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On Structured Pseudo Spectra for Polynomial Eigenvalue Problems

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Abstract

Pseudospectra associated with the generalized eigenvalue problems have been widely investigated in recent years. This paper is organized as follows. First, we prove for a $class \in \{Symmetric, Persymmetric, Toeplitz, SymToeplitz, Hankel, PersymHankel, Circulant\}$, we have the classed pseudospectrum of matrix is equal to the unclassed pseudospectrum because of the departure from singularity. Second, we prove the same results for classed pseudospectra of matrix polynomials.

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1 Introduction

In this paper, we are mainly concerned with the following linear classes {SY, PS, TO, ST, HA, PSH, CI} stand for { symmetric, persymmetic , Toeplitz matrices, symmetric Toeplitz , Hankel , persymmetric Hankel , Circulant} matrices respectivly. $\mathbb{S}(\mathbb{S}_{\mathbb{R}})$ satnd for classed complex(real) matrices in

$$\{\mathrm{SY}, \mathrm{PS}, \mathrm{TO}, \mathrm{ST}, \mathrm{HA}, \mathrm{PSH}, \mathrm{CI}\} \tag{1}$$

particularly, $A \in \mathbb{S}^{\text{SY}}(A \in \mathbb{S}^{\text{SY}}_{\mathbb{R}})$ this mean that A is the complex(real) symmetric matrix.

$$\begin{array}{lcl} A \in \mathbb{S}^{\text{SY}} & \Leftrightarrow & a_{i,j} = a_{j,i} \\ A \in \mathbb{S}^{\text{HE}} & \Leftrightarrow & A^* = A \\ A \in \mathbb{S}^{\text{SHE}} & \Leftrightarrow & A^* = -A \\ A \in \mathbb{S}^{\text{PS}} & \Leftrightarrow & A^T = JAJ \\ A \in \mathbb{S}^{\text{TO}} & \Leftrightarrow & a_{i,j} = a_{i+1,j+1} \\ A \in \mathbb{S}^{\text{ST}} & \Leftrightarrow & a_{i,j} = a_{i+1,j+1} = a_{j,i} = a_{j+1,i+1} \\ A \in \mathbb{S}^{\text{HA}} & \Leftrightarrow & a_{i-1,j+1} = a_{i,j} \\ A \in \mathbb{S}^{\text{PSH}} & \Leftrightarrow & A \in \mathbb{S}^{\text{HA}} and A^T = JAJ \end{array}$$

$A \in \mathbb{S}^{\text{CI}} \Leftrightarrow$ each row vector is rotated one element to the right relative to the preceding row vector, where $i, j = 1, 2, \ldots, k$ and J is called "flip-matrix", with ones on the anti-diagonal and zero everywhere.

Throughout the paper $\|.\|$ denotes the $\|.\|_2$, for vectors and for matrices. Let us consider A be a matrix in $\mathbb{C}^{n\times n}$. We denote its spectrum by $\Lambda_0(A)$. For a real $\epsilon > 0$, the pseudospectrum of matrix A is the set

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : \exists E \in \mathbb{C}^{n\times n}, \|E\| \le \epsilon, z \in \Lambda_0(A + E)\}$$

And the classed pseudospectrum of matrix A is the set

$$\Lambda^{\mathbb{S}}_\epsilon(A) = \{z \in \mathbb{C} : \exists E \in \mathbb{S}, \|E\| \le \epsilon, z \in \Lambda_0(A + E)\}$$

where $\mathbb{S}$ is in (1). We aim to charecterize $\Lambda^{\mathbb{S}}_\epsilon(A)$ for all classes in (1). In fact, S.M.Rump([5]) has proved $\Lambda^{\mathbb{S}}_\epsilon(A) = \Lambda_\epsilon(A)$ without using the conception of the departure from singularity and S.Graillat([1]) has proved it for $\mathbb{S} \in \{Toep, Circ\}$ becouse of the conception of the departure from singularity . We recall lemma (3.1) and lemma (3.2) in [1] and we will go to show $\Lambda^{\mathbb{S}}_\epsilon(A) = \Lambda_\epsilon(A)$ becouse of the definition of the departure from singularity.

2 Definition of the departure from singularity to prove $\Lambda^{\mathbb{S}}_\epsilon(A) = \Lambda_\epsilon(A)$

Definition 2.1. *[1] Given a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ the departure from singularity is defined as*

$$d(A) = \min\{\|E\| : A + E\, singular, E \in \mathbb{C}^{n\times n}\}.$$

Definition 2.2. *[1] Given a nonsingular matrix $A \in \mathbb{S}$ the classed departure from singularity is defined as*

$$d^{\mathbb{S}}(A) = \min\{\|E\| : A + E\, singular, E \in \mathbb{S}\}.$$

Corollary 2.3. $d(A) \leq d^{\mathbb{S}}(A)$.

