Commun. Korean Math. Soc. **23** (2008), No. 1, pp. 125–131

ON STRONG GENERALIZED NEIGHBORHOOD SYSTEMS AND *sg*-OPEN SETS

Won Keun Min

Abstract. We introduce and study the concepts of strong generalized neighborhood systems, SGNS and sg-open sets. We also introduce and investigate the concept of (ψ, ψ')-continuity and sg-continuity on SGNS's.

1. Introduction

In [1], Á. Csázár introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function(= (ψ, ψ')-continuous function) by using a closure operator defined on generalized neighborhood systems.

In this paper we introduce the strong generalized neighborhood systems which are generalizations of neighborhood systems. The strong generalized neighborhood system induces a strong generalized neighborhood space (briefly SGNS) which it implies a generalized neighborhood space. And the SGNS induces a structure(= the collection of all sg-open sets on an SGNS) which is a generalization of topology. We introduce the new concepts of interior and closure on an SGNS and investigate some properties. In particular, we introduce the concept of sg-open sets on a given SGNS and investigate some properties. We introduce the concepts of sg-continuity and (ψ, ψ')-continuity and we characterize some properties by the new interior and closure operators defined on an SGNS. Finally we show that every (ψ, ψ')-continuous function is sg-continuous but the converse is not always true.

Received August 4, 2007; Revised January 10, 2008.

2000 *Mathematics Subject Classification.* 54A10, 54A20, 54D10, 54D30.

Key words and phrases. strong generalized neighborhood systems, SGNS, sg-open set, sg-continuous, (ψ, ψ')-continuous, strong generalized interior operator, strong generalized interior.

This work was supported by a grant from Research Institute for Basic Science at Kangwon National University.

2. Preliminaries

We now recall some concepts and notations defined in [1]. Let X be a nonempty set and $\psi : X \to \exp(\exp(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* of $x \in X$ and ψ is called a *generalized neighborhood system* (briefly GNS) on X. And if ψ is a generalized neighborhood system on X and $A \subset X$, the interior and closure of A on ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as following:

$\iota_\psi(A) = \{x \in A : \text{there exists } V \in \psi(x) \text{ such that } V \subset A\}$;

$\gamma_\psi(A) = \{x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\}$.

Let ψ be a GNS on X and $G \in g_\psi$ if and only if $G \subset X$ satisfies: if $x \in G$ then there is $V \in \psi(x)$ such that $V \subset G$. Then $G \in g_\psi$ is called a generalized open set.

Let ψ and $\psi^{'}$ be generalized neighborhood systems on X and Y, respectively. Then a function $f : X \to Y$ is said to be $(\psi, \psi^{'})$-continuous if for $x \in X$ and $U \in \psi^{'}(f(x))$, there is $V \in \psi(x)$ such that $f(V) \subset U$.

Consider a function $I : 2^X \to 2^X$ satisfying the following axioms:

(**C**1) $I(A) \subset A$ for all $A \subset X$;

(**C**2) $A \subset B \Rightarrow I(A) \subset I(B)$ for all $A, B \in 2^X$;

(**C**3) $I(I(A)) = I(A)$ for all $A \subset X$;

(**C**4) $I(X) = X$.

Then the function I is called :

(a) a strong generalized interior operator (shortly *sgio*) [4] if (**C**1), (**C**2) and (**C**3) hold;

(b) a generalized interior operator (shortly *gio*)[4] if (**C**1) and (**C**2) hold;

(c) a quasi-interior operator [3] if (**C**1),(**C**2), (**C**3) and (**C**4) hold;

(d) an interior operator [2] if (**C**1), (**C**2) and (**C**4) hold.

3. Strong generalized neighborhood spaces

Consider a function $I : 2^X \to 2^X$ satisfying the following axiom:

(**C**p) $I(A) \cap I(B) \subset I(A \cap B)$, for all $A, B \in 2^X$.

Then the function I is called

(a) a *strong* generalized interior operator* (shortly s^*gio) if (**C**1), (**C**2), (**C**3), and (**C**p) hold;

(b) a *generalized* interior operator* (shortly g^*io) if (**C**1), (**C**2), and (**C**p) hold.

