On structural properties of the value function for an unbounded jump Markov process with an application to a processor sharing retrial queue

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Abstract

The derivation of structural properties for unbounded jump Markov processes cannot be done using standard mathematical tools, since the analysis is hindered due to the fact that the system is not uniformizable. We present a promising technique, a smoothed rate truncation method, to overcome the limitations of standard techniques and allow for the derivation of structural properties. We introduce this technique by application to a processor sharing queue with impatient customers that can retry if they renege. We are interested in structural properties of the value function of the system as a function of the arrival rate.

Keywords: monotonicity; processor-sharing queue; retrial queue; successive approximation.

1 Introduction

In practical applications of Markovian control problems, it is desirable that optimal strategies are well-behaved as a function of the input parameters of the problem. This facilitates computation or approximation of optimal policies. In order to check whether the model has such desirable properties, one needs the relative value function for average or discounted optimal costs to have structural properties such as monotonicity, convexity, and supermodularity (cf. [5]). A general method for checking such properties is the successive approximations method for discrete-time Markov decision chains.

When the original problem can be modeled as a continuous-time Markov decision process, application of the successive approximations method requires that the Markov decision process be uniformizable. In other words, the jump rates must be uniformly bounded as a function of action and state. In modern applications of Markov decision processes to call and contact center problems, however, the jump rates are generally unbounded functions of actions and states. Successive

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approximations therefore is not applicable, even though the naive approach of simply applying the successive approximations operator without uniformization would preserve the required structural properties. No general method has been developed so far to attack this problem. Model-dependent techniques that have been used are, for instance, sample path based interchange arguments in a competing queues setting or finite state space truncation (see, e.g., [2]). The latter requires an ad hoc choice of the transitions on the boundary of the state space in order to keep the structure that is hidden in the original model, intact. It is not clear whether such an ad hoc choice can always be made.

We develop a novel method for attacking this problem and illustrate it by applying it to a processor sharing retrial queuing model. The proposed method is called the Smoothed Rate Truncation method (abbreviated: SRT, cf. Section 3). This method uses linearly smoothed transition rates, such that the smoothed system has a finite recurrent class, and such that it retains the structural properties of the infinite system. Linear smoothing has the advantage of destroying neither convexity nor concavity, provided the smoothing occurs already at low states. In other words, the rates that would take the system to a state with more customers in the system have to be smoothed in such a way that they become linear decreasing functions of the state variable. The SRT method seems to be a flexible method that we expect to be widely applicable.

We then invoke a limiting argument to show that the relative value function for the non-smoothed model has the same desired structural properties as well. The limiting argument relies on continuity properties of a parametrized collection of Markov processes studied in [1] under the condition of uniform recurrence.

The paper is organized as follows. In the next section, we present the model to be studied. In Section 3 we set up the smoothed rate truncation method. In Section 4 we apply the smoothed rate truncation method to a processor sharing retrial queue.

2 Model description

Consider a service facility Q_1 at which customers arrive according to a Poisson process with rate λ. Customers at facility Q_1 are served according to a processor sharing discipline in which potential inter-departure times are i.i.d. exponential with mean $1/\mu$. However, customers are impatient in the sense that they renege after an exponentially distributed period with mean $1/\beta$ independently of all other customers in Q_1, whilst waiting to complete service. A customer that reneges will either abandon the system with probability $1-\psi$ or go to a retrial queue Q_2 with probability ψ. Each customer in the retrial queue independently attempts to rejoin the tail of facility Q_1 after an exponentially distributed period with mean $1/\gamma$. The retrial node will be modeled by a $\cdot/M/\infty$ queue. The inter-arrival times, potential inter-departure times, times spent in Q_1 before reneging, times spent in Q_2 before retrial, and random variables governing exits immediately after reneging constitute mutually independent sequences.

A customer is said to be using the joining rule $[s]$, for $s \in [0,1]$, if upon arrival at the system, the customer joins facility Q_1 with probability s, and balks with probability $1-s$, independently of all other customers. Note that this is equivalent to studying a system in which the arrival rate is bounded in interval $(0,\lambda)$. In the sequel we will explicitly index all variables by s to denote this dependence on the joining rule.

We can model the system in a Markov theoretic framework. For this purpose, let $(X(s),Y(s)) = \{X_t(s),Y_t(s)\}_{t\geq 0}$ be the Markov process associated with the number of customers in facilities Q_1 and Q_2, for joining rule s, Let $\mathcal{X} = \mathbb{N}_0 \times \mathbb{N}_0$ denote the state space of the system, in other words, $(X_t(s),Y_t(s)) = (x,y) \in \mathcal{X}$, if at time t there are x customers in facility Q_1 and y customers in the retrial queue Q_2.

We denote the corresponding transition rate from (x,y) to (x',y') by $q_{xy,x'y'}(s)$. Then for $(x,y),(x',y') \neq (x,y) \in \mathbb{N}_0 \times \mathbb{N}_0$ we have

$$q_{xy,x'y'}(s) = \begin{cases} \lambda s, & \text{if } (x',y') = (x+1,y) \\ \mu \mathbb{1}_{\{x>0\}} + \beta x(1-\psi), & \text{if } (x',y') = (x-1,y) \\ \beta x \psi, & \text{if } (x',y') = (x-1,y+1) \\ \gamma y, & \text{if } (x',y') = (x+1,y-1) \\ 0, & \text{otherwise} \end{cases}$$

corresponding to arrivals, service departures and reneging customers that abandon, reneging customers that go to the retrial queue, and customers leaving the retrial queue, respectively.

For each s, $(X(s),Y(s))$ is an irreducible, non-explosive, conservative, standard Markov process. We assume that it has right-continuous paths. The associated transition function is denoted by the matrix function $P_t(s) = \big(P_{t,x,y}(s)\big)_{x,y}$. Due to abandonments, it is an ergodic Markov process with stationary distribution $\pi(s)$, say, with $\pi_{xy}(s)$ denoting the stationary probability of state (x,y), provided that $\psi < 1$ or $\psi = 1$ and $\lambda < \mu$ (cf. Lemma 3.2).

We assume that the system is subject to linear rate costs in each state (x,y), given by

$$d(x,y) = d_1 x + d_2 y,$$

for some constants $d_1 \geq 0$ and $d_2 \geq 0$, independent of s. As has been mentioned in the introduction, the standard way of obtaining monotonicity properties is to uniformize the system and then apply a successive approximation scheme based on the Poisson equation for the relative value function V_s. Define $g_s = \pi(s)d = \sum_{(x,y)} \pi_{xy}(s)d(x,y)$ the expected average cost under the stationary distribution $\pi(s)$ associated with joining rule s. Then,

$$V_s(x,y) = \int_0^\infty \big(P_{t,x,y}(s)d(x,y) - g_s\big)dt.$$

The desired monotonicity and convexity properties then follow from monotonicity and convexity properties of the relative value function, given a fixed joining rule s.

There are however two problems in the present case. First, due to customers reneging, the rates are unbounded as a function of state and so uniformization is not possible. Secondly, considering

V_s as a function of s, properties of the finite horizon value function do not necessarily carry over to the properties V_s, since the constant being subtracted, g_s, is a function of s.

Before proceeding, let us first specify and define the structural properties that we are interested in. Let $f : \mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{R}$. Then f is

- *Convex(x)* if $f(x+1, y) - f(x, y) \geq f(x, y) - f(x-1, y)$ for $x, y \in \mathbb{N}_0$, $x \geq 1$;
- *Convex(y)* if $f(x, y+1) - f(x, y) \geq f(x, y) - f(x, y-1)$ for $x, y \in \mathbb{N}_0$, $y \geq 1$;
- *Supermodular(x,y)* if $f(x+1, y+1) + f(x, y) \geq f(x, y+1) + f(x+1, y)$ for $x, y \in \mathbb{N}_0$.

We are also interested in structural properties of the relative value function as a function of s. Let $f : [0, 1] \times \mathbb{N}_0 \times \mathbb{N}_0 \mapsto \mathbb{R}$. Then f is

- *Convex(s)* if $f(s+\Delta, x, y) - f(s, x, y) \geq f(s, x, y) - f(s-\Delta, x, y)$ for $x, y \in \mathbb{N}_0$;
- *Supermodular(s,x)* if $f(s+\Delta, x+1, y) + f(s, x, y) \geq f(s+\Delta, x, y) + f(s, x+1, y)$ for $x, y \in \mathbb{N}_0$;
- *Supermodular(s,y)* if $f(s+\Delta, x, y+1) + f(s, x, y) \geq f(s+\Delta, x, y) + f(s, x, y+1)$ for $x, y \in \mathbb{N}_0$.

Our main theorem is the following. Assume that $\psi < 1$ or that $\psi = 1$ and $\lambda < \mu$.

Theorem 2.1:

1. V_s is non-decreasing in x and y. Additionally, it is Convex(x), Convex(y), and Supermodular(x, y) for each s;
2. V_s is Supermodular(s, x), and Supermodular(s, y).

This theorem has the following consequences.

Corollary 2.2: g_s is monotone non-decreasing and convex in s.

3 Smoothed Rate Truncation Method

In this section we introduce a novel truncation method. As we have discussed in the introduction, in general, finite state truncations that preserve the jump structure imply that the rates become concave as a function of state. This conflicts when one want to show convexity results for the relative value function.

