ON SUBALGEBRAS OF C^*-ALGEBRAS

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Communicated by Murray Gerstenhaber, January 14, 1969

In this note we announce some new methods and results in the theory of nonnormal Hilbert space operators and nonselfadjoint operator algebras. A main difficulty in the subject has been the apparent absence of relations between, say, a nonselfadjoint algebra of operators and its generated C^*-algebra. For example, given full information about the norm-closed algebra $P(T)$ generated by all polynomials in a given (nonnormal) operator T, what can one say about the C^*-algebra $C^*(T)$ generated by T and the identity? While one cannot expect much of an answer in general, we will describe here a class of operators and operator algebras for which these relations are as simple as one could hope for.

All C^*-algebras are assumed to contain an identity (written as e), $L(H)$ denotes the algebra of all bounded operators on a Hilbert space H, and $C^*(S)$ stands for the C^*-algebra generated by S and the identity where S is either an operator or a subset of a C^*-algebra. An operator is irreducible if it commutes with no nontrivial projections.

1. **An extension theorem.** Let S be a linear subspace of a C^*-algebra B, such that S contains the identity of B. A linear map ϕ of S into another C^*-algebra is *positive* if $\phi(x) \geqq 0$ for every positive element x of S (note, however, that S may contain no positive elements other than scalars). A familiar theorem of M. Krein implies that if $S = S^*$, then every scalar-valued positive linear map of S has a positive extension to B. We first describe a generalization of Krein's theorem to operator-valued maps which is basic for virtually all of the sequel. If M_n is the algebra of all complex $n \times n$ matrices, then $B \otimes M_n$ is the *-algebra of all $n \times n$ matrices over B. There is a unique C^*-algebra norm on $B \otimes M_n$, and $S \otimes M_n$ is a linear subspace of this C^*-algebra. A linear map ϕ of S into a C^*-algebra B' induces, for every $n \geqq 1$, a linear map $\phi_n : S \otimes M_n \to B' \otimes M_n$ by applying ϕ element by element to each matrix over S. ϕ is *completely contractive* or *completely isometric* according as each ϕ_n is contractive ($\|\phi_n\| \leqq 1$) or isometric. ϕ is *completely positive* if each ϕ_n is a positive linear map.

[1] Research supported by NSF grant GP-5585 and the U.S. Army Research Office, Durham.

THEOREM 1. *Let S be a closed selfadjoint linear subspace of a C^*-algebra B, which contains the identity of B, and let H be a Hilbert space. Then every completely positive linear map of S into $L(H)$ has a completely positive extension to B.*

It is easily shown that a scalar-valued positive linear map of S is already completely positive; thus Theorem 1 generalizes Krein's theorem. We remark also that Theorem 1 is false if one deletes the adverb "completely," even when B is commutative.

2. **Boundary representations.** Let S be a closed subspace of a C^*-algebra B, such that e belongs to S. S need not be selfadjoint.

DEFINITION. *An irreducible representation π of B is called a boundary representation for S if the only completely positive linear extension of $\pi|_S$ to B is π itself.*

When $B = C(X)$ (for X a compact Hausdorff space), boundary representations correspond to points in the Choquet boundary of X relative to S. In the noncommutative cases, boundary representations occur in a variety of ways, as the following two examples show.

EXAMPLE 1. Let T be an operator on a Hilbert space. $\mathrm{sp}(T)$ and $W(T)$ will denote the spectrum of T and the numerical range of T, respectively. A *character* of a C^*-algebra is a one-dimensional representation (i.e., a complex homomorphism), necessarily irreducible. The following example seems to have independent interest.

THEOREM 2. *For every point λ in $\mathrm{sp}(T) \cap \partial W(T)$, there is a character π of $C^*(T)$ such that $\pi(T) = \lambda$. π is a boundary representation for $P(T)$.*

COROLLARY. *If $C^*(T)$ has no characters, then the spectral radius of T is less than $\|T\|$.*

EXAMPLE 2. Let ψ be an inner function in H^∞ (of the unit disc), and let Z_ψ be the "zero set" of ψ: Z_ψ consists of all zeros of ψ in the interior of the unit disc along with all boundary points λ such that ψ cannot be analytically continued from the interior to λ. S_ψ denotes the projection of the bilateral shift onto $H^2 \ominus \psi H^2$. S_ψ is an irreducible operator, so that the identity representation of $C^*(S_\psi)$ is irreducible.

