) \geq -\alpha_\varepsilon^p \|m_\varepsilon\|_{L^1(0,x_0)}$. Indeed, the strict inequality for $\varepsilon = 0$ is a direct consequence of (2.13a) and therefore it is also true for $\varepsilon > 0$ small enough.

On the other side, let w be given by (2.10) with m_ε and α_ε in place of m and α respectively. Then $-w'' \leq m_\varepsilon w^p$ in Ω_0, and as in (2.11) we have $w'(x_0) \leq -\alpha_\varepsilon^p \|m_\varepsilon\|_{L^1(0,x_0)}$. Thus, setting $\underline{u} \doteq \Phi_\varepsilon$ in $[x_0, b]$, $\underline{u} \doteq w$ in $[0, x_0]$ and extending $\underline{u}$ evenly it holds that $\underline{u}$ is a strictly positive weak subsolution for (2.1) with m_ε in place of m, satisfying $|\underline{u}'(\pm b)| \geq \varepsilon$, and Remark 1.1 provides the supersolution.

If (2.13b) is fulfilled, then we can now fix $\varepsilon > 0$ such that $\Phi_\varepsilon'(x_0) \geq -x_0^{-1}\lambda_1(m_\varepsilon, \Omega_0)^{-1/(1-p)}$. Hence as in the last paragraph of Theorem 2.3 we obtain $w'(x_0) \leq \Phi_\varepsilon'(x_0)$ and we can reason as above. □

Remark 2.6. (i) Let us note that if $m_{|[0,b]} = m^+\chi_{[0,x_0]} - m^-\chi_{(x_0,b]}$, then (2.13a) is equivalent to $\int_\Omega m > 0$.

(ii) The inequalities in Proposition 2.5 are also not comparable. Indeed, suppose for instance that $m_{|[0,b]} \doteq k\chi_{[0,x_0]} - c\chi_{(x_0,b]}$ where k, c are two positive constants. In this case (2.13a) and (2.13b) become

$$c(b - x_0) < x_0 k$$

$$c(b - x_0) < \left(\frac{2x_0}{b - x_0}\right)^p \left(\frac{2}{\pi}\right)^2 x_0 k$$

respectively (because $\lambda_1(1, \Omega_0) = (\pi/2x_0)^2$) and it is easy to find situations in which either inequality is stronger than the other, varying p and the relative sizes of x_0 and $b - x_0$.

We end the section exhibiting that similar conditions to the ones stated before must be imposed in order to have existence of solutions for (2.1). Let us observe that the bounds in (2.16) are slightly better than the corresponding ones in (2.15): the integral on M in (2.15) should be running over $(0, x_0)$ instead of $(0, b)$. Let us notice also that clearly the two inequalities in (2.15) (and (2.16)) are non-comparable. Indeed, take for instance $p = 1/2$, fix m^- and vary m^+. In order to avoid overloading the notation we write

$$\|m\|_{L^{\frac{1}{2}}(c,d)} \doteq \left(\int_c^d m^{\frac{1}{2}}\right)^2 \tag{2.14}$$

Theorem 2.7. *Let m be an even function and suppose (2.1) has a solution $u \in W^{2,2}(\Omega)$. If there exists $x_0 > 0$ such that $m, m' \le 0$ in (x_0, b), then*

$$\|m^-\|_{L^{\frac{1}{2}}(x_0,b)} \le M \quad \textit{and}$$

$$\int_{x_0}^b m^-(t) \|m^-\|_{L^{\frac{1}{2}}(t,b)}^{\frac{p}{1-p}} dt \le M^{\frac{p}{1-p}} \|m^+\|_{L^1(0,x_0)},$$

$$\textit{where} \quad M \doteq \frac{2(1+p)}{(1-p)^2} \int_0^b \|m^+\|_{L^1(0,t)} dt; \tag{2.15}$$

and if there exists $x_1 > 0$ such that $m \le 0$ and $m' \ge 0$ in $(0, x_1)$, then

$$\|m^-\|_{L^{\frac{1}{2}}(0,x_1)} \le N \quad \textit{and}$$

$$\int_0^{x_1} m^-(t) \|m^-\|_{L^{\frac{1}{2}}(0,t)}^{\frac{p}{1-p}} dt \le N^{\frac{p}{1-p}} \|m^+\|_{L^1(x_1,b)},$$

$$\textit{where} \quad N \doteq \frac{2(1+p)}{(1-p)^2} \int_{x_1}^b \|m^+\|_{L^1(x_1,t)} dt. \tag{2.16}$$