To resolve this problem, we introduce a smoothed rate truncation method. The main idea in this method is to approximate the Markov decision process by (essentially) finite state Markov decision chains. This approximating Markov decision chain is obtained by *linearly decreasing* the relevant transition rates with *increasing state variable* until these rates equal zero, hence the name *Smoothed Rate Truncation*. This smoothing provides a natural finite state recurrent class. As explained in the introduction, the local decrease should be small enough so as not to destroy the relevant properties of the model to be studied.

In our processor sharing retrial queue, the smoothed rate truncation method yields a parameterized collection of approximating Markov chains. For each N let $b(N)$ be such that $\lim_{N\to\infty} b(N) = \infty$. Set $\mathsf{A} = \{a = (u,s) \,|\, u \in \{0\} \cup \{1/N\}_{N=1,\dots}\}$. Clearly A is a compact set in the topology inherited from $\mathbb{R}^2$. We define a parametrized collection $(X(a), Y(a))$, $a \in \mathsf{A}$ by specifying their rate matrix. For $a = (0,s)$ we take $Q(0,s)$ to be the rate matrix defined earlier. In other words $(X(0,s), Y(0,s))$ is the original processor sharing retrial queue with joining rule s.

For $a = (1/N, s)$, put $l = \lambda s/N$, $k = \gamma/N$, $p = \psi\beta/b(N)$ and let for $(x,y) \in \mathcal{X}$

$$\begin{aligned}
\lambda^N_{x,y}(s) &= \max\{0, \lambda s - xl\};\\
\gamma^N_{x,y} &= \max\{0, \gamma - xk\};\\
(\beta\psi)^N_{x,y} &= \max\{0, \psi\beta - yp\}.
\end{aligned}$$

The rate matrix $Q(1/N, s)$ of the process $(X(1/N,s), Y(1/N,s))$ is defined by

$$\begin{aligned}
q_{xy,x'y'}(1/N,s) &= \lambda^N_{x,y}(s)\cdot \mathbb{1}_{\{(x+1,y)\}}(x',y') + \gamma^N_{x,y} y \cdot \mathbb{1}_{\{(x+1,y-1)\}}(x',y')\\
&\quad + (\beta\psi)^N_{x,y} x \cdot \mathbb{1}_{\{(x-1,y+1)\}}(x',y')\\
&\quad + (\mu + \beta(1-\psi)x)\cdot \mathbb{1}_{\{(x-1,y)\}}(x',y')\mathbb{1}_{\{1,\dots,N\}}(x)\mathbb{1}_{\{0,\dots,b(N)\}}(y)\\
&\quad + \xi \mathbb{1}_{\{(x,y-1)\}}(x',y')\cdot \max\{\mathbb{1}_{\{N+1,\dots\}}(x), \mathbb{1}_{\{b(N)+1,\dots\}}(y)\},
\end{aligned}$$

for $(x',y') \neq (x,y)$, where ξ will be chosen later on. The rates $q_{xy,xy}(1/N,s)$ make the row sums of $Q(1/N,s)$ equal to 0.

These down rates ξ are chosen so as to ensure that there is one closed class, and that this class is reached with probability 1 from any state outside the class. The irreducible class of states in the chain is precisely the set $\{(x,y) \in \mathcal{X} \,|\, x \le N, y \le b(N)\}$, and so essentially this is a finite state Markov chain. Note that we only smooth the rates that increase one of the components of the state variable.

The smoothing used here suffices for showing the desired convexity and supermodularity of the value function given a fixed joining rule s. However, for studying the same properties of the value function as a function of the joining rule, we need to refine the smoothing (cf. Section 4).

This method can only work when the relevant performance measures of the Markov decision process can be approximated by the corresponding performance measures of smoothed truncated Markov decision chains. We will recall the results [1] guaranteeing the validity of such approach.

To this end, let us consider a collection of parametrized countable state Markov processes, $X(a) = \{X_t(a)\}_t$, where a is a parameter from a compact parameter set A. We assume that each process is minimal and conservative with rate matrix $Q(a)$. Moreover, each process $X(a)$ has at most one closed class, plus possibly inessential states.

The function $f : E \to \mathbb{R}_+$ is said to be a *moment function* if there exists an increasing sequence of finite sets $E_n \uparrow E$, $n \to \infty$, such that $\inf\{f(x) \,|\, x \notin E_n\} \to \infty$ as $n \to \infty$. The following theorem follows from [1] Theorem 5.3, Corollary 6.5, Theorem 6.6 and [10] Lemma 2.3.

Theorem 3.1: We assume the following conditions to hold.

(i) $\{X(a)\}_{a\in\mathsf{A}}$ is f-exponentially recurrent for a moment function f. In other words, there exists a moment function $f: E \to \mathbb{R}_+$, constants $c, d > 0$, and a finite set K with

$$\sum q_{xz}(a)f(z) \le -cf(x) + d\mathbb{1}_{\{K\}}(x), \quad x \in E, a \in \mathsf{A}. \tag{1}$$

(ii) There exists a function $g: E \to \mathbb{R}_+$ and a constant c' such that

$$\sum q_{xz}(a)g(z) \le c'g(x), \quad x \in E, a \in \mathsf{A}, \tag{2}$$

with the property that there exists an increasing sequence of finite sets $E_n \uparrow E$, such that $\inf_{x\notin E_n} g(x)/f(x) \to \infty$ as $n \to \infty$.

(iii) Continuity: $a \mapsto q_{xz}(a)$, $a \mapsto \sum_z q_{xz}(a)f(z)$ are continuous functions for each $x, z \in E$.

Then the following properties hold.

(1) $\{X(a)\}_a$ is f-exponentially ergodic, i.e., there exist constants $\alpha, \kappa > 0$ such that

$$\sum_z \mid P_{t,xz}(a) - \pi(a) \mid f(z) \le \kappa e^{-\alpha t} f(x), \quad t \ge 0, a \in \mathsf{A},$$

where $\pi(a) = (\pi_x(a))_{x\in\mathcal{X}}$ is the unique stationary distribution of $X(a)$.

(2) Let $c(a) = (c(x,a))_{x\in E}$, $a \in \mathsf{A}$, be a cost function with $a \mapsto c(x,a)$ continuous for each $x \in E$. If $\sup_x |\sup_a c(x,a)|/f(x) < \infty$, then $a \mapsto g(a) = \pi(a)c(a) = \sum_x \pi_x(a)c(x,a)$ continuous and $a \mapsto V(a) = \int_0^\infty P_t(a)(c(a) - g(a))dt$ is componentwise continuous.

f-exponential recurrence is well-studied in the literature. For instance, in [4] f-exponential recurrence and f-exponential ergodicity are shown to be equivalent in the case of one Markov process ($\#\mathsf{A} = 1$, i.e., the cardinality of the set A is one). If the parameter space is such that $\{Q(a)\}_{a\in\mathsf{A}}$ has the product property, f-exponential recurrence together with f-exponential ergodicity are used for studying average cost and discounted cost optimal strategies in a Markov decision process (cf. [3]; see also [1] for a discussion concerning the relation between these properties).

We will next show that our parametrized collection is f-exponentially recurrent for a function f that grows exponentially in the state variable, provided that either $\psi < 1$ or $\psi = 1$ and $\lambda < \mu$. In case $\psi = 1$ and $\lambda \ge \mu$, the retrial queue is not positive recurrent, and so this case is not of our interest.

Lemma 3.2: $(X(0,1), Y(0,1))$ is not positive recurrent whenever $\psi = 1$ and $\lambda \ge \mu$.

Proof: Consider the process $Z = (X(0,1) + Y(0,1))$ counting the total number of customers in the system. Compare it with an M/M/1-queue with arrival rate λ and service rate μ. Start with two empty systems. It is not hard to see one can couple both systems in such a way that the state of the Z-process is always greater than or equal to the state of the corresponding M/M/1-queue. Since the latter is not positive recurrent, the expected return time to 0 is infinitely large. Hence this is true for Z. □

Theorem 3.3: Suppose that either $\psi < 1$ or $\psi = 1$ and $\lambda < \mu$. Then there exist $\epsilon > 0$ and $\xi > 0$, such that the above defined parametrized collection of Markov processes $(X(a), Y(a))$, $a \in \mathsf{A}$, is f-exponentially recurrent for the function $f : E \to \mathbb{R}_+$ given by

$$f(x, y) = \exp\{\epsilon x + \theta\epsilon y\}, \tag{3}$$

provided that $\theta \in (1, \psi^{-1})$, $b(N) \le N\psi\beta/\gamma$, if $\psi < 1$, and $\theta = 1$ if $\psi = 1$ and $\lambda < \mu$.

Additionally, for each choice of ϵ and θ satisfying the above restrictions, there exist $\epsilon' > \epsilon$ and a constant c' such that the function $g : E \to \mathbb{R}_+$ defined by

$$g(x, y) = \exp\{\epsilon' x + \theta\epsilon' y\}, \tag{4}$$

satisfies $\sum_y q_{xy,x'y'}(a) g(x', y') \le c' g(x, y)$, $a \in \mathsf{A}$, $(x, y) \in E$.

In particular, conditions (i,ii, iii) of Theorem 3.1 are satisfied.

Proof: Continuity of $a \mapsto q_{xy,x'y'}(a)$ holds by construction. Since the jump sizes are bounded this implies $a \mapsto \sum_{x'y'} q_{xy,x'y'}(a) f(x', y')$.

