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On String S-matrix, Bound States and TBA

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Abstract: The study of finite J effects for the light-cone $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring by means of the Thermodynamic Bethe Ansatz requires an understanding of a companion 2d theory which we call the mirror model. It is obtained from the original string model by the double Wick rotation. The S-matrices describing the scattering of physical excitations in the string and mirror models are related to each other by an analytic continuation. We show that the unitarity requirement for the mirror S-matrix fixes the S-matrices of both theories essentially uniquely. The resulting string S-matrix $S(z_1, z_2)$ satisfies the generalized unitarity condition and, up to a scalar factor, is a meromorphic function on the elliptic curve associated to each variable z. The double Wick rotation is then accomplished by shifting the variables z by quarter of the imaginary period of the torus. We discuss the apparent bound states of the string and mirror models, and show that depending on a choice of the physical region there are one, two or 2^{M-1} solutions of the M-particle bound state equations sharing the same conserved charges. For very large but finite values of J, most of these solutions, however, exhibit various signs of pathological behavior. In particular, they might receive a finite J correction to their energy which is complex, or the energy correction might exceed corrections arising due to finite J modifications of the Bethe equations thus making the asymptotic Bethe ansatz inapplicable.

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1. Introduction and summary

The conjectured duality between the maximally supersymmetric Yang-Mills theory in four dimensions and type IIB superstring in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [1] is the subject of active research. Integrability emerging on both sides of the gauge/string correspondence [2, 3] proved to be an indispensable tool in matching the spectra of gauge and string theories. Namely, it was shown that the problem of determining the spectra in the large volume (charge) limit, can be reduced to the problem of solving a set of algebraic (Bethe) equations. The corresponding Bethe equations are based on the knowledge of the S-matrix which describes the scattering of world-sheet excitations of the gauge-fixed string sigma-model, or alternatively, the excitations of a certain spin chain in the dual gauge theory [4]-[9].

Remarkably, the S-matrix is severely restricted by the requirement of invariance under the global symmetry of the model, the centrally extended $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ superalgebra [9, 10]. The invariance condition fixes its matrix form almost uniquely up to an overall phase [9, 11]. The constraints on the overall phase were derived in [12] by demanding the S-matrix to satisfy crossing symmetry. Recently, a physically relevant solution to the crossing relation, which interpolates between the weak (gauge) and strong (string) coupling regimes was conjectured [13, 14], building on the previous work [15]-[18]. This solution successfully passed a number of non-trivial tests [19]-[34].

So far the main focus of research was on determining the spectrum of string theory in the limiting case in which at least one of the global charges carried by a string state (and by the corresponding gauge theory operator) is large. Our ultimate goal, however, is to understand how the energies of string states (the conformal dimensions of dual gauge theory operators) depend on the coupling constant for finite values of all the other global symmetry charges. Although the conjectured S-matrix [13, 14] provides an important starting point in addressing this issue, by now there is firm evidence that the corresponding Bethe equations [8] fail to correctly reproduce the finite-size effects, neither in string [35] nor in gauge theory [36]. Indeed, already in the semi-classical string theory deviations from the controllable exact spectrum arise which are exponentially small in the effective string length playing the role of a large symmetry charge. One of the reasons behind this is that the interactions on the world-sheet are not ultra-local, typically the scattered states are solitons of finite size [37, 38]. Also, as is common to many field-theoretic models, vacuum polarization effects smear bare point-like interactions and lead to exponential corrections to the energy levels in the large-volume limit [39]. Complementary, in the spin chain description of the dual gauge theory, the obstruction to the validity of the Bethe ansatz comes from the wrapping effects. Hence, the Bethe ansatz for the planar AdS/CFT system in its present form is merely of asymptotic type. Obviously, solving a quantum sigma-model in finite volume is much harder. As an illustrative example, we mention the Sinh-Gordon model for which the equations describing the finite-size spectrum have been recently obtained in [40].

Determination of finite-size effects in the context of integrable models is a wide area of research. Basically, there are three different but related ways in which this problem could be addressed: the thermodynamic Bethe ansatz approach (TBA) [41], nonlinear integral equations (NLIE) [42] and functional relations for commuting transfer-matrices [43].

The aim of the present paper is to investigate the structure of the string S-matrix which is the necessary step for constructing the TBA equations for the quantum string sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. This would eventually allow one to describe the finite-size spectrum of the corresponding model.

The TBA approach was initially developed for studying thermodynamic properties of non-relativistic quantum mechanics in one dimension [44] and further applied to the computation of the ground state energy in integrable relativistic field theories in finite volume [41]. The method also was later extended to account for energies of excited states [45] (see also [46]).

Implementation of the TBA approach consists of several steps. The primary goal is to obtain an expression for the ground state energy of a Lorentzian theory compactified on a circle of circumference L and at zero temperature. The starting point is the *Euclidean extension* of the original theory, put on a torus generated by two orthogonal circles of circumferences L and R. The partition function of this theory can be viewed as originating from two different Lorentzian theories: the *original* one, which lives on a circle of length L at temperature T and has the Hamiltonian H, or the *mirror* theory which is defined on a circle of length $R = 1/T$ at temperature $\tilde{T} = 1/L$ and has the Hamiltonian $\tilde{H}$. For Lorentz-invariant theories, the original and the mirror Hamiltonians are the same. However, in general and in particular for the case of interest here, the two theories need not be the same. Taking the thermodynamic limit $R \to \infty$ one ends up with the mirror theory on a line and at finite temperature, for which the exact (mirror) Bethe equations can be written. Thus, computation of finite-size effects in the original theory translates into the problem of solving the infinite volume mirror theory at finite temperature.

Although taking the thermodynamic limit simplifies the system, a serious complication arises due to the fact that the mirror theory could have bound states which manifest themselves as poles of the two-particle mirror S-matrix. Thus, the complete spectrum would consist of particles and their bound states; the latter should be thought of as new asymptotic particles. Having identified the spectrum, one has to determine the S-matrix which scatters *all* asymptotic particles. It is this S-matrix which should be used to formulate the system of TBA equations.

As is clear from the discussion above, the mirror theory plays a crucial role in the TBA approach. In this paper we will analyze the mirror theory in some detail. First, we will explain its relation to the original theory. Indeed, given that the model in question is not Lorentz invariant, the mirror and the original Hamiltonians are not the same. However, since the dispersion relation and the S-matrix can be deduced from the 2-point and 4-point correlation functions on the world-sheet, and since the correlation functions in the mirror theory are inherited from the original model by performing a double Wick rotation, it follows that the mirror dispersion relation and the mirror S-matrix can be obtained from the original ones by the double Wick rotation. Here we will meet an important subtlety. To explain it, we first have to recall the basic properties of the string S-matrix.

As was shown in [47], the $\mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)$-invariant S-matrix $\mathcal{S}(p_1,p_2)$, which depends on real momenta p_1 and p_2 of scattering particles, obeys

- the Yang-Baxter equation

$$\mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12}=\mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23}$$

- the unitarity condition

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{21}(p_2,p_1)=\mathbb{I}$$

- the physical unitarity condition

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}^{\dagger}_{12}(p_1,p_2)=\mathbb{I}$$

- the requirement of crossing symmetry

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(p_1,p_2)\mathscr{C}_1\mathcal{S}_{12}(-p_1,p_2)=\mathbb{I}\,,$$

  where $\mathscr{C}$ is the charge conjugation matrix.

The first three properties naturally follow from the consistency conditions of the associated Zamolodchikov-Faddeev (ZF) algebra [48, 49], while the last one reflects the fact that the particle-to-anti-particle transformation is an automorphism of the ZF algebra [47]. The unitarity and physical unitarity conditions imply the following property

$$\mathcal{S}_{21}(p_2,p_1)=\mathcal{S}^{\dagger}_{12}(p_1,p_2)\,.$$

One should bear in mind that the S-matrix is defined up to unitary equivalence only: unitary transformations (depending on the particle momentum) of a basis of one-particle states correspond to unitary transformations of the scattering matrix without spoiling any of the properties listed above.

The mirror S-matrix $\tilde{\mathcal{S}}(\tilde{p}_1, \tilde{p}_2)$ is obtained from $\mathcal{S}(p_1, p_2)$ by the double Wick rotation. The above-mentioned subtlety lies in the fact that only for *a very special choice* of the one-particle basis the corresponding mirror S-matrix remains unitary. As we will show, this problem can be naturally attributed to the properties of the double Wick rotation for fermionic variables. Upon the basis is properly chosen to guarantee unitarity of the mirror S-matrix, the only freedom in the *matrix structure* of $\mathcal{S}(p_1, p_2)$ reduces to constant, i.e. momentum-independent, unitary transformations[1].

There is another interesting explanation of the interrelation between the original theory and its mirror. As was shown in [12], the dispersion relation between the energy and momentum of a single particle can be naturally uniformized in terms of a complex variable z living on a torus with real and imaginary periods equal to $2\omega_1$ and $2\omega_2$, respectively. Since z plays the role of the generalized rapidity variable, it is quite natural to think about the S-matrix as the function $\mathcal{S}(z_1, z_2)$, which for real values of the generalized rapidity variables coincides with $\mathcal{S}(p_1, p_2)$. In other words, the S-matrix admits an analytic continuation to the complex values of momenta. It appears that the unitary momentum-dependent freedom in the matrix structure of the S-matrix gets fixed if we require the analytic continuation to be compatible with the requirement of

- generalized unitarity

$$\mathcal{S}_{12}(z_1^*, z_2^*)\big[\mathcal{S}_{12}(z_1, z_2)\big]^\dagger = \mathbb{I}\,,$$

which can be thought of as the physical unitarity condition extended to the generalized rapidity torus. The unitarity and the generalized unitarity further imply

$$\mathcal{S}_{21}(z_2^*, z_1^*) = \big[\mathcal{S}_{12}(z_1, z_2)\big]^\dagger$$

In fact, the last equation is equivalent to the standard requirement of hermitian analyticity for an S-matrix in two-dimensional relativistic quantum field theories.

Thus, the S-matrix which admits the analytic continuation to the generalized rapidity torus compatible with the requirement of hermitian analyticity is essentially unique. Of course, it satisfies all the other properties listed above, including crossing symmetry. As we will show, the mirror S-matrix is obtained from $\mathcal{S}(z_1, z_2)$ considered for real values of z_1, z_2 by shifting z_1, z_2 by quarter of the imaginary period

$$\tilde{\mathcal{S}}(z_1, z_2) = \mathcal{S}\big(z_1 + \tfrac{\omega_2}{2}, z_2 + \tfrac{\omega_2}{2}\big)\,.$$

[1]Of course, there is always a freedom of multiplying the S-matrix by an overall (momentum-dependent) phase.

There is a close analogy with what happens in relativistic models. In the latter case the physical region is defined as the strip $0 \le \operatorname{Im}\theta \le \pi$, where $\theta = \theta_2 - \theta_1$ is the rapidity variable. A passage to the mirror theory corresponds to the shift $\theta_k \to \theta_k + i\frac{\pi}{2}$, i.e. to the shift by the quarter of imaginary period[2]. Of course, for relativistic models, due to Lorentz invariance, the S-matrix depends on the difference of rapidities and, therefore, it remains unchanged under the double Wick rotation transformation. Also, in our present case the notion of the physical region is not obvious and its identification requires further analysis of the analytic properties of the string S-matrix.

Having identified the mirror S-matrix, we can investigate the question about the bound states. We first discuss the Bethe equations for the gauge-fixed string theory where the existence of the BPS bound states is known [50]. No non-BPS bound states exist, according to [50]-[52]. We find out, however, that the number of solutions of the BPS bound state equations depends on the choice of the physical region of the model, and for a given value of the bound state momentum there could be 1, 2 or 2^{M-1} M-particle bound states sharing the same set of global conserved charges. It is unclear to us whether this indicates that the actual physical region is the one that contains only a single M-particle bound state or it hints on a hidden symmetry of the model responsible for the degeneracy of the spectrum. These solutions behave, however, differently for very large but finite values of L; most of them exhibit various signs of pathological behavior. In particular, they might have complex finite L correction to the energy, or the correction would exceed the correction due to finite L modifications of the Bethe equations thus making the asymptotic Bethe ansatz inapplicable. In the weak coupling limit, i.e. in perturbative gauge theory, and for small enough values of the bound state momentum only one solution reduces to the well-known Bethe string solution of the Heisenberg spin chain. It is also the only solution that behaves reasonably well for finite values of L. Therefore, it is tempting to identify the physical region of the string model as the one that contains this solution only.

By analyzing the Bethe equations for the mirror theory, we show that bound states exist and that they can be regarded as "mirror reflections" of the BPS bound states in the original theory. No other bound states exist, in agreement with the results by [52]. Given the knowledge of bound states, the next step would be to construct the S-matrix which describes scattering of all asymptotic particles including the ones which correspond to bound states. In principle, such an S-matrix can be obtained by the fusion procedure [53, 54] applied to the "fundamental" S-matrix we advocate here. This is the bootstrap program whose discussion we will postpone for the future.

The paper is organized as follows. The next section contains the discussion of the double Wick rotation, the mirror dispersion relation and the mirror magnon. In

[2]The shift of θ by the half-period corresponds to the crossing transformation.

section 3 we discuss the supersymmetry algebra and the construction of the mirror S-matrix. In section 4 we analyze the double Wick rotation on the generalized rapidity torus as well as various possible definitions for the physical region. In section 5 the properties of the string S-matrix defined on the generalized rapidity torus are discussed. We also prove here the unitarity of the scalar factor in the mirror theory. In section 6 we present various versions of the Bethe equations in the original and mirror theory pointing out that the Bethe equations based on the $\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$-invariant string S-matrix should be modified in the odd winding number sector since for this case the fermions of the gauge-fixed string sigma model are anti-periodic. Sections 7 and 8 contain an analysis of the bound states of the $\mathrm{AdS}_5\times\mathrm{S}^5$ gauge-fixed model and its mirror theory. Section 9 consists of several appendices.

2. Generalities

In this section we discuss how the vacuum energy of a two-dimensional field theory on a circle can be found by considering the Thermodynamic Bethe Ansatz for a mirror model obtained from the field theory by a double Wick rotation. We follow the approach developed in [41].

2.1 Double Wick rotation and mirror Hamiltonian

Consider any two-dimensional field-theoretic model defined on a circle of circumference L. Let

$$H=\int_0^L \mathrm{d}\sigma\,\mathcal{H}(p,x,x') \tag{2.1}$$

be the Hamiltonian of the model, where p and x are canonical momenta and coordinates. They may also include fermions but in this section we confine ourselves to bosonic fields only. We will refer to the action corresponding to the Hamiltonian H as to the Minkowski action, however it does not have to be relativistic invariant.

We want to compute the partition function of the model defined as follows

$$Z(R,L)\equiv\sum_n\langle\psi_n|e^{-HR}|\psi_n\rangle=\sum_n e^{-E_nR}\,, \tag{2.2}$$

where $|\psi_n\rangle$ is the complete set of eigenstates of H. By using the standard path integral representation [55], we get

$$Z(R,L)=\int\mathcal{D}p\,\mathcal{D}x\,e^{\int_0^R d\tau\int_0^L\,d\sigma(ip\dot{x}-\mathcal{H})}\,, \tag{2.3}$$

where the integration is taken over x and p periodic in both τ and σ. Formula (2.3) shows that $-\int_0^R d\tau\int_0^L\,d\sigma(ip\dot{x}-\mathcal{H})$ can be understood as the Euclidean action written in the first-order formalism. Indeed, integrating over p in the usual first-order action $\int_0^R d\tau \int_0^L d\sigma (p\dot{x} - \mathcal{H})$, we get the Minkowski-type action, and the Euclidean action is obtained from it by replacing $\dot{x} \to i\dot{x}$ which is equivalent to the Wick rotation $\tau \to -i\tau$.

Let us now take the Euclidean action, and replace $x' \to -ix'$ or, equivalently, do the Wick rotation of the σ-coordinate $\sigma \to i\sigma$. As a result we get the action where σ can be considered as the new time coordinate. Let $\widetilde{H}$ be the Hamiltonian with respect to σ

$$\widetilde{H} = \int_0^R \mathrm{d}\tau \, \widetilde{\mathcal{H}}(\widetilde{p}, x, \dot{x}) \,, \tag{2.4}$$

where $\widetilde{p}$ are canonical momenta of the coordinates x with respect to σ.

We will refer to the model with the Hamiltonian $\widetilde{H}$ as to the mirror theory. If the original model is not Lorentz-invariant then the mirror Hamiltonian is not equal to the original one, and the Hamiltonians H and $\widetilde{H}$ describe *different* Minkowski theories.

The partition function of the mirror model is given by

$$\widetilde{Z}(R, L) \equiv \sum_n \langle \widetilde{\psi}_n | e^{-\widetilde{H} L} | \widetilde{\psi}_n \rangle = \sum_n e^{-\widetilde{E}_n L} \,, \tag{2.5}$$

where $|\widetilde{\psi}_n\rangle$ is the complete set of eigenstates of $\widetilde{H}$. Again, by using the path integral representation, we obtain

$$\widetilde{Z}(R, L) = \int \mathcal{D}\widetilde{p} \, \mathcal{D}x \, e^{\int_0^R d\tau \int_0^L d\sigma (i\widetilde{p}x' - \widetilde{\mathcal{H}})} \,. \tag{2.6}$$

Finally, integrating over $\widetilde{p}$, we get the same Euclidean action and, therefore, we conclude that

$$\widetilde{Z}(R, L) = Z(R, L) \,. \tag{2.7}$$

Now, if we take the limit $R \to \infty$, then $\log Z(R, L) \sim -RE(L)$, where $E(L)$ is the ground state energy. On the other hand, $\log \widetilde{Z}(R, L) \sim -RLf(L)$, where $f(L)$ is the bulk free energy of the system at temperature $T = 1/L$ with σ considered as the time variable. This leads to the relation

$$E(L) = Lf(L) \,. \tag{2.8}$$

To find the free energy we can use the thermodynamic Bethe ansatz because $R >> 1$. This requires, however, the knowledge of the S-matrix and the asymptotic Bethe equations for the mirror system with the Hamiltonian $\widetilde{H}$. Although the light-cone gauge-fixed string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is not Lorentz invariant, $\widetilde{H} \neq H$, it is still natural to expect that there is a close relation between the two systems because their Euclidean versions coincide.

A potential problem with the proof that $\widetilde{Z}(R,L) = Z(R,L)$ is that the integration over p and $\widetilde{p}$ produces additional measure factors which may be nontrivial. The contribution of such a factor is however local, and one usually does not have to take it into account. We will assume throughout the paper that this would not cause any problem.

2.2 Mirror dispersion relation

The dispersion relation in any quantum field theory can be found by analyzing the pole structure of the corresponding two-point correlation function. Since the correlation function can be computed in Euclidean space, both dispersion relations in the original theory with H and in the mirror one with $\widetilde{H}$ are obtained from the following expression

$$H_{\rm E}^2 + 4g^2 \sin^2 \frac{p_{\rm E}}{2} + 1\,, \tag{2.9}$$

which appears in the pole of the 2-point correlation function. Here and in what follows we consider the light-cone gauge-fixed string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ which has the Euclidean dispersion relation (2.9) in the decompactification limit $L \equiv P_+ \to \infty$ [5, 6, 9, 10]. The parameter g is the string tension, and is related to the 't Hooft coupling λ of the dual gauge theory as $g = \frac{\sqrt{\lambda}}{2\pi}$.

Then the dispersion relation in the original theory follows from the analytic continuation (see also [39])

$$H_{\rm E} \to -iH\,, \qquad p_{\rm E} \to p \quad \Rightarrow \quad H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}\,, \tag{2.10}$$

and the mirror one from

$$H_{\rm E} \to \widetilde{p}\,, \qquad p_{\rm E} \to i\widetilde{H} \quad \Rightarrow \quad \widetilde{H} = 2\,\mathrm{arcsinh}\Big(\frac{1}{2g}\sqrt{1+\widetilde{p}^2}\Big)\,. \tag{2.11}$$

Comparing these formulae, we see that p and $\widetilde{p}$ are related by the following analytic continuation

$$p \to 2i\,\mathrm{arcsinh}\Big(\frac{1}{2g}\sqrt{1+\widetilde{p}^2}\Big)\,, \qquad H = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}} \to i\widetilde{p}\,. \tag{2.12}$$

We note that the plane-wave type limit corresponds to taking $g \to \infty$ with $\widetilde{p}$ fixed, in which case we get the standard relativistic dispersion relation

$$\widetilde{H}_{\rm pw} = \frac{1}{g}\sqrt{1+\widetilde{p}^2}\,. \tag{2.13}$$

The expression above suggests that in this limit it is natural to rescale $\widetilde{H}$ by $1/g$ or, equivalently, to rescale $\widetilde{\tau} = i\sigma$ by g. This also indicates that the semi-classical limit in the mirror theory should correspond to $g \to \infty$ with $\widetilde{p}/g$ fixed, so that the dispersion relation acquires the form

$$\widetilde{H}_{\rm sc} = 2\,\mathrm{arcsinh}\left(\frac{|\widetilde{p}|}{2g}\right)\,. \tag{2.14}$$

We will show in the next subsection that the mirror theory admits a one-soliton solution whose energy exactly reproduces eq.(2.14).

In what follows we need to know how the parameters $x^{\pm}$ introduced in [5] are expressed through $\widetilde{p}$. By using formulae (2.12), we find

$$x^{\pm}(p) \to \frac{1}{2g}\left(\sqrt{1+\frac{4g^2}{1+\widetilde{p}^2}} \mp 1\right)(\widetilde{p}-i) \tag{2.15}$$

and, as a consequence,

$$ix^- - ix^+ \to \frac{1}{g}\,(1+i\widetilde{p})\ .$$

Note that these relations are well-defined for real p, but one should use them with caution for complex values of p. In section 4 we introduce a more convenient parametrization of the physical quantities in terms of a complex rapidity variable z living on a torus [12]. In this parametrization the analytic continuation would simply correspond to the shift of z by the quarter of the imaginary period of the torus.

2.3 Mirror magnon

In this section we will derive the dispersion relation for the "giant magnon" in the semi-classical mirror theory. This will provide further evidence for the validity of the proposed dispersion relation (2.14).

Consider the classical string sigma-model on $\mathrm{AdS}_5\times \mathrm{S}^5$ and fix the generalized uniform light-cone gauge as in [56, 57]. The gauge choice depends continuously on a parameter a with the range $0 \le a \le 1$. The gauge-fixed Lagrangian in the generalized a-gauge can be obtained either from the corresponding Hamiltonian [59, 57] by using the canonical formalism or by T-dualizing the action in the direction canonically conjugate to the light-cone momentum P_+ [26]. Its explicit form in terms of the world-sheet fields is given in appendix 9.1. To keep the discussion simple, in what follows we will restrict our analysis to the $a = 1$ gauge[3].

We are interested in finding a soliton solution in the mirror theory, which is obtained from the original theory via the double Wick rotation with further exchange of the time and spacial directions

$$\tilde{\sigma} = -i\tau\,, \qquad\qquad \tilde{\tau} = i\sigma\,, \tag{2.16}$$

[3]Recall that unlike to the case of finite P_+ the dispersion relation of the giant magnon in the infinite volume limit $P_+ = \infty$ was shown to be gauge independent [38].

where σ, τ are the variables parametrizing the world-sheet of the original theory.

Recall that the giant magnon can be thought of as a solution of the light-cone gauge-fixed string sigma-model described by a solitonic profile $y \equiv y(\sigma - v\tau)$, where y is one of the fields parametrizing the five-sphere and v is the velocity of the soliton[4]. In the infinite P_+ limit this soliton exhibits the dispersion relation (2.10), where p coincides, in fact, with the total world-sheet momentum $p_{\rm ws}$ carried by the soliton. Owing to the same form of the dispersion relation in the dual gauge theory, this gives a reason to call this soliton a "giant magnon" [37]. For our further discussion it is important to realize that if, instead of taking the field y from the five-sphere, we would make a solitonic ansatz $z \equiv z(\sigma - v\tau)$, where z is one of the fields parametrizing AdS_5, we would find no solutions exhibiting the dispersion (2.10). As we will now show, in the mirror theory the situation is reversed: this time the giant magnon propagates in the AdS part, while there is no soliton solution associated to the five-sphere.

Take the string Lagrangian (9.1) in the gauge $a = 1$ and put all the fields to zero except a single excitation z from AdS_5. Upon making the double Wick rotation (2.16), the corresponding mirror action can be written as follows

$$S = g \int_{-r}^{r} \mathrm{d}\tilde{\sigma}\mathrm{d}\tilde{\tau} \left(-1 + \frac{\sqrt{1 + z^2 - z'^2 + (1 + z^2)\dot{z}^2}}{1 + z^2} \right) \equiv \int_{-r}^{r} \mathrm{d}\tilde{\sigma}\mathrm{d}\tilde{\tau}\, \mathscr{L} \,. \qquad (2.17)$$

Here r is an integration bound for $\tilde{\sigma}$ and $\dot{z} \equiv \partial_{\tilde{\tau}} z \,, z' \equiv \partial_{\tilde{\sigma}} z$. Although our goal is to identify the mirror magnon configuration in the decompactification limit, i.e. when $r \to \infty$, for the moment we prefer to keep r finite.

To construct a one-soliton solution of the equations of motions corresponding to the action (2.17), we make the following ansatz

$$z = z(\tilde{\sigma} - v\tilde{\tau}) \,.$$

Our further discussion follows closely [38]. Plugging the ansatz into (2.17), we obtain the reduced Lagrangian, $L_{red} = L_{red}(z, z')$, which describes a one-particle mechanical system with $\tilde{\sigma}$ treated as a time variable. Introducing the canonical momentum π conjugate to z, we construct the corresponding reduced Hamiltonian

$$H_{\rm red} = \pi z' - L_{red} \,,$$

which is a conserved quantity with respect to time $\tilde{\sigma}$. Fixing $H_{\rm red} = 1 - \omega$, where ω is a constant, we get the following equation to determine the solitonic profile

$$(z')^2 = \frac{1 + z^2 - \frac{1}{\omega^2}}{1 - v^2 - v^2 z^2} \,. \qquad (2.18)$$

[4]See [38] and appendix 9.1 for more details.

The minimal value of z corresponds to the point where the derivative of z vanishes, while the maximum value is achieved at the point where the derivative diverges

$$z_{\rm min} = \sqrt{\frac{1}{\omega^2} - 1}\,, \qquad z_{\rm max} = \sqrt{\frac{1}{v^2} - 1}\,, \quad v < \omega < 1\,.$$

The range of $\tilde{\sigma}$ is determined from the equation

$$r = \int_0^r {\rm d}\tilde{\sigma} = \int_{z_{\rm min}}^{z_{\rm max}} \frac{{\rm d}z}{|z'|} = \sqrt{1-v^2}\ \ {\rm E}\left(\arcsin(z\sqrt{\omega^2/(1-\omega^2)}), \eta\right)\Big|_{z_{\rm min}}^{z_{\rm max}}\,,$$

where we have introduced $\eta = \frac{v^2}{\omega^2}\frac{1-\omega^2}{1-v^2}$. Here E stands for the elliptic integral of the second kind. We see that the range of σ tends to infinity when $\omega \to 1$. Thus, $\omega \to 1$ corresponds to taking the decompactification limit.

The density of the world-sheet Hamiltonian is given by

$$\widetilde{\mathcal{H}} = p_z \dot{z} - \mathscr{L}\,,$$

where $p_z = \frac{\partial \mathscr{L}}{\partial \dot{z}}$ is the momentum conjugate to z with respect to time $\tilde{\tau}$. For our solution in the limiting case $\omega = 1$ we find

$$p_z = -\frac{v|z|}{\sqrt{1 - v^2(1+z^2)}}\,.$$

The energy of the soliton is then

$$\widetilde{H} = g\int_{-\infty}^{\infty} {\rm d}\sigma\, \widetilde{\mathcal{H}} = 2g\int_{z_{\rm min}}^{z_{\rm max}} \frac{{\rm d}z}{|z'|}\,\widetilde{\mathcal{H}} = 2g\,{\rm arcsinh}\frac{\sqrt{1-v^2}}{|v|}\,.$$

To find the dispersion relation, we also need to compute the world-sheet momentum $p_{\rm ws}$, the latter coincides with the momentum $\tilde{p}$ of the mirror magnon considered as a point particle. It is given by

$$\tilde{p} = p_{\rm ws} = -\int_{-\infty}^{\infty} {\rm d}\sigma\, p_z z' = 2\int_{z_{\rm min}=0}^{z_{\rm max}} {\rm d}z|p_z| = 2\frac{\sqrt{1-v^2}}{|v|}\,.$$

Finally, eliminating v from the expressions for $\widetilde{H}$ and $\tilde{p}$ we find the following dispersion relation

$$\widetilde{H} = 2g\,{\rm arcsinh}\,\frac{|\tilde{p}|}{2}\,.$$

To consider the semi-classical limit $g \to \infty$, one has to rescale the time as $\tilde{\tau} \to \tilde{\tau}/g$ so that the energy $\widetilde{H} \to g\widetilde{H}$ will be naturally measured in units of $1/g$. Under this rescaling the momentum $\tilde{p}$ scales as well, so that the dispersion relation takes the form

$$\widetilde{H} = 2\,{\rm arcsinh}\,\frac{|\tilde{p}|}{2g}\,, \tag{2.19}$$

which is precisely the previously announced expression (2.14) for the energy of the mirror magnon.

3. Mirror S-matrix and supersymmetry algebra

The S-matrix in field theory can be obtained from four-point correlation functions by using the LSZ reduction formula. Since the correlation functions can be computed by means of the Wick rotation, it is natural to expect that the mirror S-matrix is related to the original one by the same analytic continuation

$$\widetilde{S}(\widetilde{p}_1, \widetilde{p}_2) = S(p_1, p_2)\,, \tag{3.1}$$

where we replace p_i in the original S-matrix by $\widetilde{p}_i$ by using formulas (2.12). Just as the original S-matrix, the resulting mirror S-matrix should satisfy the Yang-Baxter equation, unitarity, physical unitarity, and crossing relations for real $\widetilde{p}_k$.