Lemma 2.4. *Given $\epsilon > 0$ and $A \in \mathbb{C}^{n\times n}$, the pseudospectrum satisfies*

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : d(A - zI) \leq \epsilon\}.$$

Lemma 2.5. *Given $\epsilon > 0$, $A \in \mathbb{S}$ and $\mathbb{S} \in (1)$ except Hankelmatrices, the classed pseudospectrum satisfies*

$$\Lambda^{\mathbb{S}}_\epsilon(A) = \{z \in \mathbb{C} : d^{\mathbb{S}}(A - zI) \leq \epsilon\}.$$

Proof. $(\rightarrow)$ Let $z \in \Lambda^{\mathbb{S}}_\epsilon(A)$ there exists $E \in \mathbb{S}$ such that $\|E\| \leq \epsilon$ and $z \in \Lambda_0(A + E)$. Thus $A + E - zI$ is singular. Moreover, since $zI \in \mathbb{S}$ for $z \in \mathbb{C}$ (because zI is SY, PS; $(zI)^T = J(zI)J$, ST $\subseteq \{\text{SY} \cap \text{PS}\}$, PSH $\subseteq \{\text{SY} \cap \text{PS}\}$, TO and CI) we have

$$d^{\mathbb{S}}(A - zI) \leq \epsilon$$

$(\leftarrow)$, let $z \in \mathbb{C}$ such that $d^{\mathbb{S}}(A - zI) \leq \epsilon$, then there exists $E \in \mathbb{S}$ such that $A + E - zI$ is singular; that is, $z \in \Lambda_0(A + E)$, and

$$d^{\mathbb{S}}(A - zI) = \|E\| \leq \epsilon.$$

Consequently $z \in \Lambda^{\mathbb{S}}_\epsilon(A)$. □

We have generalized some results for complex numbers which S.RUMP [6] has demonstrated for real numbers.

Lemma 2.6. *Let $x, y \in \mathbb{C}^{n\times 1}$ be given such that $\|x\| = \|y\| = 1$. Then there exists $A \in \{\mathbb{S}^{SY}, \mathbb{S}^{PS}\}$ with $y = Ax$ and $\|A\| = 1$.*

Proof. For symmetric classed the Housholder reflection H along $x + y$ satisfies $H = H^T$, $\|H\| = 1$ and $Hx = y$. Let H be Householder reflection along $x + Jy$ and set $A = JH$ then $A^T = JAJ$ is persymmetric, $\|A\| = 1$ and $Ax = JHx = JJy = y$. □

Definition 2.7. *[6] For a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ and $0 \neq x \in \mathbb{C}^{n\times 1}$ we define*

$$\varphi(A, x) = \sup\{\|A^{-1}Ex\| : E \in \mathbb{C}^{n\times n}, \|A\| \leq 1\}$$

and

$$\varphi^{\mathbb{S}}(A, x) = \sup\{\|A^{-1}Ex\| : E \in \mathbb{S}, \|A\| \leq 1\}.$$

Proposition 2.8. *[6] For a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ and $0 \neq x \in \mathbb{C}^{n\times 1}$:*

$$\varphi(A, x) = \|A^{-1}\|\|x\|$$

Lemma 2.9. *For a nonsingular matrix $A \in \mathbb{C}^{n\times n}$ and $0 \neq x \in \mathbb{C}^{n\times 1}$:*

$$\varphi^{\mathbb{S}^*}(A,x) = \varphi(A,x) = \|A^{-1}\|\|x\| \quad where\ \mathbb{S}^* \in \{\mathbb{S}^{SY}, \mathbb{S}^{PS}\}.$$

Proof. From defnition (2.7) and Proposition (2.8) we have $\varphi^{\mathbb{S}^*}(A,x) \leq \varphi(A,x) = \|A^{-1}\|\|x\|$, so it remains to show that $\varphi^{\mathbb{S}^*}(A,x) \geq \|A^{-1}\|\|x\|$. Without loss of generality, let $\|x\| = 1$ and let $\|A^{-1}\| = \|A^{-1}y\|$ for $\|y\| = 1$. Thus by lemma (2.6) there exists $E \in \{\mathbb{S}^{\mathrm{SY}}, \mathbb{S}^{\mathrm{PS}}\}$ with $\|A^{-1}\| \leq 1$ and $Ex = y$; that is,

$$\|A^{-1}\| = \|A^{-1}Ex\| \leq \varphi^{\mathbb{S}*} \leq \|A^{-1}\|.$$

□

Definition 2.10. *The classed condition number for matrix A is defiend in [6] as*

$$\kappa_E^{\mathbb{S}}(A) = \lim_{\epsilon\to 0}\sup\{\frac{\|(A+\Delta A)^{-1} - A^{-1}\|}{\epsilon\|A^{-1}\|} : \Delta A \in \mathbb{S}, \|\Delta A\| \leq \epsilon\|E\|\}.$$

