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ON STRONGLY EXTREME POINTS IN KÖTHE-BOCHNER SPACES

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ABSTRACT. Let $E(X)$ be a Köthe-Bochner space and f an element of the unit sphere of $E(X)$. Then for f to be a strongly extreme point of the unit ball of $E(X)$ it is necessary that $\|f(t)\|_X$ be a strongly extreme point of E and that $f(t)/\|f(t)\|_X$ be a strongly extreme point of X for μ almost everywhere $t \in \operatorname{supp} f$. Furthermore, if E is order continuous, then the condition is also sufficient. If E is a nonorder continuous Orlicz space, then the unit ball of $E(X)$ has no strongly extreme points which gives a negative answer to the question about the criteria for the denting points of Köthe-Bochner spaces raised by C. Castaing and R. Pluciennik.

1. Introduction. Let (T, Σ, μ) be a measure space with complete σ-finite measure μ and L^0 the space of all (equivalence classes of) μ-measurable real valued functions. For $f, g \in L^0$, $f \leq g$ means $f(t) \leq g(t)$ for μ almost everywhere $t \in T$.

A Banach subspace E of L^0 is said to be a Köthe function space, if

(i) for any $f, g \in L^0$, $|f| \leq |g|$ and $g \in E$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$;

(ii) $\operatorname{supp} E = \cup\{\operatorname{supp} f : f \in E\} = T$.

A Köthe space E is said to be order continuous provided that $x_n \downarrow 0$ implies $\|x_n\| \to 0$.

If E is a Köthe function space over (T, Σ, μ) and X is a Banach space, then by $E(X)$ we denote the Köthe-Bochner Banach space of all (equivalence classes of) strongly measurable functions $f : T \to X$ such that $\|f(\cdot)\|_X \in E$ equipped with the norm $\|f\|_{E(X)} = \|\|f(\cdot)\|_X\|_E$.

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For any Banach space X, we denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of X, respectively.

A point $x \in S(X)$ is said to be a strongly extreme point of $B(X)$ [$x \in \text{str-ext}\, B(X)$] if, for every sequence $\{x_n\} \subset X$, $\lim_{n\to\infty} \|x_n \pm x\|_X = 1$ implies $\lim_{n\to\infty} \|x_n\|_X = 0$.

Let $E(X)$ be a Köthe-Bochner space over (T, Σ, μ). For each $f \in E(X)$, we denote

$$T_f = \left\{ t \in \operatorname{supp} f : \frac{f(t)}{\|f(t)\|_X} \in \text{str-ext}\, B(X) \right\}.$$

H. Hudzik and M. Mastylo [**5**] proved the following

Theorem 1.1. *Let E be a locally uniformly rotund Köthe function space over a measure space (T, Σ, μ) and X a Banach space.*

a) *If $f \in S(E(X))$ and $f(t)/\|f(t)\|_X \in \text{str-ext}\, B(X)$ for μ almost everywhere $t \in \operatorname{supp} f$, then $f \in \text{str-ext}\, B(E(X))$;*

b) *if, in addition, X is a separable Banach space and $f \in \text{str-ext}\, B(E(X))$, then $f(t)/\|f(t)\|_X \in \text{str-ext}\, B(X)$ for μ almost everywhere $t \in \operatorname{supp} f$.*

R. Pluciennik [**7**] shows that b) is true for any Köthe function space E and any Banach space X provided that T_f is measurable. It is also asked in [**7**] that whether a) is true without the locally uniform rotundity of E.

In this paper we first prove that T_f is measurable for any element f of a Köthe-Bochner space, then present a necessary condition for $f \in \text{str-ext}\, B(E(X))$ and show that the condition is also sufficient if E is order continuous. Finally, we show that the unit ball of $E(X)$ has no strongly extreme points if E is a nonorder continuous Orlicz space. Since a denting point of a subset of a Banach space must be a strongly extreme point of the subset, our last result yields a negative answer to the question risen recently by C. Castaing and R. Pluciennik in [**1**].

2. Main results.

Definition 2.1. We say that a Köthe function space E has *Lebesgue* *dominated convergence property* if $x_n, y_n,\ y \in E,\ x_n(t) \to 0\ \mu$ almost everywhere on T and $|x_n| \le y_n \to y$ in E imply $x_n \to 0$ in E.

Lemma 2.1. *A Köthe function space E over (T, Σ, μ) has Lebesgue dominated convergence property if and only if it is order continuous.*

Proof. $\Rightarrow$. Trivial.