On the other hand, the original S-matrix is $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ invariant and the states of the light-cone gauge-fixed $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory carry unitary representations of the symmetry algebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$. Therefore, if the relation (3.1) is correct then the mirror S-matrix should possess the same symmetry, and the states of the mirror theory also should carry unitary representations of $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$. This indicates that there should exist a way to implement the double Wick rotation on the symmetry algebra level, and that is what we discuss in this section.

3.1 Double Wick rotation for fermions

It is obvious that the double Wick rotation preserves the bosonic symmetry $\mathrm{SU}(2)^4$. To understand what happens with the supersymmetry generators it is instructive to apply the double Wick rotation to fermions. We consider the quadratic part of the light-cone gauge-fixed Green-Schwarz action depending on the fermions η in the form given in [57]

$$\mathscr{L} = i\eta_a^\dagger \dot{\eta}_a - \frac{1}{2}\left(\eta_a \eta'_{5-a} - \eta_a^\dagger \eta'^\dagger_{5-a}\right) - \eta_a^\dagger \eta_a = i\eta_a^\dagger \dot{\eta}_a - \mathcal{H}\,. \tag{3.2}$$

Here, $a = 1, 2, 3, 4$, and we set $\kappa = 1$ and rescale σ in the action from [57] so that $\widetilde{\lambda}$ disappears.

Computing again the partition function of the model and using the path integral representation, we get

$$Z(R, L) = \int \mathcal{D}\eta^\dagger \mathcal{D}\eta e^{\int_0^R d\tau \int_0^L d\sigma (-\eta_a^\dagger \dot{\eta}_a - \mathcal{H})}\,. \tag{3.3}$$

We note that fermionic variables here are anti-periodic in the time direction:

$$\eta(\tau + R) = -\eta(\tau).$$

Would fermions be periodic in the time direction, the corresponding path integral would coincide with Witten's index $\mathrm{Tr}(-1)^F e^{-HR}$, where F is the fermion number

[58]. Since in the mirror model τ plays the role of the spatial direction, the mirror fermions are always anti-periodic in the spacial direction of the mirror model. On the other hand, the periodicity condition in the time direction of the mirror model coincides with a fermion periodicity condition in the spacial direction of the original model. In particular, if the fermions of the original model are periodic then the partition function of the original model is equal to the Witten's index of the mirror model.

After the first Wick rotation the Lagrangian takes the form

$$\mathscr{L} = -\eta_a^\dagger \dot{\eta}_a - \frac{1}{2}\left(\eta_a \eta_{5-a}' - \eta_a^\dagger \eta'^\dagger_{5-a}\right) - \eta_a^\dagger \eta_a \,. \tag{3.4}$$

Note that the fermions in this Euclidean action are not anymore hermitian conjugate to each other.

Let us now perform the following change of the fermionic variables

$$\eta_a = \frac{i}{\sqrt{2}}\left(\psi_{5-a}^\dagger - \psi_a\right) \,, \quad \eta_a^\dagger = \frac{i}{\sqrt{2}}\left(\psi_a^\dagger + \psi_{5-a}\right) \,. \tag{3.5}$$

Computing (3.4), we get

$$\mathscr{L} = -\psi_a^\dagger \psi_a' - \frac{1}{2}\left(\psi_a \dot{\psi}_{5-a} - \psi_a^\dagger \dot{\psi}_{5-a}^\dagger\right) - \psi_a^\dagger \psi_a \,. \tag{3.6}$$

It is the same action as (3.4) after the interchange $\tau \leftrightarrow \sigma$ and $\psi \to \eta$, and this shows that the double Wick rotation should be accompanied by the change of variables (3.5). Note, that in terms of ψ's the supersymmetry algebra has the standard form with the usual unitarity condition. Thus, we expect that the supersymmetry generators will be linear combinations of the original ones. One may assume that in the interacting theory (beyond the quadratic level) one would take the same linear combinations.

To summarize, the consideration above seems to indicate that the symmetry algebra of the mirror theory should correspond to a different real slice of the complexified $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra. Moreover, one might expect that the unitary representation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string model could be chosen in such a way that its analytic continuation by means of formulae (2.12) would produce a unitary representation of the mirror model.

3.2 Changing the basis of supersymmetry generators

Let us recall that the centrally extended $\mathfrak{su}(2|2)$ algebra consists of the bosonic rotation generators $\mathbf{L}_a{}^b$, $\mathbf{R}_\alpha{}^\beta$, the supersymmetry generators $\mathbf{Q}_\alpha{}^a$, $\mathbf{Q}_a^{\dagger\alpha}$, and three central elements $\mathbf{H}$, $\mathbf{C}$ and $\mathbf{C}^\dagger$. The algebra relations are

$$
\begin{aligned}
&\big[\mathbf{L}_a{}^b, \mathbf{J}_c\big] = \delta_c^b \mathbf{J}_a - \frac{1}{2}\delta_a^b \mathbf{J}_c\,, && \big[\mathbf{R}_\alpha{}^\beta, \mathbf{J}_\gamma\big] = \delta_\gamma^\beta \mathbf{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbf{J}_\gamma\,,\\
&\big[\mathbf{L}_a{}^b, \mathbf{J}^c\big] = -\delta_a^c \mathbf{J}^b + \frac{1}{2}\delta_a^b \mathbf{J}^c\,, && \big[\mathbf{R}_\alpha{}^\beta, \mathbf{J}^\gamma\big] = -\delta_\alpha^\gamma \mathbf{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbf{J}^\gamma\,,\\
&\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_b^{\dagger\beta}\} = \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a + \frac{1}{2}\delta_b^a\delta_\alpha^\beta \mathbf{H}\,,\\
&\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\,\mathbf{C}\,, && \{\mathbf{Q}_a^{\dagger\alpha}, \mathbf{Q}_b^{\dagger\beta}\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,\mathbf{C}^\dagger\,.
\end{aligned}
\tag{3.7}
$$

Here in the first two lines we indicate how the indices c and γ of any Lie algebra generator transform under the action of $\mathbf{L}_a{}^b$ and $\mathbf{R}_\alpha{}^\beta$. For the $\mathrm{AdS}_5\times\mathrm{S}^5$ string model the supersymmetry generators $\mathbf{Q}_\alpha{}^a$ and $\mathbf{Q}_a^{\dagger\alpha}$, and the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$ are hermitian conjugate to each other: $(\mathbf{Q}_\alpha{}^a)^\dagger = \mathbf{Q}_a^{\dagger\alpha}$. The central element $\mathbf{H}$ is hermitian and is identified with the world-sheet light-cone Hamiltonian. It was shown in [10] that the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$ are expressed through the world-sheet momentum $\mathbf{P}$ as follows

$$
\mathbf{C} = \frac{i}{2} g\,(e^{i\mathbf{P}}-1)e^{2i\xi}\,,\quad \mathbf{C}^\dagger = -\frac{i}{2} g\,(e^{-i\mathbf{P}}-1)e^{-2i\xi}\,,\quad g = \frac{\sqrt{\lambda}}{2\pi}\,. \tag{3.8}
$$

The phase ξ is an arbitrary function of the central elements, and reflects the obvious U(1) automorphism of the algebra (3.7): $\mathbf{Q}\to e^{i\xi}\mathbf{Q}\,,\ \mathbf{C}\to e^{2i\xi}\mathbf{C}$. In our previous paper [47] we fixed the phase ξ to be zero to match the gauge theory spin chain convention [9] and to simplify the comparison with the explicit string theory computation of the S-matrix performed in [26]. As we will see in a moment, if we want to implement the double Wick rotation under which $\mathbf{P}\to i\widetilde{\mathbf{H}}\,,\quad \mathbf{H}\to i\widetilde{\mathbf{P}}$ on the algebra level then we should choose $\xi = -\mathbf{P}/4$. This choice makes the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$ to be hermitian and equal to each other[5]

$$
\mathbf{C} = \mathbf{C}^\dagger = -g\sin\frac{\mathbf{P}}{2}\,. \tag{3.9}
$$

As we discussed above, the symmetry algebra of the mirror theory should correspond to a different real slice of the complexified $\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$ algebra. This means that we should give up the hermiticity condition for the algebra generators and consider a linear transformation of the generators which is an automorphism of the complexified $\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$ algebra. The transformation (3.5) suggests to consider the following change of the supersymmetry generators which manifestly preserves the bosonic $\mathrm{SU}(2)^4$ symmetry

$$
\widetilde{\mathbf{Q}}_\alpha{}^a = \frac{1}{\sqrt{2}}\left(\mathbf{Q}_\alpha{}^a - i\,\epsilon^{ac}\,\mathbf{Q}_c^{\dagger\gamma}\,\epsilon_{\gamma\alpha}\right)\,,\quad \widetilde{\mathbf{Q}}_a^{\dagger\alpha} = \frac{1}{\sqrt{2}}\left(\mathbf{Q}_a^{\dagger\alpha} - i\,\epsilon^{\alpha\beta}\,\mathbf{Q}_\beta{}^b\,\epsilon_{ba}\right)\,. \tag{3.10}
$$

[5]A possibility of this choice was noticed in [10].

Then, by using the commutation relations (3.7), we find

$$\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_b^{\dagger\beta}\} = \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a + \frac{i}{2}\delta_b^a\delta_\alpha^\beta\,(\mathbf{C}+\mathbf{C}^\dagger)\,, \tag{3.11}$$
$$\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\ \frac{1}{2}(\mathbf{C}-\mathbf{C}^\dagger + i\mathbf{H})\,, \quad \{\widetilde{\mathbf{Q}}_a^{\dagger\alpha}, \widetilde{\mathbf{Q}}_b^{\dagger\beta}\} = \epsilon_{ab}\epsilon^{\alpha\beta}\ \frac{1}{2}(\mathbf{C}^\dagger - \mathbf{C} + i\mathbf{H})\,.$$

Now we see that if we want to interpret the change of the supersymmetry generators as a result of the double Wick rotation then we should choose the central elements $\mathbf{C}\,,\mathbf{C}^\dagger$ to be of the form (3.9) because with this choice the algebra relations (3.11) take the form

$$\begin{aligned}\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_b^{\dagger\beta}\} &= \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a - \frac{1}{2}\delta_b^a\delta_\alpha^\beta\, 2ig\sin\frac{\mathbf{P}}{2}\,,\\ \{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\ \frac{i}{2}\mathbf{H}\,, \quad \{\widetilde{\mathbf{Q}}_a^{\dagger\alpha}, \widetilde{\mathbf{Q}}_b^{\dagger\beta}\} = \epsilon_{ab}\epsilon^{\alpha\beta}\ \frac{i}{2}\mathbf{H}\,,\end{aligned} \tag{3.12}$$

and performing the analytic continuation

$$\mathbf{P}\to i\widetilde{\mathbf{H}}\,, \quad \mathbf{H}\to i\widetilde{\mathbf{P}}\,,$$

we obtain the mirror algebra

$$\begin{aligned}\{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_b^{\dagger\beta}\} &= \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a + g\,\delta_b^a\delta_\alpha^\beta\,\sinh\frac{\widetilde{\mathbf{H}}}{2}\,,\\ \{\widetilde{\mathbf{Q}}_\alpha{}^a, \widetilde{\mathbf{Q}}_\beta{}^b\} &= -\epsilon_{\alpha\beta}\epsilon^{ab}\ \frac{\widetilde{\mathbf{P}}}{2}\,, \quad \{\widetilde{\mathbf{Q}}_a^{\dagger\alpha}, \widetilde{\mathbf{Q}}_b^{\dagger\beta}\} = -\epsilon_{ab}\epsilon^{\alpha\beta}\ \frac{\widetilde{\mathbf{P}}}{2}\,.\end{aligned} \tag{3.13}$$

Note that after the analytic continuation has been done we can impose on the new supersymmetry generators $\widetilde{\mathbf{Q}}$ and new central elements $\widetilde{\mathbf{H}}\,,\widetilde{\mathbf{P}}$ the same hermiticity condition as was assumed for the original generators. It is also clear that the algebra (3.13) implies the mirror dispersion relation (2.11).

3.3 Mirror S-matrix

The symmetric choice of the central charges (3.9) differs from the one we made in [47]. The S-matrix corresponding to the symmetric choice (3.9) coincides, however, with the string S-matrix in [47]. Indeed, this choice simply corresponds to multiplication of $\mathbf{Q}$ and $\mathbf{Q}^\dagger$ by $e^{-i\mathbf{P}/4}$ and $e^{i\mathbf{P}/4}$, respectively, which apparently does not change the invariance condition for the S-matrix. On the other hand, the string S-matrix also depends on the parameters η's which reflect the freedom in the choice of a basis of two-particle states. This freedom was partially fixed in [47] by requiring the string S-matrix to satisfy the standard Yang-Baxter equation. This still allows one to change the basis of one-particle states, or, in other words to change the basis of the fundamental representation of $\mathfrak{su}(2|2)$. We will see that the requirement that the representation remains unitary after the analytic continuation fixes the parameters η's basically uniquely.

To this end, we compute the action of the generators $\widetilde{\mathbf{Q}}$ and $\widetilde{\mathbf{Q}}^\dagger$ on the fundamental representation of $\mathfrak{su}(2|2)$, see [9, 11, 47] for details. Starting with

$$\begin{aligned} \mathbf{Q}_\alpha{}^a|e_b\rangle &= a\,\delta^a_b|e_\alpha\rangle\,, \qquad \mathbf{Q}_\alpha{}^a|e_\beta\rangle = b\,\epsilon_{\alpha\beta}\epsilon^{ab}|e_b\rangle\,,\\ \mathbf{Q}^{\dagger\alpha}_a|e_\beta\rangle &= d\,\delta^\alpha_\beta|e_a\rangle\,, \qquad \mathbf{Q}^{\dagger\alpha}_a|e_b\rangle = c\,\epsilon_{ab}\epsilon^{\alpha\beta}|e_\beta\rangle \end{aligned} \tag{3.14}$$

we get

$$\begin{aligned} \widetilde{\mathbf{Q}}_\alpha{}^a|e_b\rangle &= \widetilde{a}\,\delta^a_b|e_\alpha\rangle\,, \qquad \widetilde{\mathbf{Q}}_\alpha{}^a|e_\beta\rangle = \widetilde{b}\,\epsilon_{\alpha\beta}\epsilon^{ab}|e_b\rangle\,,\\ \widetilde{\mathbf{Q}}^{\dagger\alpha}_a|e_\beta\rangle &= \widetilde{d}\,\delta^\alpha_\beta|e_a\rangle\,, \qquad \widetilde{\mathbf{Q}}^{\dagger\alpha}_a|e_b\rangle = \widetilde{c}\,\epsilon_{ab}\epsilon^{\alpha\beta}|e_\beta\rangle\,, \end{aligned} \tag{3.15}$$

where

$$\widetilde{a} = \frac{1}{\sqrt{2}}(a+ic)\,,\quad \widetilde{b} = \frac{1}{\sqrt{2}}(b+id)\,,\quad \widetilde{c} = \frac{1}{\sqrt{2}}(c+ia)\,,\quad \widetilde{d} = \frac{1}{\sqrt{2}}(d+ib)\,. \tag{3.16}$$

and $\widetilde{\mathbf{Q}}_\alpha{}^a$, $\widetilde{\mathbf{Q}}^{\dagger\alpha}_a$ are defined by eqs.(3.10). The unitarity of the representation after the analytic continuation requires

$$(c+ia) = b^* - id^*\,. \tag{3.17}$$

The parameters of the original unitary representation before the analytic continuation are given by

$$\begin{aligned} a &= \sqrt{\frac{igx^- - igx^+}{2}}\,e^{i(\xi+\varphi)}\,, \qquad b = -\frac{1}{x^-}\sqrt{\frac{igx^- - igx^+}{2}}\,e^{i(\xi-\varphi)}\,,\\ d &= \sqrt{\frac{igx^- - igx^+}{2}}\,e^{-i(\xi+\varphi)}\,, \qquad c = -\frac{1}{x^+}\sqrt{\frac{igx^- - igx^+}{2}}\,e^{-i(\xi-\varphi)}\,, \end{aligned} \tag{3.18}$$

where $\xi \sim p$ and $\varphi \sim p$ are real, and the parameters $x^\pm$ satisfy the following complex conjugation rule

$$(x^+)^* = x^-\,. \tag{3.19}$$

After the analytic continuation, $\xi\,,\varphi$ and p become imaginary (so that $\widetilde{p}$ is real) and

$$(x^+)^* = \frac{1}{x^-}\,. \tag{3.20}$$

Taking this into account and computing (3.17), we find that the analytically continued representation is unitary for any choice of ξ if

$$e^{2i\varphi} = \sqrt{\frac{x^+}{x^-}} = e^{\frac{i}{2}p}\,. \tag{3.21}$$

This means that the S-matrix which is unitary for real p and real $\widetilde{p}$ is obtained from the string S-matrix, see eq. (8.7) in [47], by choosing

$$\eta_1 = \eta(p_1)e^{\frac{i}{2}p_2}\,,\quad \eta_2 = \eta(p_2)\,,\quad \widetilde{\eta}_1 = \eta(p_1)\,,\quad \widetilde{\eta}_2 = \eta(p_2)e^{\frac{i}{2}p_1}\,, \tag{3.22}$$

where we have introduced

$$\eta(p) = e^{\frac{i}{4}p}\sqrt{ix^-(p) - ix^+(p)}. \tag{3.23}$$

Up to a scalar factor the S-matrix reads as [47]

$$\begin{aligned}
S(p_1,p_2) \;=\; & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\Big(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\Big)\\
&+ \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\Big(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2\Big)\\
&- \Big(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\Big)\\
&+ \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\Big(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\Big)\\
&+ \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\frac{\eta_1}{\tilde{\eta}_1}\Big(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\Big)\\
&+ \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\frac{\eta_2}{\tilde{\eta}_2}\Big(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\Big)\\
&+ i\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde{\eta}_1\tilde{\eta}_2}\Big(E_1^4\otimes E_2^3 + E_2^3\otimes E_1^4 - E_2^4\otimes E_1^3 - E_1^3\otimes E_2^4\Big)\\
&+ i\frac{x_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\Big(E_3^2\otimes E_4^1 + E_4^1\otimes E_3^2 - E_4^2\otimes E_3^1 - E_3^1\otimes E_4^2\Big)\\
&+ \frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\frac{\eta_2}{\tilde{\eta}_1}\Big(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\Big)\\
&+ \frac{x_2^+ - x_2^-}{x_1^- - x_2^+}\frac{\eta_1}{\tilde{\eta}_2}\Big(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\Big)\,,
\end{aligned} \tag{3.24}$$

where E_i^j with $i,j = 1,\ldots,4$ are the standard 4×4 matrix unities, see appendix A of [47] for notations.

With the choice (3.22) the S-matrix (3.24) satisfies the Yang-Baxter equation and it is unitary for real p's. The analytically continued S-matrix $\tilde{S}(\widetilde{p}_1,\widetilde{p}_2)$ is then obtained from (3.24) by simply substituting

$$p \to 2i\,\mathrm{arcsinh}\frac{1}{2g}\sqrt{1+\widetilde{p}^2}\,, \tag{3.25}$$

c.f. section 2.2. One can verify that *this matrix is also unitary for real* $\widetilde{p}$*'s*:

$$\widetilde{S}(\widetilde{p}_1,\widetilde{p}_2)\widetilde{S}^\dagger(\widetilde{p}_1,\widetilde{p}_2) = \mathbb{I}\,. \tag{3.26}$$

The only subtlety here is that the string S-matrix also depends on a scalar factor, which has been omitted so far. Thus, one should separately check that this factor remains unitary after the analytic continuation. This will be discussed in section 5.2.

An exact relation between the S-matrix, $S^{\rm AFZ}$, found in [47] and the S-matrix (3.24) is given by the following transformation[6]

$$S(p_1,p_2)=G_1(p_1)G_2(p_2)S^{\rm AFZ}(p_1,p_2)G_1(p_1)^{-1}G_2(p_2)^{-1}\,,$$

where $G(p)={\rm diag}(1,1,e^{i\frac{p}{4}},e^{i\frac{p}{4}})$. It is amusing to note that a similar transformation has been recently introduced in [60], but with a very different motivation. Namely, as was shown in [60], the graded version of $S(p_1,p_2)$ coincides with the Shastry R-matrix [61] for the one-dimensional Hubbard model [62]-[64]. In section 4 we will give another interesting interpretation to our choice (3.22) which is based on the requirement of generalized unitarity. We will also show there that this choice of η's makes the S-matrix (3.24) and, therefore, the Shastry R-matrix a meromorphic function on the z-torus.

To summarize, in order to have a unified description of the symmetry algebra of the $\rm AdS_5\times S^5$ light-cone gauge-fixed string theory and its mirror sigma-model we should make the symmetric choice of the central charges (3.9), and choose the fundamental representation of the centrally-extended $\mathfrak{su}(2|2)$ with the parameters a,b,c,d given by

$$\begin{aligned} a=d&=\sqrt{\frac{igx^- - igx^+}{2}}=\sqrt{\frac{H+1}{2}}\,,\\ b=c&=-\sqrt{\frac{ig}{2x^+}-\frac{ig}{2x^-}}=-\sqrt{\frac{H-1}{2}}\,. \end{aligned} \tag{3.27}$$

Taking into account (3.16), (3.19) and (3.20), it is easy to check that both the original and the mirror (analytically-continued) representations are unitary with respect to their own reality conditions. Let us stress that the parameters a,b,c,d have the same dependence on $x^\pm$ in the original and mirror theories. We simply regard $x^\pm$ as functions of p in the original model, and as functions of $\widetilde{p}$ in the mirror one.

3.4 Hopf algebra structure

Formulas (3.27) define how the algebra generators of the original and mirror theories act on one-particle states of the theory. We also need to know their action on an arbitrary multi-particle state. The simplest way to have a unified description of their action is to use the Hopf algebra structure of the unitary graded associative algebra $\mathcal{A}$ generated by the even rotation generators ${\bf L}_a{}^b$, ${\bf R}_\alpha{}^\beta$, the odd supersymmetry generators ${\bf Q}_\alpha{}^a$, ${\bf Q}^{\dagger\alpha}_a$ and two central elements ${\bf H}$ and ${\bf P}$ subject to the algebra relations (3.7) with the central elements ${\bf C}$ and ${\bf C}^\dagger$ expressed through the world-sheet momentum ${\bf P}$ by the formula (3.9). We will be using the Hopf algebra introduced in [47] which is basically equivalent to the Hopf algebras discussed in [65], see also

[6] The finite-size correction to the dispersion relation found in [32] involves the coefficients a_1, a_2 and a_6 of $S^{\rm AFZ}$ (see [47] for notation) which are unaffected by this transformation.

[66] for further discussion of algebraic properties of the centrally extended $\mathfrak{su}(2|2)$ algebra.

Let us recall that the unit, $\epsilon : \mathcal{A} \to \mathbf{C}$, is defined by

$$\epsilon(\mathrm{id}) = 1\,, \quad \epsilon(\mathbf{J}) = 0\,, \quad \epsilon(\mathbf{Q}) = 0\,, \quad \epsilon(\mathbf{Q}^\dagger) = 0\,, \tag{3.28}$$

and the co-product is given by the following formulas[7]

$$\begin{aligned} \Delta(\mathbf{J}) &= \mathbf{J} \otimes \mathrm{id} + \mathrm{id} \otimes \mathbf{J}\,, \\ \Delta(\mathbf{Q}_\alpha{}^a) &= \mathbf{Q}_\alpha{}^a \otimes e^{i\mathbf{P}/4} + e^{-i\mathbf{P}/4} \otimes \mathbf{Q}_\alpha{}^a\,, \\ \Delta(\mathbf{Q}^{\dagger}{}_a^{\alpha}) &= \mathbf{Q}^{\dagger}{}_a^{\alpha} \otimes e^{-i\mathbf{P}/4} + e^{i\mathbf{P}/4} \otimes \mathbf{Q}^{\dagger}{}_a^{\alpha}\,, \end{aligned} \tag{3.29}$$

where $\mathbf{J}$ is any even generator. Here we use the graded tensor product, that is for any algebra elements a, b, c, d

$$(a \otimes b)(c \otimes d) = (-1)^{\epsilon(b)\epsilon(c)}(ac \otimes bd),$$

where $\epsilon(a) = 0$ if a is an even element, and $\epsilon(a) = -1$ if a is an odd element of the algebra $\mathcal{A}$.

It is interesting to note that the antipode S is trivial for *any* algebra element, that is

$$S(\mathbf{J}) = -\mathbf{J}\,, \quad S(\mathbf{Q}) = -\mathbf{Q}\,, \quad S(\mathbf{Q}^\dagger) = -\mathbf{Q}^\dagger\,. \tag{3.30}$$

This action of the antipode arises for the symmetric choice (3.9) of the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$ only.

The co-product is obviously compatible with the hermiticity conditions one imposes on the algebra generators in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory, and this ensures that the tensor product of two unitary representations is unitary. To check if the co-product is also compatible with the hermiticity conditions one imposes on the algebra generators of the mirror model we compute the co-product action on the supersymmetry generators $\widetilde{\mathbf{Q}}\,, \widetilde{\mathbf{Q}}^\dagger$

$$\begin{aligned} \Delta(\widetilde{\mathbf{Q}}_\alpha{}^a) &= \widetilde{\mathbf{Q}}_\alpha{}^a \otimes \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \widetilde{\mathbf{Q}}_\alpha{}^a \\ &\quad + i\epsilon^{ad}\widetilde{\mathbf{Q}}^{\dagger}{}_d^{\delta}\epsilon_{\delta\alpha} \otimes \sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) - i\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \epsilon^{ad}\widetilde{\mathbf{Q}}^{\dagger}{}_d^{\delta}\epsilon_{\delta\alpha}\,, \\ \Delta(\widetilde{\mathbf{Q}}^{\dagger}{}_a^{\alpha}) &= \widetilde{\mathbf{Q}}^{\dagger}{}_a^{\alpha} \otimes \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \widetilde{\mathbf{Q}}^{\dagger}{}_a^{\alpha} \\ &\quad - i\epsilon^{\alpha\delta}\widetilde{\mathbf{Q}}_\delta{}^d\epsilon_{da} \otimes \sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + i\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) \otimes \epsilon^{\alpha\delta}\widetilde{\mathbf{Q}}_\delta{}^d\epsilon_{da}\,. \end{aligned} \tag{3.31}$$

[7] To derive these expressions from the ones given in [47] one should rescale the supersymmetry generators in [47] by $e^{\pm i\mathbf{P}/4}$.

Since in the mirror theory $\widetilde{\mathbf{H}}$ is hermitian, the co-product is also compatible with the hermiticity conditions of the mirror theory. This guarantees that an $\mathfrak{su}(2|2)$-invariant S-matrix can be always chosen to be unitary.

The co-product (3.31) can be used to find the commutation relations of the supersymmetry generators with the Zamolodchikov-Faddeev (ZF) operators $A(\widetilde{p})$ and $A^\dagger(\widetilde{p})$ which create asymptotic states of the mirror model. The relations can be then used to determine the antiparticle representation, and to derive the crossing relation following the steps in [47]. A simple computation gives

$$\begin{aligned}\widetilde{\mathbf{Q}}_\alpha{}^a A^\dagger(\widetilde{p}) =\; & A^\dagger(\widetilde{p})\mathscr{Q}_\alpha{}^a \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) A^\dagger(\widetilde{p})\Sigma\widetilde{\mathbf{Q}}_\alpha{}^a \\ & + iA^\dagger(\widetilde{p})\big(\epsilon^{ad}\overline{\mathscr{Q}}_d{}^\delta\epsilon_{\delta\alpha}\big)\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) - iA^\dagger(\widetilde{p})\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right)\Sigma\big(\epsilon^{ad}\widetilde{\mathbf{Q}}^{\dagger}_d{}^{\delta}\epsilon_{\delta\alpha}\big)\,,\end{aligned} \tag{3.32}$$

$$\begin{aligned}\widetilde{\mathbf{Q}}^{\dagger}_a{}^{\alpha} A^\dagger(\widetilde{p}) =\; & A^\dagger(\widetilde{p})\,\overline{\mathscr{Q}}_a{}^\alpha \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + \cosh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) A^\dagger(\widetilde{p})\Sigma\widetilde{\mathbf{Q}}^{\dagger}_a{}^{\alpha} \\ & - iA^\dagger(\widetilde{p})\big(\epsilon^{\alpha\delta}\mathscr{Q}_\delta{}^d\epsilon_{da}\big)\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) + i\sinh\left(\frac{\widetilde{\mathbf{H}}}{4}\right) A^\dagger(\widetilde{p})\Sigma\big(\epsilon^{\alpha\delta}\widetilde{\mathbf{Q}}_\delta{}^d\epsilon_{da}\big)\,,\end{aligned} \tag{3.33}$$

where $\mathscr{Q}_\alpha{}^a$ and $\overline{\mathscr{Q}}_a{}^\alpha$ are the matrices of the symmetry algebra structure constants corresponding to the fundamental representation (3.27) and $\Sigma = \mathrm{diag}(1,1,-1,-1)$.

As was already noted, the unitarity of the mirror S-matrix can be, however, broken by a scalar factor. In section 5 we show that the physical unitarity of the mirror S-matrix (the scalar factor) follows from the crossing relations.

4. Double Wick rotation and the rapidity torus

4.1 The rapidity torus

The universal cover of the parameter space describing the fundamental representation of the centrally extended $\mathfrak{su}(2|2)$ algebra is an elliptic curve [12]. Indeed, the dispersion formula

$$H^2 - 4g^2\sin^2\frac{p}{2} = 1\,, \tag{4.1}$$

which originates from the relation between the central charges of the fundamental representation, can be naturally uniformized in terms of Jacobi elliptic functions

$$p = 2\,\mathrm{am}\,z\,, \qquad \sin\frac{p}{2} = \mathrm{sn}(z,k)\,, \qquad H = \mathrm{dn}(z,k)\,, \tag{4.2}$$

where we introduced the elliptic modulus[8] $k = -4g^2 = -\frac{\lambda}{\pi^2} < 0$. The corresponding elliptic curve (the torus) has two periods $2\omega_1$ and $2\omega_2$, the first one is real and the

[8]Our convention for the elliptic modulus is the same as accepted in the *Mathematica* program, e.g., $\mathrm{sn}(z,k) = \mathrm{JacobiSN}[z,k]$. Since the modulus is kept the same throughout the paper we will often indicate only the z-dependence of Jacobi elliptic functions.

second one is imaginary

$$2\omega_1 = 4\mathrm{K}(k)\,, \qquad\qquad 2\omega_2 = 4i\mathrm{K}(1-k) - 4\mathrm{K}(k)\,,$$

where $\mathrm{K}(k)$ stands for the complete elliptic integral of the first kind. The dispersion relation is obviously invariant under the shifts of z by $2\omega_1$ and $2\omega_2$. The torus parametrized by the complex variable z is an analog of the rapidity plane in two-dimensional relativistic models.