It therefore suffices to show that there exist $\epsilon > 0$, a finite set K, and positive constants c, d, such that $Q(u, s)$, $u = 0, 1/N$, $N = 1, \ldots$, satisfies (1) for the function f from (3). In other words, we need to show the existence of $\epsilon > 0$, a finite set K and a constant $c > 0$, such that

$$\sum_{x',y'} q_{xy,x'y'}(u, s) f(x', y')/f(x, y) \le -c, \quad (x, y) \notin K. \tag{5}$$

For $u = 0$, this reduces to finding $\epsilon > 0$, a finite set K and a constant $c > 0$, such that

$$\begin{aligned}&\lambda s(e^{\epsilon} - 1) + \gamma y(e^{\epsilon - \theta\epsilon} - 1)\mathbb{1}_{\{y>0\}} \\ &\quad + \beta\psi x(e^{-\epsilon+\theta\epsilon} - 1)\mathbb{1}_{\{x>0\}} + (\mu + \beta(1 - \psi)x)(e^{-\epsilon} - 1)\mathbb{1}_{\{x>0\}} \le -c, \quad (x, y) \notin K.\end{aligned} \tag{6}$$

Rewriting yields

$$\begin{aligned}&\lambda s(e^{\epsilon} - 1) + \gamma y(e^{\epsilon-\theta\epsilon} - 1)\mathbb{1}_{\{y>0\}} \\ &\quad + \mu(e^{-\epsilon} - 1)\mathbb{1}_{\{x>0\}} + \beta x\big(\psi e^{-\epsilon}(e^{\theta\epsilon} - 1) + (e^{-\epsilon} - 1)\big)\mathbb{1}_{\{x>0\}} \le -c.\end{aligned} \tag{7}$$

First we assume that $\psi < 1$. If the coefficients of both x and y in the above expression are both negative, then clearly a finite set K and constant c can be found for (7). The coefficient of y is negative provided $\theta > 1$. The coefficient of x equals 0 if $\epsilon = 0$. The derivative to ϵ at $\epsilon = 0$ equals

$$\theta\psi - 1,$$

which is negative if $\theta < 1/\psi$. This settles the case for all Markov processes $(X(0, s), Y(0, s))$. Next we turn to the smooth approximations $(X(1/N, s), Y(1/N, s))$. To this end we compute the difference of the resulting expression and the left-hand side of (6). This yields for $0 \le x \le N, 0 \le y \le b(N)$

$$-xl(e^{\epsilon} - 1) - kxy(e^{\epsilon-\theta\epsilon} - 1) - pxy(e^{-\epsilon+\theta\epsilon} - 1).$$

A sufficient condition for this to be non-negative is

$$k(e^{\epsilon-\theta\epsilon}-1)+p(e^{-\epsilon+\theta\epsilon}-1)\geq 0. \tag{8}$$

By a similar reasoning as before, we find that the derivative of the left-hand side above is non-negative at $\epsilon = 0$ if

$$k(1-\theta)+p(\theta-1)\geq 0.$$

In other words, if $p \geq k$, or

$$\frac{\psi\beta}{b(N)}\geq\frac{\gamma}{N},$$

that is $N \geq \gamma b(N)/\psi\beta$.

If $x > N$ and/or $y > b(N)$, then it is easily checked that one can choose ξ large enough so that (5) holds for these values of x and y and all parameters. It is easily checked that ϵ can be increased without violating (7) and (8). This yields existence of the function g with the desired properties.

In case that $\psi = 1$ and $\lambda < \mu$ we take $\theta = 1$. The rest is standard (cf. [4]). □

Corollary 3.4: For $V(a)$ the value function associated with $(X(a), Y(a))$, we have that

$$V(1/N,s)\to V(0,s),\quad N\to\infty,$$

componentwise.

Note: The constructed parametrized models are all stochastically monotonic (cf. [7]). As in [7] one can construct a function $\hat{f}$ for which this collection is $\hat{f}$-exponentially recurrent with set $K = \{0\}$. The results in [7] then imply that the $\hat{f}$-exponential ergodicity property holds with rate constant $\alpha = c$. In other words, we can explicitly compute a bound on the rate of convergence to stationarity. This means that

$$\sup_{x,y}\frac{1}{\hat{f}(x,y)}\big|V_{x,y}(a)-\int_0^T\Big(\mathbb{E}_{x,y}d(X_t(a),Y_t(a))-\mathbb{E}_{\pi(a)}d(X_t(a),Y_t(a))\Big)dt\big|\leq K\frac{e^{-cT}}{c},$$

and so it suffices to approximate the total expected cost over a finite time interval to get an approximation of the relative value function.

The following step to prove is that $V_{1/N,s}$ have the desired structural properties. This will be the subject of the next section.

4 Structural results for processor sharing retrial queue

For each N fixed, the associated smoothed rate truncation $(X(1/N,s), Y(/N,s))$ has the closed recurrent class of states

$$\mathcal{X}^N=\{0,\ldots,N\}\times\{0,\ldots,b(N)\}.$$

It is sufficient to restrict the state space to $\mathcal{X}^N$.

In this section we will provide the proof of Theorem 2.1 by using value iteration applied to approximating processes. For the proof of the first part we can use the approximations discussed in the previous section. The proof of the second part concerning the behavior of the value function as a function of the joining rule requires a more refined approximation technique. This will be discussed later on.

First we prove the following theorem.

Theorem 4.1: $V_{1/N,s}$ is non-decreasing in x and y, and, if $k_N := \gamma/N = \psi\beta/b(N) =: p_N$, it is Convex($x$), Convex($y$) and Supermodular($x,y$) on $\mathcal{X}^N$, for any $N \geq 1$.

We will show that Theorem 4.1 implies the validity of the first part of Theorem 2.1.

Proof of Theorem 2.1 (1): : Theorem 4.1 requires that $k_N = p_N$ for Convex(x), Convex(y) and
Supermodular(x,y) to hold. In turn, this requires that $b(N) = N\psi\beta/\gamma$ be integer for at least an infinite sequence of values of N. To solve this, we slightly perturb γ.

To this end, put $b(N) = \lfloor N\psi\beta/\gamma \rfloor$. Then $k_N = \gamma/N < p_N = \psi\beta/b(N)$. Next choose the perturbed parameter $\gamma(N) = Np_N$. Then

$$\gamma < \gamma(N) \leq \gamma \cdot \frac{\psi\beta}{\psi\beta - (\gamma/N)}.$$

Moreover, $\gamma(N)/N = \psi\beta/b(N)$, so that the condition of Theorem 4.1 is satisfied.

Consider the parametrized collection $(\hat{X}(u,s), \hat{Y}(u,s))$, $u \in \{0\} \cup \{1/N\}_{N \geq N_0}$, where $\hat{X}(0,s) = X(0,s)$, $\hat{Y}(0,s) = Y(0,s)$, and $(\hat{X}(1/N,s), \hat{Y}(1/N,s))$ is the SRT, with rate $\gamma(N)$ instead of γ, and closed set of states $\mathcal{X}^N$ with $b(N)$ specified as above. By inspection of the proof of Theorem 3.3, it follows that this collection satisfies the conditions of Theorem 3.1.

The result follows directly taking the limit $N \to \infty$. □

For the proof of Theorem 4.1 we fix N. We will use a standard successive approximations argument, applied to the uniformized chain with uniformization constant $\eta = \lambda + \mu + \beta b(N) + \gamma N$. One then applies successive approximations to the discrete time chain, denoted by $(X^d(1/N,s), Y^d(1/N,s))$ with transition probabilities

$$P_{xy,x'y'}(1/N,s) = \delta_{xy,x'y'} + \frac{1}{\eta} q_{xy,x'y'}(1/N,s), \quad (x,y),(x',y') \in \mathcal{X}^N.$$

Scaling the cost rates also by η, the relative value function for the associated discrete time model equals the continuous relative value function (cf. Lippman [6], Serfozo [9]). Now, rescaling the rates by a factor η and denoting these again by s, β, γ and μ, we have to slightly modify the cost function

$$d^\eta(x,y) = \frac{d_1 x + d_2 y}{\eta}.$$

Clearly, multiplying by a constant does not affect the structural properties of the relative value functions for N fixed. Hence we may assume that $\eta = 1$.

Let $V_s^0 \equiv 0$ on $\mathcal{X}^N$. The successive approximations scheme applied to discrete time approximation yields

$$\begin{aligned} V_s^{n+1}(x,y) &= \lambda_{x,y}^N(s)V_s^n(x+1,y) + \mathbb{1}_{\{x>0\}}\mu V_s^n(x-1,y) + \mathbb{1}_{\{y>0\}}\gamma_{x,y}^N V_s^n(x+1,y-1) \\ &\quad +\mathbb{1}_{\{x>0\}}\beta x(1-\psi)V_s^n(x-1,y) + \mathbb{1}_{\{x>0\}}(\beta\psi)_{x,y}^N x V_s^n(x-1,y+1) \\ &\quad +(1-\lambda_{xy}^N(s) - \mu\mathbb{1}_{\{x>0\}} - (\beta - yp)x - \gamma_{xy}^N y)V_s^n(x,y) + (d_1 x + d_2 y), \end{aligned}$$

where $V_s^n(x',y') = 0$ for $(x',y') \notin \mathcal{X}^N$. By virtue of (Lippman [6], Serfozo [9]), $V_s^n(x,y) - V_s^n(0,0) \to V_{1/N,s}(x,y) - V_{1/N,s}(0,0)$, as $n \to \infty$.

For proving the induction step, a general principle is to take out the smallest common factor in front of difference terms, or the factor in front of the largest term.

Theorem 4.2: $V_{1/N,s}^n$ and the relative value function $V_{1/N,s}$ are non-decreasing function in both variables x, y on $\mathcal{X}^N$.