$\Leftarrow$. If the "if" part of the lemma is not true, then there exist x_n, y_n, $y \in E$, $x_n(t) \to 0$ μ-almost everywhere on T and $|x_n| \le y_n \to y$ in E but $\|x_n\| > \varepsilon$ for some $\varepsilon > 0$.

It is known, for example, cf. [**6**], that if $y_n \to y$ in E, then $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ such that $|y_{n_k} - y| \le \varepsilon_k x$ for some $x \in E$ and $\varepsilon_k \downarrow 0$. So $|x_{n_k}| \le y_{n_k} \le |y| + \varepsilon_1 x$.

Let $z_k = \vee_{i=k}^{\infty} |x_{n_i}|$. Then $|x_{n_k}| \le z_k \le |y| + \varepsilon_1 x$ implies $z_k \in E$. We claim that $z_k \downarrow 0$ μ almost everywhere on T. Indeed, if this fails, then $z_k(t) > \varepsilon'$ for some $\varepsilon' > 0$ on some $G \in \Sigma$ with $\mu G > 0$. Since T is σ-finite, we may assume $\mu G < \infty$. Therefore, there exists $H \subset G$ such that $\mu H < \mu G$ and that $x_{n_k}(t) \to 0$ uniformly on $G \backslash H$. But this implies $z_k(t) < \varepsilon'$ on $G \backslash H$ for all large k contradicting that $z_k(t) > \varepsilon'$ on G. Hence, $|x_{n_k}| \le z_k \downarrow 0$ implies also a contradiction that $\varepsilon < \|x_{n_k}\| \le \|z_k\| \to 0$ since E is order continuous.

Lemma 2.2 (cf. [**6**]). *Let E be Köhle function space over a complete σ-finite measure space (T, Σ, μ) and $f_n \to f$ in E. Then $\{f_n\}$ has a subsequence convergent to f μ almost everywhere on T.*

Theorem 2.1. *Let E be a Köthe function space over (T, Σ, μ) and X a Banach space. Then for any $f \in E(X)$, the set*

$$T_f = \{t \in \operatorname{supp} f : f(t)/\|f(t)\|_X \in \text{str-ext}\, B(X)\}$$

is measurable.

Proof. For each $t \in \operatorname{supp} f$ and $n \in \mathbf{N} = \{1, 2, \dots\}$, define

$$\varepsilon_n(t) = \sup\{\|y\|_X : \|\bar{f}(t) \pm y\|_X < 1 + 1/n\} \tag{2.1}$$

where $\bar{f}(t) = f(t)/\|f(t)\|_X$. Then $1/n \le \varepsilon_n(t) \le 2 + 1/n$ and $\varepsilon_n(t) \downarrow \varepsilon(t)$ for some $\varepsilon(t) \ge 0$. Clearly

$$\varepsilon(t) \equiv \lim_n \varepsilon_n(t) = 0 \Longleftrightarrow \bar{f}(t) \in \text{str-ext}\, B(X). \tag{2.2}$$

Hence, it suffices to show that $\varepsilon(t)$ is a measurable function. Since T is σ-finite, if we are able to prove that $\varepsilon(t)$ is measurable on each subset of T with finite measure, then it is measurable on the whole space T. Due to this argument, without loss of generality, we may assume that T itself has finite measure.

Choose simple functions $f_n(t) = \sum_i a_i^n \chi_{E^n}(t)$ such that $f_n \to f$ in $E(X)$. By Lemma 2.2, we may assume $\|\bar{f}(t) - f_n(t)\|_X \to 0$ μ almost everywhere on T (passing to a subsequence if necessary) and such that the partition $\{E_i^{n+1}\}_i$ of $\operatorname{supp} f$ is finer than $\{E_i^n\}_i$ for every $i \in \mathbf{N}$. Whence by our assumption $\mu T < \infty$, for each $k \in \mathbf{N}$, there exists $T_k \in \Sigma$ with $\mu T_k < 1/k$ such that $\|\bar{f}(t) - f_n(t)\|_X \to 0$ uniformly on $T \backslash T_k$. If we are able to prove that $\varepsilon(t)$ is measurable on each $T \backslash T_k$, then it is measurable on $\cup_{k=1}^{\infty}(T \backslash T_k)$, whence it is measurable on the whole T since $\mu T_k < 1/k \to 0$ and T is complete. Hence, by this argument, without loss of generality, we may assume

$$\|\bar{f}(t) - f_n(t)\|_X < \frac{1}{3n} \tag{2.3}$$

for all $t \in \operatorname{supp} f$. Then for any $t, s \in E_i^n$, since $f_n(t) = f_n(s)$, by (2.3),

$$\|\bar{f}(t) - \bar{f}(s)\|_X \le \|\bar{f}(t) - f_n(t)\|_X + \|f_n(s) - \bar{f}(s)\|_X < \frac{2}{3n}. \tag{2.4}$$