In this parametrization the real z-axis can be called the physical one for the original string theory, because for real values of z the energy is positive and the momentum is real due to

$$1 \le \mathrm{dn}(z,k) \le \sqrt{k'}\,, \qquad\qquad z \in \mathbb{R}\,,$$

where $k' \equiv 1-k$ is the complementary modulus.

We further note that the representation parameters $x^\pm$, which are subject to the following constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}\,, \tag{4.3}$$

are expressed in terms of Jacobi elliptic functions as

$$x^\pm = \frac{1}{2g}\Big(\frac{\mathrm{cn}\, z}{\mathrm{sn}\, z} \pm i\Big)(1+\mathrm{dn}\, z)\,. \tag{4.4}$$

This form of $x^\pm$ follows from the requirement that for real values of z the absolute values of $x^\pm$ are greater than unity: $|x^\pm| > 1$ if $z \in \mathbb{R}$. Note also that for real values of z we have $\mathrm{Im}(x^+) > 0$ and $\mathrm{Im}(x^-) < 0$.

Since both the dispersion relation and the parameters $x^\pm$ are periodic with the period ω_1, the range of the variable $\mathrm{Re}\, z$ can be restricted to the interval from $-\omega_1/2$ to $\omega_1/2$ which corresponds to $-\pi \le p \le \pi$.

Postponing an extensive discussion of the bound states till section 7, we note here that the latter problem requires consideration of complex values of particle momenta. According to eq.(4.2), a rectangle $-\omega_1/2 \le \mathrm{Re}(z) \le \omega_1/2\,;\ -\omega_2/2i \le \mathrm{Im}(z) \le \omega_2/2i$ is mapped one-to-one onto the complex p-plane. By this reason, it is tempting to call this rectangle by the *physical region in the complex z-plane*,[9] and, therefore, to restrict the allowed values of the z-coordinates of the particles forming a bound state by this region. An advantage of adopting such a choice is that all the bound states

[9]In relativistic field theories treated in terms of the rapidity $\theta = \theta_2 - \theta_1$, the physical region is defined as a strip $0 < \mathrm{Im}\,\theta < \pi$ and it incorporates the bound states. Correspondingly, the physical region of an individual particle is $\mathrm{Im}\,\theta \in (-\pi/2, \pi/2)$ and it covers the complex p-plane (with a cut) through the relation $p = \sinh\theta$.

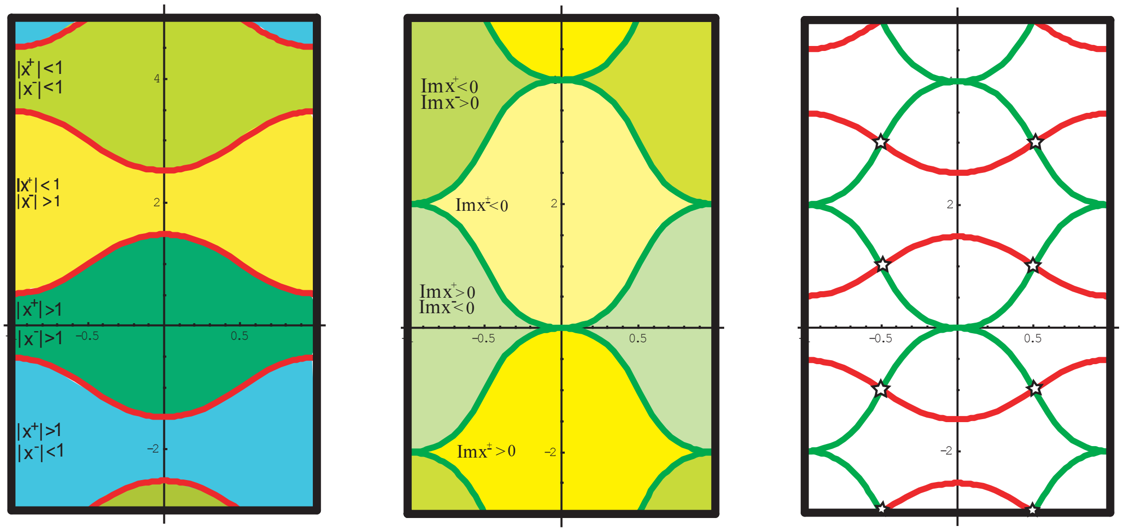

Figure 1: On the left figure the torus is divided by the curves $|x^+| = 1$ and $|x^-| = 1$ into four non-intersecting regions. The middle figure represents the torus divided by the curves $\mathrm{Im}(x^+) = 1$ and $\mathrm{Im}(x^-) = 1$, also in four regions. The right figure contains all the curves of interest.

would have positive energy. We will see, however, that this is not the only option, and there are other two regions in the complex z-plane which could equally deserve the name "*physical*". As it will become clear later on, counting the degeneracy of the bound states drastically depends on the choice of a physical region.

Each solution of eq.(4.3) corresponds to a point of the half-torus, i.e. of the rectangle[10] $-\omega_1/2 \le \mathrm{Re}(z) \le \omega_1/2\,;\ -3\omega_2/4i \le \mathrm{Im}(z) \le 5\omega_2/4i$. In what follows we will be loosely referring to this rectangle as the torus. The torus covers the complex p-plane twice. Since the space of solutions of eq.(4.3) is mapped one-to-one on the torus, the latter could be also chosen as the physical region. Such a choice is however problematic because half of all the states would have negative energy, i.e. the region would contain both particles and anti-particles, as well as bound states and anti-bound states. We point out, however, that there exist positive energy solutions of the bound state equations with some of the particles falling outside of the rectangle $-\omega_1/2 \le \mathrm{Re}(z) \le \omega_1/2\,;\ -\omega_2/2i \le \mathrm{Im}(z) \le \omega_2/2i$ that covers the complex p-plane once.

Constraint (4.3) implies that if a pair (x^+, x^-) satisfies it then $(1/x^+\,,\ x^-)$, $(x^+\,,\ 1/x^-)$ and $(1/x^+\,,\ 1/x^-)$ also do. Each of these four pairs corresponds to a different point on the torus. Taking into account that for any complex number w if $|w| > 1$ then $|1/w| < 1$, and if $\mathrm{Im}(w) > 0$ then $\mathrm{Im}(1/w) < 0$, one can divide the torus into four non-intersecting regions in the following two natural ways, see Fig.1:

[10] We made slightly asymmetric choice for $\mathrm{Im}(z)$ to achieve better visual clarity.

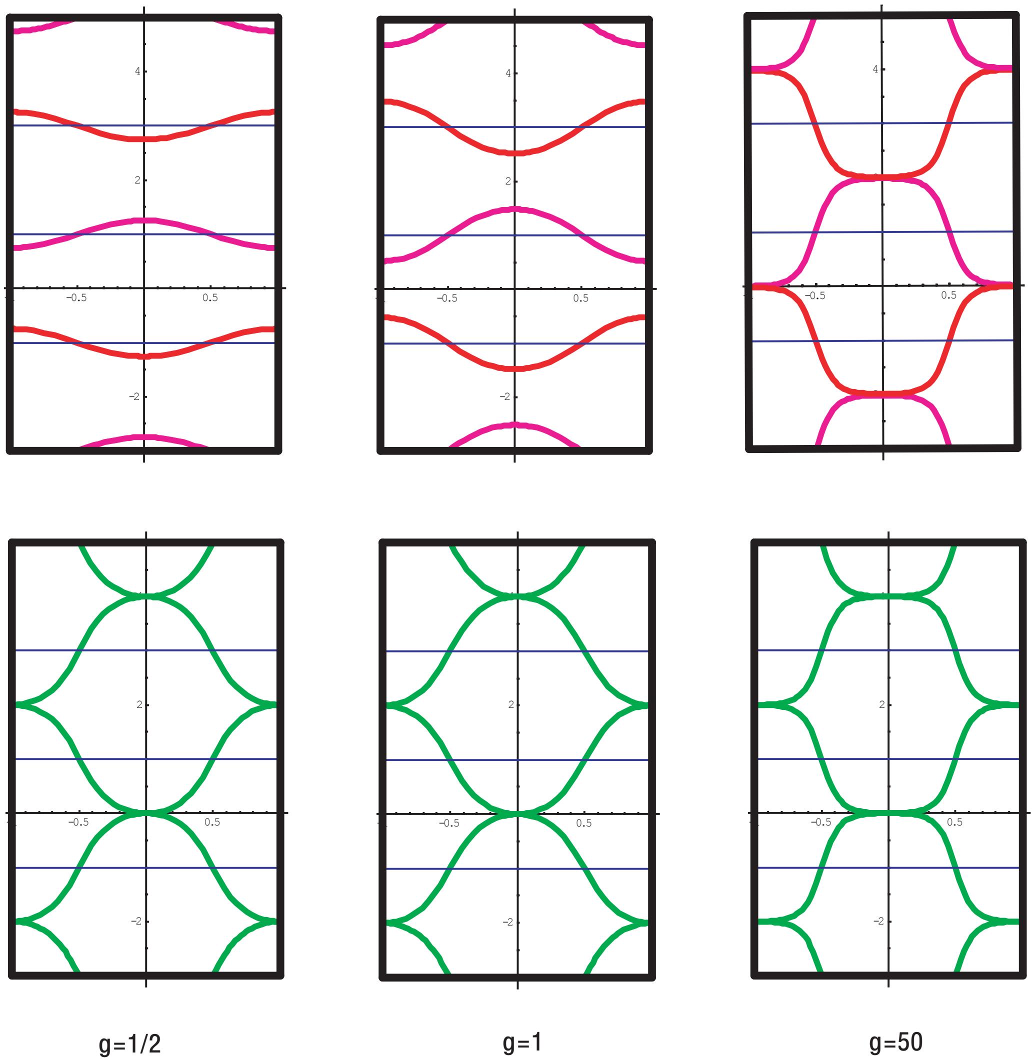

Figure 2: Divisions of the torus by the curves $|x^{\pm}| = 1$ (upper figures) and by the curves $\operatorname{Im} x^{\pm} = 0$ (lower figures) for $g = 1/2$, $g = 1$ and $g = 50$. The red curves are $|x^-| = 1$, and the pink ones are $|x^+| = 1$. The coordinates x and y are the rescaled real and imaginary parts of z: $x = \operatorname{Re}(\frac{2}{\omega_1} z)$, $y = \operatorname{Re}(\frac{4}{\omega_2} z)$. In the limit $g \to \infty$ the curves $|x^{\pm}| = 1$ and $\operatorname{Im} x^{\pm} = 0$ are related by the shift $z \to z + \frac{\omega_2}{2}$.

- $\{|x^{\pm}| > 1\}$, $\{|x^{\pm}| < 1\}$, $\{|x^+| < 1\,, |x^-| > 1\}$ and $\{|x^+| > 1\,, |x^-| < 1\}$; the division is done by the curves $|x^{\pm}| = 1$.

- $\{\operatorname{Im}(x^{\pm}) > 0\}$, $\{\operatorname{Im}(x^{\pm}) < 0\}$, $\{\operatorname{Im}(x^+) > 0\,,\ \operatorname{Im}(x^-) < 0\}$ and $\{\operatorname{Im}(x^+) < 0\,,\ \operatorname{Im}(x^-) > 0\}$; the division is done by the curves $\operatorname{Im}(x^{\pm}) = 0$.

The shape of the regions depends on the value of the coupling constant g, see Fig.2. Quite remarkably, in the strong coupling limit $g \to \infty$ two divisions of the torus produced by the red ($|x^{\pm}|=1$) and green ($\mathrm{Im}(x^{\pm})=0$) curves become related to each other through a global shift by $\omega_2/2$.

There are eight special points on the torus where the curves $|x^{\pm}|=1$ intersect with the curves $\mathrm{Im}(x^{\pm})=0$, see Fig.1. These points are $z=\pm\frac{1}{4}\omega_1+\frac{2n+1}{4}\omega_2\,,\ n=-2,-1,0,1$. It is known [18] that these points are the branch points of the one-loop correction [16] to the dressing phase. It is unclear, however, if they remain the branch points of the exact dressing phase. One could try to use the integral representation [52] of the BES dressing phase [14] to understand this issue. In fact, all currently available representations for the dressing phase are defined for $|x^{\pm}|\geq 1$, and this is another reason to figure out the location of the curves $|x^{\pm}|=1$ on the z-torus.

Both divisions play an important role in the analysis of the bound states of string and mirror theories. To understand the meaning of the equations $|x^{\pm}|=1$ and $\mathrm{Im}(x^{\pm})=0$, it is convenient to use another parameter u which is similar to the rapidity parameter of the Heisenberg spin chain. In terms of $x^{\pm}$ it is defined as follows

$$u=x^{+}+\frac{1}{x^{+}}-\frac{i}{g}=x^{-}+\frac{1}{x^{-}}+\frac{i}{g}\,. \tag{4.5}$$

By using eqs.(4.5) and (4.4), one can express the rapidity u as a meromorphic function on the torus

$$u=\frac{\mathrm{cn}\, z\, \mathrm{dn}\, z}{g\ \mathrm{sn}\, z}\,. \tag{4.6}$$

It is not difficult to check that the eight special points on the torus are mapped onto the four points on the u-plane with coordinates $u=\pm 2\pm\frac{i}{g}$, while the points $z=\pm\omega_1/2$ are mapped to $u=0$, and the points $z=\pm\omega_1/2+\omega_2/2\pm i0$ are mapped to $u=\pm\infty\ \pm\ i\,\infty$.

A special role of the points $u=\pm 2\pm\frac{i}{g}$ can be also understood by expressing $x^{\pm}$ in terms of u

$$\begin{aligned} x^{+}&=\frac{1}{2}\left(u+\frac{i}{g}\pm\sqrt{\left(u-2+\frac{i}{g}\right)\left(u+2+\frac{i}{g}\right)}\right)\,,\\ x^{-}&=\frac{1}{2}\left(u-\frac{i}{g}\pm\sqrt{\left(u-2-\frac{i}{g}\right)\left(u+2-\frac{i}{g}\right)}\right)\,. \end{aligned} \tag{4.7}$$

Thus, on the u-plane there are four branch points with coordinates $u=\pm 2\pm\frac{i}{g}$ corresponding to $x^{\pm}=\pm 1$ and $\mathrm{Im}(x^{\pm})=0$. Therefore, we can naturally choose the cuts either connecting the points $-2\pm\frac{i}{g}$ and $2\pm\frac{i}{g}$, or going from $\pm\infty$ to $\pm 2\pm\frac{i}{g}$ along the horizontal lines. Let us determine what values of $x^{\pm}$ correspond to the

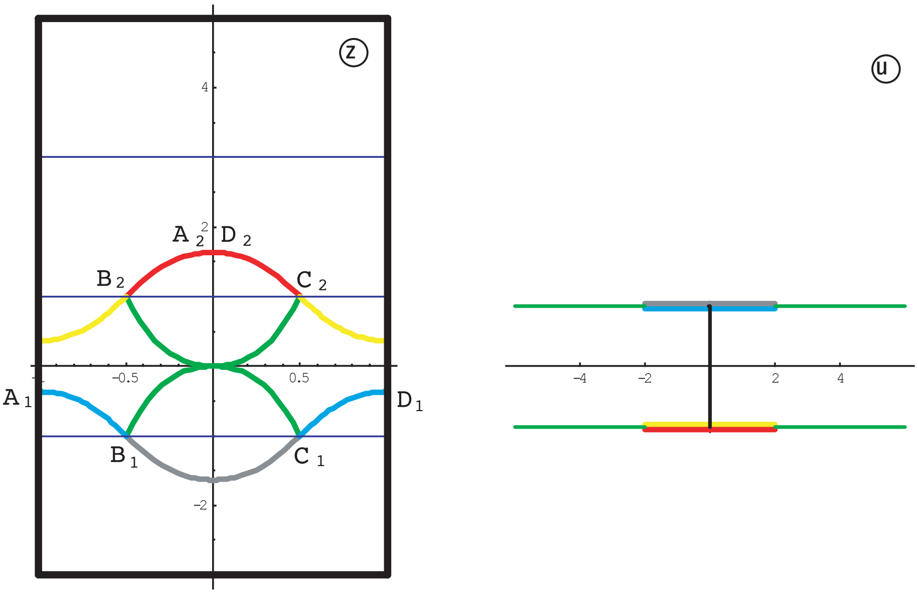

Figure 3: On the left figure the upper and lower curves correspond to $|x^+| = 1+0$ and $|x^-| = 1+0$, respectively. The map $z \to u(z)$ folds each of these curves onto the corresponding cut on the u-plane.

lines $u = u_0 \pm \frac{i}{g}$ with u_0 real. We see that

$$u_0 = x^+ + \frac{1}{x^+}\,, \quad x^+ = \frac{1}{2}\left(u_0 \pm \sqrt{u_0^2-4}\right)\,, \quad \text{if} \quad u = u_0 - \frac{i}{g}\,,$$
$$u_0 = x^- + \frac{1}{x^-}\,, \quad x^- = \frac{1}{2}\left(u_0 \pm \sqrt{u_0^2-4}\right)\,, \quad \text{if} \quad u = u_0 + \frac{i}{g}\,.$$

It is clear that points $x^\pm$ and $1/x^\pm$ of the complex $x^\pm$-plane correspond to the same point u of the u-plane. Then, the points of the circle $|x^+| = 1$ map to points u in the interval $[-2-\frac{i}{g}\,, 2-\frac{i}{g}]$, while the points of $|x^-| = 1$ correspond to $u \in [-2+\frac{i}{g}\,, 2+\frac{i}{g}]$. On the other hand, the points of the lines $\mathrm{Im}(x^+) = 0$ and $\mathrm{Im}(x^-) = 0$ correspond to points u outside the intervals $[-2-\frac{i}{g}\,, 2-\frac{i}{g}]$ and $[-2+\frac{i}{g}\,, 2+\frac{i}{g}]$, respectively. Note also that if one chooses a definite sign in eq.(4.7) then the interval $[-2 \mp \frac{i}{g}\,, 2 \mp \frac{i}{g}]$ maps onto a half of a unit circle in the $x^\pm$-plane. One has to use both signs to cover the unit circles $|x^\pm| = 1$ and real lines $\mathrm{Im}(x^\pm) = 0$.

To determine the location of the upper and lower edges of the u-plane cuts $[-2 \mp \frac{i}{g}\,, 2 \mp \frac{i}{g}]$ on the $x^\pm$-planes, we introduce a small real parameter ϵ and write

$$x^\pm = e^\epsilon e^{i\varphi}\,, \quad |x^\pm| = e^\epsilon\,, \quad \mathrm{Im}(x^\pm) = e^\epsilon \sin\varphi\,, \quad u \approx 2\cos\varphi \mp \frac{i}{g} + 2i\epsilon\sin\varphi\,. \quad (4.8)$$

We see that the upper edges $[-2 \mp \frac{i}{g} + i0\,, 2 \mp \frac{i}{g} + i0]$ are mapped either outside the upper halves or inside the lower halves of the circles $|x^\pm| = 1$, and the lower edges

$[-2 \mp \frac{i}{g} - i0\,, 2 \mp \frac{i}{g} - i0]$ are mapped either outside the lower halves or inside the upper halves of the circles $|x^{\pm}| = 1$, and vice verse:

$$[-2 \mp \frac{i}{g} + i0\,, 2 \mp \frac{i}{g} + i0] \iff \begin{cases} |x^{\pm}| = 1 + 0\,,\ \mathrm{Im}(x^{\pm}) > 0 \\ |x^{\pm}| = 1 - 0\,,\ \mathrm{Im}(x^{\pm}) < 0 \end{cases}, \tag{4.9}$$

$$[-2 \mp \frac{i}{g} - i0\,, 2 \mp \frac{i}{g} - i0] \iff \begin{cases} |x^{\pm}| = 1 + 0\,,\ \mathrm{Im}(x^{\pm}) < 0 \\ |x^{\pm}| = 1 - 0\,,\ \mathrm{Im}(x^{\pm}) > 0 \end{cases}. \tag{4.10}$$

As we discussed above, the z-torus can be divided into four non-intersecting regions by the curves $|x^{\pm}| = 1$. Now it is easy to show that each of the regions is mapped one-to-one onto the u-plane with the two cuts. Let us consider for definiteness the region with $|x^{\pm}| > 1$. Then, according to the discussion above, the boundaries of the region with $|x^{+}| = 1 + 0\,,\ \mathrm{Im}(x^{+}) > 0$ and $|x^{+}| = 1 + 0\,,\ \mathrm{Im}(x^{+}) < 0$ are mapped onto the upper and lower edges of the cut $[-2 - \frac{i}{g}\,, 2 - \frac{i}{g}]$ in the u-plane, respectively. In the same way the boundary of the region with $|x^{-}| = 1$ is mapped onto the upper and lower edges of the cut $[-2 + \frac{i}{g}\,, 2 + \frac{i}{g}]$, see Fig.3.

Another way to understand how different copies of the u-plane are glued together is to consider any of the curves $|x^{\pm}(z)| = 1$ and shift its variable z by a small positive ϵ in the imaginary direction. For the image of the corresponding shifted curve on the u-plane one obtains

$$\mathrm{Im}\ u(z + i\epsilon) = \mp\frac{1}{g} + \epsilon\ \mathrm{Re}\left(\frac{\partial u}{\partial z}\right) + \dots\,, \tag{4.11}$$

where $\mathrm{Re}\left(\frac{\partial u}{\partial z}\right)$ is computed along $|x^{\pm}| = 1$. Further analysis shows that along any of the curves $|x^{\pm}| = 1$ the expression $\mathrm{Re}\left(\frac{\partial u}{\partial z}\right)$ is positive for $-\frac{\omega_1}{4} < \mathrm{Re}\ z < \frac{\omega_1}{4}$ and negative otherwise. This determines how the edges of the cuts $|x^{\pm}| = 1$ are mapped onto the edges of the corresponding cuts on the u-plane (see Fig.3 for an example).

To summarize, any region confined between the curves $|x^{\pm}| = 1$ is mapped under $z \to u(z)$ onto a single copy of the u-plane with a point at infinity added, i.e. onto the Riemann sphere. Extended to the whole torus, *this map defines a four-fold covering of the Riemann sphere by the torus which has eight ramification points*:[11] a generic point on the u-plane has four images belonging to the four regions. There are two cuts on each copy of the u-plane

$$\begin{aligned} &1)\ \ [-2 + i/g, 2 + i/g] \\ &2)\ \ [-2 - i/g, 2 - i/g] \end{aligned}$$

which are images of the curves $|x^{-}| = 1$ and $|x^{+}| = 1$, respectively.

In the same way we can determine the images of the upper and lower edges of the u-plane cuts $(-\infty, -2 \mp \frac{i}{g}]\,, [2 \mp \frac{i}{g}, \infty)$ on the $x^{\pm}$-planes. We again introduce a

[11] In agreement with the Riemann-Hurwitz formula.

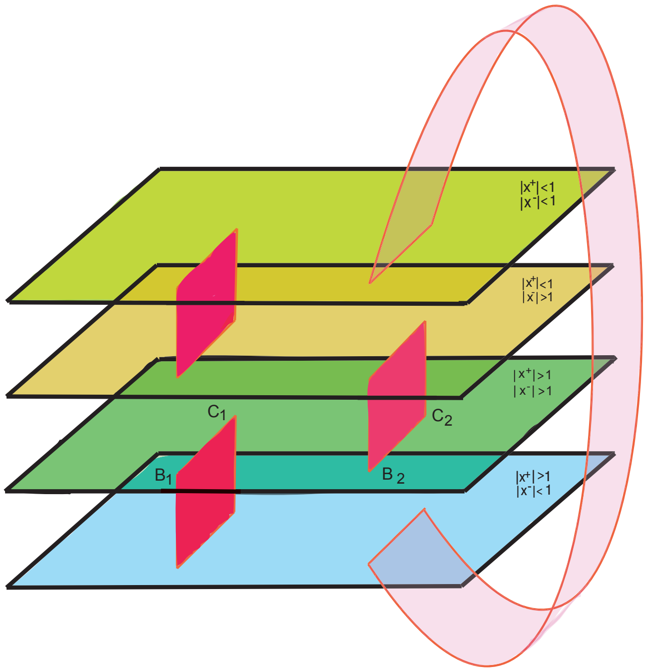

Figure 4: Four copies of the u-plane (the Riemann sphere) glued together through the cuts to produce the torus of the kinematical variable z. We indicated four branch points $\mathbf{B}_{1,2}$ and $\mathbf{C}_{1,2}$ which are images of those on Fig.3.

small real parameter ϵ and write

$$x^{\pm} = re^{i\epsilon}\,, \quad |x^{\pm}| = |r|\,, \quad \mathrm{Im}(x^{\pm}) \approx r\epsilon\,, \quad u \approx r + \frac{1}{r} \mp \frac{i}{g} + i\epsilon(r - \frac{1}{r})\,. \tag{4.12}$$

We see that the upper edges $(-\infty, -2 \mp \frac{i}{g} + i0]\,, [2 \mp \frac{i}{g} + i0, \infty)$ are mapped either onto the upper edge of the intervals $(-\infty, -1]\,, [1, \infty)$ or the lower edge of the interval $[-1, 1]$, and the lower edges $(-\infty, -2 \mp \frac{i}{g} - i0]\,, [2 \mp \frac{i}{g} - i0, \infty)$ are mapped either onto the lower edge of the intervals $(-\infty, -1]\,, [1, \infty)$ or the upper edge of the interval $[-1, 1]$ of the real lines $\mathrm{Im}(x^{\pm}) = 0$, and vice verse:

$$(-\infty, -2 \mp \frac{i}{g} + i0] \cup [2 \mp \frac{i}{g} + i0, \infty) \iff \begin{cases} \mathrm{Im}(x^{\pm}) = +0\,, \ |x^{\pm}| > 1 \\ \mathrm{Im}(x^{\pm}) = -0\,, \ |x^{\pm}| < 1 \end{cases}, \tag{4.13}$$

$$(-\infty, -2 \mp \frac{i}{g} - i0] \cup [2 \mp \frac{i}{g} - i0, \infty) \iff \begin{cases} \mathrm{Im}(x^{\pm}) = +0\,, \ |x^{\pm}| < 1 \\ \mathrm{Im}(x^{\pm}) = -0\,, \ |x^{\pm}| > 1 \end{cases}, \tag{4.14}$$

Again, dividing the z-torus into four non-intersecting regions by the curves $\mathrm{Im}(x^{\pm}) = 0$, we see that each of the regions also maps one-to-one onto the u-plane with the two cuts. This gives a different (but equivalent) four-fold covering of the Riemann sphere by the torus.

When a point on the z-plane runs along the curve $|x^+| = 1$ or $|x^-| = 1$ its image covers the corresponding interval on the u-plane twice. To appreciate this fact, let us note that if z is, e.g., on the curve $|x^+|=1$ then the points

$$z_\pm = -z \pm \frac{\omega_1}{2} + \frac{\omega_2}{2} \tag{4.15}$$

are also on this curve. Indeed, since $|x^+(z)|^2 = x^+(z)x^-(z^*)$, we have

$$|x^+(z_\pm)|^2 = x^+\Big(-z \pm \frac{\omega_1}{2} + \frac{\omega_2}{2}\Big)x^-\Big(-z^* \pm \frac{\omega_1}{2} - \frac{\omega_2}{2}\Big) = \frac{1}{|x^+(z)|^2} = 1\,,$$

where we have used the properties of Jacobi elliptic functions under the shifts by quarter-periods. In the same way one finds that if z lies on a curve $|x^-|=1$ then the points $z_\pm$ belong to another copy of $|x^-|=1$ which is obtained from the original one by the shift by ω_2. Finally, using the properties of the Jacobi elliptic functions it is easy to show that $u(z_\pm)=u(z)$, i.e. the points z and $z_\pm$ have one and the same image on the u-plane.

It is clear that the half of the torus and, therefore, the complex p-plane is mapped onto the u-plane twice. The coordinate u is real for real z, and in this case we can easily express it in terms of p [5]

$$u(p) = \frac{1}{g}\cot\frac{p}{2}\sqrt{1+4g^2\sin^2\frac{p}{2}}\,. \tag{4.16}$$

In the limit $g\to 0$ the relation (4.16) turns to the one between the rapidity and momentum variables of the Heisenberg spin chain; the latter describes the gauge theory at the one-loop level. This supports an idea that the physical region could be identified with a single copy of the u-plane, namely the one which maps to the region $|x^\pm|>1$ of the z-torus. There are certain advantages of such a choice which we will discuss later on. The main disadvantage is, however, that the region $|x^\pm|>1$ is not big enough to cover the whole complex p-plane.

It is interesting to see what happens with our three candidates for the physical region in the limits $g\to\infty$ and $g\to 0$. In the limit $g\to\infty$ the periods of the torus have the following behavior

$$\omega_1 \to \frac{\log g}{g}\,, \quad \omega_2 \to \frac{i\pi}{2g} \qquad \text{if} \quad g\to\infty\,. \tag{4.17}$$

To keep the range of $\mathrm{Im}(z)$ finite, we rescale z as $z\to z/(2g)$, and the momentum as $p\to p/g$. Then the dispersion relation (4.1) takes the relativistic form $H^2-p^2=1$, the variable z plays the role of θ because $p=\sinh z$, and we have

- The torus degenerates to the strip with $-\pi<\mathrm{Im}(z)<\pi$ and $-\infty<\mathrm{Re}(z)<\infty$
- The half-torus corresponding to the complex p-plane degenerates to the strip with $-\pi/2<\mathrm{Im}(z)<\pi/2$ and $-\infty<\mathrm{Re}(z)<\infty$

- The region $|x^{\pm}| > 1$ corresponding to the complex u-plane degenerates to the strip with $-\pi/2 < \mathrm{Im}(z) < \pi/2$ and $-\infty < \mathrm{Re}(z) < \infty$

We see that both the half-torus and the region $|x^{\pm}| > 1$ degenerate to the physical strip of a relativistic field theory.