And unclassed condition number is defined as

$$\kappa_E(A) = \lim_{\epsilon\to 0}\sup\{\frac{\|(A+\Delta A)^{-1} - A^{-1}\|}{\epsilon\|A^{-1}\|} : \Delta A \in \mathbb{C}^{n\times n}, \|\Delta A\| \leq \epsilon\|E\|\}.$$

We want prove that for nonsingular matrix $A \in \mathbb{S}$ for $\mathbb{S} \in (1)$ except for $Hankel$ matrix

$$d(A) = d^{\mathbb{S}}(A)$$

Conseqently

$$\Lambda_\epsilon^{\mathbb{S}}(A) = \Lambda_\epsilon(A).$$

From theorem Ecatr and young[3, Theorem 6.5] the departure from singularity on the 2-norm space equals the reciprocal of the norm of the inverse; that is,

$$(\kappa(A))^{-1} = d(A) = \|A^{-1}\|^{-1}$$

where norm of E in the definition(2.10)equals 1.

Theorem 2.11. *Let nonsingular $A \in \mathbb{S}$ with $\mathbb{S} \in (1)$ be given, then*

$$\kappa^{\mathbb{S}}(A) = \|A^{-1}\| \quad where\ \|E\| = 1.$$

Proof. we have

$$(A+\Delta A)^{-1} - A^{-1} = -A^{-1}\Delta A A^{-1} + O(\|A\|^2)$$

If we can show that

$$\omega^{\mathbb{S}}(A) = \sup\{\| - A^{-1}\Delta A A^{-1}\| : \Delta A \in \mathbb{S}, \|\Delta A\| \leq 1\} \geq \|A^{-1}\|$$

then we have the result will be proved. Let $x, y \in \mathbb{C}^{n\times 1}$, $\|x\| = \|y\| = 1$ be given with $A^{-1}x = \|A^{-1}\|y$. Then by definition(2.7)and lemma(2.9) we have ,for $\mathbb{S} \in \{\mathbb{S}^{\mathrm{SY}}, \mathbb{S}^{\mathrm{PS}}\}$

$$\begin{aligned} \omega^{\mathbb{S}}(A) &\geq \sup\{\| - A^{-1}\Delta A A^{-1}x\| : \Delta A \in \mathbb{S}, \|\Delta A\| \leq 1\} \\ &= \|A^{-1}\|\varphi^{\mathbb{S}}(A, y) \\ &= \|A^{-1}\|\|A^{-1}\|. \end{aligned} \tag{2}$$

For normal $A \in \mathbb{S}$, it is $A^{-1}x = \lambda x$ with $\|x\| = 1$ and $|\lambda| = \|A^{-1}\|$. Hence (2) is also proved for symmetric Topeplitz and circulant structures by using $\Delta A = I$. For $A \in \mathbb{S}^{\mathrm{PSH}}$, $AJ \in \mathbb{S}^{\mathrm{ST}}$ and $JA^{-1}x = \lambda x$ with $\|x\| = 1$, and $|\lambda| = \|JA^{-1}\| = \|A^{-1}\|$ proves (2) by using $\Delta A = J \in \mathbb{S}^{\mathrm{PSH}}$. Let A be a Hankel matriwx, A is especially(complex) symmetric. So a result by Takagi[2, Corollary 4.4.4] implies $A = U\Sigma U^T$ for nonnegative diagonal Σ and unitary U. For x denoting the nth column of U we have $A\bar{x} = \sigma_{\min}(A)x$, and therefore $A^{-1}x = \|A^{-1}\|\bar{x}$. By Lemma(3.7)below, $\exists \Delta A \in \mathbb{S}^{\mathrm{HA}}$ with $\|A^{-1}\| \leq 1$ and $\Delta A\bar{x} = x$, so that $A^{-1}\Delta A A^{-1}x = \|A^{-1}\|^2 x$ and we have the result. Finally, for $A \in \mathbb{S}^{\mathrm{TO}}$ then we have $H = JA \in \mathbb{S}^{\mathrm{HA}}$, as above, we concule that there is x and $\Delta H \in \mathbb{S}^{\mathrm{HA}}$ with $H^{-1}\Delta H H^{-1}x = \|H^{-1}\|^2 x$. Then $\Delta A = J\Delta H \in \mathbb{S}^{\mathrm{TO}}$ with $\|A^{-1}\| \leq 1$ and $y = Jx$ with $\|y\| = \|Jx\| = 1$ yields inequality(2) □