Proof: Here, as in the proofs of the next theorems, we suppress the dependence on the parameter N in our notation.

Since $V_s^0 \equiv 0$, this function is clearly non-decreasing in both x and y. Therefore, assume that V_s^n is non-decreasing in x and y. We first show that V_s^{n+1} is non-decreasing in x. To this end, define $V_s^n(x,y) = V_s^n(0,y)$ for $x < 0$ and $V_s^n(x,y) = V_s^n(x,0)$ for $y < 0$. Then, consider $0 \le x < N-1$ and $0 \le y < b(N)$. Then,

$$\begin{aligned} & V_s^{n+1}(x+1,y) - V_s^{n+1}(x,y) = d_1[(x+1) - x] \\ &\qquad +(\lambda s - (x+1)l)\, V_s^n(x+2,y) - (\lambda s - xl)V_s^n(x+1,y) \\ &\qquad +\mu\,[V_s^n(x,y) - V_s^n(x-1,y)] \\ &\qquad +(\gamma - (x+1)k)y\, V_s^n(x+2,y-1) - (\gamma - xk)y\, V_s^n(x+1,y-1) \\ &\qquad +(x+1)\beta(1-\psi)\, V_s^n(x,y) - x\beta(1-\psi)\, V_s^n(x-1,y) \\ &\qquad +(x+1)(\beta\psi - yp)\, V_s^n(x,y+1) - x(\beta\psi - yp)\, V_s^n(x-1,y+1) \\ &\qquad +(1 - (\lambda s - (x+1)l) - \mu - (\beta - yp)(x+1) - (\gamma - (x+1)k)y)\, V_s^n(x+1,y) \\ &\qquad -(1 - (\lambda s - xl) - \mu - (\beta - yp)x - (\gamma - xk)y)\, V_s^n(x,y) \\ \ge\;& (\lambda s - (x+1)l)\,[V_s^n(x+2,y) - V_s^n(x+1,y)] \\ &\qquad +(\gamma - (x+1)k)y[V_s^n(x+2,y-1) - V_s^n(x+1,y-1)] \\ &\qquad +\beta x(1-\psi)\,[V_s^n(x,y) - V_s^n(x-1,y)] + \beta(1-\psi)\, V_s^n(x,y) \\ &\qquad +x(\beta\psi - yp)\,[V_s^n(x,y+1) - V_s^n(x-1,y+1)] + (\beta\psi - yp)\, V_s^n(x,y+1) \\ &\qquad +(1 - (\lambda s - xl) - \mu - (\beta - yp)(x+1) - y(\gamma - xk))\,[V_s^n(x+1,y) - V_s^n(x,y)] \\ &\qquad\qquad -(\beta - yp)\, V_s^n(x,y) \\ \ge\;& d_1 + (\beta\psi - yp)\,[V_s^n(x,y+1) - V_s^n(x,y)] \\ \ge\;& 0. \end{aligned}$$

For the first inequality we interchange the terms $(\lambda s - xl)V_s^n(x+1,y)$ and $(\lambda s-(x+1)l)V_s^n(x+1,y)$; we use non-decreasingness in x, y for the first and second inequalities.

At the boundary $x = N-1$ and $y = b(N)$ the induction step follows in the same way as the above, since naturally the transition rates leading outside $\mathcal{X}^N$ are 0 and so one may define V_s^n outside $\mathcal{X}^N$ as one likes.

We now continue the proof by showing that V_s^{n+1} is increasing in y. To this end, consider $0 \le x < N$ and $0 \le y < b(N)-1$. Then,

$$
\begin{aligned}
& V_s^{n+1}(x,y+1) - V_s^{n+1}(x,y) \\
&= d_2[(y+1)-y] + (\lambda s - xl)[V_s^n(x+1,y+1) - V_s^n(x+1,y)] \\
&\quad +\mu\,[V_s^n(x-1,y+1) - V_s^n(x-1,y)] \\
&\quad +\beta x(1-\psi)\,V_s^n(x-1,y+1) - \beta x(1-\psi)V_s^n(x-1,y) \\
&\quad +x(\beta\psi-(y+1)p)\,V_s^n(x-1,y+2) - x(\beta\psi - yp)\,V_s^n(x-1,y+1) \\
&\quad +(y+1)(\gamma - xk)\,V_s^n(x+1,y) - y(\gamma-xk)\,V_s^n(x+1,y-1) \\
&\quad +(1-(\lambda s - xl) - \mu - (\beta-(y+1)p)x - (y+1)(\gamma-xk))\,V_s^n(x,y+1) \\
&\quad -(1-(\lambda s - xl) - \mu - (\beta - yp)x - y(\gamma - xk))\,V_s^n(x,y) \\
&\ge y(\gamma-xk)\,[V_s^n(x+1,y) - V_s^n(x+1,y-1)] + (\gamma-xk)\,(V_s^n(x+1,y) - V_s(x,y) \\
&\quad +px\,(V_s^n(x,y+1) - V_s^n(x-1,y+1)) \\
&\quad +(1-(\lambda s - xl) - \mu - (\beta - yp)x - (y+1)(\gamma - xk)\,[V_s^n(x,y+1) - V_s^n(x,y)] \\
&\ge (\gamma - xk)\,[V_s^n(x+1,y) - V_s^n(x,y)] + px\,(V_s^n(x,y+1) - V_s^n(x-1,y+1)) \\
&\ge 0.
\end{aligned}
$$

Both inequalities follows from the induction hypothesis by using increasingness in x and y. The last inequality uses increasingness in x, y. One can easily check that in a similar way the result also holds for the boundaries corresponding to $x = N$ and $y = b(N)-1$. The proof is finally concluded by taking the limit as $n \to \infty$. □

Let us now move on to second-order properties of $V_{1/N,s}$.

Theorem 4.3: $V_{1/N,s}^n, V_{1/N,s}$ are Convex(x), Convex(y) and Supermodular(x,y) on $\mathcal{X}^N$, if $k_N = \gamma/N = p_N = \psi\beta/b(N)$.

Proof: $V_s^0 \equiv 0$ has the three properties Convex(x), Convex(y), and Supermodular(x, y). Assume that these properties hold for V_s^n. We first show that V_s^{n+1} satisfies Convex(x). To this end, define $V_s^n(x,y) = V_s^n(0,y)$ for $x < 0$ and $V_s^n(x,y) = V_s^n(x,0)$ for $y < 0$. Then, consider $1 \le x < N-1$

and $0 \leq y < b(N)$. Then,

$$
\begin{aligned}
& V_s^{n+1}(x+1,y) - 2\,V_s^{n+1}(x,y) + V_s^{n+1}(x-1,y) \\
&= (\lambda s - (x+1)l)V_s^n(x+2,y) - 2(\lambda s - xl)V_s^n(x+1,y) + (\lambda s - (x-1)l)V_s^n(x,y) \\
&\quad +\mu[V_s^n(x,y) - 2V_s^n(x-1,y) + V_s^n(x-2,y)] \\
&\quad +\beta(1-\psi)\,[(x+1)\,V_s^n(x,y) - 2x\,V_s^n(x-1,y) + (x-1)\,V_s^n(x-2,y)] \\
&\quad +(\beta\psi - yp)\,[(x+1)\,V_s^n(x,y+1) - 2x\,V_s^n(x-1,y+1) + (x-1)\,V_s^n(x-2,y+1)] \\
&\quad +(\gamma - (x+1)k)yV_s^n(x+2,y-1) - 2(\gamma - xk)yV_s^n(x+1,y-1) \\
&\qquad\qquad +(\gamma - (x-1)k)yV_s^n(x,y-1) \\
&\quad +(1 - (\lambda s - (x+1)l) - \mu - (\beta - yp)(x+1) - y(\gamma - (x+1)k))V_s^n(x+1,y) \\
&\quad -2(1 - (\lambda s - xl) - \mu - (\beta - yp)x - y(\gamma - xk))V_s^n(x,y) \\
&\quad +(1 - (\lambda s - (x-1)l) - \mu - (\beta - yp)(x-1) - y(\gamma - (x-1)k)V_s^n(x-1,y) \\
&\geq (\lambda s - (x+1)l)[V_s^n(x+2,y) - 2V_s^n(x+1,y) + V_s^n(x,y)] \\
&\quad +2l[V_s^n(x,y) - V_s^n(x+1,y)] \\
&\quad +\beta(1-\psi)(x-1)[V_s^n(x,y) - 2V_s^n(x-1,y) + V_s^n(x-2,y)] \\
&\quad +2\beta(1-\psi)\,[V_s^n(x,y) - V_s^n(x-1,y)] \\
&\quad +(\beta\psi - yp)(x-1)[V_s^n(x,y+1) - 2V_s^n(x-1,y+1) + V_s^n(x-2,y+1] \\
&\quad +2(\beta\psi - yp)\,[V_s^n(x,y+1) - V_s^n(x-1,y+1)] \\
&\quad +(\gamma - (x+1)k)y[V_s^n(x+2,y-1) - 2V_s^n(x+1,y-1) + V_s^n(x,y-1)] \\
&\quad +2ky[V_s^n(x,y-1) - V_s^n(x+1,y-1)] \\
&\quad +(1 - (\lambda s - (x-1)l) - \mu - (\beta - yp)(x+1) \\
&\qquad\qquad -y(\gamma - (x-1)k))[V_s^n(x+1,y) - 2V_s^n(x,y) + V_s^n(x-1,y)] \\
&\quad +2l[V_s^n(x+1,y) - V_s^n(x,y)] - 2(\beta - yp)[V_s^n(x,y) - V_s^n(x-1,y)] \\
&\quad +2ky[V_s^n(x+1,y) - V_s^n(x,y)] \\
&\geq 0.
\end{aligned}
$$

The first inequality follows from the induction hypothesis on the terms having the factor μ. In the second inequality the terms with factor $2l$ cancel; we use Supermodular$(x-1,y)$ for the terms with factor $2(\beta\psi - yp)$, so that the terms with factor 2β all cancel; we use Supermodular$(x,y-1)$ for the terms with factor $2ky$; for the remaining terms evidently one should use the appropriate convexity properties.