Pick arbitrarily $t_i^n \in E_i^n$; then, by (2.1), there exists $y_i^n \in X$ such that

$$\|y_i^n\|_X > \left(1 - \frac{1}{n}\right)\varepsilon_n(t_i^n); \qquad \|\bar{f}(t_i^n) \pm y_i^n\|_X < 1 + \frac{1}{n}. \tag{2.5}$$

Define

$$\varepsilon_n'(t) = \sum_i \|y_i^n\|_X \chi_{E_i^n}(t); \qquad g_n(t) = \sum_i y_i^n \chi_{E_i^n}(t). \tag{2.6}$$

Then, by (2.1) and (2.5),

$$\varepsilon_n(t_i^n) \geq \varepsilon'_n(t_i^n) = \|y_i^n\|_X \geq (1-1/n)\varepsilon_n(t_i^n). \tag{2.7}$$

For any $t \in \operatorname{supp} f$, $t \in E_i^{3n} \subseteq E_j^n$ for some $i, j \in \mathbf{N}$. By (2.1), we may find $y \in X$ such that

$$\|y\|_X > \left(1 - \frac{1}{n}\right)\varepsilon_{3n}(t); \qquad \|\bar{f}(t) \pm y\|_X < 1 + \frac{1}{3n}. \tag{2.8}$$

Then, by (2.8) and (2.4),

$$\begin{aligned} 1 + \frac{1}{3n} &> \|\bar{f}(t) \pm y\|_X \\ &\geq \|\bar{f}(t_i^{3n}) \pm y\|_X - \|\bar{f}(t_i^{3n}) - \bar{f}(t)\|_X \\ &\geq \|\bar{f}(t_i^{3n}) \pm y\|_X - \frac{2}{3n}, \end{aligned}$$

i.e.,

$$\|\bar{f}(t_i^{3n}) \pm y\|_X \leq 1 + 1/n.$$

Therefore, (2.8), (2.1), (2.7), (2.6) and $t \in E_i^{3n} \subseteq E_j^n$ imply

$$\begin{aligned} \left(1 - \frac{1}{n}\right)\varepsilon_{3n}(t) &< \|y\|_X \leq \varepsilon_n(t_i^{3n}) \\ &\leq \frac{n}{n-1}\varepsilon'_n(t_i^{3n}) = \frac{n}{n-1}\|y_j^n\|_X \\ &= \frac{n}{n-1}\varepsilon'_n(t). \end{aligned} \tag{2.9}$$

On the other hand, by (2.5) and (2.4) we have

$$\begin{aligned} \|\bar{f}(t) \pm y_i^{3n}\|_X &\leq \|\bar{f}(t) - \bar{f}(t_i^{3n})\|_X + \|\bar{f}(t_i^{3n}) \pm y_i^{3n}\|_X \\ &\leq \frac{2}{3n} + 1 + \frac{1}{3n} \\ &= 1 + \frac{1}{n}, \end{aligned}$$

which implies, by (2.1),

$$\varepsilon'_{3n}(t) = \|y_i^{3n}\|_X \leq \varepsilon_n(t). \tag{2.10}$$

(2.9) and (2.10) show that

$$\varepsilon(t) = \lim_n \varepsilon_n(t) = \lim \varepsilon'_n(t),$$

which is measurable since, by (2.6), each $\varepsilon'_n(t)$ is measurable.

Corollary 2.1. *If $f \in \text{str-ext}\, B(E(X))$, then for μ almost everywhere $t \in \text{supp}\, f$, $f(t)/\|f(t)\|_X \in \text{str-ext}\, B(X)$.*

Theorem 2.2. *Let $E(X)$ be a Köthe-Bochner space, $f \in S(E(X))$.*

(i) *If $f \in \text{str-ext}\, B(E(X))$, then*

(a) *for μ almost everywhere $t \in \text{supp}\, f$, $f(t)/\|f(t)\|_X \in \text{str-ext}\, B(X)$;*

(b) $\|f(\cdot)\|_X \in \text{str-ext}\, B(E)$.

(ii) *If E is order continuous, then* (a) *and* (b) *in* (i) *imply* $f \in \text{str-ext}\, B(E(X))$.

Proof. (i) (a) follows immediately from Corollary 2.1.