Theorem 2.12. *For nonsingular matrix $A \in \mathbb{S}$ we have*

$$d^{\mathbb{S}}(A) = d(A) = (\kappa(A))^{-1} = (\|A^{-1}\|)^{-1} = (\kappa^{\mathbb{S}}(A))^{-1} \quad \textit{where } \mathbb{S} \in (1).$$

Proof. From theorem Ecatr and young[3, Theorem 6.5] we have the departure from singularity on the 2-norm space equals the reciprocal of the norm of the inverse; that is,

$$(\kappa(A))^{-1} = d(A) = \|A^{-1}\|^{-1}$$

where norm of E in the definition(2.10)equals 1 and from Theorem(2.11) we have

$$d^{\mathbb{S}}(A) = (\kappa^{\mathbb{S}}(A))^{-1} = (\|A^{-1}\|)^{-1}.$$

Hence it remais to show that $d^{\mathbb{S}}(A) = d(A)$.

From Corollary(2.3) we have $d^{\mathbb{S}}(A) \geq d(A)$. If we show that $(A+\Delta A)x = 0$ for some $0 \neq x \in \mathbb{C}^{n\times 1}$ and $\Delta A \in \mathbb{S}$ with $\|\Delta A\| = \sigma_{\min}(A)$. we obtain our result. For $A \in \mathbb{S}^{\mathrm{SY}}$, $\exists \lambda \in \mathbb{C}$ and $0 \neq x \in \mathbb{C}^{n\times 1}$ such that $Ax = \lambda x$ and $|\lambda| = \sigma_{\min}(A)$. Suppose $\Delta A = -\lambda I \in \mathbb{S}^{\mathrm{SY}}$. Thus we conclue the proof of matrix $A \in \{\mathbb{S}^{\mathrm{SY}}, \mathbb{S}^{\mathrm{ST}}\}$ because $\mathbb{S}^{\mathrm{ST}} \subseteq \mathbb{S}^{\mathrm{SY}}$. For $A \in \mathbb{S}^{\mathrm{PS}}$ we have $JA \in \mathbb{S}^{\mathrm{SY}}$ and $JAx = \lambda x$ for $0 \neq x \in \mathbb{C}^{n\times 1}$ and $|\lambda| = \sigma_{\min}(JA) = \sigma_{\min}(A)$. Therefore $det(J(A+\Delta A)) = 0 = det(A+\Delta A)$ for $\Delta A = -\lambda J$. And because of $\mathbb{S}^{\mathrm{PSH}} \subseteq \mathbb{S}^{\mathrm{PS}}$ this process above is true for $A \in \mathbb{S}^{\mathrm{PSH}}$. For $A \in \{\mathbb{S}^{\mathrm{HA}}, \mathbb{S}^{\mathrm{TO}}, \mathbb{S}^{\mathrm{CI}}\}$ see proof of theorem[6, theorem 12.2]. □

From Theorem(2.12)and Lemma(2.4,2.5) we have

$$\Lambda_\epsilon^{\mathbb{S}}(A) = \Lambda_\epsilon(A)$$

where $\mathbb{S} \in (1)$.

3 Classed Pseudospectrum of matrix polynomials

We prove an analogous results of [1, Theorem(2.1)] for the classed pseudospectrum of matrix polynomials.

The polynomial eigenvalue problem is to find the solutions $x \in \mathbb{C}^{n\times 1}$ and $\lambda \in \mathbb{C}$ of

$$P(\lambda)x = 0$$

where

$$P(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \lambda^{m-2} A_{m-2} + \cdots + A_0,$$

with $A_k \in \mathbb{C}^{n\times n}$, $k = 0, 1, \cdots, m$. If $x \neq 0$ then λ is called an eigenvalue and x the corresponding eigenvector. The set of eigenvalues of P is denoted by $\Lambda_0(P)$. We assume that P has only finit eigenvalue(that is, A_m is non singular matrix[4]). Concerning notation we denote by $\bar{x} \in \mathbb{C}^{n\times 1}$ the conjugate of $x \in \mathbb{C}^{n\times 1}$.
Let us define

$$\Delta P(\lambda) = \lambda^m \Delta A_m + \lambda^{m-1} \Delta A_{m-1} + \lambda^{m-2} \Delta A_{m-2} + \cdots + \Delta A_0$$

where $\Delta A_k \in \mathbb{C}$. We define the pseudospectra of $P = P(\lambda)$ as the set

$$\Lambda_\epsilon(P) = \{\lambda \in \mathbb{C} : \quad (P(\lambda) + \Delta P(\lambda))x = 0 \,\text{for some}\; x \neq 0, \|A_k\| \leq \alpha_k \epsilon, k = 0, \ldots, m \,\text{and}\, \alpha_k \geq 0\}$$