Proof. Suppose first that $m, m' \le 0$ in (x_0, b). Then, multiplying (2.1) by u' we get $\left((u')^2/2\right)' \le \left(m^- u^{1+p}/(1+p)\right)'$ in (x_0, b). Let $x \in (x_0, b)$. Integrating over (x, b) we find that $(u'(x))^2/2 \ge m^-(x) u(x)^{1+p}/(1+p)$. Observe next that proceeding as in (2.6) we find that u is convex in (x_0, b) because $m \le 0$ there, and hence since $u(b) = 0$ we must have $u' \le 0$ in (x_0, b). Therefore, taking square root in the above inequality, dividing by u^{1+p} and again integrating over (x, b) we obtain $u(x)^{(1-p)/2} \ge \frac{1-p}{2}\left(\frac{2}{1+p}\right)^{1/2} \int_x^b (m^-)^{1/2}$ and hence

$$u(x) \ge \left[\frac{(1-p)^2}{2(1+p)} \|m^-\|_{L^{1/2}(x,b)}\right]^{1/(1-p)} \qquad \text{for all } x \in [x_0, b]. \tag{2.17}$$

Also from (2.1) we get $-u'(x_0) \ge \int_{x_0}^b m^-(t) u^p(t)\, dt$ and so (2.17) implies that

$$u'(x_0) \le -\left[\frac{(1-p)^2}{2(1+p)}\right]^{p/(1-p)} \int_{x_0}^b m^-(t) \|m^-\|_{L^{1/2}(t,b)}^{p/(1-p)} dt. \tag{2.18}$$

On the other side, let $x \in (0, x_0)$ and $t \in (0, b)$. Since u es even (by uniqueness, because m is even) it holds that $u'(0) = 0$. Thus, integrating (2.1) over $(0,t)$ we get $-u'(t) \leq \int_0^t m^+ u^p$ and so integrating now over (x,b) we deduce that

$$0 \leq u(x) \leq \int_x^b \int_0^t m^+(r)\, u^p(r)\, dr dt \leq \|u\|^p_{L^\infty(0,x_0)} \int_0^b \left\| m^+ \right\|_{L^1(0,t)} dt$$

because $m \leq 0$ in (x_0, b). Hence,

$$\|u\|_{L^\infty(0,x_0)} \leq \left[\int_0^b \left\| m^+ \right\|_{L^1(0,t)} dt \right]^{1/(1-p)}. \tag{2.19}$$

In particular, (2.19) and (2.17) with $x = x_0$ prove the first inequality in (2.15). Furthermore, since $-u'(x_0) \leq \int_0^{x_0} m^+ u^p$ we have

$$u'(x_0) \geq -\left[\int_0^b \left\| m^+ \right\|_{L^1(0,t)} dt \right]^{p/(1-p)} \left\| m^+ \right\|_{L^1(0,x_0)} \tag{2.20}$$

and thus (2.18) yields the remaining inequality.

In order to prove (2.16) we may proceed almost exactly as before. As above, u is convex in $(-x_1, x_1)$. Also, since u is even and C^1 it follows that $u' \geq 0$ in $(0, x_1)$. Define $z(x) \doteq u(x) - u(0)$. Then $z, z' \geq 0$ in $(0, x_1)$. Moreover, if $z \equiv 0$ in $(0, \delta)$ for some $\delta > 0$, (2.1) says that $m \equiv 0$ in that interval. But $m \leq 0$ and $m' \geq 0$ in $(0, x_1)$ imply that $m \equiv 0$ in $(0, x_1)$ and in this case we have nothing to prove. Thus we assume that $z > 0$ in $(0, x_1)$. Now, (2.1) gives that $z'' = m^-(z + u(0))^p \geq m^- z^p$ in $(0, x_1)$ and so taking into account the above mentioned facts one we can reason as in the first part of the proof and deduce lower bounds for $z(x)$ $(x \in [0, x_1])$ and $z'(x_1)$ analogous to those in (2.17) and (2.18). Also one can argue as in the preceding paragraph and derive upper bounds for $\|u\|_{L^\infty(x_1,b)}$ and $u'(x_1)$ similar to those in (2.19) and (2.20) and this ends the proof. □