As before the boundaries $x = N-1$ and $y = b(N)$ follow in the same manner, by the usual argument that the rates leading out of $\mathcal{X}^N$ are zero.

We will now proceed to prove Convex(y). The arguments will not be given quite as elaborately as in the previous case, since they are quite similar. Consider $0 \leq x < N$ and $1 \leq y < b(N)-1$.

Then

$$
\begin{aligned}
&V_s^{n+1}(x,y+1) - 2\,V_s^{n+1}(x,y) + V_s^{n+1}(x,y-1) \\
&\quad\geq\ \mu[V_s^n(x-1,y+1) - 2V_s^n(x-1,y) + V_s^n(x-1,y-1)] \\
&\qquad +x[(\beta\psi - (y+1)p)V_s^n(x-1,y+2) - 2(\beta\psi - yp)V_s^n(x-1,y+1) \\
&\qquad\qquad +(\beta\psi - (y-1)p)V_s^n(x-1,y)] \\
&\qquad +\beta(1-\psi)x[V_s^n(x-1,y+1) - 2V_s^n(x-1,y) + V_s^n(x-1,y-1)] \\
&\qquad +(\gamma - xk)\,[(y+1)\,V_s^n(x+1,y) - 2y\,V_s^n(x+1,y-1) + (y-1)\,V_s^n(x+1,y-2)] \\
&\qquad +(1-(\lambda s - xl) - \mu - (\beta - (y+1)p)x - (y+1)(\gamma - xk))V_s^n(x,y+1) \\
&\qquad\quad -2(1-(\lambda s - xl) - \mu - (\beta - yp)x - y(\gamma - xk))V_s^n(x,y) \\
&\qquad +(1-(\lambda s - xl) - \mu - (\beta - (y-1)p)x - (y-1)(\gamma - xk))V_s^n(x,y-1) \\
&\quad\geq\ x(\beta\psi - (y+1)p)[V_s^n(x-1,y+2) - 2V_s^n(x-1,y+1) + V_s^n(x-1,y)] \\
&\qquad +2(\gamma - xk)\,[V_s^n(x+1,y) - V_s^n(x+1,y-1) - V_s^n(x,y) + V_s^n(x,y-1)] \\
&\qquad +(1-(\lambda s - xl) - \mu - (\beta - (y-1)p)x \\
&\qquad\qquad\qquad -(y+1)(\gamma - xk))[V_s^n(x,y+1) - 2V_s^n(x,y) + V_s^n(x,y-1)] \\
&\quad\geq\ 0.
\end{aligned}
$$

The first inequality follows from Convex(y) for the term with $(\lambda s - xl)$. The second from Convex(y) for the terms with μ and $\beta(1-\psi)x$; from Convex($y-1$) for the term with $\gamma - xk$, and by rearranging terms such that Supermodular $(x-1,y)$ parts of the induction hypothesis can be used. The second inequality follows from Convex(y), Convex($y+1$) and Supermodular($x, y-1$). The result also holds trivially at the boundaries $x = N$ and $y = b(N) - 1$ by the same arguments as before.

We continue the proof by showing Supermodular(x, y). Consider $0 \leq x < N - 1$ and $0 \leq y <$

$b(N)-1$. Then

$$
\begin{aligned}
&V_s^{n+1}(x+1,y+1)+V_s^{n+1}(x,y)-V_s^{n+1}(x+1,y)-V_s^{n+1}(x,y+1)\\
&= (\lambda s-(x+1)l)V_s^n(x+2,y+1)+(\lambda s-xl)V_s^n(x+1,y)\\
&\quad -(\lambda s-(x+1)l)V_s^n(x+2,y)-(\lambda s-xl)V_s^n(x+1,y+1)\\
&\quad +\mu[V_s^n(x,y+1)+V_s^n(x-1,y)-V_s^n(x,y)-V_s^n(x-1,y+1)]\\
&\quad +\beta(1-\psi)\,(x+1)\,V_s^n(x,y+1)+\beta(1-\psi)\,x\,V_s^n(x-1,y)\\
&\quad -\beta(1-\psi)(x+1)\,V_s^n(x,y)-\beta(1-\psi)x\,V_s^n(x-1,y+1)]\\
&\quad +(\beta\psi-(y+1)p)\,(x+1)\,V_s^n(x,y+2)+(\beta\psi-yp)x\,V_s^n(x-1,y+1)\\
&\quad -(\beta\psi-yp)(x+1)\,V_s^n(x,y+1)-(\beta\psi-(y+1)p)x\,V_s^n(x-1,y+2)]\\
&\quad +(\gamma-(x+1)k)\,(y+1)\,V_s^n(x+2,y)\\
&\quad +(\gamma-xk)\,y\,V_s^n(x+1,y-1)-(\gamma-(x+1)k)\,y\,V_s^n(x+2,y-1)\\
&\quad -(\gamma-xk)\,(y+1)\,V_s^n(x+1,y)\\
&\quad +(1-(\lambda s-(x+1)l)-\mu-(\beta-(y+1)p)(x+1)-(y+1)(\gamma-(x+1)k))\,V_s^n(x+1,y+1)\\
&\quad +(1-(\lambda s-xl)-\mu-(\beta-yp)x-y(\gamma-xk))\,V_s^n(x,y)\\
&\quad -(1-(\lambda s-(x+1)l)-\mu-(\beta-yp)(x+1)-y(\gamma-(x+1)k))\,V_s^n(x+1,y)\\
&\quad -(1-(\lambda s-xl)-\mu-(\beta-(y+1)p)x-(y+1)(\gamma-xk))\,V_s^n(x,y+1)\\
&\geq (\lambda s-(x+1)l)[V_s^n(x+2,y+1)+V_s^n(x+1,y)-V_s^n(x+2,y)-V_s^n(x+1,y+1)]\\
&\quad +\beta(1-\psi)x\,[V_s^n(x,y+1)+V_s^n(x-1,y)-V_s^n(x,y)-V_s^n(x-1,y+1)]\\
&\quad +(\beta\psi-(y+1)p)x\,[V_s^n(x,y+2)+V_s^n(x-1,y+1)-V_s^n(x,y+1)-V_s^n(x-1,y+2)]\\
&\quad +\beta(1-\psi)[V_s^n(x,y+1)-V_s(x,y)]+(\beta\psi-(y+1)p)[V_s^n(x,y+2)-V_s^n(x,y+1)]\\
&\qquad\quad +pxV_s^n(x-1,y+1)-p(x+1)V_s^n(x,y+1)\\
&\quad +(\gamma-(x+1)k)y\,[V_s^n(x+2,y)+V_s^n(x+1,y-1)-V_s^n(x+2,y-1)-V_s^n(x+1,y)]\\
&\quad (\gamma-(x+1)k)V_s^n(x+2,y)+kyV_s^n(x+1,y-1)-k(y+1)V_s^n(x+1,y)\\
&\qquad\quad -(\gamma-(x+1)k)V_s^n(x+1,y)\\
&\quad +(1-(\lambda s-xl)-\mu-(\beta-yp)(x+1)-(y+1)(\gamma-xk))[V_s^n(x+1,y+1)+V_s^n(x,y)\\
&\qquad\quad -V_s^n(x+1,y)-V_s^n(x,y+1)]\\
&\quad +p(x+1)V_s^n(x+1,y+1)+(\beta-yp)V_s^n(x,y)-pxV_s^n(x,y+1)-(\beta-yp)V_s^n(x,y+1)\\
&\quad +k(y+1)V_s^n(x+1,y+1)+(\gamma-xk)V_s^n(x,y)-(\gamma-(x+1)k)V_s^n(x+1,y)\\
&\qquad\quad -k(y+1)V_s^n(x+1,y)\\
&\geq p(V_s^n(x+1,y+1)+V_s^n(x,y)-2V_s^n(x,y+1))+k(V_s^n(x+1,y+1)+V_s^n(x,y)-2V_s^n(x+1,y))\\
&\geq 0,
\end{aligned}
$$

if $k_N=p_N$! The first inequality follows from the induction hypothesis on the term with factor

μ. The second inequality follows by rearranging terms such that the induction hypothesis can be used again. For the final inequality we use Supermodular(x, y). The result also holds for the boundaries. Finally, the proof is concluded by taking the limit as $n \to \infty$. □

Proof of Theorem 4.1 : By combination of Theorems 4.2 and 4.3. □

As has been mentioned in Section 2, structural properties of the value function as a function of the joining rule s (or, equivalently, the arrival rate λ) are much harder to obtain by value iteration. Comparing systems with different joining rules, and smoothing as we did before, has the effect that the smoothed arrival rates for the system with the larger joining rule decrease *faster* than for the system with a smaller joining rule. This destroys propagation of the induction step.