Lemma 3.1. *[1]* $\Lambda_\epsilon(P) = \{\lambda \in \mathbb{C} : d(P(\lambda)) \leq \epsilon.p(|\lambda|)\}$ *where* $p(\lambda) = \sum_{k=0}^{m} \lambda_k x^k$

Definition 3.2. *[1] Let consider* $\mathbb{S} \in (1)$*, we define the classed pseudospectra of* P *as*

$$\Lambda_\epsilon^{\mathbb{S}}(P) = \{\lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \,\textit{for some}\, x \neq 0, \Delta A_k \in \mathbb{S} \, , \|\Delta A_k\| \leq \alpha_k \epsilon, k = 0, \ldots, m \,\textit{and}\, \alpha_k \geq 0\}$$

Lemma 3.3. *For* $\mathbb{S} \in (1)$ *we have* $\Lambda_\epsilon^{\mathbb{S}}(P) = \{\lambda \in \mathbb{C} : d^{\mathbb{S}}(P(\lambda)) \leq \epsilon.p(|\lambda|)\}$

Proof. ($\rightarrow$) Let $\lambda \in \Lambda_\epsilon^{\mathbb{S}}(P)$ there exists $\Delta P(\lambda) \in \mathbb{S}$ such that $\Delta A_k \in \mathbb{S}$, $\|\Delta A_k\| \leq \alpha_k \epsilon$, $k = 0, 1, \ldots, m$ and $P(\lambda) + \Delta P(\lambda)$ is singular. Thus we have

$$d^{\mathbb{S}}(P(\lambda)) \leq \|\Delta P(\lambda)\|.$$

And

$$\|\Delta P(\lambda)\| \leq \sum_{k=0}^{m} |\lambda|^k \alpha_k \epsilon = \epsilon p(|\lambda|).$$

Thus we have

$$d^{\mathbb{S}}(P(\lambda)) \leq \epsilon p(|\lambda|).$$

($\leftarrow$), let $\lambda \in \mathbb{C}$ such that $d^{\mathbb{S}}(P(\lambda)) \leq \epsilon p(|\lambda|)$, that is, there exists $E \in \mathbb{S}$ such that $d^{\mathbb{S}}(P(\lambda)) = \|E\| \leq \epsilon p(|\lambda|)$ and $P(\lambda) + E$ is singular. We define for each $k = 0, 1, \ldots, m$

$$\Delta A_k = \operatorname{sign}(\lambda^k) \alpha_k p(|\lambda|)^{-1} E \in \mathbb{S}$$

such that $\|\Delta A_k\| \leq \alpha_k \epsilon$, where the sign of the complex λ is defined[1] as

$$\operatorname{sign}(\lambda) = \{ \begin{array}{ll} \frac{\bar{\lambda}}{|\lambda|}, & \lambda \neq 0 \\ 0, & \lambda = 0. \end{array}$$

Thus

$$\Delta P(\lambda) = \sum_{k=0}^{m} \lambda^k \Delta A_k = \{\sum_{k=0}^{m} \lambda^k \operatorname{sign}(\lambda^k) \alpha_k p(|\lambda|)^{-1} E\} = E.$$

Hence $\lambda \in \Lambda_\epsilon^{\mathbb{S}}(P)$ □

Lemma 3.4. *[4]* $\Lambda_\epsilon(P) = \{\lambda \in \mathbb{C} : \|P(\lambda)^{-1}\| \geq \epsilon . p(|\lambda|)\}$

Lemma 3.5. *Let* $\mathbb{S} \in (1)$. *Suppose for* $\lambda \in \Lambda_\epsilon(P)$ *and* $s = d(P\lambda)$, *there existe* $\Delta P \in \mathbb{S}$ *and* $0 \neq x \in \mathbb{C}^{n \times 1}$ *with* $\|\Delta P(\lambda)\| \leq 1$ *and* $Px = s\Delta Px$. *Then* $\lambda \in \Lambda_\epsilon^{\mathbb{S}}(P)$

Proof. Let $\lambda \in \Lambda_\epsilon(P)$ and $s = d(P(\lambda))$. If $\lambda \in \Lambda_0(P)$ then the zero matrix which is in $\mathbb{S}$ does the job. If $\lambda \notin \Lambda_0(P)$ this means that $P(\lambda)$ is nonsingular, and we have $d(P(\lambda)) = \|P(\lambda)^{-1}\|^{-1} \leq \epsilon . p(|\lambda|)$, define $E = -s\Delta P$ then $E \in \mathbb{S}$, $\|E\| = s\epsilon . p(|\lambda|)$ and $(P+E)x = 0$ Suppose that $\Delta A_k = \operatorname{sign}(\lambda^k) \alpha_k p(|\lambda|)^{-1} E \in \mathbb{S}$, $\|\Delta A_k\| \leq \alpha_k \epsilon$ and

$$\sum_{k=0}^{m} \lambda^k \Delta A_k = \{\sum_{k=0}^{m} \lambda^k \operatorname{sign}(\lambda^k) \alpha_k p(|\lambda|)^{-1} E\} = E$$