3. The N-dimensional problem in a ball

For $0 < R_0 < R$ and $N \geq 2$ we denote

$$\begin{aligned} B_{R_0} &\doteq \left\{ x \in \mathbb{R}^N : |x| < R_0 \right\}, \\ A_{R_0,R} &\doteq \left\{ x \in \mathbb{R}^N : R_0 < |x| < R \right\}, \\ \omega_{N-1} &\doteq \text{surface area of the unit sphere } \partial B_1 \text{ in } \mathbb{R}^N. \end{aligned}$$

When f is a radial function we shall write (with a slight abuse of notation) $f(x) \doteq f(|x|) \doteq f(r)$. Here we shall denote by P° the interior of the positive cone of $C^{1+\theta}\left(\overline{B_R}\right)$. Let $m : B_R \to \mathbb{R}$ with $m^- \not\equiv 0$. In this section we consider the problem

$$\begin{cases} -\Delta u = m u^p & \text{in } B_R \\ u > 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{3.1}$$

In the first two theorems we study the cases in which m is nonnegative in some $B_{R_0} \subset B_R$. The following one corresponds to the case treated in Proposition 2.5. We note that here the proof becomes quite more technical. The main reason is that there is no simple function which can play the role of Φ_ε in the aforementioned proposition. Observe also that we cannot prove an analogous condition to (2.13b) since the positive principal eigenfunction with respect to a nonnegative weight is not necessarily concave.

Theorem 3.1. (i) *Let $m \in L^q(B_R)$, $q > N$, be a radial function. Suppose there exists $R_0 > 0$ such that $m \geq 0$ in B_{R_0} and*

$$\left\|m^-\right\|_{L^1\left(A_{R_0,R}\right)} < \|m\|_{L^1\left(B_{R_0}\right)}. \tag{3.2}$$

Then there exists $u \in W^{2,q}(B_R) \cap P^\circ$ solution of (3.1).

(ii) *Let $m \in L^q(B_R)$, $q > N$, and suppose there exists $\underline{m} \leq m$ and satisfying* (i)*. Then the same conclusion of* (i) *holds.*

Proof. (ii) follows from Remark 1.1 as in the previous section. In order to prove (i) we shall first assume that $\omega_{N-1}R_0^{N-1} = 1$, and we shall prove it for some

$$m_\delta \doteq \frac{\omega_{N-1}}{(1+\delta)\int_{R_0}^R t^{-(N-1)}\left\|m^-\right\|_{L^1(A_{t,R})}\,dt} m, \qquad \delta > 0. \tag{3.3}$$

Since $m^- \not\equiv 0$ in $A_{R_0,R}$, m_δ is well defined. Let $w_\delta \in W^{2,q}(B_R) \cap P^\circ$ be the unique radial solution of $-\Delta w_\delta = m_\delta^-$ in B_R, $w_\delta = 0$ on ∂B_R, and let Φ be the fundamental solution of Laplace's equation. For $r \in (0,R)$ and $\varepsilon \geq 0$ we set

$$u_{\delta,\varepsilon}(r) \doteq \left(\left\|m_\delta^-\right\|_{L^1(B_R)} + \varepsilon\omega_{N-1}R^{N-1}\right)(\Phi(r) - \Phi(R)) - w_\delta(r).$$

It is clear that $\Delta u_{\delta,\varepsilon} = m_\delta^-$ in $B_R - \{0\}$ and $u_{\delta,\varepsilon}(R) = 0$. Furthermore, since $w_\delta \in W^{2,q}(B_R)$ with $q > 1$ we may apply the divergence theorem (as stated e.g., in [6, p. 742]) and get $-w_\delta'(R)\,\omega_{N-1}R^{N-1} = \int_{B_R} m_\delta^-$, and it also holds that $\Phi'(R) = -\left(R^{N-1}\omega_{N-1}\right)^{-1}$. Therefore $u_{\delta,\varepsilon}'(R) = -\varepsilon$.