Definition 3.1. Let $\psi : X \to \exp(\exp(X))$. Then ψ is called a *strong generalized neighborhood system* on X if it satisfies the following:

(1) $x \in V$ for $V \in \psi(x)$;

(2) for $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.

Then the pair (X, ψ) is called a *strong generalized neighborhood space* (briefly SGNS) on X. Then $V \in \psi(x)$ is called a *strong generalized neighborhood* of $x \in X$.

It is obvious that every strong generalized neighborhood system is a generalized neighborhood system but the converse may not be true as the following example.

Example 3.2. Let $X = \{a,b,c\}$ and $\psi : X \to \exp(\exp(X))$ be a strong generalized neighborhood system defined as $\psi(a) = \{\{a,b\},\{a,c\}\}$, $\psi(b) = \{\{a,b\},\{b,c\}\}$, and $\psi(c) = \{\emptyset\}$; then ψ is a generalized neighborhood system but it is not a strong generalized neighborhood system on X.

Definition 3.3. Let (X,ψ) be an SGNS on X and $A \subset X$, the interior and closure of A on ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as follows:
$\iota_\psi(A) = \{x \in A :$ there exists $V \in \psi(x)$ such that $V \subset A\}$;
$\gamma_\psi(A) = \{x \in X : V \cap A \neq \emptyset$ for all $V \in \psi(x)\}$.

Theorem 3.4. *Let (X,ψ) be an SGNS on X. We obtain the following.*
(1) $\iota_\psi(A) \subset A$ *for all* $A \subset X$*;*
(2) $\iota_\psi(A \cap B) = \iota_\psi(A) \cap \iota_\psi(B)$ *for all* $A, B \in 2^X$*.*

Proof. (1) Obvious.
(2) Let $x \in \iota_\psi(A) \cap \iota_\psi(B)$; then there are U, V in $\psi(x)$ such that $U \subset A, V \subset B$. Since $\psi(x)$ is a strong generalized neighborhood, $U \cap V \in \psi(x)$ and $U \cap V \subset A \cap B$. Thus $x \in \iota_\psi(A \cap B)$.
The converse inclusion is obvious. □

Example 3.5. Let $X = \{a,b,c,d\}$ and $\psi : X \to \exp(\exp(X))$ be a strong generalized neighborhood system defined as $\psi(a) = \{\{a,c\}\}$, $\psi(b) = \{\{b,c\}\}$, $\psi(c) = \emptyset$, and $\psi(d) = \emptyset$; then for $A = \{a,b,c\} \subset X$, $\iota_\psi(A) = \{a,b\}$ but $\iota_\psi(\iota_\psi(A)) = \emptyset$, and so $\iota_\psi(A) \neq \iota_\psi(\iota_\psi(A))$.

Theorem 3.6. *Let (X,ψ) be an SGNS on X. We obtain the following.*
(1) $A \subset \gamma_\psi(A)$ *for all* $A \subset X$*.*
(2) $\gamma_\psi(A \cup B) = \gamma_\psi(A) \cup \gamma_\psi(B)$ *for all* $A, B \in 2^X$*.*
(3) $\gamma_\psi(A) = X - \iota_\psi(X - A)$, $\iota_\psi(A) = X - \gamma_\psi(X - A)$*.*

Proof. By Definition 3.3, it is obvious. □

From Theorem 3.4 and definition of g^*io, we obtain the following:

Theorem 3.7. (1) *Let (X,ψ) be an SGNS on X and $I : 2^X \to 2^X$ be defined as $I(A) = \iota_\psi(A)$ for each $A \subset X$. Then I is a g^*io.*
(2) *Let $I : 2^X \to 2^X$ be a g^*io and $\phi_I : X \to \exp(\exp(X))$ be defined as $\phi_I(x) = \{I(A) : x \in I(A)$ for $A \subset X\}$; then ϕ_I is a strong generalized neighborhood system induced by I.*

Let (X,ψ) be an SGNS on X and $I : 2^X \to 2^X$ be defined as $I(A) = \iota(A)$ for each $A \subset X$. Then for a strong generalized neighborhood system ϕ_ι induced by $I = \iota$ there is no relation between ϕ_ι and ψ.