In the limit $g \to 0$ the periods of the torus have the following behavior

$$\omega_1 \to \pi\,, \quad \omega_2 \to 2i \log g \qquad \text{if} \quad g \to 0\,. \tag{4.18}$$

We see that all the three regions degenerate into the strip with $-\pi/2 < \mathrm{Re}(z) < \pi/2$ and $-\infty < \mathrm{Im}(z) < \infty$. The properties of the S-matrix arising in the limit $g \to 0$ will be discussed in appendix 9.2.

4.2 Double Wick rotation

The z-torus can be also used to describe the mirror model. Since we know the relation between $p = 2\,\mathrm{am}\,z$ and the mirror momentum $\tilde{p}$, we can express $\tilde{p}$ in terms of z. Indeed, the equality

$$2\,\mathrm{am}\,z = 2i\,\mathrm{arcsinh}\frac{1}{2g}\sqrt{1+\widetilde{p}^2} \tag{4.19}$$

implies

$$\widetilde{p} = -i\,\mathrm{dn}\,z\,. \tag{4.20}$$

The energy in the mirror theory takes the form

$$\widetilde{H} = 2\,\mathrm{arccoth}\frac{\sqrt{k'}}{\mathrm{dn}\,z}\,. \tag{4.21}$$

The formulae above show that real values of z correspond to imaginary $\tilde{p}$. Now we would like to understand for which values of z the corresponding values of $\widetilde{p}$ are real. One can see that if we shift the variable z by $\omega_2/2$, $z \to z + \omega_2/2$, that is if we write

$$\widetilde{p} = -i\,\mathrm{dn}\left(z + \frac{\omega_2}{2}, k\right) \equiv \sqrt{k'}\,\frac{\mathrm{sn}\,z}{\mathrm{cn}\,z}\,, \tag{4.22}$$

then for real values of the shifted variable z the corresponding values of $\widetilde{p}$ are real as well. We also recognize here a close analogy with the relativistic case – making the double Wick rotation corresponds to the shift by a quarter-period on the rapidity plane. The function $\mathrm{cn}(z,k)$ has zeroes at $z = -\frac{1}{2}\omega_1$ and $z = \frac{1}{2}\omega_1$ (and $\mathrm{dn}(z,k)$ has poles at $z = \frac{1}{2}(-\omega_1 + \omega_2)$ and $z = \frac{1}{2}(\omega_1 + \omega_2)$) which explains the apparent absence of the periodicity in $\widetilde{p}$. Thus, when the shifted variable z runs from $-\frac{1}{2}\omega_1$ to $\frac{1}{2}\omega_1$ the momentum $\tilde{p}$ monotonically increases from $-\infty$ to $+\infty$ and it passes though zero for $z = 0$.

One further finds that the parameters $x^\pm$ are expressed in terms of the shifted parameter z of the mirror model as follows

$$x^\pm = -i\frac{\sqrt{k'} \mp \mathrm{dn}\, z}{\sqrt{-k}\,\mathrm{dn}\, z}\left(1 + i\sqrt{k'}\frac{\mathrm{sn}\, z}{\mathrm{cn}\, z}\right). \tag{4.23}$$

We can now find how $x^\pm$ are expressed in terms of the mirror momentum. Indeed, since

$$\left(\frac{\mathrm{cn}\, z}{\mathrm{sn}\, z}\right)^2 = -1 + \frac{k}{1 - \mathrm{dn}^2\, z},$$

we deduce from eq.(4.23) that

$$x^\pm = \frac{1}{2g}\left(\sqrt{1 + \frac{4g^2}{1 + \widetilde{p}^2}} \mp 1\right)(\widetilde{p} - i).$$

This, of course, agrees with the formula (2.15).

The variables $x^\pm$ of the mirror theory obey a relation

$$x^+ x^- = \frac{\widetilde{p} - i}{\widetilde{p} + i}$$

which implies that $|x^+ x^-| = 1$ for $\widetilde{p}$ real.

It is also not difficult to show that the dispersion relation in the mirror theory takes the form (2.11)

$$\widetilde{H} = 2\,\mathrm{arccoth}\frac{\sqrt{k'}}{\mathrm{dn}\, z} = 2\,\mathrm{arccoth}\sqrt{1 - \frac{k}{1 + \widetilde{p}^2}} = 2\,\mathrm{arcsinh}\frac{1}{\sqrt{-k}}\sqrt{1 + \widetilde{p}^2}.$$

This completes the proof that the double-Wick rotation corresponds to a shift of the z variable by a quarter of the imaginary period of the torus, and the real axes of the shifted z corresponds to real values of the momentum of the mirror theory.[12]

Finally, it is useful to express the rapidity u in terms of the shifted parameter z of the mirror model and $\widetilde{p}$. We have

$$u = \frac{2\,\mathrm{cn}\left(z + \frac{\omega_2}{2}, k\right)\mathrm{dn}\left(z + \frac{\omega_2}{2}, k\right)}{\sqrt{-k}\,\mathrm{sn}\left(z + \frac{\omega_2}{2}, k\right)} = -\frac{2i\sqrt{k'}\,\mathrm{dn}\left(z + \frac{\omega_2}{2}, k\right)}{\sqrt{-k}\,\mathrm{dn}\left(z, k\right)}.$$

Then one can check that the points $z = \pm\omega_1/2 \pm i0$ are mapped to $u = \pm\infty \ \pm\ i\infty$. The coordinate u is real for real z, and in this case we can express it in terms of $\widetilde{p}$

$$u = \frac{2\widetilde{p}}{\sqrt{-k}}\sqrt{1 - \frac{k}{1 + \widetilde{p}^2}} = \frac{\widetilde{p}}{g}\sqrt{1 + \frac{4g^2}{1 + \widetilde{p}^2}}.$$

[12] After having performed the shift, one can do various physically equivalent transformations of the shifted z-variable preserving the axes of real z. Particular useful examples of these transformations are $z \to z + \frac{\omega_1}{2}$, $z \to -z + \frac{\omega_1}{2}$, $z \to -z \pm \frac{\omega_1}{2}$.

Again, there are three choices of the physical region. It is the half-torus corresponding to the complex $\widetilde{p}$-plane, the whole torus, and the region $\mathrm{Im}(x^{\pm}) < 0$ which is mapped onto the u-plane. The third choice is different from the one made for the string theory, and is motivated by the analysis of the bound states of the mirror model.

5. S-matrix on elliptic curve

5.1 Elliptic S-matrix and its properties

The dispersion relation (4.1) is naturally parametrized by the elliptic curve. Without imposing the unitarity condition for the S-matrix, the phase η in (3.23) can be chosen in an arbitrary way, for instance, $\eta(p) = 1$. In the latter case, the S-matrix (3.24) is well defined on the elliptic curve but it is non-unitary. It is therefore tempting to assume that the unitary S-matrix also admits an analytic continuation into the complex z-plane. To find such a continuation one has to resolve the branch cut ambiguities arising due to the η-factor in the S-matrix (3.24): $\eta(p) = e^{\frac{i}{4}p}\sqrt{ix^-(p) - ix^+(p)}$.

This can be done in the following way. First, we recall the elliptic parametrization (4.4) which gives

$$\eta(p) = e^{\frac{i}{4}p}\sqrt{ix^-(p) - ix^+(p)} = \frac{1}{\sqrt{g}} e^{\frac{i}{2}\mathrm{am}\, z}\sqrt{1+\mathrm{dn}\, z} = \\ = \frac{1}{\sqrt{g}}\sqrt{(1+\mathrm{dn}\, z)(\mathrm{cn}\, z + i\mathrm{sn}\, z)}\,. \qquad (5.1)$$

Second, by using the following formulae (recall $k = -4g^2$)

$$1+\mathrm{dn}\, z = \frac{2\,\mathrm{dn}^2\frac{z}{2}}{1-k\,\mathrm{sn}^4\frac{z}{2}}\,, \qquad \mathrm{cn}\, z + i\,\mathrm{sn}\, z = \frac{\left(\mathrm{cn}\frac{z}{2} + i\,\mathrm{sn}\frac{z}{2}\mathrm{dn}\frac{z}{2}\right)^2}{1-k\,\mathrm{sn}^4\frac{z}{2}}$$

relating elliptic functions to those of the half argument, we can resolve the branch cut ambiguities by means of the relation

$$e^{\frac{i}{4}p}\sqrt{ix^-(p) - ix^+(p)} = \frac{\sqrt{2}}{\sqrt{g}}\frac{\mathrm{dn}\frac{z}{2}\left(\mathrm{cn}\frac{z}{2} + i\,\mathrm{sn}\frac{z}{2}\mathrm{dn}\frac{z}{2}\right)}{1+4g^2\,\mathrm{sn}^4\frac{z}{2}} \equiv \eta(z) \qquad (5.2)$$

valid in the region $-\frac{\omega_1}{2} < \mathrm{Re}\, z < \frac{\omega_1}{2}$ and $-\omega_2/i < \mathrm{Im}\, z < \omega_2/i$. Further, we notice that the non-local dependence of η's on the momentum of another particle enters as $e^{\frac{i}{2}p} = e^{i\mathrm{am}\, z}$ and, therefore, can be naturally treated as $e^{\frac{i}{2}p} = \mathrm{cn}\, z + i\,\mathrm{sn}\, z$.

Thus, we define an analytic continuation of the S-matrix onto the rapidity torus for each of the complex variables z_1 and z_2 by means of eq.(3.24), where the variables $\eta_{1,2}$ and $\tilde{\eta}_{1,2}$ are given by

$$\eta_1 = \eta(z_1)(\mathrm{cn}\, z_2 + i\,\mathrm{sn}\, z_2)\,, \qquad \eta_2 = \eta(z_2)\,,$$
$$\tilde{\eta}_2 = \eta(z_2)(\mathrm{cn}\, z_1 + i\,\mathrm{sn}\, z_1)\,, \qquad \tilde{\eta}_1 = \eta(z_1)\,.$$

In this way we completely resolved the branch cut ambiguities of the S-matrix (3.24) and defined it as a meromorphic function on the elliptic curve (for each z-variable). It is remarkable to observe that such a continuation becomes possible due to additional phase factors, $e^{\frac{i}{4}p}$, introduced in the previous section to guarantee unitarity of the mirror theory.

Let us now analyze the basic properties of the elliptic S-matrix. One can check that it satisfies the Yang-Baxter equation and the usual unitarity requirement

$$S_{12}(z_1, z_2)S_{21}(z_2, z_1) = \mathbb{I}\,. \tag{5.3}$$

Further, it obeys the *generalized unitarity condition*:

$$S_{12}(z_1^*, z_2^*)\big[S_{12}(z_1, z_2)\big]^\dagger = \mathbb{I}\,. \tag{5.4}$$

Here "†" means hermitian conjugation. For z_1 and z_2 real the last condition reduces to the requirement of physical unitarity. In fact, one can see that the elliptic S-matrix is compatible with the generalized unitarity condition only due to our specific choice for the phase factors discussed above. Then, unitarity and generalized unitarity imply hermitian analyticity: $S_{21}(z_2^*, z_1^*) = \big[S_{12}(z_1, z_2)\big]^\dagger$.

Let us now compute monodromies of the S-matrix (3.24) over the real and imaginary periods. We find

$$\begin{aligned} S(z_1 + 2\omega_1, z_2) &= \Sigma_1\, S(z_1, z_2)\Sigma_1 = \Sigma_2\, S(z_1, z_2)\Sigma_2\,, \\ S(z_1 + 2\omega_2, z_2) &= \Sigma_1\, S(z_1, z_2)\Sigma_1 = \Sigma_2\, S(z_1, z_2)\Sigma_2\,. \end{aligned} \tag{5.5}$$

Hence, the S-matrix exhibits the same monodromies over real and imaginary cycles and it is a periodic function on a double torus with periods $4\omega_1$ and $4\omega_2$. Here $\Sigma_1 = \Sigma \otimes \mathbb{I}$ and $\Sigma_2 = \mathbb{I} \otimes \Sigma$, where Σ is defined in section 3.4, and the S-matrix commutes with the product $\Sigma \otimes \Sigma$. Note that Σ is in the center of the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Second, we establish the monodromy properties w.r.t. shifts by half-periods. Under the shift by the real half-period we get

$$S(z_1 + \omega_1, z_2) = \big(V \otimes \Sigma\big)\, S(z_1, z_2)\big(V^{-1} \otimes \mathbb{I}\big)\,, \tag{5.6}$$

where $V = \mathrm{diag}\big(e^{-\frac{i\pi}{4}}, e^{-\frac{i\pi}{4}}, e^{\frac{i\pi}{4}}, e^{\frac{i\pi}{4}}\big)$.

The shift by the imaginary half-period corresponds to the crossing symmetry transformation [12]. To discuss it, we multiply the S-matrix (3.24) with a scalar factor S_0 to produce the string S-matrix obeying crossing symmetry

$$\mathcal{S}(z_1, z_2) = S_0(z_1, z_2)S(z_1, z_2)\,. \tag{5.7}$$

We then find that with a proper choice for $S_0(z_1, z_2)$ the string S-matrix exhibits the following crossing symmetry relations

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(z_1, z_2)\mathscr{C}_1\mathcal{S}_{12}(z_1 + \omega_2, z_2) = \mathbb{I}\,, \quad \mathscr{C}_1\mathcal{S}_{12}^{t_1}(z_1, z_2)\mathscr{C}_1^{-1}\mathcal{S}_{12}(z_1 - \omega_2, z_2) = \mathbb{I}\,, \tag{5.8}$$

and also

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(z_1,z_2)\mathscr{C}_1\mathcal{S}_{12}(z_1,z_2-\omega_2)=\mathbb{I}\,,\quad \mathscr{C}_1\mathcal{S}_{12}^{t_1}(z_1,z_2)\mathscr{C}_1^{-1}\mathcal{S}_{12}(z_1,z_2+\omega_2)=\mathbb{I}\,. \quad (5.9)$$

Here t_1 denotes transposition in the first matrix space and $\mathscr{C}$ is a *constant*[13] charge conjugation matrix

$$\mathscr{C}=\begin{pmatrix}\sigma_2 & 0\\ 0 & i\,\sigma_2\end{pmatrix}\,, \quad (5.10)$$

where σ_2 is the Pauli matrix. The compatibility of eqs.(5.8) and (5.9) with (5.5) is guaranteed by the identity $\mathscr{C}\Sigma=\mathscr{C}^{-1}$ which is equivalent to $\mathscr{C}^2=\Sigma$.

The crossing symmetry relations lead to the following equations for the scalar factor S_0 [12]

$$S_0(z_1,z_2)\,S_0(z_1+\omega_2,z_2)=f(z_1,z_2)\,,\quad S_0(z_1,z_2)\,S_0(z_1,z_2-\omega_2)=f(z_1,z_2)\,, \quad (5.11)$$

where the function f is expressed through $x^\pm$ as follows

$$f(z_1,z_2)=\frac{\left(\frac{1}{x_1^-}-x_2^-\right)(x_1^--x_2^+)}{\left(\frac{1}{x_1^+}-x_2^-\right)(x_1^+-x_2^+)}\,. \quad (5.12)$$

One can easily check that the function $f(z_1,z_2)$ obeys the following properties

$$f(z_2,z_1)f(z_1+\omega_2,z_2)=1=f(z_2,z_1)f(z_1,z_2+\omega_2)\,,\quad f(z_1+\omega_2,z_2+\omega_2)=f(z_1,z_2)\,,$$

which are, however, incompatible with the assumption that the scalar factor is an analytical function of z_1,z_2.

Another important property of the string S-matrix (5.7) is that it remains invariant under the simultaneous shift of z_1 and z_2 by ω_2:

$$\mathcal{S}(z_1+\omega_2,z_2+\omega_2)=\mathcal{S}(z_1,z_2)\,. \quad (5.13)$$

This follows from the fact that both the S-matrix (3.24) and the scalar factor S_0 are invariant under the shift. This property together with the crossing relations (5.8), (5.9) implies

$$\mathcal{S}^{t_1,t_2}(z_1,z_2)=\mathscr{C}_1\mathscr{C}_2\mathcal{S}(z_1,z_2)\mathscr{C}_1^{-1}\mathscr{C}_2^{-1}=\mathscr{C}_1^{-1}\mathscr{C}_2^{-1}\mathcal{S}(z_1,z_2)\mathscr{C}_1\mathscr{C}_2\,,$$

where t_1 and t_2 mean the transposition in the first and in the second matrix spaces, respectively.

[13]This is in opposite to [47], where the charge conjugation matrix was found to depend on the sign of the particle momentum. This dependence is, in fact, spurious and it gets removed by a proper resolution of the branch cut ambiguities we propose here.

Assuming that the above-mentioned properties of the S-matrix (3.24) are shared by $\mathcal{S}$, we can now see that the string S-matrix allows one to define consistently an elliptic analog of the ZF algebra, i.e.

$$\begin{aligned} A_1(z_1)A_2(z_2) &= \mathcal{S}_{12}(z_1,z_2)A_2(z_2)A_1(z_1)\,, \\ A_1^\dagger(z_1)A_2^\dagger(z_2) &= A_2^\dagger(z_2)A_1^\dagger(z_1)\mathcal{S}_{12}(z_1,z_2)\,, \end{aligned} \tag{5.14}$$

where the creation and annihilation ZF operators are now functions of the complex variable z. In addition, away from the line $z_1 = z_2$ we can impose the following relation between the creation and annihilation operators

$$A_1(z_1)A_2^\dagger(z_2) = A_2^\dagger(z_2)\mathcal{S}_{21}(z_2,z_1)A_1(z_1)\,. \tag{5.15}$$

As usual, the absence of cubic and higher relations for the ZF operators is guaranteed by the Yang-Baxter equation for $\mathcal{S}$. Furthermore, the validity of relations (5.14), (5.15) for all values of z_1 and z_2 is due to unitarity condition (5.3).

Transposing the second equation in (5.14) in the first matrix space we get

$$(A_1^\dagger(z_1))^{t_1}A_2^\dagger(z_2) = A_2^\dagger(z_2)\mathcal{S}_{12}^{t_1}(z_1,z_2)(A_1^\dagger(z_1))^{t_1}\,,$$

On the other hand, shifting in eq.(5.15) the variable z_1 by the imaginary half-period we obtain

$$A_1(z_1+\omega_2)A_2^\dagger(z_2) = A_2^\dagger(z_2)\mathcal{S}_{12}(z_1+\omega_2,z_2)^{-1}A_1(z_1+\omega_2)\,.$$

Since the string S-matrix satisfies the crossing relation we see that the algebra structure is compatible with the following identification

$$A(z+\omega_2) = \mathscr{C}^{-1}A^\dagger(z)^t\,, \quad A^\dagger(z-\omega_2) = -A(z)^t\mathscr{C}\,. \tag{5.16}$$

Analogously, we establish

$$A(z-\omega_2) = \mathscr{C}A^\dagger(z)^t\,, \qquad A^\dagger(z+\omega_2) = -A(z)^t\mathscr{C}^{-1}\,. \tag{5.17}$$

These relations together with the monodromy properties (5.5) of the S-matrix further imply

$$\begin{aligned} A(z+2\omega_1) &= \Sigma A(z)\,, \quad A^\dagger(z+2\omega_1) = A^\dagger(z)\Sigma\,, \\ A(z+2\omega_2) &= \Sigma A(z)\,, \quad A^\dagger(z+2\omega_2) = A^\dagger(z)\Sigma\,. \end{aligned}$$

This means that the bosonic operators are unchanged under the shift around the torus while fermionic ones acquire the minus sign. Thus, the monodromy properties of the S-matrix imply the spin structure $(-,-)$ for the fermionic ZF operators.

Finally, the generalized unitarity condition (5.4) allows one to impose the following hermiticity conditions on the ZF operators:

$$
\begin{aligned}
&[A^i(z)]^\dagger = A_i^\dagger(z^*) \qquad &&\text{for} \quad 0 < |\mathrm{Im}\, z| < \frac{\omega_2}{2i};\\
&[A^i(z)]^\dagger = -A_i^\dagger(z^*) \qquad &&\text{for} \quad \frac{\omega_2}{2i} < |\mathrm{Im}\, z| < \frac{\omega_2}{i}\,.
\end{aligned} \tag{5.18}
$$

The hermiticity condition for the ZF creation and annihilation operators in the anti-particle region $\omega_2/2i < |\mathrm{Im}\, z| < \omega_2/i$ is compatible with the hermiticity condition for the ZF operators in the particle region $0 < |\mathrm{Im}\, z| < \omega_2/2i$ and the identifications (5.16) and (5.17).

5.2 Unitarity of the scalar factor in mirror theory

It is clear from the discussion above that the S-matrix of the mirror theory is obtained from the string S-matrix just by the shift of the z-variables by $\omega_2/2$

$$
\widetilde{\mathcal{S}}(z_1,z_2) = \mathcal{S}(z_1+\frac{\omega_2}{2}, z_2+\frac{\omega_2}{2})\,. \tag{5.19}
$$

The momentum of the mirror theory is expressed in terms of the variable z by eq.(4.20) and is real for real values of z, and the generalized unitarity of the mirror S-matrix in terms of the shifted coordinates z takes the usual form

$$
\Big[\widetilde{\mathcal{S}}(z_1,z_2)\Big]^\dagger\,\widetilde{\mathcal{S}}(z_1^*,z_2^*) = \mathbb{I}\,. \tag{5.20}
$$

This just follows from the generalized unitarity of the string S-matrix and relation (5.13) which is a consequence of the crossing equations

$$
\Big[\widetilde{\mathcal{S}}_{12}(z_1,z_2)\Big]^\dagger = \mathcal{S}_{21}(z_2^*-\frac{\omega_2}{2}, z_1^*-\frac{\omega_2}{2}) = \mathcal{S}_{21}(z_2^*+\frac{\omega_2}{2}, z_1^*+\frac{\omega_2}{2}) = \widetilde{\mathcal{S}}_{21}(z_2^*,z_1^*)\,.
$$

In fact, since both the S-matrix (3.24) and the scalar factor S_0 satisfy the generalized unitarity condition and relation (5.13), the same holds for the mirror theory.

It is of interest to understand how the dressing factor of the mirror theory transforms under the complex conjugation. To this end we recall that in the $a=0$ light-cone gauge[14] the scalar factor of the string S-matrix can be written in the form [18]

$$
S_0(z_1,z_2)^2 = s(z_1,z_2)\,\sigma(z_1,z_2)\,, \quad s(z_1,z_2) = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\,. \tag{5.21}
$$

Here the gauge-independent dressing factor $\sigma(z_1,z_2)$ has the following structure [6]

$$
\frac{1}{i}\ln\sigma(z_1,z_2) \equiv \theta(z_1,z_2) = \sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} c_{r,s}(g)\Big[q_r(z_1)q_s(z_2) - q_r(z_2)q_s(z_1)\Big]\,, \tag{5.22}
$$

[14] It is easy to check that the additional a-dependent factor does not break any of the properties of the S-matrix.

where $q_r(z)=\frac{i}{r-1}\left[(x^+)^{1-r}-(x^-)^{1-r}\right]$ are the local conserved charges. At any order of the perturbative expansion in powers of $1/g$ the sums in r and s define the convergent series for $|x_1^\pm|>1$ and $|x_2^\pm|>1$. *Thus, the S-matrix is by construction well-defined only in the region where $|x^\pm|>1$ and it should be analytically continued for other values of $x^\pm$.*

The string theory dressing factor satisfies the generalized unitarity condition that follows from the fact that under the complex conjugation the variables $x^\pm$ transform as $[x^\pm(z)]^\dagger = x^\mp(z^*)$. In the mirror theory the variables $x^\pm$ depend on the shifted coordinate z and, as a result, satisfy the following complex conjugation rule

$$\left[x^\pm(z+\frac{\omega_2}{2})\right]^\dagger = \frac{1}{x^\mp(z^*+\frac{\omega_2}{2})}\,.$$

By using this rule one can easily check that the factor $s(z_1,z_2)$ in (5.21) transforms under the complex conjugation as follows

$$[s(z_1^*,z_2^*)]^\dagger\, s(z_1,z_2) = \left(\frac{x_1^- x_2^+}{x_1^+ x_2^-}\right)^2\,, \tag{5.23}$$

where $x_i^\pm = x^\pm(z_i+\frac{\omega_2}{2})$. Taking into account that the scalar factor S_0 of the mirror theory satisfies the generalized unitarity condition, we find the complex conjugation rule for the dressing factor of the mirror theory

$$[\sigma(z_1^*,z_2^*)]^\dagger\, \sigma(z_1,z_2) = \left(\frac{x_1^+ x_2^-}{x_1^- x_2^+}\right)^2\,. \tag{5.24}$$

In particular, for real values of z's corresponding to real $\widetilde{p}$'s the dressing factor of the mirror theory is not unitary.

It is interesting to note that the scalar factor can be split into a product of two factors satisfying the generalized unitarity condition in both string and mirror theory

$$S_0(z_1,z_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{x_1^+ x_2^- - 1}{x_1^- x_2^+ - 1} \times \frac{x_1^- x_2^+}{x_1^+ x_2^-}\,\sigma(z_1,z_2)\,. \tag{5.25}$$

Another interesting splitting is given by

$$S_0(z_1,z_2)^2 = \frac{u_1-u_2-\frac{2i}{g}}{u_1-u_2+\frac{2i}{g}} \times \left(\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\right)^2 \sigma(z_1,z_2)\,. \tag{5.26}$$

This splitting is useful for analyzing the bound state spectrum of the mirror model.

Knowing the series representation for the dressing phase in the original theory [13], it is interesting to understand what is precisely the source of its unitarity breakdown in the mirror theory.

To clarify this issue, we recall that the dressing phase can be conveniently written in terms of a single function $\chi(x_1, x_2)$ [18]

$$\begin{aligned}\theta(z_1, z_2) = \;& \chi(x_1^+, x_2^+) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+) + \chi(x_1^-, x_2^-) - \\ & - \chi(x_2^+, x_1^+) + \chi(x_2^-, x_1^+) + \chi(x_2^+, x_1^-) - \chi(x_2^-, x_1^-)\,,\end{aligned}$$

which admits the following strong coupling expansion

$$\chi(x_1, x_2) = g \sum_{n=0}^{\infty} \chi^{(n)}(x_1, x_2) \left(\frac{g}{2}\right)^{-n}.$$

Here

$$\chi^{(0)}(x_1, x_2) = -\frac{1}{x_2} - \frac{x_1 x_2 - 1}{x_2} \log \frac{x_1 x_2 - 1}{x_1 x_2}$$

is the leading AFS factor [6]. The next-to-leading contribution is [16]:

$$\begin{aligned}\chi^{(1)}(x_1, x_2) = & - \frac{1}{2\pi} \mathrm{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} - \frac{1}{2\pi} \mathrm{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \\ & + \frac{1}{2\pi} \mathrm{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{1}{2\pi} \mathrm{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}}\,. \end{aligned} \tag{5.27}$$

All higher terms are rational functions of x_1, x_2 [13]. As we will now show, the unitarity breakdown of the dressing phase is due to the leading AFS contribution only, the Hernández-López term (5.27), as well as all higher order terms do not influence the unitarity condition.

To simplify the notations in what follows we only consider the case of real z's in the mirror theory. It is easy to see that the complex conjugate of the function $\chi^{(0)}$ is given by

$$\left[\chi^{(0)}(x_1^\pm, x_2^\pm)\right]^* = -\chi^{(0)}(x_2^\mp, x_1^\mp) - \frac{i\pi + 1}{x_1^\mp} + (i\pi - 1)x_2^\mp - \left(\frac{1}{x_1^\mp} - x_2^\mp\right) \log x_1^\mp x_2^\mp\,.$$

Using this formula for computing the leading value $\theta_{\rm AFS}$, we find that the contribution of non-logarithmic terms cancels out and we get

$$\begin{aligned}\theta^*_{\rm AFS} = \theta_{\rm AFS} \,&+ g\frac{x_1^- - x_2^-}{x_1^- x_2^-}(1 - x_1^- x_2^-) \log x_1^- x_2^- + g\frac{x_1^+ - x_2^+}{x_1^+ x_2^+}(1 - x_1^+ x_2^+) \log x_1^+ x_2^+ \\ &- g\frac{x_1^- - x_2^+}{x_1^- x_2^+}(1 - x_1^- x_2^+) \log x_1^- x_2^+ + g\frac{x_2^- - x_1^+}{x_1^+ x_2^-}(1 - x_1^+ x_2^-) \log x_1^+ x_2^-\,.\end{aligned}$$

Using identity (4.3), it is easy to show that all logarithmic terms are neatly combined to produce the following answer

$$\theta^*_{\rm AFS} = \theta_{\rm AFS} + i \log \left(\frac{x_1^+ x_2^-}{x_1^- x_2^+}\right)^2, \tag{5.28}$$

which coincides with the logarithmic form of eq.(5.24).

Since we have shown that the shift of the phase under the complex conjugation occurs due to the leading contribution, all the higher order terms in the expansion of θ must be real functions. To convince oneself that this is indeed the case, we consider the next-to-leading term in the strong coupling expansion of θ. As was shown in [13], this term admits the following representation

$$\theta_{\rm HL} = \psi(q_1^+ - q_2^+) - \psi(q_1^+ - q_2^-) - \psi(q_1^- - q_2^+) + \psi(q_1^- - q_2^-)\,. \tag{5.29}$$

Here the function $\psi(q)$ is

$$\psi(q) = \frac{1}{2\pi}{\rm Li}_2(1-e^{iq}) - \frac{1}{2\pi}{\rm Li}_2(1-e^{iq+i\pi}) - \frac{i}{2}\log(1-e^{iq+i\pi}) + \frac{\pi}{8}\,, \tag{5.30}$$

where the variables $q^\pm$ are related to $x^\pm$ through

$$e^{iq^\pm} = \frac{x^\pm + 1}{x^\pm - 1}\,. \tag{5.31}$$

Taking into account the conjugation rule in the mirror theory, eq.(3.20), we obtain

$$(q^\pm)^* = -q^\mp - \pi\,. \tag{5.32}$$

Since $\theta_{\rm HL}$ depends on the difference of two $q's$, the shift by π arising upon the complex conjugation will cancel out. Thus, taking the complex conjugate we obtain

$$\theta^*_{\rm HL} = \bar\psi(q_1^- - q_2^-) - \bar\psi(q_1^- - q_2^+) - \bar\psi(q_1^+ - q_2^-) + \bar\psi(q_1^+ - q_2^+)\,, \tag{5.33}$$

where the function $\bar\psi(q)$ is defined as

$$\bar\psi(q) = \frac{1}{2\pi}{\rm Li}_2(1-e^{iq}) - \frac{1}{2\pi}{\rm Li}_2(1-e^{iq-i\pi}) - \frac{i}{2}\log(1-e^{iq-i\pi}) + \frac{\pi}{8}\,. \tag{5.34}$$

Taking into account the following transformation property of the dilogarithm function

$${\rm Li}_2(1-e^{iq-i\pi}) = {\rm Li}_2(1-e^{iq+i\pi-2\pi i}) = {\rm Li}_2(1-e^{iq+i\pi}) + 2\pi i\log(1-e^{iq})\,,$$

we find that

$$\bar\psi(q) = \psi(q) + \pi\,.$$

Since the shift by π in the previous formula does not contribute to $\theta^*_{\rm HL}$, we conclude that $\theta^*_{\rm HL} = \theta_{\rm HL}$. Finally, by working out several higher order terms $\chi^{(k)}$, one can easily check that they always lead to the real functions θ, in accord with eqs.(5.24) and (5.28).