We next observe that $r \to u_{\delta,\varepsilon}(r)$ is decreasing in (R_0,R) if $\varepsilon > 0$. Indeed, for $R_0 < r < R$ the divergence theorem yields $\int_{\partial A_{r,R}} \partial u_{\delta,\varepsilon}/\partial\nu = \int_{A_{r,R}} m_\delta^- \geq 0$ and thus $r^{N-1}u_{\delta,\varepsilon}'(r) \leq R^{N-1}u_{\delta,\varepsilon}'(R) < 0$. It follows that $u_{\delta,\varepsilon} > 0$ in $A_{R_0,R}$ because $u_{\delta,\varepsilon}(R) = 0 > u_{\delta,\varepsilon}'(R)$ (if $\varepsilon > 0$).

Claim. There exist $\delta, \varepsilon > 0$ small enough such that

$$u_{\delta,\varepsilon}(r) \leq 1 \text{ for all } r \in [R_0,R] \text{ and } u_{\delta,\varepsilon}'(R_0) \geq -u_{\delta,\varepsilon}^p(R_0)\,\|m_\delta\|_{L^1\left(B_{R_0}\right)}. \tag{3.4}$$

We prove the claim. By (3.2) we may first choose $\delta > 0$ such that

$$\left\|m^-\right\|_{L^1\left(A_{R_0,R}\right)}(1+\delta)^p < \|m\|_{L^1\left(B_{R_0}\right)}. \tag{3.5}$$

Since $\Delta u_{\delta,\varepsilon} = m_\delta^-$ in $B_R - \{0\}$ and $u_{\delta,\varepsilon}'(R) = -\varepsilon$, from the divergence theorem we obtain

$$-u_{\delta,\varepsilon}'(r) = \left[\varepsilon\omega_{N-1}R^{N-1} + \int_{A_{r,R}} m_\delta^-\right] / \left(\omega_{N-1}r^{N-1}\right) \tag{3.6}$$

for every $r \in (0, R)$ and so integrating over (R_0, R) we find that

$$u_{\delta,\varepsilon}(R_0) = \int_{R_0}^{R} \left[\varepsilon\omega_{N-1}R^{N-1} + \left\|m_\delta^-\right\|_{L^1(A_{r,R})}\right] / \left(\omega_{N-1}r^{N-1}\right) dr. \tag{3.7}$$

Taking into account (3.3), (3.7) says that $u_{\delta,0}(R_0) = 1/(1+\delta)$ and so the first inequality in (3.4) is true for $\varepsilon > 0$ sufficiently small. In particular, for such ε's, $-\Delta u_{\delta,\varepsilon} = -m_\delta^- \leq m_\delta u_{\delta,\varepsilon}^p$ in $A_{R_0,R}$. Moreover, since we are assuming $\omega_{N-1}R_0^{N-1} = 1$, from (3.5) and (3.6) with R_0 in place of r we deduce that

$$\begin{aligned} -u_{\delta,0}'(R_0) &= \left[\int_{A_{R_0,R}} m_\delta^-\right] / \left(\omega_{N-1}R_0^{N-1}\right) = \left\|m_\delta^-\right\|_{L^1\left(A_{R_0,R}\right)} \\ &< \frac{1}{(1+\delta)^p}\left\|m_\delta\right\|_{L^1\left(B_{R_0}\right)} = u_{\delta,0}^p(R_0)\left\|m_\delta\right\|_{L^1\left(B_{R_0}\right)} \end{aligned}$$

and so the second condition in (3.4) is also fulfilled for $\varepsilon > 0$ small enough. We fix for the rest of the proof ε and δ satisfying (3.4).

On the other side, let $0 < v \in W^{2,q}(B_{R_0})$ be the unique solution of

$$\begin{cases} -\Delta v = m_\delta v^p & \text{in } B_{R_0} \\ v = u_{\delta,\varepsilon}(R_0) & \text{on } \partial B_{R_0}. \end{cases} \tag{3.8}$$

Such a v exists; in fact, it can be constructed (with the obvious changes) exactly as in (2.10). Furthermore, since the sub- and supersolutions that are used in this case are radial, $m_\delta \geq 0$ in B_{R_0} and one can obtain such a v with a standard iterative procedure, we see that v is a radial function (let us note that the radial symmetry of v also follows from the uniqueness of the solution and the fact that $v(Rx)$ is also a solution when R is an isometry of $\mathbb{R}^N$). Also arguing as in the first part of the proof one can verify that $r \to v(r)$ is nonincreasing in $(0, R_0)$ because m_δ is nonnegative in B_{R_0}. Hence, $v(r) \geq u_{\delta,\varepsilon}(R_0)$ for all $r \in (0, R_0)$ and thus

$$v'(R_0) = \omega_{N-1}R_0^{N-1}v'(R_0) = -\int_{B_{R_0}} m_\delta v^p \leq -u_{\delta,\varepsilon}(R_0)\left\|m_\delta\right\|_{L^1\left(B_{R_0}\right)}. \tag{3.9}$$