Example 3.8. Let $X = \{a, b, c\}$ and $\psi : X \to \exp(\exp(X))$ be a strong generalized neighborhood system defined as $\psi(a) = \{\{a\}, \{a, c\}\}$, $\psi(b) = \{\{b\}, \{b, c\}\}$ $\psi(c) = \emptyset$; then $\phi_\iota : X \to \exp(\exp(X))$ is a strong generalized neighborhood system induced by $I = \iota$ as the following: $\phi_\iota(a) = \{\{a\}, \{a, b\}\}$, $\phi_\iota(b) = \{\{b\}, \{a, b\}\}$ $\phi_\iota(c) = \emptyset$.

Definition 3.9. Let (X, ψ) and (Y, ϕ) be two SGNS's. Then $f : X \to Y$ is said to be *(ψ, ϕ)-continuous* if for $x \in X$ and $V \in \phi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subset V$.

Theorem 3.10. *Let $f : X \to Y$ be a function on two SGNS's (X, ψ) and (Y, ϕ). Then the following statements are equivalent:*
(1) *f is (ψ, ϕ)-continuous.*
(2) *$f(\gamma_\psi(A)) \subset \gamma_\phi f(A)$ for $A \subset X$.*
(3) *$\gamma_\psi f^{-1}(B) \subset f^{-1}(\gamma_\phi(B))$ for $B \subset Y$.*
(4) *$f^{-1}(\iota_\phi(B)) \subset \iota_\psi f^{-1}(B)$ for $B \subset Y$*

Proof. (1) $\Rightarrow$ (2) Let $x \in \gamma_\psi(A)$. If $f(x)$ is not in $\gamma_\phi f(A)$, then there exists $V \in \phi(f(x))$ such that $V \cap \gamma_\phi f(A) = \emptyset$. By the (ψ, ϕ)-continuity, there is $U \in \psi(x)$ such that $f(U) \subset V$, and so $f(U) \cap \gamma_\phi f(A) = \emptyset$. Consequently, $U \cap A = \emptyset$: a contradiction.

(2) $\Rightarrow$ (3) Let $A = f^{-1}(B)$ for $B \subset Y$; then by (2) $f(\gamma_\psi(A)) \subset \gamma_\phi f(A) = \gamma_\phi f(f^{-1}(B)) \subset \gamma_\phi(B)$. Thus $\gamma_\psi f^{-1}(B) \subset f^{-1}(\gamma_\phi(B))$.

(3) $\Rightarrow$ (4) By Theorem 3.6, it is obvious.

(4) $\Rightarrow$ (1) For $x \in X$, let $V \in \phi(f(x))$; then $f(x) \in \iota_\phi(V)$. By (4), $x \in f^{-1}(\iota_\phi(V)) \subset \iota_\psi f^{-1}(V)$. From definition of interior, it follows that there exists $U \in \psi(x)$ such that $U \subset \iota_\psi f^{-1}(V)$. Thus we get the result. □

4. *sg*-open sets and *sg*-continuity

Definition 4.1. Let (X, ψ) be an SGNS on X and $G \subset X$. Then G is called an *sg_ψ-open* set if for each $x \in G$, there is $V \in \psi(x)$ such that $V \subset G$.

Let us denote $sg_\psi(X)$ the collection of all *sg_ψ-open* sets on an SGNS (X, ψ). The complements of *sg_ψ-open* sets are called *sg_ψ-closed* sets.

Theorem 4.2. *Let (X, ψ) be an SGNS on X.*
(1) *The empty set is an sg_ψ-open set;*
(2) *The intersection of two sg_ψ-open sets is an sg_ψ-open set;*
(3) *The arbitrary union of sg_ψ-open sets is an sg_ψ-open set.*

Proof. (1) Obvious.

(2) Let A, B be two sg_ψ-open sets; then for each $x \in A \cap B$ there exist $U, V \in \psi(x)$ such that $U \subset A, V \subset B$. Since $\psi(x)$ is a strong generalized neighborhood, $U \cap V \in \psi(x)$ and $U \cap V \subset A \cap B$. Thus $A \cap B$ is also sg_ψ-open.