Thus, we have shown that under the double Wick rotation the scalar factor remains unitary, while the dressing factor does not; the non-unitarity of the dressing factor is only due to the leading contribution $\theta_{\rm AFS}$, which is another distinguished property of $\theta_{\rm AFS}$. Concluding this section, we note that it would be interesting to understand whether the BES factor [14] exhibits the same kind of non-unitarity behavior in the mirror theory.

6. Bethe ansatz equations

In this section we discuss the nested Bethe equations for the light-cone string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and its mirror model. These equations based on the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant string S-matrix [47] were recently derived by using the algebraic [60] and the coordinate [67] Bethe ansatz approaches. In the sector with even winding number, i.e. with the total momentum P satisfying $e^{iP/2} = 1$, the set of equations found in these papers coincides with the one previously obtained in [8, 9] by using the spin chain description of the gauge theory. It appears, however, that in the sector with odd winding number, where $e^{iP/2} = -1$, the Bethe equations by [60, 67] differ from the ones derived from the gauge theory. The origin of this disagreement can be traced back to the fact that in the light-cone gauge the fermions of the string sigma model are *anti-periodic in the odd winding number sector* [68, 56], and this changes the periodicity conditions for wave functions which one imposes to get the Bethe equations. Indeed, in the light-cone gauge one of the fields, an angle ϕ which parametrizes the five-sphere, appears to be unphysical and it is solved in terms of (transversal) physical fields. In particular, the equation of motion for ϕ implies

$$\phi(2\pi) - \phi(0) = P\,.$$

Since ϕ enters into parametrization of the five-sphere via $e^{i\phi}$, the closed string periodicity condition for physical fields gives rise to the winding sectors

$$\phi(2\pi) - \phi(0) = 2\pi m\,,$$

where m is an integer. Now, we recall that fermions of the original string sigma-models are charged w.r.t. the U(1) isometry acting on ϕ as $\phi \to \phi + \text{const}$. Also, the Wess-Zumino term in the sigma-model action contains $e^{i\phi}$, i.e. it is non-local in terms of physical fields. To uncharge the fermions under the U(1) isometry, as well as to make the Wess-Zumino term local, one has to redefine the fermions as

$$\psi \to e^{\frac{i}{2}\phi}\psi\,.$$

Thus, the redefined fermions acquire the periodicity properties which do depend on the winding sector

$$\psi(0) = e^{\frac{i}{2}P}\psi(2\pi) = e^{i\pi m}\psi(2\pi)\,,$$

i.e. they are periodic in the even winding sector and they are ant-periodic in the odd winding sector [68, 56].

As a result, the Bethe equations obtained in [60, 67] correctly describe the light-cone string theory in the sector with periodic fermions only. Changing the boundary conditions for fermions to anti-periodic, one derives a new set of Bethe equations which does agree with the gauge theory one for physical states satisfying $e^{iP/2} = -1$.

6.1 BAE for a model with the $\mathfrak{su}(2|2)$-invariant S-matrix

The asymptotic states of both the original and the mirror theory are constructed by applying the ZF operators $A^\dagger_{M\dot{M}}$ to the vacuum state. The indices M and $\dot{M}$ are associated with two factors of the centrally-extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra; the latter being the symmetry algebra of the light-cone string theory [9, 10]. For our present purpose it is convenient to think about the ZF operator as being a product of two (anti)commuting operators each transforming in a fundamental representation of $\mathfrak{su}(2|2)$: $A^\dagger_{M\dot{M}} \sim A^\dagger_M A^\dagger_{\dot{M}}$. Since the string S-matrix is a tensor product of two $\mathfrak{su}(2|2)$-invariant S-matrices, the Bethe equations for the string model are, in a sense, the square of the Bethe equations for a model with the $\mathfrak{su}(2|2)$-invariant S-matrix. We start with discussing the Bethe equations for such a model.

The multi-particle wave function which satisfies the Bethe periodicity conditions can be written as a superposition of the asymptotic states (see appendix 9.3.1 for details)

$$|\Psi\rangle = \sum \Psi^{M_1\cdots M_{K^{\rm I}}} A^\dagger_{M_1}(p_1)\cdots A^\dagger_{M_{K^{\rm I}}}(p_{K^I})|0\rangle\,, \tag{6.1}$$

where $K^{\rm I}$ is a number of particles in the asymptotic state and p_i are their momenta. Denote by $N(M)$ the number of particles of type M (that is number of indices of type M) occurring in the wave function (6.1). Obviously,

$$K^{\rm I} = N(1) + N(2) + N(3) + N(4)\,.$$

Since the scattering is elastic, the number of particles $K^{\rm I}$ is a conserved quantity.

The form of the Bethe equations derived through the nesting procedure of the coordinate Bethe ansatz depends on the choice of the initial reference state. Due to the $\mathfrak{su}(2)^2$ bosonic symmetry there are two inequivalent choices for a model with the $\mathfrak{su}(2|2)$-invariant S-matrix. This is obviously related to the two forms of the Dynkin graph for $\mathfrak{su}(2|2)$.

First, one can choose a "bosonic" reference state which is created by acting with $K^{\rm I}$ bosonic operators $A^\dagger_1$ on the vacuum:

$$A^\dagger_1(p_1)\ldots A^\dagger_1(p_{K^{\rm I}})|0\rangle\,.$$

Then, we define

$$K^{\rm II}_+ = 2N(2) + N(3) + N(4)\,,\quad K^{\rm III} = N(2) + N(4)\,.$$

It appears that in the scattering process not only $K^{\rm I}$ but also these numbers are conserved [9]. By this reason, the values of $K^{\rm II}_+$ and $K^{\rm III}$ are the *same* for any term in the sum (6.1). In particular, $K^{\rm II}_+$ plays the role of the fermionic number, because in the background of the $A^\dagger_1$-particles $A^\dagger_2$ counts for two fermions. The number $K^{\rm III}$ has a similar meaning.

Then the asymptotic Bethe equations based on the $\mathfrak{su}(2|2)$-invariant S-matrix for a sigma model on a circle of length R and with (anti)-periodic fermions can be written in the form [9, 60, 67]

$$
\begin{aligned}
e^{ip_k R} &= \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{I}}} S_0(p_k,p_l)\frac{x_k^+ - x_l^-}{x_k^- - x_l^+}\sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}}\prod_{l=1}^{K_+^{\mathrm{II}}}\frac{x_k^- - y_l}{x_k^+ - y_l}\sqrt{\frac{x_k^+}{x_k^-}} \\
(-1)^\epsilon &= \prod_{l=1}^{K^{\mathrm{I}}}\frac{y_k - x_l^+}{y_k - x_l^-}\sqrt{\frac{x_l^-}{x_l^+}}\prod_{l=1}^{K^{\mathrm{III}}}\frac{v_k - w_l + \frac{i}{g}}{v_k - w_l - \frac{i}{g}} \\
1 &= \prod_{l=1}^{K_+^{\mathrm{II}}}\frac{w_k - v_l - \frac{i}{g}}{w_k - v_l + \frac{i}{g}}\prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}}\frac{w_k - w_l + \frac{2i}{g}}{w_k - w_l - \frac{2i}{g}}.
\end{aligned}
\tag{6.2}
$$

Here $\epsilon = 0$ for a sector with periodic fermions and $\epsilon = 1$ for a sector with anti-periodic fermions, $x_k^\pm$ depend on the momentum p_k of the model, y_l and w_l are auxiliary roots of the second and third levels, respectively, and $v = y + \frac{1}{y}$.

On the other hand, if one chooses a "fermionic" reference state created by K^{I} fermionic operators $A_3^\dagger$:

$$A_3^\dagger(p_1)\dots A_3^\dagger(p_{K^{\mathrm{I}}})|0\rangle\,,$$

then, one should define

$$K_-^{\mathrm{II}} = 2N(4) + N(1) + N(2)\,,\quad K^{\mathrm{III}} = N(2) + N(4)\,,$$

because these numbers are also conserved in the scattering process. Then, K_-^{II} plays the role of the bosonic number, because in the background of the $A_3^\dagger$-particles $A_4^\dagger$ counts for two bosons.

Then the corresponding Bethe equations take the following form

$$
\begin{aligned}
e^{ip_k R} &= (-1)^\epsilon \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{I}}} S_0(p_k,p_l)\prod_{l=1}^{K_-^{\mathrm{II}}}\frac{x_k^+ - y_l}{x_k^- - y_l}\sqrt{\frac{x_k^-}{x_k^+}} \\
(-1)^\epsilon &= \prod_{l=1}^{K^{\mathrm{I}}}\frac{y_k - x_l^+}{y_k - x_l^-}\sqrt{\frac{x_l^-}{x_l^+}}\prod_{l=1}^{K^{\mathrm{III}}}\frac{v_k - w_l + \frac{i}{g}}{v_k - w_l - \frac{i}{g}} \\
1 &= \prod_{l=1}^{K_-^{\mathrm{II}}}\frac{w_k - v_l - \frac{i}{g}}{w_k - v_l + \frac{i}{g}}\prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}}\frac{w_k - w_l + \frac{2i}{g}}{w_k - w_l - \frac{2i}{g}}.
\end{aligned}
\tag{6.3}
$$

Equations (6.3) can be derived either by using the nesting procedure of the coordinate Bethe ansatz (see appendix 9.3.2 for an example) or by applying the duality transformation discussed in [8] to eqs.(6.2). Comparing the two sets of Bethe equations (6.2) and (6.3), we see that only the first lines in two sets are different. Let us stress, however, that in general $K^{\rm II}_- \neq K^{\rm II}_+$. We further note that the bosonic reference state corresponds to, say, $\eta_1 = 1$ and the fermionic one to $\eta_1 = -1$, where η_1 and η_2 are the gradings introduced in [8].

6.2 BAE based on the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant string S-matrix

The Bethe equations based on the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant string S-matrix for both string and mirror models can be now easily written by taking a "product" of two copies of the Bethe equations for the $\mathfrak{su}(2|2)$-invariant model. Since any of the two sets, (6.2) and (6.3), can be used there are four different forms of the asymptotic Bethe equations based on the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant S-matrix [8]. The corresponding bosonic reference states of the coordinate Bethe ansatz are of the form

$$A^\dagger_{1\dot 1}(z_1)\dots A^\dagger_{1\dot 1}(z_{K^{\rm I}})|0\rangle\,, \quad \eta_1 = \eta_2 = 1\,; \quad A^\dagger_{3\dot 3}(z_1)\dots A^\dagger_{3\dot 3}(z_{K^{\rm I}})|0\rangle\,, \quad \eta_1 = \eta_2 = -1\,,$$

and fermionic reference states are

$$A^\dagger_{1\dot 3}(z_1)\dots A^\dagger_{1\dot 3}(z_{K^{\rm I}})|0\rangle\,, \quad \eta_1 = -\eta_2 = 1\,; \quad A^\dagger_{3\dot 1}(z_1)\dots A^\dagger_{3\dot 1}(z_{K^{\rm I}})|0\rangle\,, \quad \eta_1 = -\eta_2 = -1\,,$$

where for the original theory the z-variables lie on the real line, while for the mirror theory they have $\mathrm{Im}\, z = \omega_2/2i$, and we also indicated the corresponding gradings.

To discuss the bound states of the light-cone string sigma model, it is convenient to choose as the reference state the one created by the bosonic operators $A^\dagger_{1\dot 1}$. These reference states are dual to gauge theory operators from the $\mathfrak{su}(2)$ sector. Then the corresponding Bethe equations based on the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$-invariant string S-matrix can be written in the form [8, 9, 60, 67]

$$\begin{aligned}
e^{ip_k J} &= \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}} \left[S_0(p_k,p_l)\frac{x_k^+ - x_l^-}{x_k^- - x_l^+}\sqrt{\frac{x_l^+ x_k^-}{x_l^- x_k^+}}\right]^2 \prod_{\alpha=1}^{2}\prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}}\sqrt{\frac{x_k^+}{x_k^-}} \\
(-1)^\epsilon &= \prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-}\sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K^{\rm III}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}} \\
1 &= \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}\,.
\end{aligned} \tag{6.4}$$

Here we take into account that the string sigma model in the $a = 0$ light-cone gauge is defined on a circle of length J, $\alpha = 1, 2$ reflects the two copies of $\mathfrak{su}(2|2)$ and $y_l^{(\alpha)}$ and $w_l^{(\alpha)}$ are auxiliary roots of the second and third levels, respectively, and $v = y + \frac{1}{y}$.

For the reader's convenience we point out that the excitation numbers in the set of Bethe equations are related to the ones used in [8] as follows

$$(K^{\rm III}_{(1)}\,,\ K^{\rm II}_{(1)}\,,\ K^{\rm I}\,,\ K^{\rm II}_{(2)}\,,\ K^{\rm III}_{(2)}) = (K_2, K_1+K_3, K_4, K_5+K_7, K_6)\,,$$

and the Dynkin labels $[q_1, p, q_2]$ of $\mathfrak{su}(4)$ and $[s_1, s_2]$ of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\subset\mathfrak{su}(2,2)$ are expressed in terms of the excitation numbers by the following formulas[15]

$$\begin{aligned}
q_1 &= K^{\rm I}-K^{\rm II}_{(1)}\,, & s_1 &= K^{\rm II}_{(1)}-2K^{\rm III}_{(1)}\,,\\
p &= J-K^{\rm I}+\frac{1}{2}(K^{\rm II}_{(1)}+K^{\rm II}_{(2)})\,, & s_2 &= K^{\rm II}_{(2)}-2K^{\rm III}_{(2)}\,,\\
q_2 &= K^{\rm I}-K^{\rm II}_{(2)}\,.
\end{aligned} \tag{6.5}$$

To analyze the bound states of the mirror theory, it is more convenient, however, to choose as an initial reference state the one created by the operators $A^\dagger_{3\dot{3}}$. The reason is that the operators $A^\dagger_{3\dot{3}}$ create states from the $\mathfrak{sl}(2)$ sector, and, as we have seen, it is this sector which gives rise to mirror magnons. Analogously, there are M-particle bound states made only out of the $A^\dagger_{3\dot{3}}$-type particles.

If we choose in the mirror theory the above-described reference state then the corresponding Bethe equations take the form

$$\begin{aligned}
e^{i\widetilde{p}_kR} &= \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}}\left[S_0(\widetilde{p}_k,\widetilde{p}_l)\right]^2\prod_{\alpha=1}^{2}\prod_{l=1}^{K^{\rm II}_{(\alpha)}}\frac{x_k^+-y_l^{(\alpha)}}{x_k^--y_l^{(\alpha)}}\sqrt{\frac{x_k^-}{x_k^+}}\\
-1 &= \prod_{l=1}^{K^{\rm I}}\frac{y_k^{(\alpha)}-x_l^+}{y_k^{(\alpha)}-x_l^-}\sqrt{\frac{x_l^-}{x_l^+}}\prod_{l=1}^{K^{\rm III}_{(\alpha)}}\frac{v_k^{(\alpha)}-w_l^{(\alpha)}+\frac{i}{g}}{v_k^{(\alpha)}-w_l^{(\alpha)}-\frac{i}{g}}\\
1 &= \prod_{l=1}^{K^{\rm II}_{(\alpha)}}\frac{w_k^{(\alpha)}-v_l^{(\alpha)}-\frac{i}{g}}{w_k^{(\alpha)}-v_l^{(\alpha)}+\frac{i}{g}}\prod_{\substack{l=1\\ l\neq k}}^{K^{\rm III}_{(\alpha)}}\frac{w_k^{(\alpha)}-w_l^{(\alpha)}+\frac{2i}{g}}{w_k^{(\alpha)}-w_l^{(\alpha)}-\frac{2i}{g}}\,.
\end{aligned} \tag{6.6}$$

Note that in the mirror model we do not have $(-1)^\epsilon$ in the middle equation because the fermions are always anti-periodic[16] with respect to $\tilde{\sigma}$. In terms of excitation

[15] Let us note in passing that in recent papers [70, 71] the anomalous dimension of the operator $\mathrm{Tr}\mathcal{F}^L$ was computed by using the asymptotic Bethe ansatz with an understanding that in the large L limit one may trust the corresponding result to an arbitrary loop order. One can notice, however, that the excitation pattern of Bethe roots for the operator is $(K^{\rm III}_{(1)}\,,\ K^{\rm II}_{(1)}\,,\ K^{\rm I}\,,\ K^{\rm II}_{(2)}\,,\ K^{\rm III}_{(2)}) = (0, 2L-3, 2L-2, 2L-4, L-2)$ with $J=\frac{3}{2}$, and, therefore, one would expect the breakdown of the asymptotic ansatz due to the finite size effects already at two loops. It may happen that the asymptotic ansatz could still be used to determine the leading L behavior of the anomalous dimension of $\mathrm{Tr}\mathcal{F}^L$ if the finite-size corrections are subleading at large L, but this is currently unknown.

[16] We are grateful to R. Janik and M. Martins for drawing our attention to this point.

numbers, the Dynkin labels read now as follows

$$\begin{aligned} q_1 &= K^{\rm II}_{(1)} - K^{\rm III}_{(1)}\,, & s_1 &= K^{\rm I} - K^{\rm II}_{(1)}\,, \\ p &= J - \frac{1}{2}(K^{\rm II}_{(1)} + K^{\rm II}_{(2)}) + K^{\rm III}_{(1)} + K^{\rm III}_{(2)}\,, & s_2 &= K^{\rm I} - K^{\rm II}_{(2)}\,, \\ q_2 &= K^{\rm II}_{(2)} - K^{\rm III}_{(2)}\,. \end{aligned} \tag{6.7}$$

7. Bound states of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ gauge-fixed model

Bound states arise as poles of the multi-particle S-matrix corresponding to complex values of the particle momenta, see e.g. [69]. In the thermodynamic limit they are described by string-like solutions known as "Bethe strings". In this section we discuss in detail the bound states of the string sigma-model. They have been already analyzed in [50, 51]. The main outcome of this analysis is that the M-particle bound states comprise into short (BPS) multiplets of the centrally extended $\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$ symmetry algebra. Although the S-matrix exhibits additional simple and double poles beyond those corresponding to the BPS multiplets, these singularities do not lead however to the appearance of new bound states [52]. In other words, the only bound states in the theory are the BPS ones. As we will see, they exist for all values of the (real) bound state momentum $-\pi \le p \le \pi$, but have a rather intricate structure. Moreover, depending on the choice of the physical region for a given value of the bound state momentum there could be 1, 2 or 2^{M-1} M-particle bound states sharing the same set of global conserved charges: $Q_r = \sum_{i=1}^M q_r(z_i)$. It is unclear to us whether this indicates that the actual physical region is the one that contains only a single M-particle bound state (it is the one with $|x^\pm| > 1$) or it is a sign of a hidden symmetry of the model responsible for the degeneracy of the spectrum.

7.1 Two-particle bound states

Let us consider a bound state made of two excitations from the $\mathfrak{su}(2)$ sector of the string sigma-model. In terms of the ZF creation operators we can think about this state as

$$A^\dagger_{1\dot{1}}(p_1)A^\dagger_{1\dot{1}}(p_2)|0\rangle\,,$$

where the particle momenta p_1 and p_2 are complex. We find it convenient to parametrize the momenta as follows

$$p_1 = \frac{p}{2} - iq\,, \qquad p_2 = \frac{p}{2} + iq\,, \qquad \mathrm{Re}\, q > 0\,, \tag{7.1}$$

where p is the real total momentum of the bound state. When q is real then p_1 and p_2 are complex conjugate to each other and the energy of the corresponding bound state being the sum of the (complex) energies of individual particles is obviously real. Interestingly, as we will show below, there necessarily exists a branch of BPS bound states which corresponds to complex values of q with $\mathrm{Re}\, q > 0$. Such solutions can be reinterpreted as solutions parametrized by a new real variable q: $q \to \mathrm{Re}\, q$ and for which the real parts of p_1 and p_2 are not anymore equal to each other. Of course, one has to check that the energy of these solutions is real.

The first equation in the set of the Bethe equations [6] takes the form

$$e^{i(p/2+\mathrm{Im}\, q)L} e^{\mathrm{Re}\, q\, L} = e^{iP} \prod_{l=2}^{K^{\mathrm{I}}} \frac{x_1^+ - x_l^-}{x_1^- - x_l^+} \frac{1 - \frac{1}{x_1^+ x_l^-}}{1 - \frac{1}{x_1^- x_l^+}} \sigma_{1l} \,, \tag{7.2}$$

where $P = p_1 + p_2 + \cdots + p_{K^{\mathrm{I}}}$ and $L = J + K^{\mathrm{I}}$ with J being one of the global charges corresponding to the isometries of the five-sphere.

We see that for large L the l.h.s. is exponentially divergent. Then, there should exist a root p_2 such that for $\mathrm{Re}\, q > 0$ we have[17]

$$\left(x_1^- - x_2^+\right)\left(1 - \frac{1}{x_1^- x_2^+}\right) \sim e^{-\mathrm{Re}\, q\, L} \,. \tag{7.3}$$

In the infinite L limit eq.(7.3) becomes

$$\left(x_1^- - x_2^+\right)\left(1 - \frac{1}{x_1^- x_2^+}\right) = 0 \,, \tag{7.4}$$

which is equivalent to

$$x_1^- - x_2^+ = 0 \qquad \text{or} \qquad 1 - \frac{1}{x_1^- x_2^+} = 0 \,. \tag{7.5}$$

The first equation

$$x_1^- - x_2^+ = 0 \tag{7.6}$$

implies that the central charges corresponding to the two-particle bound state saturate the BPS condition [50]

$$H^2 = 2^2 + 4g^2 \sin^2 \frac{p}{2} \,. \tag{7.7}$$

On the contrary, solutions of the second equation in (7.5) do not saturate the BPS bound, and as was argued in [52], this pole of the S-matrix does not correspond to a bound state.

It is easy to see that equation (7.6) is equivalent to vanishing the following fourth order polynomial[18] in the variable $t = \cos \frac{p}{2}$

$$4e^q(t - e^q)(1 - e^q t) + g^2(t^2 - 1)(1 - 2e^q t + e^{2q})^2 = 0 \,. \tag{7.8}$$

[17]We assume here and in what follows that the dressing factor σ_{12} is non-singular on solutions of the bound state equation.

[18]For any p there are two solutions for x^- and, therefore, for $x^+ = e^{ip} x^-$. The fourth order polynomial is universal and it does not depend on which solution for x^- we take.

The equation has four solutions which can be cast to the following simple form

$$e^q = \frac{\left(\sqrt{1+g^2\sin^2\frac{p}{2}}\pm 1\right)\left(\cos\frac{p}{2}\sqrt{1+g^2\sin^2\frac{p}{2}}\pm\sqrt{\cos^2\frac{p}{2}-g^2\sin^4\frac{p}{2}}\right)}{g^2\sin^2\frac{p}{2}}\,, \tag{7.9}$$

where any choice of the $\pm$ sign is possible.

Analysis of eq.(7.9) immediately shows that solutions corresponding to real values of q exist if and only if the total momentum p does not exceed a critical value $p_{\rm cr}$ determined by

$$\sin^2\frac{p_{\rm cr}}{2} = \frac{1}{2g^2}\left(\sqrt{1+4g^2}-1\right)\,. \tag{7.10}$$

For any given p obeying $|p| < p_{\rm cr}$ equation (7.8) has four real roots for q, two of them are positive and the other two are negative. According to our assumption ${\rm Re}\, q > 0$, only positive roots are acceptable.[19] They are given by the formula

$$e^{q_\pm} = \frac{\left(\sqrt{1+g^2\sin^2\frac{p}{2}}+1\right)\left(\cos\frac{p}{2}\sqrt{1+g^2\sin^2\frac{p}{2}}\pm\sqrt{\cos^2\frac{p}{2}-g^2\sin^4\frac{p}{2}}\right)}{g^2\sin^2\frac{p}{2}}\,. \tag{7.11}$$

Various expansions of eq.(7.11) for small and large values of g can be found in Appendix 9.4.

It turns out that from the two positive roots only the smaller one, q_-, falls inside the region confined by the curves $|x^\pm| = 1$. We therefore arrive at the conclusion that inside the region $|x^\pm| > 1$ there is a unique solution with real p and q, and it exists if and only if [20]

$$|p| < p_{\rm cr}\,, \qquad 0 \le q < \log\frac{2g+\sqrt{2\sqrt{1+4g^2}-2}}{\sqrt{1+4g^2}-1}\,. \tag{7.12}$$

The second solution with $q = q_+$ lies outside the region with $|x^\pm| > 1$ but inside the region with $-\omega_2/2i < {\rm Im}(z) < \omega_2/2i$; the latter maps onto the complex p-plane, see section 4. Both solutions have the same values of all global conserved charges $Q_r = q_r(z_1) + q_r(z_2) = \frac{i}{r-1}\left[(x_1^+)^{1-r} - (x_2^-)^{1-r}\right]$ because x_1^+ and x_2^- are the same on both solutions.

We see that if we choose the physical region to be the one with $|x^\pm| > 1$ then there is a unique bound state with $|p| < p_{\rm cr}$. This region, however, does not cover the whole complex p-plane. One the other hand, if the physical region is the half of the torus corresponding to the p-plane, then there are two solutions with the same

[19] The solutions with negative q correspond to bound states of anti-particles with negative energy.

[20] The energy of the corresponding bound state is $E < \sqrt{2\sqrt{1+4g^2}+2}\,$.

energy and other conserved charges. Finally, if one considers solutions on the z-torus then there are four solutions but only two of them have positive energy.

Continuing above the critical value, $|p| > p_{\rm cr}$, two solutions (7.11) acquire imaginary parts and become complex-conjugate to each other, or, equivalently, the real parts of p_1 and p_2 become different. Thus, we see that *the BPS bound states naturally split into two families depending on whether the total momentum is below (the first family) or above (the second family) the critical value* $p_{\rm cr}$.

The two complex conjugate roots give two different solutions beyond criticality:

$$p_1^{\pm} = \frac{p}{2} \pm {\rm Im}\, q - i\, {\rm Re}\, q\,, \qquad p_2^{\pm} = \frac{p}{2} \mp {\rm Im}\, q + i\, {\rm Re}\, q\,, \qquad {\rm Re}\, q > 0\,. \tag{7.13}$$

We can choose either (p_1^+, p_2^+) or (p_1^-, p_2^-) as a possible solution of the BPS condition (7.6). Note that the second solution is the complex conjugate of the first one. A remarkable fact to be proven below is that both solutions lie precisely on the boundary of the region defined by the curves $|x^{\pm}| = 1$.

Now if we adopt the physical region (sheet) to be $|x^{\pm}| > 1$ with the boundary $|x^{\pm}| = 1$, then it should contain only one solution from the second BPS family. Indeed, we do not expect the doubling of the number of BPS bound states moving beyond the critical point. The second solution can be then naturally interpreted as lying on the boundary of another (unphysical) sheet joint to the physical one through the cut. It is unclear however what is the precise origin for such an asymmetry. A possible explanation would be the absence of parity invariance of the string sigma-model but a concrete implication of this fact is unknown to us.

To visualize the singularities of the string S-matrix and also to verify that energy is real for the second BPS family, it is instructive to analyze eqs.(7.5) in terms of the generalized rapidity variables z_1 and z_2 associated to the first and the second particles, respectively. It is not hard to see that the first family of the BPS states corresponds to imposing the reality condition $z_2^* = z_1$. In this case, eqs.(7.5) are equivalent to

$${\rm Im}(x_1^-) = 0 \quad \text{or} \quad |x_1^-| = 1\,, \tag{7.14}$$

where the first equation defines the first BPS family. Solving the bound state equation for z_1, one gets a curve in the torus. The part of the curve that lies inside the region $|x^{\pm}| > 1$ is represented in Fig.5 by the green curve $\mathbf{B}_1\mathbf{OC}_1$, and the corresponding momentum p_1 has ${\rm Im}(p_1) = -q_-$. The variable $z_2 = z_1^*$ of the second particle runs along another (conjugate) green curve $\mathbf{B}_2\mathbf{OC}_2$, which can be also viewed as describing solutions of the equation ${\rm Im}(x_2^+) = 0$ for z_2. The dashed curves on Fig.5a, which are outside the region $|x^{\pm}| > 1$, represent solutions of the equations ${\rm Im}(x_1^-) = {\rm Im}(x_2^+) = 0$ for z_1, z_2 corresponding to the momentum p_1 with ${\rm Im}(p_1) = -q_+$.