Now we set $\underline{u} \doteq u_{\delta,\varepsilon}$ in the annulus $\overline{A_{R_0,R}}$ and $\underline{u} \doteq v$ in B_{R_0}. Using again the divergence theorem and taking into account (3.4) and (3.9) we verify that $\underline{u}$ is a strictly positive weak subsolution for (3.1) satisfying $\partial u/\partial\nu < 0$ on ∂B_R, and therefore (i) follows recalling Remark 1.1.

To end the proof we notice that the assumption $\omega_{N-1}R_0^{N-1} = 1$ can be easily removed. Indeed, pick s such that $\omega_{N-1}(sR_0)^{N-1} = 1$, and let $\overline{u}(x) \doteq u(x/s)$ and $\overline{m}(x) \doteq m(x/s)$. Then (3.1) has a solution if and only if $\overline{u}$ solves

$$\begin{cases} -\Delta \overline{u} = \frac{1}{s^2}\overline{m}\,\overline{u}^p & \text{in } B_{sR} \\ \overline{u} > 0 & \text{in } B_{sR} \\ \overline{u} = 0 & \text{on } \partial B_{sR}. \end{cases} \tag{3.10}$$

But (3.10) has a solution by the above part of the proof, because $m \geq 0$ in B_{R_0} and (3.2) imply $\overline{m} \geq 0$ in B_{sR_0} and $\left\|\overline{m}^-\right\|_{L^1\left(A_{sR_0,sR}\right)} < \|\overline{m}\|_{L^1\left(B_{sR_0}\right)}$ respectively. □

The next result is the analogous of the first bound that comes along in Theorem 2.3. We observe that here we have to impose the restriction

$$\frac{1+p}{1-p} > \frac{(N-1)(R-R_0)}{R_0}$$

which did not appear before. Furthermore, we assume that m is radial in the whole B_R while in Theorem 2.3 m was even only in Ω_0. Let us also point out that if $p \in (1-2/N, 1)$ and $R_0 \geq R/2$ then the bound in (3.11) corresponds to the first inequality in the aforementioned theorem.

Theorem 3.2. (i) *Let $m \in L^q(B_R)$ be a radial function with $q > N$ and $m^- \in L^\infty(B_R)$. Suppose there exists $R_0 > 0$ such that $m \geq 0$ in B_{R_0} and assume that*

$$\left\|m^-\right\|_{L^\infty(B_R)} \leq \frac{(1+p)R_0 - (1-p)(N-1)(R-R_0)}{\omega_{N-1}(1-p)(R-R_0)R_0^N}\|m\|_{L^1\left(B_{R_0}\right)}. \tag{3.11}$$

Then there exists $u \in W^{2,q}(B_R)$ solution of (3.1).

(ii) *Let $m \in L^q(B_R)$, $q > N$, and suppose there exists $\underline{m} \leq m$ and satisfying* (i). *Then the same conclusion of* (i) *holds.*

Proof. We only sketch the proof since it is similar to one part of the proof of Theorem 2.3. Let

$$\beta \doteq \frac{2}{1-p}, \qquad \lambda \doteq \left[\frac{(1-p)^2 R_0 \left\|m^-\right\|_{L^\infty(B_R)}}{2\left[(1+p)R_0 - (1-p)(N-1)(R-R_0)\right]}\right]^{\beta/2} \tag{3.12}$$

and $\Phi(r) \doteq \lambda(R-r)^\beta$ for $r \in (0, R]$. Using (3.12) we can verify that

$$\begin{aligned} \Delta\Phi(r) &= \lambda\beta(R-r)^{\beta-2}\left(\beta - 1 - (N-1)(R-r)/r\right) \\ &\geq \lambda\beta(R-r)^{\beta-2}\left(\beta - 1 - (N-1)(R-R_0)/R_0\right) \\ &\geq m^-(r)\,\Phi^p(r), \qquad \text{for } r \in [R_0, R). \end{aligned}$$