A solution to this problem is by smoothing the arrivals for different joining rules at the same rate, independently of the γ-transfers. This solves the problem up to the x-boundary where the smoothed arrival rates for the approximation with a smaller initial joining rule becomes 0. For larger values of x we have the same problem that we indicated above. As in the previous analysis (where we used compensation of γ-transitions by $\beta\psi$-transitions or vice versa in showing supermodularity), this can be solved by introducing transitions in the opposite direction.

First fix $s \in (0, 1)$, and Δ, such that $s + \Delta, s - \Delta \in [0, 1]$. Hence we restrict to the joining rules $[s']$, $s' \in \{s - \Delta, s, s + \Delta\}$. It is convenient to choose s, Δ such that λs, Δs are rational. Further, we choose a sequence $\{N_t\}_t \subset \mathbb{N}_0$, such that $N_t(s - \Delta)/(s + \Delta)$, $N_t\Delta/(\Delta + s)$ are integer. Further we put $l_t = \lambda(s + \Delta)/N_t$ and put $M_t = \lambda(s - \Delta)/l_t$. By our condition M_t is integer.

Next we construct the processes $(X(1/N_t, s'), Y(1/N, s')$, $s' \in \{s - \Delta, s, s + \Delta\}$ in the same way as in Section 4, except that we smooth the arrival rates differently. For $x \le N_t$, $y \le b(N_t)$

$$\begin{aligned}
\lambda^{N_t}_{x,y}(s - \Delta) &= \max\{\lambda(s - \Delta) - xl_t, 0\}, \\
\lambda^{N_t}_{x,y}(s) &= \max\{(\lambda s - xl_t)\, 0\}, \\
\lambda^{N_t}_{x,y}(s - \Delta) &= \lambda(s + \Delta) - xl_t.
\end{aligned}$$

We further increase the departure rates by putting

$$\begin{aligned}
\mu^{N_t}_{x,y}(s - \Delta) &= \mu + \beta(1 - \psi)x + l_t x, \\
\mu^{N_t}_{x,y}(s + \Delta) = \mu^{N_t}_{s,x,y} &= \mu + \beta(1 - \psi)x + \min\{l_t x, M_t l_t\}.
\end{aligned}$$

The N-approximations $(X(1/N_t, s'), Y(1/N_t, s')$, $s' \in \{s - \Delta, s, s + \Delta\}$ live on the finite space $\mathcal{X}^{N_t}$. As in Theorem 3.3 it can be checked that the collection $\{X(1/N_t, s'), Y(1/N_t, s')$, $s' \in \{s - \Delta, s, s + \Delta\}$, $t = 1, \ldots$ is f-uniformly ergodic, with f a function that increases exponentially quickly in both variables. The intuition is that the arrival rates are still smoothed linearly, and increasing the departure rates, only makes the system 'more stable'.

Without loss of generality we may suppress the dependence on t in our notation. We index the corresponding finite horizon expected total cost functions in the same manner as before. It is straightforward to show that $V^n_{1/N,s'}$ is increasing in x and y on $\mathcal{X}^N$, for $n, N = 1, \ldots,$ $s' \in \{s-\Delta, s, s+\Delta\}$. Moreover, $V^n_{1/N,s-\Delta}$ is Convex(x), Convex(y) and Supermodular(x, y). This can be seen directly from the equations used for showing these properties for the approximations from Section 4. Note that a problem may only occur at the boundary $x = M$. We will show the following theorem.

Theorem 4.4: $V_{1/N,s'}$ is non-decreasing in $s' \in \{s-\Delta, s, s+\Delta\}$, and, if $k_N := \gamma/N = \psi\beta/b(N) =: p_N$, it is Convex($s'$) on $s' \in \{s-\Delta, s, s+\Delta\}$ and Supermodular(s', x) on Supermodular(s', y) on $s' \in \{s-\Delta, s\}$ or $s' \in \{s-\Delta, s+\Delta\}$, all on $\mathcal{X}^N$, for any $N \geq 1$.

Proof of Theorem 2.1 (2): : This is a copy of the proof of Theorem 2.1 (1). □

Let us now proceed to proving Theorem 4.4. It consists of a number of steps.

Theorem 4.5: $V^n_{1/N,s'}$ is componentwise non-decreasing in $s' \in \{s-\Delta, s, s+\Delta\}$ on $\mathcal{X}^N$, for $n = 1, \ldots,$, and so is $V_{1/N,s'}$.

Proof: The statement clearly holds for $V^0_{s'} \equiv 0$. Assume hence that the statement is true upto value n.

Let first $0 \leq x \leq M$, $0 \leq y < b(N)$. Then,

$$\begin{aligned}
&V^{n+1}_s(x,y) - V^{n+1}_{s-\Delta}(x,y) \\
&\quad= (\lambda s - xl)\, V^n_s(x+1,y) - (\lambda(s-\Delta) - xl)\, V^n_{s-\Delta}(x+1,y) \\
&\qquad +\mu\,[V^n_s(x-1,y) - V^n_{s-\Delta}(x-1,y)] + (\beta x(1-\psi) + lx)\,[V^n_s(x-1,y) - V^n_{s-\Delta}(x-1,y)] \\
&\qquad +x(\beta\psi - yp)\,[V^n_s(x-1,y+1) - V^n_{s-\Delta}(x-1,y+1)] \\
&\qquad +y(\gamma - xk)\,[V^n_s(x+1,y-1) - V^n_{s-\Delta}(x+1,y-1)] \\
&\qquad +(1 - (\lambda s - xl) - \mu - (\beta - yp)x - lx - y(\gamma - xk))\, V^n_s(x,y) \\
&\qquad -(1 - (\lambda(s-\Delta) - xl) - \mu - (\beta - yp)x - lx - y(\gamma - xk))\, V^n_{s-\Delta}(x,y) \\
&\quad\geq (\lambda(s-\Delta) - xl)\,[V^n_s(x+1,y) - V^n_{s-\Delta}(x+1,y)] + \lambda\Delta[V^n_s(x+1,y) - V^n_s(x,y)] \\
&\qquad +(1 - (\lambda(s-\Delta) - xl) - \mu - (\beta - yp)x - lx - y(\gamma - xk))\,[V^n_s(x,y) - V^n_{s-\Delta}(x,y)] \\
&\quad\geq \lambda\Delta[V^n_s(x+1,y) - V^n_s(x,y)] \\
&\quad\geq 0.
\end{aligned}$$

The first two inequalities follow from applying the induction hypothesis by using increasingness in s'. The last inequality follows from increasingness in x.

Suppose that $M < x \leq N$, $y \leq b(N)$. First suppose that $\lambda^N_{x,y}(s) > 0$. The only λ-terms occurring are factors of $V^n_s(x,y)$ and $V^n_s(x+1,y)$. This is easily seen to give a contribution $(\lambda s - xl)[V^n_s(x+1,y) - V^n_s(x,y)]$. Furthermore, there is a difference in the departure terms: rewriting in terms of factors with $xl - Ml$ and Ml, we get the extra term $(xl - Ml)[V^n_{s-\Delta}(x, y) - V^n_{s-\Delta}(x-1, y)]$ By increasingness in x of $V^n_{s-\Delta}$, this contribution is non-negative.

The case of $\lambda^N_{x,y}(s) = 0$ is similar.

For the boundaries $y = b(N)$ and $x = N$, the argument is precisely the same as in the above, since the rates leading out of $\mathcal{X}^N$ are 0.

A similar derivation holds, when comparing V^n_s and $V^n_{s+\Delta}$. The proof is concluded by taking the limit as $n \to \infty$. □

The extension of Theorem 4.3 to our adapted SRT provides the necessary ingredients for showing convexity in s'. Note that, similar to the case of the first-order properties, the proof of this result only depends on the properties of the facility Q_1, i.e., convexity in x in this case. However, convexity in x depends on convexity in y for queue Q_2. For the convexity in s' we also need supermodularity in s' in combination with the other variables.

Theorem 4.6: $V^n_{1/N,s'}$ is Convex(s') on $\{s-\Delta, s, s+\Delta\}$, and $V^n_{1/N,s'}, V_{1/N,s'}$ are Supermodular($s-\Delta, x$) and Supermodular($s-\Delta, y$) on $\{s-\Delta, s\}$ and $\{s-\Delta, s+\Delta\}$.

Proof: For $V^0_{s'}(x, y) \equiv 0$, clearly this function satisfies Convex(s') on $\{s-\Delta, s, s+\Delta\}$, as well as Supermodular($s-\Delta, x$) and Supermodular($s-\Delta, y$) on $\{s-\Delta, s\}$ and $\{s-\Delta, s+\Delta\}$. Therefore, assume that these properties hold for $V^n_{s'}$. We first show that $V^{n+1}_{s'}$ satisfies Convex(s'). To this end, consider $0 \le x \le M$ and $0 \le y < b(N)$. Then,

$$
\begin{aligned}
&V^{n+1}_{s+\Delta}(x, y) - 2\,V^{n+1}_s(x, y) + V^{n+1}_{s-\Delta}(x, y) \\
&\quad \ge (\lambda(s+\Delta) - xl)\,V^n_{s+\Delta}(x+1, y) - 2(\lambda s - xl)\,V^n_s(x+1, y) + (\lambda(s-\Delta) - xl)V_{s-\Delta}(x+1, y) \\
&\qquad + (1 - (\lambda(s+\Delta) - xl) - \mu - lx - (\beta - py)x - y(\gamma - xk))\,V^n_{s+\Delta}(x, y) \\
&\qquad - 2(1 - (\lambda s - xl)s - \mu - lx - (\beta - py)x - y(\gamma - xk))\,V^n_s(x, y) \\
&\qquad + (1 - (\lambda(s-\Delta) - xl) - \mu - lx - (\beta - py)x - y(\gamma - xk))\,V^n_{s-\Delta}(x, y) \\
&\quad \ge 2\lambda\Delta\,[V^n_s(x+1, y) - V^n_{s-\Delta}(x+1, y) - V^n_s(x, y) + V^n_{s-\Delta}(x, y)] \\
&\quad \ge 0.
\end{aligned}
$$

The first inequality follows from the induction hypothesis on the terms having factors with no s. The second inequality follows by rearranging terms such that the induction hypothesis can be used again on terms with factor $\lambda(s+\Delta) - xl$ and $1 - (\lambda(s+\Delta) - xl) - \mu - lx - (\beta - py)x - y\gamma$. The final inequality follows from Supermodular($s-\Delta, x$) on $\{s-\Delta, s\}$.