(3) Let $\mathbf{C} = \{A_\alpha : A_\alpha$ is an sg_ψ-open set $\}$; then for each $x \in \mathbf{C}$, there exists $A_\alpha \in \mathbf{C}$ such that $x \in A_\alpha$, and so there is $V \in \psi(x)$ such that $V \in \psi(x) \subset A_\alpha \subset \cup\mathbf{C}$. Thus $\cup\mathbf{C}$ is sg_ψ-open. □

Remark 4.3. In Theorem 4.2, X may not be sg_ψ-open as shown in the next example.

Example 4.4. Let $X = \{a, b, c, d\}$ and let ψ be a strong generalized neighborhood system defined as in Example 3.5. Then X is not an sg_ψ-open set.

Definition 4.5. Let (X, ψ) be an SGNS on X and $A \subset X$. The *sg_ψ-interior* of A (denoted by $i_{g_\psi}(A)$) is the union of all $G \subset A$, $G \in sg_\psi(X)$, and the *sg_ψ-closure* of A (denoted by $c_{g_\psi}(A)$) is the intersection of all sg_ψ-closed sets containing A.

Theorem 4.6. *Let (X, ψ) be an SGNS on X and $A \subset X$.*
(1) $i_{g_\psi}(A) = X - c_{g_\psi}(X - A)$*;*
(2) $c_{g_\psi}(A) = X - i_{g_\psi}(X - A)$*;*
(3) $i_{g_\psi}(A) \subset \iota_\psi(A)$*;*
(4) $\gamma_\psi(A) \subset c_{g_\psi}(A)$*.*

Proof. (1), (2) Obvious.
(3) For $x \in i_{g_\psi}(A)$, there exists sg_ψ-open set G such that $x \in G \subset A$. By definition of sg_ψ-open sets, there exists $V \in \psi(x)$ such that $x \in V \subset G \subset A$. Thus $x \in \iota_\psi(A)$.
(4) It follows obviously from (1), (3) and Theorem 3.6. □

Example 4.7. Let $X = \{a, b, c, d\}$ and $\psi : X \to \exp(\exp(X))$ be a strong generalized neighborhood system defined as $\psi(a) = \{\{a, c\}\}$, $\psi(b) = \{\{b\}\}$, $\psi(c) = \emptyset$, and $\psi(d) = \emptyset$; then for $A = \{a, b, c\} \subset X$, $\iota_\psi(A) = \{a, b\}$ but $i_{g_\psi}(A) = \{b\}$, and so $\iota_\psi(A) \neq i_{g_\psi}(A)$.

Theorem 4.8. *Let (X, ψ) be an SGNS on X and $A \subset X$.*
(1) *A is sg_ψ-open if and only if $i_{g_\psi}(A) = A$;*
(2) *A is sg_ψ-closed if and only if $c_{g_\psi}(A) = A$.*

Proof. By Definition 4.1 and Theorem 4.2, it is obvious. □

Theorem 4.9. *Let (X, ψ) be an SGNS on X and $A \subset X$. Then $\iota_\psi(A) = A$ if and only if A is sg_ψ-open.*

Proof. For each $x \in \iota_\psi(A)$, there exists $V_x \in \psi(x)$ such that $x \in V_x \subset A = \iota_\psi(A)$, and so $\iota_\psi(A)$ is an sg_ψ-open set.
For the converse, let A be sg_ψ-open; then from Theorem 4.6(3) and $i_{g_\psi}(A) = A$, it follows $A = i_{g_\psi}(A) \subset \iota_\psi(A) \subset A$, i.e. $\iota_\psi(A) = A$. □

Theorem 4.10. *Let (X, ψ) be an SGNS on X.*
(**C***1*) $i_{g_\psi}(A) \subset A$ *for all* $A \subset X$*;*
(**C***2*) $i_{g_\psi}(A \cap B) = i_{g_\psi}(A) \cap i_{g_\psi}(B)$ *for all* $A, B \in 2^X$*;*
(**C***3*) $i_{g_\psi}(i_{g_\psi}(A)) = i_{g_\psi}(A)$ *for all* $A \subset X$*;*