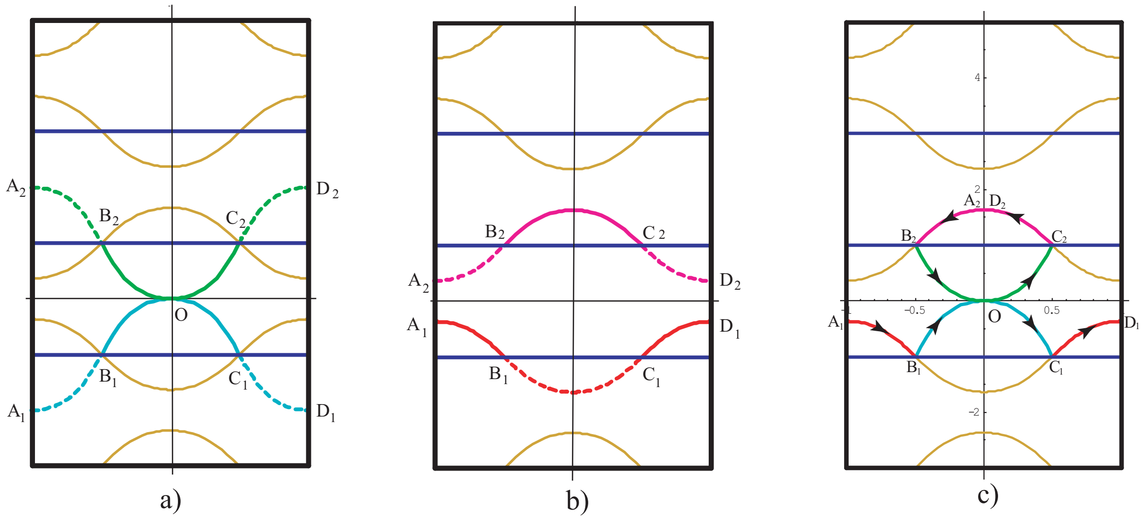

Figure 5: Two-particle bound states of string theory. Figure a) describes the first BPS family corresponding to $p < p_{\rm cr}$. The green curves are $\mathrm{Im}(x^-) = 0$ for $\mathrm{Im}(z) < 0$ and $\mathrm{Im}(x^+) = 0$ for $\mathrm{Im}(z) > 0$. For any $p < p_{\rm cr}$ there are two solutions: the first one is represented by the continuous curves $\mathbf{B}_1\mathbf{O}\mathbf{C}_1$ (1st particle) and $\mathbf{B}_2\mathbf{O}\mathbf{C}_2$ (2nd particle), the second one corresponds to the dashed curves $\mathbf{A}_1\mathbf{B}_1\cup\mathbf{C}_1\mathbf{D}_1$ (1st particle) and $\mathbf{A}_2\mathbf{B}_2\cup\mathbf{C}_2\mathbf{D}_2$ (2nd particle). Figure b) describes the second BPS family corresponding to $p > p_{\rm cr}$. Again, for any $p > p_{\rm cr}$ there are two solutions $\mathbf{B}_2\mathbf{C}_2 \cup \mathbf{A}_1\mathbf{B}_1 \cup \mathbf{C}_1\mathbf{D}_1$ and $\mathbf{B}_1\mathbf{C}_1 \cup \mathbf{A}_2\mathbf{B}_2 \cup \mathbf{C}_2\mathbf{D}_2$. Figure c) represents one of the four possibilities to smoothly connect solutions from the first and the second BPS families. Here the variable z_1 of the 1st particle is on the curve $\mathbf{A}_1\mathbf{B}_1\mathbf{O}\mathbf{C}_1\mathbf{D}_1$. When z_1 runs along the curve from $\mathbf{A}_1$ to $\mathbf{D}_1$ the real part of the momentum $\mathrm{Re}(p_1)$ increases monotonically from $-\pi$ to π. At the same time, the variable z_2 corresponding to the 2nd particle encloses the curve $\mathbf{A}_2\mathbf{B}_2\mathbf{O}\mathbf{C}_2\mathbf{D}_2$.

To describe the second family of the BPS states corresponding to the complex values of q one has to take

$$z_2 = -z_1^* + \frac{\omega_1}{2} + \frac{\omega_2}{2}\,. \tag{7.15}$$

In this case

$$x^+(z_2) = x^+\Big(-z_1^* + \frac{\omega_1}{2} + \frac{\omega_2}{2}\Big) = \frac{1}{x^+(z_1^*)} = \frac{1}{[x^-(z_1)]^*}\,, \tag{7.16}$$

where we have used the properties of Jacobi elliptic functions under the shifts by quarter-periods. Hence, due to the BPS equation $x_1^- = x_2^+$, the points z_1 and z_2 lie on the curves $|x^-| = 1$ and $|x^+| = 1$, respectively.

As was discussed above, there are two different ways to choose the second BPS family which is equivalent to deciding what is the physical sheet. Consider the point

z_1 corresponding to the first particle, Fig.5c. When it moves along the curve $\mathbf{B}_1\mathbf{O}\mathbf{C}_1$ corresponding to the first BPS family and reaches, e.g., the point $\mathbf{C}_1$ then there are two possibilities to continue its path along the curve $|x^-| = 1$: either one moves along $\mathbf{C}_1\mathbf{D}_1$ or along $\mathbf{C}_1\mathbf{B}_1$. In the case when z_1 moves along the curve $\mathbf{C}_1\mathbf{D}_1$, the second point z_2 follows the path $\mathbf{C}_2\mathbf{D}_2$. In the opposite situation, when z_1 moves along $\mathbf{C}_1\mathbf{B}_1$, the point z_2 follows $\mathbf{C}_2\mathbf{B}_2$. Similar discussion applies to continuing the first BPS family beyond $\mathbf{B}_1$. Obviously, for the second family z_1 and z_2 are not complex conjugate anymore, rather they obey the relation (7.15). The bound state energy $H = ig(x_2^- - x_1^+) - 2$ is however real, as one can also check by using the shift/reflection properties of the elliptic functions.

Our discussion reveals that there are four special points on the z-plane

$$z_{\rm cr} = \pm\frac{\omega_1}{4} \pm \frac{\omega_2}{4} \tag{7.17}$$

where both equations $\mathrm{Im}(x_1^-) = 0$ and $|x_1^-| = 1$ or $\mathrm{Im}(x_2^+) = 0$ and $|x_2^+| = 1$ are simultaneously satisfied. These are the critical points where two BPS families meet.

The most transparent description of the bound states is achieved in terms of the rapidity variable u introduced in section 4, rather than in terms of momentum p or the variable z. Indeed, in terms of u eq.(7.3) becomes

$$\left(x_1^- - x_2^+\right)\left(1 - \frac{1}{x_1^- x_2^+}\right) = u_1 - u_2 - \frac{2i}{g} = 0\,, \tag{7.18}$$

i.e. the rapidity variables u_1 and u_2 of the first and the second particle lie on a straight line running parallel to the imaginary axis. Moreover, for the first BPS family the variables $u_{1,2}$ are subject to the following conjugation rule $u_1^* = u_2$ which, together with eq.(7.18) allows one to conclude that

$$u_{1,2} = u_0 \pm \frac{i}{g}\,, \qquad u_0 \in \mathbb{R}\,. \tag{7.19}$$

This is a typical pattern of "Bethe string". One can further see that for the first BPS family corresponding to $p \leq p_{\rm cr}$ the variable u_0 is restricted to satisfy

$$|u_0| \geq 2\,, \qquad u_{1,\rm cr} = \pm 2 + \frac{i}{g}\,, \tag{7.20}$$

where $u_{1,\rm cr}$ is a critical value of rapidity u_1 for which the first BPS family ceased to exist. Under the map to the u-plane the four critical points $z_{\rm cr}$ are mapped to the four branch points on the u-plane (see Fig.3 in section 4)

$$u_{\rm cr} = \pm 2 \pm \frac{i}{g}\,. \tag{7.21}$$

Let us now turn to the second BPS family. First, by using eq.(4.5) and the properties of the elliptic functions, we derive

$$u_1^* = x^-\Big(- z_2 + \frac{\omega_1}{2} + \frac{\omega_2}{2}\Big) + \frac{1}{x^-\Big(- z_2 + \frac{\omega_1}{2} + \frac{\omega_2}{2}\Big)} + \frac{i}{g} = u_2\,. \tag{7.22}$$

We see that for both families of BPS states the conjugation rule for u's is the one and the same. By this reason, a solution to the BPS condition is always represented by the Bethe string (7.19). However, one finds that for the second family a solution exists for $|u_0| \leq 2$ only. Thus, on the u-plane both families of BPS states admit a uniform description in terms of the Bethe string with u_0 running over the whole real line.

7.2 Multi-particle bound states

The consideration of the two-particle bound states can be easily generalized to the M-particle case. The corresponding set of bound state equations reads [50]

$$x_j^- - x_{j+1}^+ = 0, \qquad j = 1, \ldots, M\,. \tag{7.23}$$

The total momentum of a state satisfying these equations is given by

$$e^{ip} = \frac{x_1^+}{x_1^-}\frac{x_2^+}{x_2^-}\cdots\frac{x_M^+}{x_M^-} = \frac{x_1^+}{x_M^-}\,,$$

and the energy of the state is

$$H_M = \sum_{i=1}^{M}\left(-1 - igx_i^+ + igx_i^-\right) = -M - igx_1^+ + igx_M^-\,. \tag{7.24}$$

Both the energy and momentum depend on the values of x_1^+ and x_M^- only. Since the energy is real, x_M^- must be the complex conjugate of x_1^+: $(x_M^-)^* = x_1^+$. In fact, a simple but important observation is that any global conserved charge of a state obeying (7.23) depends only on x_1^+ and x_M^-:

$$Q_r = \sum_{i=1}^{M} q_r(z_i) = \sum_{i=1}^{M}\frac{i}{r-1}\Big[(x_i^+)^{1-r} - (x_i^-)^{1-r}\Big] = \frac{i}{r-1}\Big[(x_1^+)^{1-r} - (x_M^-)^{1-r}\Big]\,.$$

Another important consequence of eqs.(7.23) is that the coordinates x_1^+ and x_M^- satisfy the following quadratic constraint

$$x_1^+ + \frac{1}{x_1^+} - x_M^- - \frac{1}{x_M^-} = \frac{2M}{g}i\,. \tag{7.25}$$

This is the same constraint as the one satisfied by $x^\pm$ (4.3) with $g \to g/M$, and we get immediately the dependence of x_1^+ and x_M^- on the total real momentum p [21]

$$\begin{aligned} x^+ &= \frac{e^{i\frac{p}{2}}}{2g\sin\frac{p}{2}}\left(M + \sqrt{M^2 + 4g^2\sin^2\frac{p}{2}}\right), \\ x^- &= \frac{e^{-i\frac{p}{2}}}{2g\sin\frac{p}{2}}\left(M + \sqrt{M^2 + 4g^2\sin^2\frac{p}{2}}\right), \end{aligned} \tag{7.26}$$

and, using (7.24), the BPS energy formula

$$H_M^2 = M^2 + 4g^2\sin^2\frac{p}{2}\,.$$

Moreover, we see that the set of global conserved charges Q_r is the same for any solution of (7.23) with a given total momentum p.

It is also easy to see that the number of different solutions with a real momentum p and positive energy is equal to 2^{M-1} because for a given x^+ there are two different x^- solving the constraint (4.3), see the diagram below for $M = 4$

$$x_1^+ \longrightarrow \left\{ \begin{array}{l} x_1^- = x_2^+ \longrightarrow \left\{ \begin{array}{l} x_2^- = x_3^+ \longrightarrow \left\{ \begin{array}{l} x_3^- = x_4^+ \\ x_3^- = x_4^+ \end{array} \right. \\ x_2^- = x_3^+ \longrightarrow \left\{ \begin{array}{l} x_3^- = x_4^+ \\ x_3^- = x_4^+ \end{array} \right. \end{array} \right. \\ x_1^- = x_2^+ \longrightarrow \left\{ \begin{array}{l} x_2^- = x_3^+ \longrightarrow \left\{ \begin{array}{l} x_3^- = x_4^+ \\ x_3^- = x_4^+ \end{array} \right. \\ x_2^- = x_3^+ \longrightarrow \left\{ \begin{array}{l} x_3^- = x_4^+ \\ x_3^- = x_4^+ \end{array} \right. \end{array} \right. \end{array} \right\} \longrightarrow x_4^-$$

To have all these solutions one would have to allow the parameters z_i of the particles to be anywhere on the z-torus, in particular, some of them would be in the anti-particle region with $|x^\pm| < 1$.

However, if we require that all the constituent particles of the bound state belong to the region $|x^\pm| > 1$ then we are left with a unique solution because for a given x^+ only one solution for x^- satisfies the condition $|x^-| \ge 1$. For M even it is also necessary to specify what parts of the boundaries $|x^\pm| = 1$ belong to the region because if the momentum of a bound state exceeds a critical, $g-$ and M-dependent, value then there are several solutions of the bound state equations with $|x_{M/2}^-| = |x_{M/2+1}^+| = 1$.

Finally, if the parameters z_i of the particles belong to the half of the torus corresponding to the complex p-plane, then one can show that for any M there are two solutions of the bound state equations.

[21] In general for a given momentum p there are two solutions of the constraint (7.25), and there could be any sign in front of the square root in (7.26). The positive sign guarantees the positivity of the energy.

Just as for the case of two-particle bound states, the simplest description of M-particle bound states is provided by the u-plane where a solution is given by the Bethe string

$$u_j = u_0 + (M - 2j + 1)\frac{i}{g}\,, \qquad j = 1, \ldots, M\,. \tag{7.27}$$

We can choose one and the same map of the u-plane with the cuts described in section 4 onto the region of the z-torus with $|x^\pm| > 1$ for all the particles. It is then obvious that for a given momentum p there is just a single M-particle bound state that falls inside the physical region. Its structural description however becomes rather involved.

7.3 Finite-size corrections to the bound states

It is of interest to analyze finite-size corrections to the energy of the BPS bound states, and to see what restrictions on the dressing factor could be derived from the condition that the energy corrections are real. To this end, we consider two-particle states in the $\mathfrak{su}(2)$ sector described by the following two equations, see (7.2)

$$\left(\frac{x_1^+}{x_1^-}\right)^J = \Sigma_{12}\,\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\,, \qquad \left(\frac{x_1^+ x_2^+}{x_1^- x_2^-}\right)^J = 1\,, \tag{7.28}$$

where $\Sigma_{12} = \frac{x_1^- x_2^+}{x_1^+ x_2^-}\sigma_{12}$ is the unitary factor that appeared in the splitting (5.25) of the scalar factor, and J is one of the global charges corresponding to the isometries of the five-sphere. The variables $x_i^\pm$ also satisfy the constraint (4.3). These equations are supposed to be valid asymptotically for large values of J, and have to be modified for finite J.

We will analyze these equations for large values of J in the vicinity of a bound state satisfying the bound state equation $x_1^- = x_2^+$ and having a fixed total momentum $p = \frac{2\pi m}{J}$ where m is an integer. The quantization condition for the total momentum follows from the second equation in (7.28).

Let $\mathbf{x}_i^\pm$ denote the values of $x_i^\pm$ satisfying the bound state equation and the second equation in (7.28). Then, $\left(\frac{\mathbf{x}_1^-}{\mathbf{x}_1^+}\right)^J \sim e^{-qJ}$ exponentially decreases at large J, and we can look for a solution of the form

$$x_i^\pm = \mathbf{x}_i^\pm\left(1 + \left(\frac{\mathbf{x}_1^-}{\mathbf{x}_1^+}\right)^J y_i^\pm\right)\,.$$

Expanding the equations (7.28) and the constraint (4.3) in powers of $y_i^\pm$, we find a system of linear equations for $y_i^\pm$. The solution of the system is given below

$$
\begin{aligned}
y_1^- &= \frac{4\Sigma_{12}(ig(\mathbf{x}_2^- - \mathbf{x}_2^+) + \mathbf{x}_2^-\mathbf{x}_2^+)\left(2i\mathbf{x}_1^+\mathbf{x}_2^+ + g\left(\mathbf{x}_1^+\left((\mathbf{x}_2^+)^2+1\right) - 2\mathbf{x}_2^+\right)\right)}{g^2(g\mathbf{x}_2^- - (g+2i\mathbf{x}_2^-)\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)^2} \\
y_1^+ &= \frac{4\Sigma_{12}\mathbf{x}_1^+(ig(\mathbf{x}_2^- - \mathbf{x}_2^+) + \mathbf{x}_2^-\mathbf{x}_2^+)}{g(g\mathbf{x}_2^- - (g+2i\mathbf{x}_2^-)\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)} \\
y_2^- &= -\frac{4i\Sigma_{12}\mathbf{x}_2^-(g(\mathbf{x}_1^+ - \mathbf{x}_2^+) + i\mathbf{x}_1^+\mathbf{x}_2^+)}{g((g+2i\mathbf{x}_2^-)\mathbf{x}_1^+ - g\mathbf{x}_2^-)\left((\mathbf{x}_2^+)^2-1\right)} \\
y_2^+ &= \frac{4\Sigma_{12}(g(\mathbf{x}_1^+ - \mathbf{x}_2^+) + i\mathbf{x}_1^+\mathbf{x}_2^+)\left(g\mathbf{x}_2^-(\mathbf{x}_2^+)^2 - 2(g+i\mathbf{x}_2^-)\mathbf{x}_2^+ + g\mathbf{x}_2^-\right)}{g^2(-ig(\mathbf{x}_2^- - \mathbf{x}_1^+) - 2\mathbf{x}_2^-\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)^2}\,,
\end{aligned}
$$

where Σ_{12} is evaluated on the solution to the bound state equation. The leading correction to the energy of the state is easily found by expanding

$$
E = E_1 + E_2\,, \quad E_i = 1 + \frac{ig}{x_i^+} - \frac{ig}{x_i^-} = -1 - igx_i^+ + igx_i^-\,. \tag{7.29}
$$

By using $E_i = 1 + \frac{ig}{x_i^+} - \frac{ig}{x_i^-}$, we obtain

$$
\delta E = \left(\frac{\mathbf{x}_1^-}{\mathbf{x}_1^+}\right)^J \Sigma_{12}\frac{4i(\mathbf{x}_2^-(2\mathbf{x}_1^+ - \mathbf{x}_2^+) - \mathbf{x}_1^+\mathbf{x}_2^+)}{(g(\mathbf{x}_2^- - \mathbf{x}_1^+) - 2i\mathbf{x}_2^-\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)}\,. \tag{7.30}
$$

On the other hand by using $E_i = -1 - igx_i^+ + igx_i^-$, we get

$$
\delta E = \left(\frac{\mathbf{x}_1^-}{\mathbf{x}_1^+}\right)^J \Sigma_{12}\frac{4i\mathbf{x}_2^-\mathbf{x}_1^+\mathbf{x}_2^+(\mathbf{x}_2^- + \mathbf{x}_1^+ - 2\mathbf{x}_2^+)}{(g(\mathbf{x}_2^- - \mathbf{x}_1^+) - 2i\mathbf{x}_2^-\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)}\,. \tag{7.31}
$$

Even though the expressions look different they coincide on solutions to the bound state equation. In what follows we will be using the simpler eq.(7.31). Note also that the perturbation theory breaks down at $p = p_{\rm cr}$. Due to the quantization condition for the momentum p it may happen only at special values of the coupling constant g depending on m/J.

It is clear that the energy correction cannot be real for any choice of the dressing factor Σ_{12}. The imaginary part of the correction depends on the branch of the bound state under consideration.

In the first case with $\mathrm{Im}(\mathbf{x}_1^-) = \mathrm{Im}(\mathbf{x}_2^+) = 0$ and the total momentum smaller than the critical value (7.10), the parameters $\mathbf{x}_i^\pm$ satisfy the complex conjugation rule $(\mathbf{x}_1^\pm)^* = \mathbf{x}_2^\mp$, and we get

$$
\delta E - \delta E^* = \left(\left(\frac{\mathbf{x}_1^-}{\mathbf{x}_1^+}\right)^J \Sigma_{12} - \left(\frac{\mathbf{x}_2^+}{\mathbf{x}_2^-}\right)^J \Sigma_{12}^*\right)\frac{4i\mathbf{x}_2^-\mathbf{x}_1^+\mathbf{x}_2^+(\mathbf{x}_2^- + \mathbf{x}_1^+ - 2\mathbf{x}_2^+)}{(g(\mathbf{x}_2^- - \mathbf{x}_1^+) - 2i\mathbf{x}_2^-\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)}\,.
$$

Taking into account that $\left(\frac{\mathbf{x}_1^-}{\mathbf{x}_1^+}\right)^J = \left(\frac{\mathbf{x}_2^+}{\mathbf{x}_2^-}\right)^J$, we conclude that in this case the correction is real only if the dressing factor is real $\Sigma_{12} = \Sigma_{12}^*$. This property of the dressing factor can be easily shown by using the representation (5.22) for the dressing phase.

In the second case with $|\mathbf{x}_1^-| = |\mathbf{x}_2^+| = 1$ and the total momentum exceeding the critical value (7.10), the parameters $\mathbf{x}_i^\pm$ satisfy the complex conjugation rule $(\mathbf{x}_1^+)^* = \mathbf{x}_2^-$, $(\mathbf{x}_1^-)^* = 1/\mathbf{x}_2^+$, and we obtain

$$\delta E - \delta E^* = \\ = \frac{4\mathbf{x}_2^-\mathbf{x}_1^+\left(\Sigma_{12}^*(\mathbf{x}_2^+\mathbf{x}_2^-)^{-J}(2-(\mathbf{x}_2^-+\mathbf{x}_1^+)\mathbf{x}_2^+) - \Sigma_{12}(\mathbf{x}_1^+)^{-J}(\mathbf{x}_2^+)^{J+1}(\mathbf{x}_2^-+\mathbf{x}_1^+-2\mathbf{x}_2^+)\right)}{(ig(\mathbf{x}_2^- - \mathbf{x}_1^+) + 2\mathbf{x}_2^-\mathbf{x}_1^+)\left((\mathbf{x}_2^+)^2-1\right)} .$$

We see that the imaginary part of the correction would vanish only if

$$\Sigma_{12}^* = \Sigma_{12}(\mathbf{x}_2^+)^{2J+1}\frac{(\mathbf{x}_2^-+\mathbf{x}_1^+-2\mathbf{x}_2^+)}{(2-(\mathbf{x}_2^-+\mathbf{x}_1^+)\mathbf{x}_2^+)} .$$

Since the last equation depends on J and on a particular bound state solution, it cannot be satisfied for any choice of the dressing factor. The complex energy of the state would mean that the Hamiltonian of the model is not hermitian for finite J.

One might conclude from this result that the S-matrix poles with $|\mathbf{x}_1^-| = |\mathbf{x}_2^+| = 1$ are spurious and do not correspond to bound states, and, therefore, should be omitted. That would mean, however, that for any value of the total momentum the bound states satisfying the equations $\mathrm{Im}(\mathbf{x}_1^-) = \mathrm{Im}(\mathbf{x}_2^+) = 0$ would disappear as soon as the coupling constant g reaches a critical (momentum-dependent) value. This seems to contradict to the statement that the bound states are BPS. We believe that such a conclusion might be erroneous and the result above indicates, in fact, that the asymptotic Bethe ansatz cannot be used to analyze the finite-size corrections to the energy of bound states with the total momentum exceeding the critical value (7.10).

To show that this is indeed the case, let us recall that, as was shown in [38], at large values of the string tension g and the charge J the dispersion relation receives finite-size corrections of the order $e^{-J/(g\sin p/2)}$. On the other hand, the energy correction we computed above is of the order e^{-qJ} where q is the imaginary part of the momentum p_2. It depends on the total momentum p and the string tension g. By using eq.(9.35), it is not difficult to determine the large g dependence of the momenta p_1 and p_2 of a bound state

$$p_1 = \frac{\cos\frac{p}{2}}{2g^2\sin^3\frac{p}{2}} - \frac{i}{g\sin\frac{p}{2}} + \mathcal{O}(\frac{1}{g^3}) , \quad p_2 = p - \frac{\cos\frac{p}{2}}{2g^2\sin^3\frac{p}{2}} + \frac{i}{g\sin\frac{p}{2}} + \mathcal{O}(\frac{1}{g^3}) . \tag{7.32}$$

The second solution of eq.(7.8) (with $q>0$) is related to (7.32) as $p_1 \to p_2^*$, $p_2 \to p_1^*$ that is one exchanges the real parts of momenta p_i. A surprising result of the computation is that q is equal to $\frac{1}{g\sin\frac{p}{2}}$, and, therefore, e^{-qJ} is exactly equal to the magnitude of the finite-size correction to the dispersion relation. That means that computing the finite J correction to the energy of such a bound state one has to take into account the necessary modifications of the asymptotic Bethe ansatz. As a result of these modifications, one should be able to get a real finite-size correction to the energy of a bound state carrying momentum exceeding the critical value. In fact, this would be a non-trivial check of finite J "Bethe" equations.

The analysis performed above raises the question if one can use the asymptotic Bethe ansatz to compute the corrections to the energy of the bound states with momenta smaller than the critical value. At large g we can again compare the value of q with $\frac{1}{g\sin\frac{p}{2}}$. If q is less than $\frac{1}{g\sin\frac{p}{2}}$ then the energy correction (7.31) is bigger than the correction due to finite J modifications of the asymptotic Bethe ansatz, and we can trust (7.31). Since $p_{\rm cr}=2/\sqrt{g}$ at large values of g one should consider a bound state with momentum p of the order $1/\sqrt{g}$. The large g dependence of the momenta p_1 and p_2 of a bound state is easily found by using eq.(9.36)

$$p_1=\frac{p}{2}-2i\frac{1\pm\sqrt{1-\frac{p^4g^2}{16}}}{gp}+\cdots\,,\quad p_2=\frac{p}{2}+2i\frac{1\pm\sqrt{1-\frac{p^4g^2}{16}}}{gp}+\cdots\,, \tag{7.33}$$

leading for $p<p_{\rm cr}$ to the following two real solutions for q

$$q_\pm=2\frac{1\pm\sqrt{1-\frac{p^4g^2}{16}}}{gp}\,. \tag{7.34}$$

Comparing these values with $\frac{1}{g\sin\frac{p}{2}}\approx\frac{2}{gp}$, we see that $q_-<\frac{2}{gp}$ and $q_+>\frac{2}{gp}$. Thus, the asymptotic Bethe ansatz can be used to analyze finite J corrections to the energy of a bound state with momentum smaller than $p_{\rm cr}$ for the bound state with q_- only.

Actually, the fact that the energy correction (7.31) to the bound state with q_+ is smaller than the corrections due to finite J modifications of the asymptotic Bethe ansatz raises a question if these solutions correspond to the actual bound states. It may happen that finite J Bethe equations would not have any solution that would reduce to the solution with q_+ in the limit $J\to\infty$.

A similar analysis can also be performed for small values of g. Then we expect that the finite J effects (in gauge theory they are due to the wrapping interactions) become important at order g^{2J}, and therefore we could trust the asymptotic Bethe ansatz and the energy correction (7.31) only if $q<-2\log g$.

The leading small g dependence of q of the bound state solutions with the momentum smaller than $p_{\rm cr}$ is given by eqs.(9.37), (9.38)

$$q_+=-2\log g+\cdots\,,\quad q_-=-\log\cos\frac{p}{2}+\cdots\,. \tag{7.35}$$

We see immediately that again only the solution with the smaller imaginary part of the momentum q_- satisfies the necessary condition. The energy correction to the state with q_+ is of order g^{2J} that is exactly the order of wrapping interactions, and the asymptotic Bethe ansatz again breaks down for the state.

Finally, the leading small g dependence of q of the bound state solutions with the momentum exceeding $p_{\rm cr}$ is given by (9.39)

$$q_\pm = -\log\frac{g}{2} \pm i\alpha + \cdots\,, \tag{7.36}$$

where α is related to the momentum p as follows $p = \pi - 2g\cos\alpha$.

We see that the real part of $q_\pm$ is smaller than $-2\log g$, and therefore one could conclude that one might use the asymptotic Bethe ansatz for the states in this regime. This, however, leads to the problem of the complex energy of these states discussed above. As before the only resolution of the problem we see is the breakdown of the asymptotic Bethe ansatz. This would imply, however, that for these states the wrapping interactions become important already at the order g^J. The fact that in gauge theory these states are not dual to gauge-invariant operators does not seem to be important for this conclusion. One could for example scatter a bound state carrying momentum $p = \pi$ which always exceeds the critical momentum $p_{\rm cr}$ with an elementary one carrying momentum $-\pi$ so that the total momentum would be zero, and such a state would be dual to a gauge-invariant operator. We would still expect the finite J corrections to this state to be of the order g^J. Another puzzling property of the states with $p > p_{\rm cr}$ is that in the limit $g \to 0$ the states are pushed away from the spectrum because $p_{\rm cr} = \pi$, and cannot be seen in the perturbative gauge theory.

8. Bound states of the mirror model

Let us now consider in a similar fashion bound states of the mirror model. In this case one should consider mirror particles of type $A^\dagger_{3\dot{3}}$.

We begin our consideration with two-particle bound states, and let the complex momenta of the two particles be

$$\widetilde{p}_1 = \frac{p}{2} - iq\,, \quad \widetilde{p}_2 = \frac{p}{2} + iq\,, \quad {\rm Re}\, q > 0\,,$$

where p is the total momentum of the mirror bound state.

The first equation in (6.6) takes the form

$$e^{ipR/2}e^{qR} = \sigma_{12}\,\frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\,, \tag{8.1}$$

where we set all auxiliary roots to 0. Assuming that the dressing factor does not vanish, we conclude that for ${\rm Re}\, q > 0$ and in the limit $R \to \infty$ the following bound state equation should hold

$$x_1^+ - x_2^- = 0\,. \tag{8.2}$$

The second factor in the denominator of the Bethe equation (8.1) may also vanish but the energy of the corresponding state does not satisfy the BPS condition. We expect that, just as a similar factor in the string theory, the pole due to this factor does not correspond to a bound state.

By using eqs. (2.15) which express $x^\pm$ as functions of $\widetilde{p}$, we find that eq.(8.2) is equivalent to

$$-4g^2q^2 + t^3 - 2q^2t\,(2-t) + q^4t = 0\,, \tag{8.3}$$

where $t \equiv 1 + \frac{p^2}{4}$. This equation gives the following two solutions with a positive real part of q:

$$q = \sqrt{1+\frac{g^2}{t}} \pm \sqrt{1-t+\frac{g^2}{t}} = \sqrt{1+\frac{4g^2}{4+p^2}} \pm \sqrt{-\frac{p^2}{4}+\frac{4g^2}{4+p^2}}\,. \tag{8.4}$$

Solutions for q are real provided the expression under the square root is nonnegative, and this implies the following restriction on the total momentum of the bound state

$$|p| \le p_{\rm cr} \equiv \sqrt{2}\sqrt{-1+\sqrt{1+4g^2}}\,. \tag{8.5}$$

For an exact inequality we have two positive solutions $q_\pm$, and when the bound on the momentum is saturated the solution is obviously unique[22]

$$q_- < q_{\rm cr} < q_+\,, \quad q_{\rm cr} = \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+4g^2}}\,. \tag{8.6}$$

It is interesting to notice that the dependence of $q_\pm$ on the momentum of the bound state is smoother at $p=0$ than the one for string theory bound states. We see from eq.(8.4) that q_- reaches its minimum, and q_+ reaches its maximum at $p=0$

$$q_-^{\rm min} = \sqrt{1+g^2} - g\,, \quad q_+^{\rm max} = \sqrt{1+g^2} + g\,, \quad p = 0\,. \tag{8.7}$$

In string theory the corresponding values are 0 and ∞.