On the other hand, let $0 < v \in W^{2,q}(B_{R_0})$ be the solution of (3.8) with m and $\Phi(R_0)$ in place of m_δ and $u_{\delta,\varepsilon}(R_0)$ respectively. Taking into account (3.11), after some computations we find that

$$\begin{aligned} v'(R_0)\,\omega_{N-1}R_0^{N-1} &= \int_{B_{R_0}} \Delta v = -\int_{B_{R_0}} m v^p \\ &\leq -\|m\|_{L^1\left(B_{R_0}\right)}\Phi(R_0)^p \leq \Phi'(R_0)\,\omega_{N-1}R_0^{N-1} \end{aligned}$$

and the theorem follows. □

Theorem 3.3. (i) *Let* $m \in L^q(B_R)$, $q > N$, *be a radial function. Suppose there exists* $R_1 > 0$ *such that* $m \geq 0$ *in* $A_{R_1,R}$ *and*

$$\int_0^R \frac{1}{\omega_{N-1} r^{N-1}} \left\| m^- \right\|_{L^1(B_r)} dr \leq \frac{1}{\lambda_1(m, A_{R_1,R})}. \tag{3.13}$$

Then there exists $u \in W^{2,q}(B_R) \cap P^\circ$ *solution of* (3.1).
(ii) *Let* $m \in L^q(B_R)$, $q > N$, *and suppose there exists* $\underline{m} \leq m$ *and satisfying* (i). *Then the same conclusion of* (i) *holds.*

Proof. The proof follows the lines of the proof of the first part of the proof of Proposition 2.5 with some changes. Let us indicate them. We prove (i) for

$$\widetilde{m} \doteq \frac{m}{\int_0^R \frac{1}{\omega_{N-1} r^{N-1}} \left\| m^- \right\|_{L^1(B_r)} dr}.$$

For $\beta > 0$, let Φ be the solution of $\Delta\Phi = \widetilde{m}^-$ in B_R, $\Phi = \beta$ on ∂B_R. Adding a constant if necessary we may assume that $\Phi(0) = 0$. The divergence theorem tells us that $r \to \Phi(r)$ is nondecreasing in $(0, R)$. Moreover,

$$\Phi(r) = \int_0^r \Phi'(t)\, dt = \int_0^r \frac{1}{\omega_{N-1} t^{N-1}} \left\| \widetilde{m}^- \right\|_{L^1(B_t)} dt$$

and hence $\|\Phi\|_\infty = 1$.

On the other side, let $\varphi \in W^{2,q}(B_R) \cap P^\circ$ and $\lambda_1(\widetilde{m}, A_{R_1,R})$ be satisfying

$$\begin{cases} -\Delta\varphi = \lambda_1(\widetilde{m}, A_{R_1,R})\, \widetilde{m}\varphi & \text{in } A_{R_1,R} \\ \varphi = 0 & \text{on } \partial A_{R_1,R} \end{cases}$$

with $\|\varphi\|_\infty = 1$. Taking into account (3.13) we establish that $\lambda_1(\widetilde{m}, A_{R_1,R}) \leq 1$ and therefore $-\Delta\varphi \leq \widetilde{m}\varphi^p$ in $A_{R_1,R}$.

Let $\rho \doteq \min\{r \in (R_1, R) : \varphi(r) = 1\}$. It holds that $\Phi(\rho) < 1 = \varphi(\rho)$ and $\Phi(R_1) > 0 = \varphi(R_1)$. It follows that the graph of $\Phi(r)$ intersects the graph of $\varphi(r)$ at some point with abscissa in (R_1, ρ). Let γ_M be the maximum $\gamma > 0$ such that the graph of $(\Phi + \gamma)_{|(R_1,\rho)}$ intersects the graph of $\varphi_{|(R_1,\rho)}$. Clearly such a γ_M exists (in fact, $\gamma_M < 1$). If $\overline{r}$ denotes the abscissa of the intersection point between the graphs of $(\Phi + \gamma_M)_{|(R_1,\rho)}$ and $\varphi_{|(R_1,\rho)}$, then we must have $(\Phi + \gamma_M)(\overline{r}) = \varphi(\overline{r})$ and $(\Phi + \gamma_M)'(\overline{r}) = \varphi'(\overline{r})$. Moreover, $\Delta(\Phi + \gamma_M) = \widetilde{m}^- \geq -\widetilde{m}(\Phi + \gamma_M)^p$ in $B_{\overline{r}}$ because $\Phi + \gamma_M < \varphi(\rho) = 1$ in $B_{\overline{r}}$. Defining now $\underline{u}$ by $\underline{u} \doteq \Phi + \gamma_M$ in $B_{\overline{r}}$ and $\underline{u} \doteq \varphi$ in $\overline{A_{\overline{r},R}}$ we obtain a weak subsolution satisfying $\partial u/\partial\nu < 0$ on ∂B_R. □