Next consider the case $M \le x \le N$, with $\lambda^N_{x,y}(s) > 0$. Separating the departure rates xl into terms with factor $Ml = \lambda(s-\Delta)$ and with factor $xl - Ml = xl - \lambda(s-\Delta)$, we get

$$
\begin{aligned}
&(\lambda(s+\Delta) - xl)\,V^n_{s+\Delta}(x+1, y) - 2(\lambda s - xl)\,V^n_s(x+1, y) \\
&\quad +(xl - Ml)[V^n_{s-\Delta}(x-1, y) - V^n_{s-\Delta}(x, y)] + Ml\,V^n_{s-\Delta}(x, y) \\
&\quad +(1 - (\lambda(s+\Delta) - xl) - \mu - Ml - (\beta - py)x - y(\gamma - xk))\,V^n_{s+\Delta}(x, y)
\end{aligned}
$$

$$
\begin{aligned}
&-2(1-(\lambda s-xl)s-\mu-Ml-(\beta-py)x-y(\gamma-xk))\,V_s^n(x,y)\\
&+(1-\mu-Ml-(\beta-py)x-y(\gamma-xk))\,V_{s-\Delta}^n(x,y)\\
\geq\;& (\lambda(s+\Delta)-xl)\,V_{s+\Delta}^n(x+1,y)-2(\lambda s-xl)\,V_s^n(x+1,y)\\
&+(xl-\lambda(s-\Delta))[V_{s-\Delta}^n(x,y)-V_{s-\Delta}^n(x+1,y)]\\
&+(1-(\lambda(s+\Delta)-xl)-\mu-Ml-(\beta-py)x-y(\gamma-xk))\,V_{s+\Delta}^n(x,y)\\
&-2(1-(\lambda s-xl)s-\mu-Ml-(\beta-py)x-y(\gamma-xk))\,V_s^n(x,y)\\
&+(1-\mu-Ml-(\beta-py)x-y(\gamma-xk))\,V_{s-\Delta}^n(x,y),
\end{aligned}
$$

where we have used that $V_{s-\Delta}^n$ is Convex(x). The rest is analogous to the previous.

Finally, we consider the case that also $\lambda_{x,y}^N(s)=0$. The only terms of interest to consider are

$$(\lambda(s+\Delta)-xl)[V_{s+\Delta}^n(x+1,y)-V_{s+\Delta}^n(x,y)]-(xl-lM)[V_{s-\Delta}^n(x,y)-V_{s-\Delta}^n(x-1,y)],$$

which is non-negative by first applying Supermodular($s-\Delta,x$) (w.r.t $\{s-\Delta,s+\Delta\}$) to the first term and then Convex(x) for $V_{s-\Delta}^n$. Note that $\lambda(s+\Delta)\geq lM$. The result also holds at the boundaries corresponding to $x=N$ and $y=b(N)$.

We next proceed to prove Supermodular($s-\Delta,x$) on $\{s-\Delta,s\}$. The proof on $\{s-\Delta,s+\Delta\}$ is completely analogous. To this end, consider $0\leq x<M$ and $0\leq y<b(N)$.

$$
\begin{aligned}
&V_s^{n+1}(x+1,y)+V_{s-\Delta}^{n+1}(x,y)-V_s^{n+1}(x,y)-V_{s-\Delta}^{n+1}(x+1,y)\\
&\geq\; (\lambda s-(x+1)l)\,V_s^n(x+2,y)+(\lambda(s-\Delta)-xl)\,V_{s-\Delta}^n(x+1,y)\\
&\quad -(\lambda s-xl)\,V_s^n(x+1,y)-(\lambda(s-\Delta)-(x+1)l)\,V_{s-\Delta}^n(x+2,y)\\
&\quad +(\beta(1-\psi)+l)\,[(x+1)\,V_s^n(x,y)+x\,V_{s-\Delta}^n(x-1,y)\\
&\quad -x\,V_s^n(x-1,y)-(x+1)\,V_{s-\Delta}^n(x,y)]\\
&\quad +(\beta\psi-yp)\,[(x+1)\,V_s^n(x,y+1)\\
&\quad +x\,V_{s-\Delta}^n(x-1,y+1)-x\,V_s^n(x-1,y+1)-(x+1)\,V_{s-\Delta}^n(x,y+1)]\\
&\quad +(\gamma-(x+1)k)\,y\,V_s^n(x+2,y-1)+(\gamma-xk)\,y\,V_{s-\Delta}^n(x+1,y-1)\\
&\quad -(\gamma-xk)\,y\,V_s^n(x+1,y-1)-(\gamma-(x+1)k)\,y\,V_{s-\Delta}^n(x+2,y-1)\\
&\quad +(1-(\lambda s-(x+1)l)-\mu-(x+1)l-(\beta-yp)(x+1)-y(\gamma-(x+1)k))\,V_s^n(x+1,y)\\
&\quad +(1-(\lambda(s-\Delta)-xl)-\mu-xl-(\beta-yp)x-y(\gamma-xk))\,V_{s-\Delta}^n(x,y)\\
&\quad -(1-(\lambda s-xl)-\mu-xl-(\beta-yp)x-y(\gamma-xk))\,V_s^n(x,y)\\
&\quad -(1-(\lambda(s-\Delta)-(x+1)l)-\mu-(x+1)l-(\beta-yp)(x+1)-y(\gamma-(x+1)k))\,V_{s-\Delta}^n(x+1,y)\\
&\geq\; \Delta\lambda[V_{s-\Delta}^n(x+2,y)-2V_{s-\Delta}^n(x+1,y)+V_{s-\Delta}^n(x,y)]\\
&\quad +l[V_{s-\Delta}^n(x+1,y)-V_s^n(x+1,y)+V_s^n(x+1,y)-V_{s-\Delta}^n(x+1,y)]\\
&\quad +(\beta(1-\psi)+l)\,[V_s^n(x,y)-V_{s-\Delta}^n(x,y)+V_{s-\Delta}^n(x,y)-V_s^n(x,y)]\\
&\quad +(\beta\psi-yp)\,[V_s^n(x,y+1)-V_{s-\Delta}^n(x,y+1)+V_{s-\Delta}^n(x,y)-V_s^n(x,y)]\\
&\quad +ky(V_s^n(x+1,y)+V_{s-\Delta}^n(x+1,y-1)-V_s^n(x+1,y-1)-V_{s-\Delta}^n(x+1,y))
\end{aligned}
$$

$$\geq \quad 0.$$

The first inequality follows from the induction hypothesis on terms having only the factor μ. The second inequality follows by rearranging terms such that the induction hypothesis can be used on terms with factors $\lambda s - (x+1)l$, $(\beta(1-\psi)+l)x$, $(\beta\psi - yp)x$, $(\gamma - (x+1)k)y$, and $1-(\lambda s - xl) - \mu - (x+1)l - (\beta - yp)(x+1) - y(\gamma - xk)$. The final inequality follows from Convex$(x+1)$ (w.r.t. $s-\Delta$) for the first; and Supermodular$(s-\Delta, y)$ and Supermodular$(s-\Delta, y-1)$ for the two last lines.

Suppose that $x \geq M$ with $\lambda^N(x,y)(s) > 0$. We only consider the crucial terms and get

$$\begin{aligned}
&(\lambda s - (x+1)l)V_s^n(x+2,y) - (\lambda s - xl)V_s^n(x+1,y) + MlV_s^n(x,y) - MlV_s^n(x-1,y) \\
&\quad +(1-(\lambda s - (x+1)l) - Ml)V_s^n(x+1,y) - (1-(\lambda s - xl) - Ml)V_s^n(x,y) \\
&\quad +xlV_{s-\Delta}^n(x-1,y) - (x+1)lV_{s-\Delta}^n(x,y) \\
&\quad +(1-xl)[V_{s-\Delta}^n(x,y) - (1-(x+1)l)V_{s-\Delta}^n(x+1,y) \\
\geq\ & (\lambda s - (x+1)l)[V_s^n(x+2,y) - V_s^n(x+1,y)] + Ml\,[V_{s-\Delta}^n(x,y) - V_{s-\Delta}^n(x-1,y)] \\
&\quad +(1-(\lambda s - xl) - lM)[V_s^n(x+1,y) - V_s^n(x,y)] \\
&\quad +xlV_{s-\Delta}^n(x-1,y) - (x+1)lV_{s-\Delta}^n(x,y) \\
&\quad +(1-xl)V_{s-\Delta}^n(x,y) - (1-(x+1)l)V_{s-\Delta}^n(x+1,y) \\
\geq\ & (\lambda s - (x+1)l)[V_s^n(x+2,y) - V_s^n(x+1,y)] + (1-(\lambda s - xl) - lM)[V_s^n(x+1,y) - V_s(x,y)] \\
&\quad +(xl - Ml)[V_{s-\Delta}^n(x,y) - V_{s-\Delta}^n(x-1,y)] - (1-xl)[V_{s-\Delta}^n(x+1,y) - V_{s-\Delta}^n(x,y)] \\
&\quad +l\,[V_{s-\Delta}^n(x+1,y) - V_{s-\Delta}^n(x,y)] \\
\geq\ & (\lambda s - (x+1)l)[V_s^n(x+2,y) - V_s^n(x+1,y)] + (1-(\lambda s - xl) - lM)[V_s^n(x+1,y) - V_s(x,y)] \\
&\quad -(1-Ml)[V_{s-\Delta}^n(x+1,y) - V_{s-\Delta}^n(x,y)] \\
&\quad +l\,[V_{s-\Delta}^n(x+1,y) - V_{s-\Delta}^n(x,y)] \geq 0.
\end{aligned}$$

In the first inequality we use Supermodular$(s-\Delta, x-1)$, in the third we use Convex(x) w.r.t $s-\Delta$ and the final inequality we use Supermodular$(s-\Delta, x+1)$ and Convex$(x+1)$ (w.r.t $s-\Delta$) for the first term and Supermodular$(s-\Delta, x)$ for the second.