Thus $\lambda \in \mathbb{C}$ such that

$$(P(\lambda) + \sum_{k=0}^{m} \lambda^k \Delta A_k)x = 0$$

for some $x \neq 0$, $\Delta A_k \in \mathbb{S}$ and $\|\Delta A_k\| \leq \alpha_k \epsilon$. □

Lemma 3.6. *For $P(\lambda) \in \mathbb{S}^{SY} \cap \mathbb{S}^{PS}$ and $\lambda \in \Lambda_0(A)$ there exists an eigenvector x to λ with $x = \alpha Jx$ and $\alpha \in \{-1, 1\}$. If $P(\lambda)$ is real, x can be chosen real.*

Lemma 3.7. *Let $x \in \mathbb{C}^{n\times 1}$ be given. Then there exists $H \in \mathbb{S}^{HA}$ with $Hx = \bar{x}$ and $\|H\| = 1$. If x is real H can be chosen real so that $Hx = x$.*

Lemma 3.8. *Let $x \in \mathbb{C}^{n\times 1}$ with $x = \alpha Jx$ and $\alpha \in \{-1, 1\}$ be given. Then there exists a symmetric Toeplitz matrix $T \in \mathbb{S}^{ST}$ with $Tx = \bar{x}$ and $\|T\| = 1$. If x is real T can be chosen real with $\|T\| = 1$.*

Lemma 3.9. *Let $P(\lambda) \in \mathbb{C}^{n\times n}$ and $s = d(P\lambda)$ then:*

1. *$P(\Lambda) \in \mathbb{S}^{SY} \Longrightarrow \exists 0 \neq x \in \mathbb{C}^{n\times 1} : P(\lambda)x = s\bar{x}$.*
2. *$P(\Lambda) \in \mathbb{S}^{PS} \Longrightarrow \exists 0 \neq x \in \mathbb{C}^{n\times 1} : P(\lambda)x = sJ\bar{x}$.*
3. *$P(\Lambda) \in \mathbb{S}^{SY} \cap \mathbb{S}^{PS} \Longrightarrow \exists 0 \neq x \in \mathbb{C}^{n\times 1} : P(\lambda)x = s\bar{x}, x = \alpha Jx$ and $\alpha \in \{-1, 1\}$.*

Theorem 3.10. *Let $\epsilon > 0$. If $P(\lambda) \in \mathbb{S}$ for each $\mathbb{S} \in (1)$ then*

$$\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P) \tag{3}$$

Proof. We nkown that $\Lambda_\epsilon^{\mathbb{S}}(P) \leq \Lambda_\epsilon(P)$. We have to prove that if $\lambda \in \Lambda_\epsilon(P)$ then $\lambda \in \Lambda_\epsilon^{\mathbb{S}}(P)$. If $\lambda \in \Lambda_0(P)$ then $\lambda \in \Lambda_0^{\mathbb{S}}(P)$ which is contained in $\Lambda_\epsilon^{\mathbb{S}}(P)$. If $\lambda \notin \Lambda_\epsilon^{\mathbb{S}}(P)$ that is $P(\lambda)$ is nonsingular. Let $P(\lambda) \in \mathbb{S}^{\mathrm{SY}}$, by lemma (3.9.1)$\exists 0 \neq x \in \mathbb{C}^{n\times 1}$ such that $P(\lambda)x = s\bar{x}$ and by lemme(3.7)there is $\Delta P(\lambda) \in \mathbb{S}^{\mathrm{HA}} \subseteq \mathbb{S}^{\mathrm{SY}}$ with $\Delta P(\lambda)x = \bar{x}$ and $\|\Delta P(\lambda)\| = 1$. Hence $P(\lambda)x = s\bar{x} = s\Delta P(\lambda)x$ and lemma(3.5) proves $\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P)$. Now $\mathbb{S}^{\mathrm{HA}} \subseteq \mathbb{S}^{\mathrm{SY}}$ and $\Delta P(\lambda) \in \mathbb{S}^{\mathrm{HA}}$ proves $\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P)$.

Let $P(\lambda) \in \mathbb{S}^{\mathrm{PS}}$, by lemma(3.9.2)$\exists 0 \neq x \in \mathbb{C}^{n\times 1}$ with $P(\lambda)x = sJ\bar{x}$.By lemme(3.7)there is $H \in \mathbb{S}^{\mathrm{HA}}$ with $Hx = \bar{x}$ and $\|H\| = 1$. Then $\Delta P(\lambda) = JH \in \mathbb{S}^{\mathrm{TO}} \subseteq \mathbb{S}^{\mathrm{PS}}$, $\|\Delta P(\lambda)\| = 1$ and $P(\lambda)x = sJ\bar{x} = sJHx = s\Delta P(\lambda)x$ and by lemma(3.5) we obtain $\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P)$. Futhermore, $\mathbb{S}^{\mathrm{TO}} \subseteq \mathbb{S}^{\mathrm{PS}}$, so $\Delta P(\lambda) \in \mathbb{S}^{\mathrm{TO}}$ and consequently we have $\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P)$.