The necessary conditions are a straightforward adaptation of the ones for the one-dimensional problem. We note however that the upper bounds for u derived in the proof of Theorem 2.7 are sharper than the ones that we can use here (see also Remark 3.5 below). In order to avoid overloading the notation we leave the $L^{1/2}$-norms (see (2.14)) in terms of $|x| = r$.

Theorem 3.4. *Let* $m \in L^q(B_R)$, $q > N$, *be a radial function and suppose* (3.1) *has a radial solution* $u \in W^{2,q}(B_R)$. *If there exists* $R_0 > 0$ *such that*

$m(r), m'(r) \leq 0$ in (R_0, R), then

$$\left\| m^- (r) \right\|_{L^{1/2}(R_0,R)} \leq \frac{(1+p)\, R^2}{(1-p)^2 N} \left\| m^+ \right\|_{L^\infty(B_{R_0})}; \tag{3.14}$$

and if there exists $R_1 > 0$ such that $m(r) \leq 0$ and $m'(r) \geq 0$ in $(0, R_1)$, then

$$\left\| m^- (r) \right\|_{L^{1/2}(0,R_1)} \leq \frac{(1+p)\left(R^2 - R_1^2\right)}{(1-p)^2 N} \left\| m^+ \right\|_{L^\infty\left(A_{R_1,R}\right)}. \tag{3.15}$$

Proof. Suppose that there exists $R_0 > 0$ such that $m(r) \leq 0$ in (R_0, R), and let u be the radial solution of (3.1). As in the previous theorems, the divergence theorem gives that $r \to u(r)$ is nonincreasing in (R_0, R). Therefore from (3.1) it follows that

$$u''(r) \geq u''(r) + (N-1)\, u'(r)/r = m^-(r)\, u^p(r) \qquad \text{for } r \in (R_0, R),$$

and hence if $m'(r) \leq 0$ in (R_0, R) we may proceed as in the first paragraph of the proof of Theorem 2.7 and get

$$u(R_0)^{1-p} \geq \frac{(1-p)^2}{2(1+p)} \left\| m^-(r) \right\|_{L^{1/2}(R_0,R)}. \tag{3.16}$$

On the other side, recalling that $r \to u(r)$ is nonincreasing in (R_0, R), from (3.1) we also derive

$$\begin{aligned} 0 \leq u &\leq (-\Delta)^{-1}\left(m^+ u^p\right) \leq \left\| m^+ \right\|_{L^\infty(B_R)} \|u\|^p_{L^\infty(B_R)} (-\Delta)^{-1}(1) \\ &= \left\| m^+ \right\|_{L^\infty\left(B_{R_0}\right)} \|u\|^p_{L^\infty\left(B_{R_0}\right)} \left(R^2 - r^2\right)/2N \qquad \text{in } B_R. \end{aligned}$$

Therefore

$$\|u\|^{1-p}_{L^\infty\left(B_{R_0}\right)} \leq \left\| m^+ \right\|_{L^\infty\left(B_{R_0}\right)} R^2/2N \tag{3.17}$$

and thus (3.16) proves (3.14). We finally observe that (3.15) follows similarly. □

Remark 3.5. If in the first part of (i) one also knows that $m(r) \geq 0$ in $(0, R_0)$ one can deduce the exact analogous bounds of Theorem 2.7. Indeed, in this case $r \to u(r)$ is nonincreasing in $(0, R_0)$ and so

$$-u''(r) \leq -u''(r) - (N-1)\, u'(r)/r = m^+(r)\, u^p(r) \qquad \text{for } r \in (0, R_0)$$

and then the upper bounds for u can be proved as in 2.19, and also the rest of the results. The same observation applies if $m(r) \geq 0$ in (R_1, R).

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