To find what curves in the z-torus correspond to the two solutions with real q we take into account that in this case $\widetilde{p}_1{}^* = \widetilde{p}_2$, and the reality condition for $x^\pm$ in the mirror theory is $\left(x_1^\pm\right)^* = 1/x_2^\mp$. Thus the bound state equation (8.2) reduces to the following equivalent conditions

$$|x_1^+| = 1 \quad \Longleftrightarrow \quad |x_2^-| = 1\,,$$

being represented by the two curves in the z-torus that bound the yellow region with $|x^+| < 1\,,\ |x^-| > 1$ in Figure 1. Note that the curves are symmetric about

[22]The energy of the bound state is $\widetilde{\mathcal{E}}_{\rm cr} = 2\,{\rm arcsinh}\frac{\sqrt{2}}{g}\sqrt{1+1\sqrt{1+4g^2}}\,.$

the horizontal line passing through the point $z = \frac{\omega_2}{2}$. Let us recall that hermitian conjugation in the mirror theory is defined with respect to this line, see section 4. It is not difficult to check that the parts of the curves $|x_1^+| = 1\,,\ |x_2^-| = 1$ that are inside the region $\mathrm{Im}(x^\pm) < 0$ correspond to the smaller root q_- of eq. (8.2). The other parts of the curves correspond to the second solution with $q = q_+$, see Figure 1. Just as it was for string theory bound states, both solutions have the same values of all global conserved charges $Q_r = q_r(z_1) + q_r(z_2) = \frac{i}{r-1}\big[-(x_1^-)^{1-r} + (x_2^+)^{1-r}\big]$.

We see that if we want to have only one bound state with $|p| < p_{\rm cr}$ in a physical region, then we should choose the physical region to be the one with $\mathrm{Im}(x^\pm) < 0$ but not the one bounded by the curves $|x^\pm| = 1$ as it is for string theory. We will see in a moment that the region $\mathrm{Im}(x^\pm) \le 0$ also contains bound states with $|p| > p_{\rm cr}$ described by the solutions with complex q.

Above the critical value, $|p| > p_{\rm cr}$, the two solutions (8.4) acquire imaginary parts and become complex conjugate to each other. It is convenient to denote the corresponding solutions as follows

$$q_\pm = \sqrt{1 + \frac{4g^2}{4+p^2}} \pm i\,\frac{p}{2}\sqrt{1 - \frac{16g^2}{p^2(4+p^2)}}\,. \tag{8.8}$$

We see that the real part of $q_\pm$ is a decreasing function of p, and its minimum value is 1. On the contrary the imaginary part of $q_\pm$ is an increasing function of p and it behaves as $\pm p/2$ at large values of p. As a result, the two complex momenta

$$\widetilde{p}_1{}^\pm = \frac{p}{2} \pm \mathrm{Im}\, q - i\,\mathrm{Re}\, q\,, \qquad \widetilde{p}_2{}^\pm = \frac{p}{2} \mp \mathrm{Im}\, q + i\,\mathrm{Re}\, q\,, \qquad \mathrm{Re}\, q > 0\,,$$

have the following large p behavior

$$\widetilde{p}_1{}^+ = p - i\,, \quad \widetilde{p}_2{}^+ = i\,; \qquad \widetilde{p}_1{}^- = -i\,, \quad \widetilde{p}_2{}^- = p + i\,.$$

A remarkable fact is that both solutions lie precisely on the boundary of the region $\mathrm{Im}(x^\pm) \le 0$. To see this we notice that, just as it was for string theory bound states, the coordinates z_1 and z_2 of the solutions with the complex values of q are related by eq.(7.15)

$$z_2 = -z_1^* + \frac{\omega_1}{2} + \frac{\omega_2}{2}\,. \tag{8.9}$$

Then, one can easily show that

$$x^-(z_2) = x^-\Big(-z_1^* + \frac{\omega_1}{2} + \frac{\omega_2}{2}\Big) = x^-(z_1^*) = [x^+(z_1)]^*\,,$$

and, therefore, the bound state equation $x_1^+ = x_2^-$ is equivalent to $\mathrm{Im}(x^+(z_1)) = \mathrm{Im}(x^-(z_2)) = 0$. We plot the corresponding curves in Figure 6.

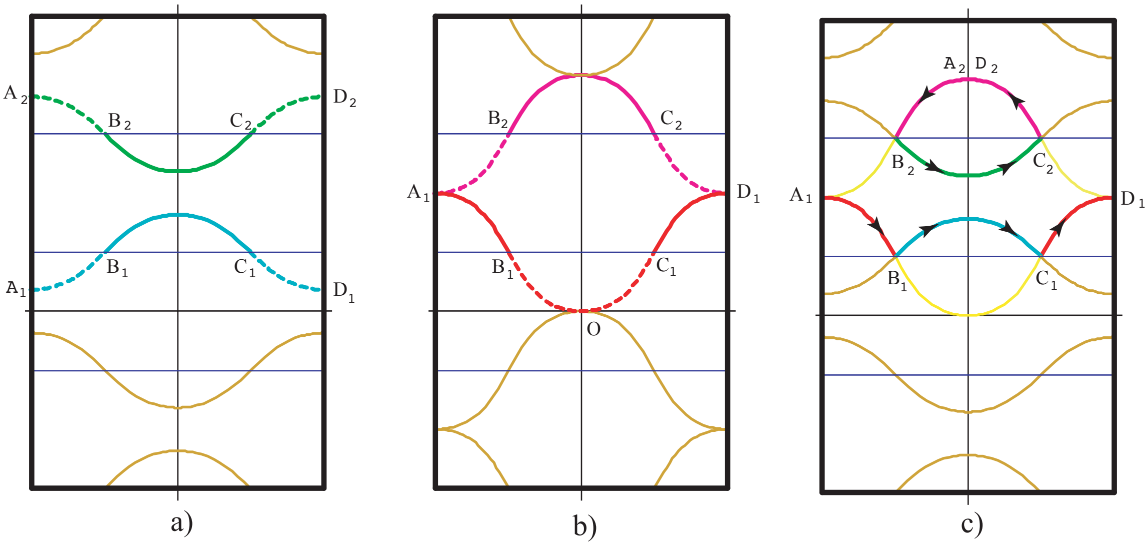

Figure 6: Bound states of the mirror theory. Figure a) represents the first BPS family: for any p with $|p| < p_{\rm cr}$ there are two solutions corresponding to q_- (the curves $\mathbf{B}_1\mathbf{C}_1$ for the 1st particle and $\mathbf{B}_2\mathbf{C}_2$ for the 2nd one, respectively) and to q_+ (the curves $\mathbf{A}_1\mathbf{B}_1 \cup \mathbf{C}_1\mathbf{D}_1$ for the 1st particle and $\mathbf{A}_2\mathbf{B}_2 \cup \mathbf{C}_2\mathbf{D}_2$ for the 2nd one, respectively). Figure b) represents the second BPS family which is also doubly degenerate: it is given by either $\mathbf{A}_1\mathbf{B}_1 \cup \mathbf{C}_1\mathbf{D}_1 \cup \mathbf{B}_2\mathbf{D}_2$ or by $\mathbf{B}_1\mathbf{C}_1 \cup \mathbf{A}_1\mathbf{B}_2 \cup \mathbf{D}_1\mathbf{C}_2$. Figure c) corresponds to one of the four possibilities to connect the first and the second BPS family: when the variable z_1 of the 1st particle runs along the curve $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1\mathbf{D}_1$ the real part of its momentum increases from $-\infty$ to $+\infty$. At the same time, the variable z_2 of the 2nd particle encloses the curve $\mathbf{A}_2\mathbf{B}_2\mathbf{C}_2\mathbf{D}_2$.

Thus, we have shown that these solutions lie on the boundary of the region $\mathrm{Im}(x^{\pm}) \leq 0$, and, therefore, the region contains bound states with any value of the total momentum and could be considered as the physical one for the mirror model. It is also necessary to specify what part of the boundary of the region $\mathrm{Im}(x^{\pm}) \leq 0$ belongs to the physical region, and this can be done by choosing properly the cuts in the u-plane where the bound state equation reduces to

$$x_1^+ - x_2^- = 0 \quad \Longrightarrow \quad u_2 - u_1 = \frac{2i}{g}\,.$$

As was discussed in section 4, eqs. $|x_1^+| = |x_2^-| = 1$ describing a bound state with the momentum not exceeding the critical value $p_{\rm cr}$ and with a real q give a Bethe string solution with the real part of u lying in the interval $[-2, 2]$

$$u_{1,2} = u_0 \mp \frac{i}{g}\,, \qquad -2 \leq u_0 \leq 2\,.$$

On the other hand, values of u_0 lying outside the interval $[-2, 2]$ correspond to solutions of eqs. $\mathrm{Im}(x_1^+) = \mathrm{Im}(x_2^-) = 0$. The momentum $\widetilde{p} = \widetilde{p}(u)$ is a multi-valued function of u, and one should choose a proper branch of the function to get the right values of the momenta $\widetilde{p}_1\,,\ \widetilde{p}_2$ of the bound state. This fixes the cuts in the u-plane which run from $\pm\infty$ to $\pm 2 \mp \frac{i}{g}$, and also the boundaries of the region $\mathrm{Im}(x^\pm) \leq 0$ in the z-plane which is mapped onto the u-plane with these cuts.

The discussion of bound states of M particles of type $A^\dagger_{3\dot{3}}$ basically repeats the one in section 7. One finds a system of equations

$$x_j^+ - x_{j+1}^- = 0\,, \qquad j = 1, \ldots, M-1\,. \tag{8.10}$$

In terms of the variable u the Bethe string solution reads as

$$u_j = u_0 - (M - 2j + 1)\frac{i}{g}\,, \qquad j = 1, \ldots, M\,, \tag{8.11}$$

and has the energy

$$\mathcal{E} = \log \frac{x_1^-}{x_M^+} = 2\,\mathrm{arcsinh}\, \frac{1}{2g}\sqrt{M^2 + \widetilde{p}^2}\,, \tag{8.12}$$

where $\widetilde{p} = \widetilde{p}_1 + \ldots + \widetilde{p}_M$ is a total (real) momentum of the bound state.

Depending on a choice of the physical region, the system (8.10) could have one, two or 2^{M-1} solutions. All solutions have the same global conserved charged. They behave, however, differently for very large but finite values of R, and the solutions which are not in the region $\mathrm{Im}(x^\pm) < 0$ show various signs of pathological behavior. In particular, they might have complex finite R correction to the energy, or the correction would exceed the correction due to finite R modifications of the Bethe equations thus making the asymptotic Bethe ansatz inapplicable.

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9. Appendices

9.1 Gauge-fixed Lagrangian.

The Lagrangian density of the gauge-fixed sigma-model in the generalized a-gauge [56, 57] can be written in the following form [26]

$$\mathscr{L} = -\frac{\sqrt{G_{\varphi\varphi}G_{tt}}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \sqrt{\mathcal{W}} + \frac{a\ G_{tt} + (1-a)\ G_{\varphi\varphi}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}}\,, \tag{9.1}$$

where

$$\begin{aligned}\mathcal{W} \equiv 1 - \frac{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}}{2} \Big[\Big(1 + \frac{1}{G_{\varphi\varphi}G_{tt}}\Big) \partial_\alpha X \cdot \partial^\alpha X \\ - \Big(1 - \frac{1}{G_{\varphi\varphi}G_{tt}}\Big) \Big(\dot{X}\cdot\dot{X} + X' \cdot X'\Big) \Big] \\ + \frac{((1-a)^2 G_{\varphi\varphi} - a^2 G_{tt})^2}{2 G_{\varphi\varphi} G_{tt}} \big((\partial_\alpha X \cdot \partial^\alpha X)^2 - (\partial_\alpha X \cdot \partial_\beta X)^2 \big)\,.\end{aligned}$$

Here $X = (y^i, z^i)$, where y^i, $i = 1, \ldots, 4$ are four fields parametrizing five-sphere, while z^i are fields parametrizing four directions in AdS_5. The fields X in the Lagrangian above are contracted with the help of the metric

$$ds^2 = -G_{tt}dt^2 + G_{zz}dz^2 + G_{\varphi\varphi}d\varphi^2 + G_{yy}dy^2\,.$$

Here

$$G_{tt} = \left(\frac{1+z^2}{1-z^2}\right)^2, \quad G_{zz} = \frac{1}{\left(1-z^2\right)^2}, \quad G_{\varphi\varphi} = \left(\frac{1-y^2}{1+y^2}\right)^2, \quad G_{yy} = \frac{1}{\left(1+y^2\right)^2},$$

where we had used the notation $z^2 \equiv z^i z^i$ and $y^2 \equiv y^i y^i$.

9.2 One-loop S-matrix

Here we describe the properties of the "one-loop" S-matrix which is obtained from the S-matrix (3.24) upon taking the limit $g \to 0$. We will work in the elliptic parametrization discussed in section 4.1. According to eq.(4.18), in this limit Jacobi elliptic functions degenerate into the corresponding trigonometric ones and we find the following trigonometric S-matrix:

$$
\begin{aligned}
S(z_1,z_2) = {} & e^{-i(z_1-z_2)}\frac{\cot z_1-\cot z_2+2i}{\cot z_1-\cot z_2-2i} && \left(E_1^1\otimes E_1^1+E_2^2\otimes E_2^2+E_1^1\otimes E_2^2+E_2^2\otimes E_1^1\right)\\
& -e^{-i(z_1-z_2)}\frac{2i}{\cot z_1-\cot z_2-2i} && \left(E_1^1\otimes E_2^2+E_2^2\otimes E_1^1-E_1^2\otimes E_2^1-E_2^1\otimes E_1^2\right)\\
& - && \left(E_3^3\otimes E_3^3+E_4^4\otimes E_4^4+E_3^3\otimes E_4^4+E_4^4\otimes E_3^3\right)\\
& -\frac{2i}{\cot z_1-\cot z_2-2i} && \left(E_3^3\otimes E_4^4+E_4^4\otimes E_3^3-E_3^4\otimes E_4^3-E_4^3\otimes E_3^4\right)\\
& +e^{iz_2}\frac{\cot z_1-\cot z_2}{\cot z_1-\cot z_2-2i} && \left(E_1^1\otimes E_3^3+E_1^1\otimes E_4^4+E_2^2\otimes E_3^3+E_2^2\otimes E_4^4\right)\\
& +e^{-iz_1}\frac{\cot z_1-\cot z_2}{\cot z_1-\cot z_2-2i} && \left(E_3^3\otimes E_1^1+E_4^4\otimes E_1^1+E_3^3\otimes E_2^2+E_4^4\otimes E_2^2\right)\\
& +e^{-i(z_1-z_2)}\frac{2i}{\cot z_1-\cot z_2-2i} && \left(E_1^3\otimes E_3^1+E_1^4\otimes E_4^1+E_2^3\otimes E_3^2+E_2^4\otimes E_4^2\right)\\
& +e^{-i(z_1-z_2)}\frac{2i}{\cot z_1-\cot z_2-2i} && \left(E_3^1\otimes E_1^3+E_4^1\otimes E_1^4+E_3^2\otimes E_2^3+E_4^2\otimes E_2^4\right).
\end{aligned}
\tag{9.2}
$$

The relations between the z-variable, momentum and the rescaled rapidity $u \to gu$ transform in the limit $g \to 0$ into

$$
p = 2z\,, \qquad u = \cot z = \cot\frac{p}{2}\,. \tag{9.3}
$$

Surprisingly enough, this S-matrix cannot be written in the difference form, i.e. as a function of one variable being the difference of a properly introduced spectral parameter. By construction, this S-matrix satisfies the usual Yang-Baxter equation

$$
S_{23}(z_2,z_3)S_{13}(z_1,z_3)S_{12}(z_1,z_2) = S_{12}(z_1,z_2)S_{13}(z_1,z_3)S_{23}(z_2,z_3)\,, \tag{9.4}
$$

as one can also verify by direct calculation. On the other hand, at one-loop there is another "canonical" S-matrix which is a linear combination of the graded identity and the usual permutation:

$$
S_{12}^{\rm can} = \frac{u_1-u_2}{u_1-u_2-2i}I_{12}^g + \frac{2i}{u_1-u_2-2i}P_{12}\,. \tag{9.5}
$$

This S-matrix satisfies the same Yang-Baxter equation (9.4).

The results of [47] imply that the two one-loop S-matrices, (9.2) and (9.5) are related through the following transformation

$$
S^{\rm can}(z_1,z_2) = U_2(z_1)\Big[V_1(z_1)V_2(z_2)S_{12}(z_1,z_2)V_1^{-1}(z_1)V_2^{-1}(z_2)\Big]U_1^{-1}(z_2)\,, \tag{9.6}
$$

where we have introduced the diagonal matrices

$$
U(z) = {\rm diag}(e^{iz}, e^{iz}, 1, 1)\,, \tag{9.7}
$$

$$
V(z) = {\rm diag}(e^{i\frac{z}{4}}, e^{i\frac{z}{4}}, e^{-i\frac{z}{4}}, e^{-i\frac{z}{4}})\,. \tag{9.8}
$$

The transformation by V is just a gauge transformation which always preserves the Yang-Baxter equation. On the other hand, transformation by U is a twist, that generically transforms the usual Yang-Baxter equation into the twisted one and vice versa [47]. Indeed, $S_{12}^{\rm can}$ is nothing else but the one-loop limit of the spin chain S-matrix [9]; the latter obeys the twisted Yang-Baxter equation [47]. Note also that the twist U does not belong to the symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of the string S-matrix.

To understand why at one loop the Yang-Baxter equation is preserved under the twisting, we first write the Yang-Baxter equation for $S^{\rm can}$ by using the relation[23] (9.5)

$$\begin{aligned} &U_3(z_2)S_{23}U_2^{-1}(z_3)U_3(z_1)S_{13}U_1^{-1}(z_3)U_2(z_1)S_{12}U_1^{-1}(z_2) = \\ &\qquad = U_2(z_1)S_{12}U_1^{-1}(z_2)U_3(z_1)S_{13}U_1^{-1}(z_3)U_3(z_2)S_{23}U_2^{-1}(z_3)\,, \end{aligned} \tag{9.9}$$

which can be reshuffled as follows

$$\begin{aligned} &U_3(z_2)S_{23}U_2(z_1)U_3(z_1)S_{13}U_1^{-1}(z_3)U_2^{-1}(z_3)S_{12}U_1(z_2) = \\ &\qquad = U_2(z_1)U_3(z_1)S_{12}U_1^{-1}(z_2)S_{13}U_3(z_2)S_{23}U_1^{-1}(z_3)U_2^{-1}(z_3)\,. \end{aligned} \tag{9.10}$$

It is clear now that we will get the usual Yang-Baxter equation for S provided it obeys the following relation

$$[S, U \otimes U] = 0\,, \tag{9.11}$$

where U is an *arbitrary diagonal matrix*. One can easily verify that both S-matrices, (9.2) and (9.5), do indeed satisfy this relation. At higher orders in g the relation (9.11) does not hold anymore. The corresponding "all-loop" S-matrix (3.24) satisfies only a weaker condition

$$[S, G \otimes G] = 0\,, \qquad G \in \mathrm{SU}(2) \times \mathrm{SU}(2)\,, \tag{9.12}$$

which is nothing else but the invariance condition for the string S-matrix. As a consequence, the Yang-Baxter equation is preserved by the twist transformation only at the one-loop order.

As a final remark, we note that it would be interesting to understand how the derivation of the Hirota difference equations for the canonical S-matrix [72] could be extended to the "twisted" S-matrix (9.2). This might shed some light on construction the fusion procedure for the all-loop S-matrix (3.24).

[23]The gauge transformation by the matrix V decouples from the Yang-Baxter equation.

9.3 BAE with nonperiodic fermions

9.3.1 Bethe wave function and the periodicity conditions

In any asymptotic domain $\mathcal{Q}$ with $x_{\mathcal{Q}_1} \ll x_{\mathcal{Q}_2} \ll \cdots \ll x_{\mathcal{Q}_N}$ where $N \equiv K^I$ and $\mathcal{Q}_1, \ldots, \mathcal{Q}_N$ is a permutation of $1, 2, \ldots, N$, the wave function of N particles with flavors $i_1, i_2, \ldots, i_N$ can be written as a superposition of plane waves with momenta $p_1 > p_2 > \cdots > p_N$

$$\Psi^{\mathcal{Q}}_{i_1 \cdots i_N}(x_1, \ldots, x_N) = \sum_{\mathcal{P}} \mathcal{A}^{\mathcal{P}|\mathcal{Q}}_{i_1 \cdots i_N}\, e^{i\, p_{\mathcal{P}} \cdot x_{\mathcal{Q}}}\,, \tag{9.13}$$

where the sum runs over all permutations of the momenta p_i. The scalar product $p_{\mathcal{P}} \cdot x_{\mathcal{Q}}$ is defined as $p_{\mathcal{P}} \cdot x_{\mathcal{Q}} \equiv \sum_{k=1}^N p_{\mathcal{P}_k} x_{\mathcal{Q}_k}$, and for any two permutations $\mathcal{P}$ and $\mathcal{Q}$ it satisfies $p_{\mathcal{P}} \cdot x_{\mathcal{Q}} = p_{\mathcal{P}\mathcal{Q}^{-1}} \cdot x_{\mathcal{I}} = \sum_{k=1}^N p_{(\mathcal{P}\mathcal{Q}^{-1})_k} x_k$ where $\mathcal{I}$ is the trivial permutation.

The amplitude $\mathcal{A}^{\mathcal{P}|\mathcal{Q}}_{i_1 \cdots i_N}$ is related to the probability of finding the particle with the flavor i_k (the i_k-th particle in what follows) carrying the momentum $p_{(\mathcal{P}\mathcal{Q}^{-1})_k}$ at the position x_k. That means that the index i_k is attached to the coordinate x_k. As a result the wave function (9.13) should satisfy the following symmetry condition for any two indices k, m

$$\begin{aligned} &\Psi^{\mathcal{Q}}_{i_1 \cdots i_k \cdots i_m \cdots i_N}(x_1, ..., x_k, ..., x_m, ..., x_N) = \\ &\qquad = (-)^{\epsilon_{i_k} \epsilon_{i_m}}\, \Psi^{\mathcal{P}_{km}\mathcal{Q}}_{i_1 \cdots i_m \cdots i_k \cdots i_N}(x_1, ..., x_m, ..., x_k, ..., x_N)\,, \end{aligned} \tag{9.14}$$

where $\mathcal{P}_{km}$ is the permutation of k and m, and $\epsilon_i = 0$ if the i-th particle is boson and $\epsilon_i = 1$ if the i-th particle is fermion, that is one takes the minus sign if both the i_k-th and i_m-th particles are fermions, and the plus sign otherwise.

In any two domains $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ the amplitudes $\mathcal{A}^{\mathcal{P}|\mathcal{Q}}_{i_1 \cdots i_N}$ and $\mathcal{A}^{\overline{\mathcal{P}}|\overline{\mathcal{Q}}}_{i_1 \cdots i_N}$ of the same plane wave (that is $p_{\mathcal{P}} \cdot x_{\mathcal{Q}} = p_{\overline{\mathcal{P}}} \cdot x_{\overline{\mathcal{Q}}}$) are related through the S-matrix. The relation can be easily found by representing the amplitudes as the following products of the ZF operators

$$\mathcal{A}^{\mathcal{P}|\mathcal{Q}}_{i_1 \cdots i_N} \sim \pm A^\dagger_{i_{\mathcal{Q}_1}}(p_{\mathcal{P}_1}) \cdots A^\dagger_{i_{\mathcal{Q}_N}}(p_{\mathcal{P}_N})\,, \tag{9.15}$$

and then by using the ZF algebra to relate the amplitudes in the domains $\mathcal{Q}$ and $\overline{\mathcal{Q}}$. The $+/-$ sign in this formula is related to the even/odd number of permutations of fermions by the permutation $\mathcal{Q}$. To understand the origin of this formula let us recall that the indices i_k are attached to the coordinates x_k which explains the order of $A^\dagger_{i_{\mathcal{Q}_k}}$. The dependence of $A^\dagger_{i_{\mathcal{Q}_k}}$ of the momentum follows from the coupling $p_{\mathcal{P}_k} x_{\mathcal{Q}_k}$ in the exponential of the wave function (9.13).

To proceed it is convenient to use matrix notations. We introduce the simple permutation $P_{12} = E^i_j \otimes E^j_i$ which permutes the spaces V_1 and V_2 but does not touch the momenta p_i so that $S_{21} = P_{12} S(p_2, p_1) P_{12}$, the graded permutation $P^g_{12} =$

$(-1)^{\epsilon_i\epsilon_j} E^i_j \otimes E^j_i$, and the graded two-particle S-matrix S^g_{12} which can be written in the form $S^g_{12} = I^g_{12} S_{12}$ where $I^g_{12} = (-1)^{\epsilon_i\epsilon_j} E^i_i \otimes E^j_j$ is the graded identity. We also define $S^g_{21} = P_{12} S(p_2,p_1) P_{12} I^g_{12} = P_{12} S(p_2,p_1) P^g_{12}$ so that the unitarity condition $S^g_{12} S^g_{21} = I$ is fulfilled.

Then we multiply the wave function (9.13) and (9.15) by the tensor product of N rows $E^{i_1} \otimes E^{i_2} \otimes \cdots \otimes E^{i_N} \equiv \left(E^1 E^2 \cdots E^N\right)^{i_1 i_2 \cdots i_N}$, and (9.13) takes the form

$$\Psi^{\mathcal{Q}}(x_1,\ldots,x_N) = \sum_{\mathcal{P}} \mathcal{A}^{\mathcal{P}|\mathcal{Q}}\, e^{i\,p_{\mathcal{P}}\cdot x_{\mathcal{Q}}}\,, \tag{9.16}$$

where

$$\mathcal{A}^{\mathcal{P}|\mathcal{Q}} \sim A^\dagger_{\mathcal{Q}_1}(p_{\mathcal{P}_1}) \cdots A^\dagger_{\mathcal{Q}_N}(p_{\mathcal{P}_N}) I^g_{\mathcal{Q}}\,,$$

and the index $\mathcal{Q}_k$ refers to the location of the row $E^{\mathcal{Q}_k}$, and $I^g_{\mathcal{Q}}$ is the product of graded identities which can be found by representing the permutation $\mathcal{Q}$ as a product of Y simple permutations $\mathcal{P}_{km}$: $Q = \mathcal{P}_{k_1m_1} \cdots \mathcal{P}_{k_Ym_Y}$, and then $I^g_{\mathcal{Q}} = I^g_{k_1m_1} \cdots I^g_{k_Ym_Y}$.

Now the ZF algebra can be used to express the amplitudes $\mathcal{A}^{\mathcal{P}|\mathcal{Q}}$ with $\mathcal{Q}_0 \equiv \mathcal{P}^{-1}\mathcal{Q}$ fixed in terms of the amplitude $\mathcal{A}^{\mathcal{I}|\mathcal{Q}_0}$. In particular the amplitudes $\mathcal{A}^{\mathcal{P}|\mathcal{P}}$ are expressed in terms of the incoming amplitude $\mathcal{A}^{\mathcal{I}|\mathcal{I}} \sim A^\dagger_1(p_1)\cdots A^\dagger_N(p_N)$. The corresponding terms in the wave function can be used to derive the periodicity conditions.

To find the relations, it is convenient to represent

$$\begin{aligned}
\mathcal{A}^{\mathcal{P}|\mathcal{Q}} &\sim A^\dagger_{\mathcal{Q}_1}(p_{\mathcal{P}_1}) \cdots A^\dagger_{\mathcal{Q}_N}(p_{\mathcal{P}_N}) I^g_{\mathcal{Q}} = A^\dagger_{\mathcal{P}_1}(p_{\mathcal{P}_1}) \cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N}) (\mathcal{Q}\mathcal{P}^{-1})_{1\cdots N} I^g_{\mathcal{Q}}\,,\\
\mathcal{A}^{\mathcal{I}|\mathcal{Q}_0} &\sim A^\dagger_1(p_1)\cdots A^\dagger_N(p_N)\cdot(\mathcal{Q}_0)_{1\cdots N} I^g_{\mathcal{Q}_0} = A^\dagger_1(p_1)\cdots A^\dagger_N(p_N)(\mathcal{P}^{-1}\mathcal{Q})_{1\cdots N} I^g_{\mathcal{Q}_0}\,,
\end{aligned}$$

where $(\mathcal{Q}\mathcal{P}^{-1})_{1\cdots N}$ is the permutation matrix that acting on the tensor product $E^{\mathcal{P}_1} \otimes \cdots \otimes E^{\mathcal{P}_N}$ produces $E^{\mathcal{Q}_1} \otimes \cdots \otimes E^{\mathcal{Q}_N}$. Now we use the ZF algebra to find the relation

$$\begin{aligned}
\mathcal{A}^{\mathcal{P}|\mathcal{Q}} &= A^\dagger_1 \cdots A^\dagger_N \cdot S_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N}) (\mathcal{Q}\mathcal{P}^{-1})_{1\cdots N} I^g_{\mathcal{Q}} = \\
&\qquad \mathcal{A}^{\mathcal{I}|\mathcal{Q}_0}\, I^g_{\mathcal{Q}_0} (\mathcal{Q}^{-1}\mathcal{P})_{1\cdots N} S_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N}) (\mathcal{Q}\mathcal{P}^{-1})_{1\cdots N} I^g_{\mathcal{Q}}\,,
\end{aligned}$$

where $S_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N})$ is the multi-particle S-matrix.