Let $P(\lambda) \in \mathbb{S}^{\mathrm{ST}} \subseteq \mathbb{S}^{\mathrm{SY}} \cap \mathbb{S}^{\mathrm{PS}}$ so by lemma(3.9.3)$\exists 0 \neq x \in \mathbb{C}^{n\times 1}$ such that $P(\lambda)x = s\bar{x}$, $x = \alpha Jx$ and $\alpha^2 = 1$. Now by lemme(3.8)there is $\Delta P(\lambda) \in \mathbb{S}^{\mathrm{ST}}$ with $\Delta P(\lambda)x = \bar{x}$ and $\|\Delta P(\lambda)\| = 1$. Therefore, $P(\lambda)x = s\bar{x} = s\Delta P(\lambda)x$ and by lemma(3.5) we have $\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P)$.

Let $P(\lambda) \in \mathbb{S}^{\mathrm{PSH}} \subseteq \mathbb{S}^{\mathrm{SY}} \cap \mathbb{S}^{\mathrm{PS}}$ so by lemma(3.9.3)$\exists 0 \neq x \in \mathbb{C}^{n\times 1}$ with $P(\lambda)x = s\bar{x}$, $x = \alpha Jx$ and $\alpha^2 = 1$ and by lemme(3.8)there is $H \in \mathbb{S}^{\mathrm{ST}}$ with $Hx = \bar{x}$ and $\|H\| = 1$. Then $\Delta P(\lambda) = \alpha HJ \in \mathbb{S}^{\mathrm{PSH}}$ implies $\|\Delta P(\lambda)\| = 1$

with $P(\lambda)x = s\bar{x} = sH\alpha Jx = s\Delta P(\lambda)x$ and by lemma(3.5) we obtain $\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P)$.

Finally, Let $P(\lambda) \in \mathbb{S}^{\mathrm{CI}}$ implies $P(\lambda)$ is normal. So there is $\exists 0 \neq x \in \mathbb{C}^{n\times 1}$ with $P(\lambda)x = s\beta x$, $\|\beta\| = 1$. Then $\Delta P(\lambda) = \beta I \in \mathbb{S}^{\mathrm{CI}}$, $\|\Delta P(\lambda)\| = 1$ and$P(\lambda)x = s\beta x = s\Delta P(\lambda)x$ finish this part and the proof. □

Remark 3.11. 1. *When we siad that there is (for example)*$\Delta P(\lambda) \in \mathbb{S}^{HA}$ *with* $\Delta P(\lambda)x = \bar{x}$, $\|\Delta P(\lambda)\| = 1$, *in fact there is* $H \in \mathbb{S}^{HA}$ *such that* $H = \Delta P(\lambda)$ *and for finding* $\Delta A_k \in \mathbb{S}^{HA}$ *for each* $k = 0, 1, \ldots, m$ *we define*

$$\Delta A_k = sign(\lambda^k)\alpha_k p(|\lambda|)^{-1}H$$

such that $\|\Delta A_k\| \leq \alpha_k \epsilon$. *Thus*

$$H = \Delta P(\lambda) = \sum_{k=0}^{m} \lambda^k \Delta A_k = \{\sum_{k=0}^{m} sign(\lambda^k)\alpha_k p(|\lambda|)^{-1}H\}.$$

2. *We can prove that theorem(3.10) by applying theorem(2.12)because we have proved if* $\lambda \in \Lambda_0(P)$ *then* $\lambda \in \Lambda_0^{\mathbb{S}}(P)$ *which is contained in* $\Lambda_\epsilon^{\mathbb{S}}(P)$ *but if* $\lambda \notin \Lambda_\epsilon^{\mathbb{S}}(P)$ *that is* $P(\lambda)$ *is nonsingular and we can apply Theorem(2.12) and Lemma(3.1, 3.3).*

Remark 3.12. *To prove lemma(2.5), lemma3.3 and Theorem (2.11) for complex classed in (1) we used the same proof line of lemma[1, Lemma 3.2], lemma[1, lemma 5.2] and Lemma[6, Lemma (11.1),(11.2)] respectively.*