Proof. By Definition 4.5 and Theorem 4.8, it is obvious. □

Theorem 4.11. *Let* (X, ψ) *be an SGNS on* X *and* $I : 2^X \to 2^X$ *be defined as* $I(A) = i_{g_\psi}(A)$ *for each* $A \subset X$*. Then* $I = i_{g_\psi}$ *is an* s^**gio.*

Proof. By Theorem 4.10, it is obvious. □

Theorem 4.12. *Let* $I : 2^X \to 2^X$ *be an* s^**gio. Then there exists a strong generalized neighborhood system* ψ *induced by the* s^**gio* I *such that* A *is* sg_ψ*-open if and only if* $I(A) = A$*.*

Proof. Let us define $\psi : X \to \exp(\exp(X))$ as the following: for $x \in X$, $\psi(x) = \{V : I(V) = V \text{ and } x \in V\}$. Then ψ is an SGNS. Now A is sg_ψ-open if and only if $A = \cup_{x\in A} V$ where $V \in \psi(x)$. Since $I(V) = V$ for $V \in \psi(x)$, $A = \cup_{x\in A} V = \cup_{x\in A} I(V) \subset I(\cup_{x\in A} V) = I(A)$. Consequently, we can say A is sg_ψ-open if and only if $I(A) = A$. □

Definition 4.13. Let $f : X \to Y$ be a function on two SGNS's (X, ψ) and (Y, ϕ). Then f is said to be *sg-continuous* if for every $A \in sg_\phi(Y)$, $f^{-1}(A)$ is in $sg_\psi(X)$.

Theorem 4.14. *Let* $f : X \to Y$ *be a function on two SGNS's* (X, ψ) *and* (Y, ϕ)*. Then the following things are equivalent:*

(1) f *is sg-continuous*
(2) *For each* sg_ϕ*-closed set* F *in* Y, $f^{-1}(F)$ *is* sg_ψ*-closed in* X.
(3) $f(c_{g_\psi}(A)) \subset c_{g_\phi}(f(A))$ *for all* $A \subset X$.
(4) $c_{g_\psi}(f^{-1}(B)) \subset f^{-1}(c_{g_\phi}(B))$ *for all* $B \subset Y$.
(5) $f^{-1}(i_{g_\phi}(B)) \subset i_{g_\psi}(f^{-1}(B))$ *for all* $B \subset Y$.

Proof. Obvious □

Theorem 4.15. *Let* $f : X \to Y$ *be a function on two SGNS's* (X, ψ) *and* (Y, ϕ)*. Then if* f *is* (ψ, ϕ)*-continuous, then it is also sg-continuous.*

Proof. Let $A \in sg_\phi(Y)$; then from Theorem 4.9, it follows $\iota_\phi(A) = A$. By Theorem 3.10(4), we get $f^{-1}(A) = f^{-1}(\iota_\phi(A)) \subset \iota_\psi f^{-1}(A)$. Thus $f^{-1}(A)$ is sg_ψ-open by Theorem 4.9. □

In the following example, we show that the converse is not true in Theorem 4.15.

Example 4.16. Let $X = \{a, b, c, d\}$ and $\psi : X \to \exp(\exp(X))$ be a strong generalized neighborhood system defined as Example 3.5. Let us define the function $f : (X, \psi) \to (X, \psi)$ as the following: $f(a) = a, f(b) = b, f(c) = d, f(d) = c$; then f is sg-continuous, but it is not (ψ, ψ)-continuous because for $a \in X$ and $V = \{a, c\} \in \psi(f(a))$, since $\{a, c\} \in \psi(a)$ and $f(\{a, c\}) = \{a, d\}$, there is not $U \in \psi(a)$ such that $f(U) \subset V$.

References

[1] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar. **96** (2002), 351–357.
[2] D. C. Kent and W. K. Min, *Neighborhood spaces*, International Journal of Mathematics and Mathematical Sciences **32** (2002), no. 7, 387–399.
[3] A. S. Mashhour, A. A. Allam, F. S. Mahmoud, and F. H. Khedr, *On supratopological spaces*, Indian J. pure appl. Math. **14** (1983), no. 4, 502–510.
[4] W. K. Min, *Some results on generalized topological spaces and generalized systems*, Acta Math. Hungar. **108** (2005), no. 1-2, 171–181.

Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea
E-mail address: wkmin@kangwon.ac.kr