In particular, we find that

$$\mathcal{A}^{\mathcal{P}|\mathcal{P}} = \mathcal{A}^{\mathcal{I}|\mathcal{I}}\, S_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N}) I^g_{\mathcal{P}} \equiv \mathcal{A}^{\mathcal{I}|\mathcal{I}}\, S^g_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N})\,,$$

where $S^g_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N})$ is the graded multi-particle S-matrix. Note that it is not a product of two-particle graded S-matrices.

This formula can be used to find the set of periodicity conditions. We write the part of the wave function with the plane wave $e^{ip_kx_k}$

$$\begin{aligned}
\Psi(x_1,\ldots,x_N) &= \sum_{\mathcal{P}} \mathcal{A}^{\mathcal{P}|\mathcal{P}}\, e^{i\,p_{\mathcal{P}}\cdot x_{\mathcal{P}}} \theta(x_{\mathcal{P}_1} < \ldots < x_{\mathcal{P}_N}) \\
&= \mathcal{A}^{\mathcal{I}|\mathcal{I}} \sum_{\mathcal{P}} S^g_{\mathcal{P}_1\cdots\mathcal{P}_N}(p_{\mathcal{P}_1},\ldots,p_{\mathcal{P}_N})\, e^{i\,p_{\mathcal{P}}\cdot x_{\mathcal{P}}} \theta(x_{\mathcal{P}_1} < \ldots < x_{\mathcal{P}_N})\,.
\end{aligned} \tag{9.17}$$

The periodicity conditions read

$$\Psi(x_1,\dots,x_k=0,\dots,x_N)=\Psi(x_1,\dots,x_k=L,\dots,x_N)W_k\,,$$

where the diagonal matrix W is equal to the identity matrix if the fermions are periodic, and it is $W=(-1)^{\epsilon_i}E^i_i$ if the fermions are anti-periodic. For the $\mathfrak{su}(2|2)$ case we have $W=\Sigma=\mathrm{diag}(1,1,-1,-1)$ for anti-periodic fermions.

By using eq.(9.17), we get

$$\begin{aligned}
&\Psi(x_1,\dots,x_k=0,\dots,x_N)=\\
&\qquad\mathcal{A}^{\mathcal{I}|\mathcal{I}}\sum_{\mathcal{P}:\mathcal{P}_1=p_k}S^g_{k\mathcal{P}_2\cdots\mathcal{P}_N}(p_k,p_{\mathcal{P}_2},\dots,p_{\mathcal{P}_N})\,e^{i\,p_{\mathcal{P}}\cdot x_{\mathcal{P}}}\theta(x_{\mathcal{P}_2}<\ldots<x_{\mathcal{P}_N})\,,\\
&\Psi(x_1,\dots,x_k=L,\dots,x_N)=\\
&e^{ip_kL}\mathcal{A}^{\mathcal{I}|\mathcal{I}}\sum_{\mathcal{P}:\mathcal{P}_N=p_k}S^g_{\mathcal{P}_1\cdots\mathcal{P}_{N-1}k}(p_{\mathcal{P}_1},\dots,p_{\mathcal{P}_{N-1}},p_k)W_k\,e^{i\,p_{\mathcal{P}}\cdot x_{\mathcal{P}}}\theta(x_{\mathcal{P}_1}<\ldots<x_{\mathcal{P}_{N-1}})\,,
\end{aligned}$$

Comparing the terms, we obtain

$$\mathcal{A}^{\mathcal{I}|\mathcal{I}}\left(S^g_{k\mathcal{P}_2\cdots\mathcal{P}_N}(p_k,p_{\mathcal{P}_2},\dots,p_{\mathcal{P}_N})-e^{ip_kL}S^g_{\mathcal{P}_2\cdots\mathcal{P}_Nk}(p_{\mathcal{P}_2},\dots,p_{\mathcal{P}_N},p_k)W_k\right)=0\,. \quad (9.18)$$

To compute the S-matrices, we use their definitions

$$\begin{aligned}
A^\dagger_k(p_k)A^\dagger_{\mathcal{P}_2}(p_{\mathcal{P}_2})\cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N}) &= A^\dagger_1\cdots A^\dagger_N\cdot S_{k\mathcal{P}_2\cdots\mathcal{P}_N}(p_k,p_{\mathcal{P}_2},\dots,p_{\mathcal{P}_N})\,,\\
A^\dagger_{\mathcal{P}_2}(p_{\mathcal{P}_2})\cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N})A^\dagger_k(p_k) &= A^\dagger_1\cdots A^\dagger_N\cdot S_{\mathcal{P}_2\cdots\mathcal{P}_Nk}(p_{\mathcal{P}_2},\dots,p_{\mathcal{P}_N},p_k)\,.
\end{aligned}$$

Then we use the ZF algebra to order the product $A^\dagger_{\mathcal{P}_2}(p_{\mathcal{P}_2})\cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N})$

$$A^\dagger_{\mathcal{P}_2}(p_{\mathcal{P}_2})\cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N})=A^\dagger_1\cdots A^\dagger_{k-1}A^\dagger_{k+1}\cdots A^\dagger_N\cdot S_{\mathcal{P}_2\cdots\mathcal{P}_N}(p_{\mathcal{P}_2},\dots,p_{\mathcal{P}_N})\,,$$

and finally we get the multi-particle S-matrices

$$\begin{aligned}
A^\dagger_kA^\dagger_{\mathcal{P}_2}(p_{\mathcal{P}_2})\cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N}) &= A^\dagger_1\cdots A^\dagger_N\cdot S_{k,k-1}S_{k,k-2}\cdots S_{k1}\cdot S_{\mathcal{P}_2\cdots\mathcal{P}_N}\\
A^\dagger_{\mathcal{P}_2}(p_{\mathcal{P}_2})\cdots A^\dagger_{\mathcal{P}_N}(p_{\mathcal{P}_N})A^\dagger_k &= A^\dagger_1\cdots A^\dagger_N\cdot S_{k+1,k}S_{k+2,k}\cdots S_{Nk}\cdot S_{\mathcal{P}_2\cdots\mathcal{P}_N}\,.
\end{aligned}$$

Thus, for $S_{\mathcal{P}_2\cdots\mathcal{P}_N}=1$ eq.(9.18) takes the form

$$\mathcal{A}^{\mathcal{I}|\mathcal{I}}\left(S_{k,k-1}\cdots S_{k1}I^g_{k,k-1}\cdots I^g_{k1}-e^{ip_kL}S_{k+1,k}\cdots S_{Nk}I^g_{k+1,k}\cdots I^g_{Nk}W_k\right)=0 \quad (9.19)$$

or, equivalently,

$$\mathcal{A}^{\mathcal{I}|\mathcal{I}}\left(e^{ip_kL}-S_{k,k-1}\cdots S_{k1}I^g_{k,k-1}\cdots I^g_{k1}W_kI^g_{kN}\cdots I^g_{k,k+1}S_{kN}\cdots S_{k,k+1}\right)=0\,. \quad (9.20)$$

It is possible to show that the same equations follow if $S_{\mathcal{P}_2\cdots\mathcal{P}_N}\neq 1$ which uses the identity $S_{km}I^g_{kn}I^g_{mn}=I^g_{kn}I^g_{mn}S_{km}$, and also that the terms in the wave function with the plane wave $e^{i\,p_{\mathcal{P}}\cdot x_{\mathcal{Q}}}$ lead to the same equations.

The consistency condition for the system of equations (9.20) requires that the matrices

$$T_k \equiv S_{k,k-1}\cdots S_{k1} I^g_{k,k-1}\cdots I^g_{k1} W_k I^g_{kN}\cdots I^g_{k,k+1} S_{kN}\cdots S_{k,k+1}$$

mutually commute. Naturally, we expect that the matrices T_k should be related to the monodromy matrix

$$T(p_A) = -\mathrm{Str}_A\, W_A S^f_{AN}(p_A,p_N) S^f_{A,N-1}(p_A,p_{N-1})\cdots S^f_{A1}(p_A,p_1)\,, \tag{9.21}$$

where S^f_{jk} is the fermionic R-operator defined, e.g., in eq.(102) of [73]. The authors of [73] use index notations to define the operator. It is more convenient, however, to use the matrix notations and the usual convention for S_{jk} to work with the operator. One can check that it can be written in the following form

$$S^f_{jk}(p_j,p_k) = \begin{cases} I^g_{j\cdots N} I^g_{k\cdots N}\, I^g_{jk} S_{jk}(p_j,p_k)\, I^g_{j\cdots N} I^g_{k\cdots N} & \text{if} \;\; j<k\,; \\ I^g_{j\cdots N} I^g_{k\cdots N}\, S_{jk}(p_j,p_k) I^g_{jk}\, I^g_{j\cdots N} I^g_{k\cdots N} & \text{if} \;\; j>k\,. \end{cases} \tag{9.22}$$

Here I^g_{jk} is the graded identity and

$$I^g_{j\cdots N} \equiv I^g_{j,j+1} I^g_{j,j+2}\cdots I^g_{jN}\,.$$

To prove the formula, one should use the following representation for the graded projection operators $\widetilde{E}^\beta_{j\alpha}$ eq.(28) of [73]

$$\widetilde{E}^\beta_{j\alpha} = I^g_{j\cdots N}\, E^\beta_{j\alpha}\, I^g_{j\cdots N}\,.$$

There are two natural choice for the index A in (9.21), that is $A=0$ or $A=N+1$. The choice leading to T_k appears to be $A=N+1>k$. Then we get

$$S^f_{Ak}(p_A,p_k) = I^g_{k\cdots N}\, S_{Ak}(p_A,p_k) I^g_{Ak}\, I^g_{k\cdots N}\,.$$

Now we compute the following product

$$\begin{aligned} & S^f_{Ak}(p_A,p_k) S^f_{A,k-1}(p_A,p_{k-1}) = \\ & I^g_{k\cdots N} S_{Ak}\, I^g_{Ak}\, I^g_{k\cdots N} I^g_{k-1\cdots N}\, S_{A,k-1}\, I^g_{A,k-1}\, I^g_{k-1\cdots N} = \\ & I^g_{k\cdots N} I^g_{k-1\cdots N} I^g_{k-1,k}\, S_{Ak}\, I^g_{Ak} I^g_{k-1,k} S_{A,k-1}\, I^g_{A,k-1}\, I^g_{k\cdots N}\, I^g_{k-1\cdots N} = \\ & I^g_{k\cdots N} I^g_{k-1\cdots N} I^g_{k-1,k}\, S_{Ak} S_{A,k-1}\, I^g_{Ak}\, I^g_{A,k-1}\, I^g_{k-1,k}\, I^g_{k\cdots N}\, I^g_{k-1\cdots N}\,, \end{aligned} \tag{9.23}$$

where we used the identity

$$S_{A,k-1}(p_A,p_k)\, I^g_{k-1,k}\, I^g_{Ak} = I^g_{k-1,k}\, I^g_{Ak}\, S_{A,k-1}(p_A,p_k)\,.$$

The following generalization of the formula (9.23) can be proven by using the mathematical induction

$$
\begin{aligned}
& S^f_{Ak}(p_A,p_k)S^f_{A,k-1}(p_A,p_{k-1})\cdots S^f_{A,k-n}(p_A,p_{k-n}) = \\
& \qquad = I^g_{k\cdots N}\cdots I^g_{k-n\cdots N}\, I^g_{k-1,k}I^g_{k-2\cdots k}\cdots I^g_{k-n\cdots k}\times \\
& \qquad \times S_{Ak}\cdots S_{A,k-n}\, I^g_{Ak}\cdots I^g_{A,k-n}\, I^g_{k\cdots N}\cdots I^g_{k-n\cdots N}\, I^g_{k-1,k}I^g_{k-2\cdots k}\cdots I^g_{k-n\cdots k}\,.
\end{aligned}
\tag{9.24}
$$

To get the monodromy matrix, we set $k=N$ and $n=N-1$ in this formula, and using the identity

$$
I^g_{N-1\cdots N}\cdots I^g_{1\cdots N}\, I^g_{N-1,N}I^g_{N-2\cdots N}\cdots I^g_{1\cdots N} = I\,,
$$

we find the following drastic simplification

$$
T(p_A) = -\mathrm{Str}_A\, W_A\, S_{AN}\cdots S_{A1}\, I^g_{AN}\cdots I^g_{A1}\,.
$$

Now we choose $p_A=p_k$ and use the fact that $S_{Ak}(p_k,p_k)=-P_{Ak}$. Recalling that our goal is to show that $T(p_k)=T_k$, we have

$$
\begin{aligned}
T(p_k) &= \mathrm{Str}_A\, W_A\, S_{AN}\cdots S_{A,k+1}\, P_{Ak}\, S_{A,k-1}\cdots S_{A1}\, I^g_{AN}\cdots I^g_{A1} \\
&= \mathrm{Str}_A\, P_{Ak}\, W_k\, S_{kN}\cdots S_{k,k+1}\cdot S_{A,k-1}\cdots S_{A1}\, I^g_{AN}\cdots I^g_{A1} \\
&= \mathrm{Str}_A\, P_{Ak}\, S_{A,k-1}\cdots S_{A1}\, I^g_{A,k-1}\cdots I^g_{A1}\cdot W_k\, S_{kN}\cdots S_{k,k+1}\cdot I^g_{AN}\cdots I^g_{Ak}\,.
\end{aligned}
$$

Now we use that

$$
S_{kN}\cdots S_{k,k+1}\cdot I^g_{AN}\cdots I^g_{Ak} = I^g_{AN}\cdots I^g_{Ak}\cdot S_{kN}\cdots S_{k,k+1}
$$

to get

$$
T(p_k) = \mathrm{Str}_A\, S_{k,k-1}\cdots S_{k1}\, I^g_{k,k-1}\cdots I^g_{k1}I^g_{kN}\cdots I^g_{k,k+1}\, P_{Ak}I^g_{Ak}\, W_k\, S_{kN}\cdots S_{k,k+1}\,.
$$

The supertrace can be easily taken

$$
\begin{aligned}
\mathrm{Str}_A\, P_{Ak}I^g_{Ak} &= \mathrm{Tr}_2\left((-1)^{\epsilon_c} I\otimes E^c_c\right)\left(E^a_b\otimes E^b_a\right)\left((-1)^{\epsilon_f\epsilon_g}E^f_f\otimes E^g_g\right) \\
&= (-1)^{\epsilon_a+\epsilon_a^2}E^a_a = I\,,
\end{aligned}
$$

and, therefore, we show that $T(p_k)=T_k$. Since $T(u)T(v)=T(v)T(u)$ for any u and v, we have shown that the periodicity equations (9.20) are consistent.[24]

[24]In framework of the algebraic Bethe Ansatz twisted boundary conditions for Hubbard-like models have been studied in [63].

9.3.2 Two-particle Bethe equations

To see how the formulas of the previous subsection work let us consider a two-particle wave function given by

$$\Psi_{ij}(x_1,x_2)=\begin{cases}\mathcal{A}_{ij}^{12|12}\,e^{i\,p_1x_1+i\,p_2x_2}+\mathcal{A}_{ij}^{21|12}\,e^{i\,p_2x_1+i\,p_1x_2} & \text{if}\quad x_1<x_2\\ \mathcal{A}_{ij}^{12|21}\,e^{i\,p_1x_2+i\,p_2x_1}+\mathcal{A}_{ij}^{21|21}\,e^{i\,p_2x_2+i\,p_1x_1} & \text{if}\quad x_2<x_1\end{cases}. \tag{9.25}$$

According to (9.15), we can identify

$$\mathcal{A}_{ij}^{12|12}\sim A_i^\dagger(p_1)A_j^\dagger(p_2)\,,\quad \mathcal{A}_{ij}^{21|21}\sim(-)^{\epsilon_i\epsilon_j}A_j^\dagger(p_2)A_i^\dagger(p_1)\,.$$

It is clear that the amplitudes $\mathcal{A}_{ij}^{12|12}$ and $\mathcal{A}_{ij}^{21|21}$ correspond to the in- and out-states, respectively. By using the ZF algebra we find

$$A_j^\dagger(p_2)A_i^\dagger(p_1)=S_{ji}^{lk}(p_2,p_1)A_k^\dagger(p_1)A_l^\dagger(p_2)\quad\Rightarrow\quad\mathcal{A}_{ij}^{21|21}=(-)^{\epsilon_i\epsilon_j}S_{ji}^{lk}(p_2,p_1)\mathcal{A}_{kl}^{12|12}\,.$$

In a similar way we get

$$\mathcal{A}_{ij}^{21|12}\sim A_i^\dagger(p_2)A_j^\dagger(p_1)\,,\quad \mathcal{A}_{ij}^{12|21}\sim(-)^{\epsilon_i\epsilon_j}A_j^\dagger(p_1)A_i^\dagger(p_2)\,,$$

and

$$A_j^\dagger(p_1)A_i^\dagger(p_2)=S_{ji}^{lk}(p_1,p_2)A_k^\dagger(p_2)A_l^\dagger(p_1)\quad\Rightarrow\quad\mathcal{A}_{ij}^{12|21}=(-)^{\epsilon_i\epsilon_j}S_{ji}^{lk}(p_1,p_2)\mathcal{A}_{kl}^{21|12}\,.$$

The amplitudes $\mathcal{A}_{ij}^{12|12}$ and $\mathcal{A}_{ij}^{21|12}$ are not independent. By the symmetry condition (9.14) they are related to each other as follows

$$\mathcal{A}_{ij}^{12|12}=(-)^{\epsilon_i\epsilon_j}\mathcal{A}_{ji}^{12|21}=S_{ij}^{lk}(p_1,p_2)\mathcal{A}_{kl}^{21|12}\quad\Rightarrow\quad\mathcal{A}_{ij}^{21|12}=S_{ij}^{lk}(p_2,p_1)\mathcal{A}_{kl}^{12|12}\,.$$

The wave function (9.25) can be written in the matrix form by multiplying it by the row $E^i\otimes E^j$ and summing over i,j. Then we get

$$\Psi(x_1,x_2)=\begin{cases}\mathcal{A}\,(e^{i\,p_1x_1+i\,p_2x_2}+S_{21}P_{12}\,e^{i\,p_2x_1+i\,p_1x_2}) & \text{if}\quad x_1<x_2\\ \mathcal{A}\,(P_{12}^ge^{i\,p_1x_2+i\,p_2x_1}+S_{21}^g\,e^{i\,p_2x_2+i\,p_1x_1}) & \text{if}\quad x_2<x_1\end{cases}, \tag{9.26}$$

where we recall that $P_{12}^g=(-1)^{\epsilon_i\epsilon_j}E_j^i\otimes E_i^j$ is the graded permutation and S_{21}^g is the graded S-matrix $S_{21}^g=P_{12}S(p_2,p_1)P_{12}I_{12}^g=P_{12}S(p_2,p_1)P_{12}^g$.

The (quasi)-periodicity condition can be easily imposed

$$\Psi(x_1,x_2)=\Psi(x_1+L,x_2)W_1\,,\quad\Psi(x_1,x_2)=\Psi(x_1,x_2-L)W_2\,,\quad x_1<x_2\,,$$

where the matrix W is equal to I for periodic boundary conditions and to Σ for anti-periodic boundary conditions for fermions. By using the wave $e^{ip_kx_k}$ this leads to the following equations

$$\mathcal{A}\left(1-e^{ip_1L}S_{21}^gW_1\right)=0\,,\quad\mathcal{A}\left(1-e^{-ip_2L}S_{21}^gW_2\right)=0\,,$$

or by using the wave $e^{ip_2x_2+ip_1x_2}$ to

$$\mathcal{A}\left(S_{21}P_{12}-e^{ip_2L}P^g_{12}W_1\right)=0\,,\quad \mathcal{A}\left(S_{21}P_{12}-e^{-ip_1L}P^g_{12}W_2\right)=0\,.$$

These two sets of the periodicity conditions are obviously equivalent because $P^g_{12}W_1 = W_2P^g_{12}$. Let us also mention that the equations are compatible if the matrices $W_1S^g_{12}$ and $S^g_{21}W_2$ commute, and this follows from unitarity $S^g_{12}S^g_{21}=I$ and

$$W_1W_2S^g_{12}=S^g_{12}W_1W_2\,.$$

Let us now see how the nesting procedure works for the case of one $A_1^\dagger$ boson and one $A_3^\dagger$ fermion. Consider the system of equations

$$\begin{aligned}\mathcal{A}^{21|12}_{13} &= S^{13}_{13}(p_2,p_1)\mathcal{A}^{12|12}_{31}+S^{31}_{13}(p_2,p_1)\mathcal{A}^{12|12}_{13}\,,\\ \mathcal{A}^{21|12}_{31} &= S^{13}_{31}(p_2,p_1)\mathcal{A}^{12|12}_{31}+S^{31}_{31}(p_2,p_1)\mathcal{A}^{12|12}_{13}\,.\end{aligned}\tag{9.27}$$

Assuming that S^{kl}_{ij} are matrix elements of the string S-matrix $\mathcal{S}$, we get

$$\begin{aligned}\mathcal{A}^{21|12}_{13} &= S_0(p_2,p_1)\Big[\frac{x_1^- - x_2^-}{x_1^+ - x_2^-}e^{\frac{i}{2}p_1}\mathcal{A}^{12|12}_{31}-\frac{x_1^+ - x_1^-}{x_1^+ - x_2^-}\frac{\eta(p_2)}{\eta(p_1)}\frac{e^{\frac{i}{2}p_1}}{e^{\frac{i}{2}p_2}}\mathcal{A}^{12|12}_{13}\Big]\,,\\ \mathcal{A}^{21|12}_{31} &= S_0(p_2,p_1)\Big[\frac{x_2^+ - x_2^-}{x_2^- - x_1^+}\frac{\eta(p_1)}{\eta(p_2)}\mathcal{A}^{12|12}_{31}+\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}e^{-\frac{i}{2}p_2}\mathcal{A}^{12|12}_{13}\Big]\,,\end{aligned}\tag{9.28}$$

where $S_0(p_1,p_2)$ is the scalar prefactor.

For the amplitudes of interest the general Bethe equations

$$e^{-ip_1L}\mathcal{A}^{12|12}_{ij}=(-1)^{\epsilon\epsilon_i+\epsilon_i\epsilon_j}S^{lk}_{ji}(p_2,p_1)\mathcal{A}^{12|12}_{kl}\tag{9.29}$$

read as follows

$$\begin{aligned}e^{-ip_1L}\mathcal{A}^{12|12}_{13} &= S^{13}_{31}(p_2,p_1)\mathcal{A}^{12|12}_{31}+S^{31}_{31}(p_2,p_1)\mathcal{A}^{12|12}_{13}=\mathcal{A}^{21|12}_{31}\,,\\ e^{-ip_1L}\mathcal{A}^{12|12}_{31} &= (-1)^\epsilon\Big[S^{13}_{13}(p_2,p_1)\mathcal{A}^{12|12}_{31}+S^{31}_{13}(p_2,p_1)\mathcal{A}^{12|12}_{13}\Big]=(-1)^\epsilon\mathcal{A}^{21|12}_{13}\,,\end{aligned}\tag{9.30}$$

where we have used eqs.(9.27). Note that in eq.(9.29) the multiplier $(-1)^{\epsilon\epsilon_i}$ takes into account the boundary conditions for fermions: $\epsilon=0$ for periodic fermions and $\epsilon=1$ for anti-periodic ones, respectively.

The system (9.28) can be solved in two different ways depending on the choice of the first level vacuum [9]. Below we present both solutions.

- Regarding $A_1\ldots A_1$ as the first level vacuum, we first choose the following ansatz

$$\begin{aligned}\mathcal{A}^{12|12}_{13} &= f(p_2)S(p_1)\,, & \mathcal{A}^{12|12}_{31} &= f(p_1)\,,\\ \mathcal{A}^{21|12}_{13} &= S^{11}_{11}(p_2,p_1)f(p_1)S(p_2)\,, & \mathcal{A}^{21|12}_{31} &= S^{11}_{11}(p_2,p_1)f(p_2)\,,\end{aligned}\tag{9.31}$$

where $S^{11}_{11}(p_1,p_2)$ is the corresponding element of the string S-matrix. One can easily show that this ansatz indeed solves the system (9.28) provided we take

$$f(p)=\frac{e^{i\frac{p}{2}}}{\eta(p)}\frac{x^+-x^-}{y-x^-}\,,\qquad S(p)=e^{i\frac{p}{2}}\frac{y-x^-}{y-x^+}\,.$$

According to eqs.(9.31), the last formulae give

$$\begin{aligned}e^{-ip_1L}f(p_2)S(p_1)&=\mathcal{S}^{11}_{11}(p_2,p_1)f(p_2)\,,\\ e^{-ip_1L}f(p_1)&=(-1)^{\epsilon}\mathcal{S}^{11}_{11}(p_2,p_1)f(p_1)S(p_2)\end{aligned}$$

and we derive the corresponding Bethe equations

$$\begin{aligned}e^{ip_1L}&=S^{11}_{11}(p_1,p_2)S(p_1)\,,\\ (-1)^{\epsilon}&=S(p_1)S(p_2)\,.\end{aligned}$$

- If we choose $A_3\ldots A_3$ as the first level vacuum, we modify the ansatz for the corresponding amplitudes as follows

$$\begin{aligned}\mathcal{A}^{12|12}_{13}&=f(p_1)\,, & \mathcal{A}^{12|12}_{31}&=f(p_2)S(p_1)\,.\\ \mathcal{A}^{21|12}_{13}&=S^{33}_{33}(p_2,p_1)f(p_2)\,, & \mathcal{A}^{21|12}_{31}&=S^{33}_{33}(p_2,p_1)f(p_1)S(p_2)\,.\end{aligned}\tag{9.32}$$

Note that $S^{33}_{33}(p_1,p_2)=-S_0(p_1,p_2)$. This time satisfaction of eqs.(9.28) requires one to choose

$$f(p)=\eta(p)e^{-i\frac{p}{2}}\frac{y}{y-x^-}\,,\qquad S(p)=-e^{-i\frac{p}{2}}\frac{y-x^+}{y-x^-}\,.$$

The Bethe equations (9.30) read

$$\begin{aligned}e^{-ip_1L}f(p_1)&=S^{33}_{33}(p_2,p_1)f(p_1)S(p_2)\,,\\ e^{-ip_1L}f(p_2)S(p_1)&=(-1)^{\epsilon}S^{33}_{33}(p_2,p_1)f(p_2)\,,\end{aligned}$$

and, therefore, we find

$$\begin{aligned}&e^{ip_1L}=(-1)^{\epsilon}S^{33}_{33}(p_1,p_2)S(p_1)\equiv(-1)^{\epsilon}S_0(p_1,p_2)\frac{x_1^+-y}{x_1^--y}e^{-i\frac{p_1}{2}}\,,\\ &(-1)^{\epsilon}=S(p_1)S(p_2)\,.\end{aligned}$$

This completes consideration of our simple example illustrating the dependence of the Bethe equations on the periodicity conditions for fermions.

9.4 Large/small g expansions of solutions to the bound state equation

The four general solutions of the bound state equation (7.8) are

$$e^q = \frac{\left(\sqrt{g^2\sin^2\frac{p}{2}+1}+1\right)\left(\cos\frac{p}{2}\sqrt{g^2\sin^2\frac{p}{2}+1}\pm\sqrt{\cos^2\frac{p}{2}-g^2\sin^4\frac{p}{2}}\right)}{g^2\sin^2\frac{p}{2}}\,, \qquad (9.33)$$

$$e^q = \frac{\left(\sqrt{g^2\sin^2\frac{p}{2}+1}-1\right)\left(\cos\frac{p}{2}\sqrt{g^2\sin^2\frac{p}{2}+1}\pm\sqrt{\cos^2\frac{p}{2}-g^2\sin^4\frac{p}{2}}\right)}{g^2\sin^2\frac{p}{2}}\,, \qquad (9.34)$$

where only the first two solutions (9.33) correspond to states with positive energy.

The large g dependence of q of the bound state solutions with momentum exceeding $p_{\rm cr}$ is obtained by expanding (9.33) in powers of $1/g$ with the bound state momentum p kept fixed[25]

$$q_\pm = \frac{1}{g\sin\frac{p}{2}} - \frac{1}{6g^3\sin^3\frac{p}{2}} \pm i\left(\frac{p}{2} - \frac{\cos\frac{p}{2}}{2g^2\sin^3\frac{p}{2}}\right) + \mathcal{O}(\frac{1}{g^4})\,. \qquad (9.35)$$

To find the large g dependence of q of the bound state solutions with momentum smaller than $p_{\rm cr}$ one should take into account that $p_{\rm cr} \to 2/\sqrt{g}$ as $g \to \infty$, and therefore one should consider a bound state with momentum p of the order $1/\sqrt{g}$ and keep the product $p\sqrt{g}$ fixed in the large g expansion

$$q_\pm = 2\frac{1\pm\sqrt{1-\frac{p^4g^2}{16}}}{gp} - \frac{4}{3g^3p^3}\left(1-\frac{p^4g^2}{16} \pm \frac{1}{\sqrt{1-\frac{p^4g^2}{16}}}\right)\,. \qquad (9.36)$$

The small g dependence of q of the bound state solutions with momentum smaller than $p_{\rm cr}$ is obtained by expanding (9.33) at small g with the bound state momentum p kept fixed

$$q_+ = -2\log g + \log\frac{4\cos\frac{p}{2}}{\sin^2\frac{p}{2}} + \frac{g^2}{8}(1+3\cos p)\tan^2\frac{p}{2} + \mathcal{O}(g^4)\,, \qquad (9.37)$$

$$q_- = -\log\cos\frac{p}{2} + \frac{g^2}{4}\sin^2\frac{p}{2}\tan^2\frac{p}{2} + \mathcal{O}(g^4)\,. \qquad (9.38)$$

To find the small g dependence of q of the bound state solutions with the momentum exceeding $p_{\rm cr}$, one should take into account that $p_{\rm cr} \to \pi - 2g$ as $g \to 0$. Then, one can parametrize p as follows

$$p = \pi - 2g\cos\alpha\,,$$

and keep α fixed in the expansion. Then we get

$$q_\pm = -\log\frac{g}{2} + \frac{g^2}{4}(2+\cos 2\alpha) \pm i\left(\alpha + \frac{g^2}{6}\sin 2\alpha - \frac{5g^2}{6}\cot\alpha\right) + \mathcal{O}(g^4)\,. \qquad (9.39)$$

[25] We assume here that $p \in (0,\pi)$.

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