4 Pseudospectrum of matrix polynomials with backward error

A nuatural of the normwise backward error of an approximate eigenpair (x, λ) of $P(\lambda)x = 0$ is

$$\begin{aligned} \eta(x,\lambda) = \ & \min\{\epsilon : (P(\lambda) + \Delta P(\lambda))\, x = 0, \\ & \|\Delta A_k\| \leq \epsilon\alpha_k, k = 0, 1, \ldots, m\}, \end{aligned} \tag{4}$$

and the backward error for an approximate eigenvalue λ is given by

$$\eta(\lambda) = \min_{x\neq 0} \eta(x, \lambda). \tag{5}$$

Remark 4.1. *A polynomial eigenvalue problem is in (1) when all matrices* A_k *are in (1).*

Lemma 4.2. *[4] The normwise backward error $\eta(x,\lambda)$ is given for $x \neq 0$ by*

$$\eta(x,\lambda) = \frac{\|r\|}{p(|\lambda|)\|x\|}, \tag{6}$$

where $r = P(\lambda)x$ and $p(x) = \sum_{k=0}^{m} \alpha_k x^k$. If λ is not an eigenvalue of P then

$$\eta(\lambda) = \frac{1}{p(|\lambda|)\|P(\lambda)^{-1}\|}. \tag{7}$$

Now, the normwise backward error of an approximate eigenpair (x,λ) of classed polynomial eigenvalue problems are

$$\begin{aligned} \eta^{\mathbb{S}}(x,\lambda) = \quad & \min\{\epsilon : (P(\lambda) + \Delta P(\lambda))\, x = 0, \Delta A_k \in \mathbb{S}, \\ & \|\Delta A_k\| \leq \epsilon\alpha_k, k = 0, 1, \ldots, m\}, \end{aligned} \tag{8}$$

and the backward error for an approximate eigenvalue λ is given by

$$\eta^{\mathbb{S}}(\lambda) = \min_{x\neq 0} \eta^{\mathbb{S}}(x,\lambda). \tag{9}$$

Corollary 4.3. *We have $\eta^{\mathbb{S}}(x,\lambda) \geq \eta(x,\lambda)$.*

Proposition 4.4. *If P is Hermitian matrix polynomial and $\lambda \in \mathbb{R}$ then*

$$\eta^{HE}(x,\lambda) = \eta(x,\lambda)$$

And consequently, $\eta^{HE}(\lambda) = \eta(\lambda)$

Proof. Let $r = P(\lambda)x$ such that $\|x\| = 1$, we are looking for a Hermitian matrix $\Delta P(\lambda)$ such that $\eta^{\mathrm{HE}}(\lambda) = \dfrac{\|r\|}{p(|\lambda|)}$. We take $\Delta P(\lambda) = \|r\|I$.

Let ΔA_k be Hermitian matrices defined by

$$\Delta A_k = \mathrm{sign}(\lambda^k)\alpha_k p(|\lambda|)^{-1}\|r\|I \tag{10}$$

such that $\|\Delta A_k\| \leq \alpha_k\epsilon$ for $i, j = 1, \ldots, k$. Using the equality in Lemma(4.2), we get

$$\|\Delta P(\lambda)\| = \|r\| \leq \eta(x,\lambda)p(\|\lambda\|)^{-1}$$

From equation(10) we deduce $\eta^{\mathrm{HE}}(\tilde{x},\tilde{\lambda}) \leq \eta(\tilde{x},\tilde{\lambda})$. Consequently, from corollary (4.3) we have $\eta^{\mathrm{HE}}(\tilde{x},\tilde{\lambda}) = \eta(\tilde{x},\tilde{\lambda})$. □

Proposition 4.5. *If P is in $\mathbb{S}^{SHE}$ and $\lambda \in \mathbb{R}$ then*

$$\eta^{SHE}(x,\lambda) = \eta(x,\lambda)$$

And consequently, $\eta^{SHE}(\lambda) = \eta(\lambda)$

Proof. By taking $\Delta P(\lambda) = \|r\|\sqrt{-1}I$ we will finish ours proof. □

Proposition 4.6. *[4] The pseudospectrum can be expressed in term of the backward error of λ as*

$$\Lambda_\epsilon(P) = \{\lambda \in \mathbb{C} : \eta(\lambda) \le \epsilon\}. \tag{11}$$

Corollary 4.7. *If $P \in \mathbb{S} = \{\mathbb{S}^{HE}, \mathbb{S}^{SHE}\}$ and $\lambda \in \mathbb{R}$ then*

$$\Lambda_\epsilon^{\mathbb{S}}(P) = \Lambda_\epsilon(P) \cap \mathbb{R}.$$

Corollary 4.8.

$$\begin{aligned} \Lambda_\epsilon(P) &= \{\lambda \in \mathbb{C} : d(P(\lambda)) \le \epsilon.p(|\lambda|)\} \\ &= \{\lambda \in \mathbb{C} : \eta(\lambda) \le \epsilon\}. \end{aligned}$$

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