(b) Suppose $\varphi_n \in E$ satisfying

$$\| \|f(\cdot)\|_X \pm \varphi_n(\cdot)\|_E \longrightarrow 1.$$

Define

$$g_n(t) = \begin{cases} \varphi_n(t) f(t)/\|f(t)\|_X & t \in \text{supp}\, f, \\ \varphi_n(t) e & \text{otherwise}, \end{cases}$$

where $e \in X$ and $\|e\|_X = 1$. Then $g_n \in E(X)$ and

$$\begin{aligned} \|f \pm g_n\|_{E(X)} &= \| \|f(t) \pm g_n(t)\|_X \|_E \\ &= \| \|f(t)\|_X \pm \varphi_n(t)\|_E \longrightarrow 1. \end{aligned}$$

Since $f \in \text{str-ext}\, B(E(X))$, we have $\|\varphi_n(\cdot)\|_E = \|g_n\|_{E(X)} \to 0$. This shows that $\|f(\cdot)\|_X \in \text{str-ext}\, B(E)$.

(ii) Assume that E is order continuous. Let $f \in S(E(X))$ satisfy (a) and (b) in (i). For any $g_n \in E(X)$ satisfying

$$\|f \pm g_n\|_{E(X)} \longrightarrow 1,$$

we define

$$\|\varphi_n(t) = \|f(t) + g_n(t)\|_X - \|f(t)\|_X.$$

Then

$$\| \|f(t)\|_X + \varphi_n(t)\|_E = \|f + g_n\|_{E(X)} \longrightarrow 1 \tag{2.11}$$

and

$$\begin{aligned} \| \|f(t)\|_X - \varphi_n(t)\|_E &= \| \|2f(t)\|_X \\ &\qquad - \|f(t) + g_n(t)\|_X\|_E \\ &\le \| \|f(t) - g_n(t)\|_X\|_E \\ &= \|f - g_n\|_{E(X)} \longrightarrow 1. \end{aligned} \tag{2.12}$$

By (2.13), (2.14) and (b), we deduce that $\varphi_n \to 0$ in E, i.e., $\|f(\cdot) + g_n(\cdot)\|_X \to \|f(\cdot)\|_X$ in E. Similarly, we can prove $\|f(\cdot) - g_n(\cdot)\|_X \to \|f(\cdot)\|_X$ in E.

Since $\{g_n\}$ is arbitrarily given, by Lemma 2.2, passing to a subsequence if necessary, we may assume

$$\|f(t) \pm g_n(t)\|_X \longrightarrow \|f(t)\|_X \tag{2.13}$$

for μ almost everywhere $t \in \operatorname{supp} f$. Therefore, for μ almost everywhere $t \in \operatorname{supp} f$, we have

$$\left\| \frac{f(t)}{\|f(t)\|_X} \pm \frac{g_n(t)}{\|f(t)\|_X} \right\|_X \longrightarrow 1.$$

By condition (a), for μ almost everywhere, $t \in \operatorname{supp} f$, $g_n(t) \to 0$ in X. But this is also true for μ almost everywhere $t \in T \backslash (\operatorname{supp} f)$ according to (2.15). Thus, $g_n(t) \to 0$ for μ almost everywhere $t \in T$. Since

$$\|2g_n(\cdot)\|_X \le \|g_n(\cdot) - f(\cdot)\|_X + \|g_n(\cdot) + f(\cdot)\|_X \longrightarrow 2\|f(\cdot)\|_X,$$

by Lemma 2.1, we find that $g_n \to 0$ in $E(X)$ and, thus, $f \in$ str-ext $B(E(X))$.

Corollary 2.2. *If E is a nonorder continuous Orlicz space, then $B(E(X))$ has no strongly extreme points.*

Proof. By [**2**], $B(E)$ has no strongly extreme points if E is not order continuous. Therefore, $B(E(X))$ has no strongly extreme points by Theorem 2.2.

Remark. C. Castaing and R. Pluciennik proved in [**1**] that

(a) For a given locally uniformly rotund Köthe function space E over a measure space (T, Σ, μ) and a Banach space X, if $f \in S(E(X))$ is such that $f(t)/\|f(t)\|$ is a denting point of $B(X)$ for μ almost everywhere $t \in \operatorname{supp} f$, then f is a denting point of $B(E(X))$.

(b) If, in addition, X is separable, then for each denting point f of $B(E(X))$, $f(t)/\|f(t)\|$ is a denting point of $B(X)$ for μ almost everywhere $t \in \operatorname{supp} f$.

Then they asked a question about whether the above two results still hold without requiring that E be locally uniformly rotund. From our Corollary 2.2, if E is a nonorder continuous Orlicz space, then $B(E(X))$ has no strongly extreme points and, of course, it has no denting points. This answers their question negatively.

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