The argument is similar for the case that $M < x < N$ with $\lambda_{x,y}^N(s) = 0$: the λ terms in the above are absent.

We continue the proof by showing Supermodular(s, y). Consider $0 < x \leq M$ and $0 < y < b(N) - 1$. Then

$$\begin{aligned}
&V_s^{n+1}(x,y+1) + V_{s-\Delta}^{n+1}(x,y) - V_s^{n+1}(x,y) - V_{s-\Delta}^{n+1}(x,y+1) \\
&\geq\ (\lambda s - xl)\,[\,V_s^n(x+1,y+1) + V_{s-\Delta}^n(x+1,y) - \ V_s^n(x+1,y) - V_{s-\Delta}^n(x+1,y+1)] \\
&\quad +\lambda\Delta[V_s^n(x+1,y+1) - V_s^n(x+1,y)] \\
&\quad +(\beta\psi - (y+1)p)xV_s^n(x-1,y+2)
\end{aligned}$$

$$
\begin{aligned}
& +(\beta\psi - yp)xV^n_{s-\Delta}(x-1,y+1) - (\beta\psi - yp)xV^n_s(x-1,y+1) + \\
& -(\beta\psi - (y+1)p)xV^n_{s-\Delta}(x-1,y+2) \\
& +(\gamma - xk)\,[(y+1)\,V^n_s(x+1,y) + y\,V^n_{s-\Delta}(x+1,y-1) \\
& -y\,V^n_s(x+1,y-1) - (y+1)\,V^n_{s-\Delta}(x+1,y)] \\
& +(1-(\lambda s - xl) - \mu - lx - (\beta - (y+1)p)x - (y+1)(\gamma - xk))\,V^n_s(x,y+1) \\
& +(1-(\lambda(s-\Delta) - xl) - \mu - lx - (\beta - yp)x - y(\gamma - xk))\,V^n_{s-\Delta}(x,y) \\
& -(1-(\lambda s - xl) - \mu - lx - (\beta - yp)x - y(\gamma - xk))\,V^n_s(x,y) \\
& -(1-(\lambda(s-\Delta) - xl) - \mu - lx - (\beta - (y+1)p)x - (y+1)(\gamma - xk))\,V^n_{s-\Delta}(x,y+1) \\
\geq\ & \lambda\Delta\,[V^n_s(x+1,y+1) - V^n_s(x+1,y) - V^n_{s-\Delta}(x,y+1) + V^n_{s-\Delta}(x,y)] \\
& +x(\beta\psi - (y+1)p)[V^n_s(x-1,y+2) + V^n_{s-\Delta}(x-1,y+1) \\
& \quad -V^n_s(x-1,y+1) - V^n_{s-\Delta}(x-1,y+2)] \\
& +(\gamma - xk)\,[V^n_s(x+1,y) - V^n_{s-\Delta}(x+1,y) + V^n_s(x,y) - V^n_s(x,y)] \\
& +px[V^n_s(x,y+1) + V^n_{s-\Delta}(x-1,y+1) - V^n_s(x-1,y+1) - V^n_{s-\Delta}(x,y+1)] \\
\geq\ & 0.
\end{aligned}
$$

The first inequality follows from the induction hypothesis on the terms having factors with no s or y in them. The second inequality follows by rearranging terms such that the induction hypothesis can be used on terms with factors λs, $(\gamma - xk)y$, $1 - (\lambda s - xk) - \mu - xl - (\beta - yp)x - (y+1)(\gamma - xk)$. The final inequality follows from Supermodular(x, y) [Theorem 4.3], Supermodular(s, x) and Supermodular(s, y).

We now consider $M < x$ and $\lambda^N_{x,y}(s) > 0$. The relevant terms become:

$$
\begin{aligned}
(\lambda s - xl)[V^n_s(x+1,y+1) - V^n_s(x+1,y)] + Ml[V^n_s(x-1,y+1) - V^n_s(x-1,y)] & \\
+(1-(\lambda s - xl) - lM)[V^n_s(x,y+1) - V^n_s(x,y)] & \\
-xl[V^n_{s-\Delta}(x-1,y+1) - V^n_{s-\Delta}(x-1,y)] & \\
-(1-xl)[V^n_{s-\Delta}(x,y+1) - V^n_{s-\Delta}(x,y)] & \\
\geq\ (\lambda s - xl)[V^n_s(x+1,y+1) - V^n_s(x+1,y)] & \\
+(1-(\lambda s - xl) - lM)[V^n_s(x,y+1) - V^n_s(x,y)] & \\
-(xl - Ml)[V^n_{s-\Delta}(x-1,y+1) - V^n_{s-\Delta}(x-1,y)] & \\
-(1-xl)[V^n_{s-\Delta}(x,y+1) - V^n_{s-\Delta}(x,y)] & \\
\geq\ (\lambda s - xl)[V^n_{s-\Delta}(x,y+1) - V^n_{s-\Delta}(x,y)] & \\
+(1-(\lambda s - xl) - lM)[V^n_s(x,y+1) - V^n_s(x,y)] & \\
-(1-Ml)[V^n_{s-\Delta}(x,y+1) - V^n_{s-\Delta}(x,y)] & \\
\geq\ 0. &
\end{aligned}
$$

In the first inequality we use Supermodular$(s-\Delta, y)$ for the first and second terms; in the second we use Supermodular $(x-1, y)$.

Let finally $M < x$ and $\lambda^N_{s,x,y} = 0$. We get the same derivation, except for the fact that the λ

terms are absent.

The result also holds at the boundaries corresponding to $x = b(N)$ and $y = N - 1$.

Since $V^n_{u,s}(x,y) - V^n_{u,s}(x',y') \to V_{u,s}(x,y) - V_{u,s}(x',y')$, the supermodularity properties of $V^n_{1/N}$ carry over to analogous supermodularity properties of the value functions. $\square$

Proof of Theorem 4.4: : By combination of Theorems 4.5 and 4.6, using that the value function V_s is componentwise continuous in s. $\square$

The proofs of Corollary 2.2 has already been given at the end of Section 3, henceforth we omit these here.

5 Discussion

It is natural to ask to what extent the usual finite state truncation really destroys structural properties. The usual finite state truncation limits the state space to a finite set parameterized by N, and the transition probabilities leading towards states out of this set and mapped back to some state inside the set. Sennott ([8], Section C4 and C5) shows that the average expected cost under the truncated model converges to the average expected cost of the original model as N tends to infinity. However, the following examples show that the structural properties of the original model are lost in the finite-state truncations.

In the following, we calculate and plot the system delay cost, g_s, as a function of $s \in (0,1]$. We truncate the state space to the set $\{0,\dots,20\} \times \{0,\dots,50\}$. In the two examples we see that g_s is increasing in s, however, g_s lacks convexity. As seen, convexity is not preserved under this truncation. In some cases, ad-hoc choices of the transition rates on the boundaries do preserve structural properties of the relative value function (see, e.g., Down et al. [2]). Their approach does not apply to our model.

Example 1

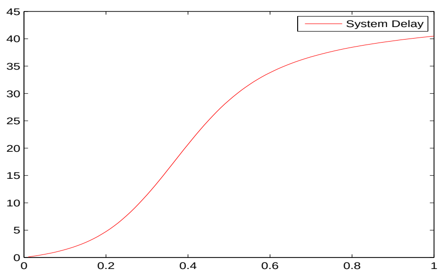

Figure 1: $\lambda = 50, \quad \mu = 10, \quad \beta = 10, \quad \gamma = 100, \quad \psi = 0.9, \quad \text{and} \quad R = 1.$

Example 2

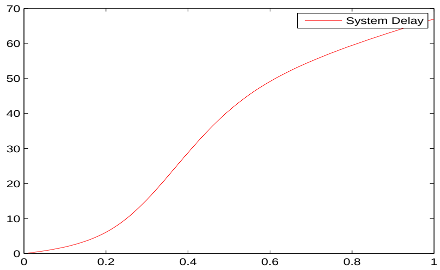

Figure 2: $\lambda = 50, \quad \mu = 10, \quad \beta = 10, \quad \gamma = 10, \quad \psi = 0.9, \quad \text{and} \quad R = 1.$

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