THEOREM 3. *If $Z_\psi \cap \{|z| = 1\}$ has (linear) Lebesgue measure zero, then the identity representation is a boundary representation for $P(S_\psi)$. If, on the other hand, Z_ψ contains the entire unit circle, then the identity representation is not a boundary representation for $P(S_\psi)$.*

We remark that the proof of the first half of Theorem 3 utilizes machinery we have not discussed, and the result is a principal one in this theory.

The following theorem gives the key property of boundary representations.

THEOREM 4. *Let S_i be a linear subspace of a C^*-algebra B_i, $i=1, 2$, each containing the respective identity e_i. Assume $B_i = C^*(S_i)$, and let ϕ be a linear completely isometric map of S_1 on S_2 such that $\phi(e_1)=e_2$. Then for every boundary representation π of B_1 for S_1 there is a (unique) boundary representation π' of B_2 for S_2 such that $\pi' \circ \phi = \pi$ on S_1.*

Theorem 4 has a number of general consequences, relating to "Silov boundaries," of which we mention only the following.

COROLLARY. *Let $S_i \subseteq B_i$, $i=1, 2$, be as above. Assume that the intersection of the kernels of all boundary representations of B_i for S_i is 0. Then every completely isometric linear map of S_1 on S_2, which preserves the identity, is implemented by a *-isomorphism of B_1 on B_2.*

This corollary implies that certain subspaces (and subalgebras) of C^*-algebras completely determine the structure of the C^*-algebra they generate. It can be shown that the hypotheses are satisfied if there are enough unitary elements in the closure of $S_i + S_i^*$ to generate B_i as a C^*-algebra, $i=1, 2$.

3. **Some applications.** We describe three applications of the preceding in the theory of nonnormal operators on Hilbert space. Let T be a nonscalar simple algebraic operator (i.e., $C^*(T)$ is simple and T satisfies a polynomial equation $p(T)=0$). Assume $\|T\|=1$, and let $p(z)=(z-a_1)^{n_1}(z-a_2)^{n_2} \cdots (z-a_k)^{n_k}$ be the minimum polynomial of T. The corollary of Theorem 2 implies that $|a_i|<1$, $1 \leqq i \leqq k$, and so the Blaschke product B_T which has p as its numerator is in H^∞. T is called *maximal* if $\|B(T)\|=1$ for every proper Blaschke divisor B of B_T. Thus, if $p(z)=z^n$, then T is maximal iff $\|T\| = \cdots = \|T^{n-1}\| = 1$. In general, simple algebraic operators can generate complicated C^*-algebras; for example, there are operators T having cube zero for which $C^*(T)$ is an (antiliminal) UHF algebra. Nevertheless, we have

THEOREM 5. *If T is a maximal simple algebraic operator, then $C^*(T)$ is *-isomorphic with M_n, n being the order of the minimum polynomial of T. An irreducible maximal simple algebraic operator is finite-dimensional, and two such are unitarily equivalent if, and only if, they have the same minimum polynomial.*

The second application is similar to the preceding, for operators having an "infinite" minimum polynomial. Let T be an arbitrary contraction, and let A be the norm-closure (in $C(T)$, T being the unit circle) of the polynomials in $e^{i\theta}$. Because the unit disc is a spectral set for T, we have $\|p(T)\| \leqq \|p\|$ for every polynomial $p \in A$, and thus there is a unique contractive homomorphism ϕ of A such that $\phi(p) = p(T)$ for every polynomial p. Assume that ϕ has *nonzero* kernel. Then $\ker \phi$ has the form $\psi \cdot A_K$, where K is a closed set in $\boldsymbol{T}$ of Lebesgue measure zero, ψ is an inner function for which $Z_\psi \cap \boldsymbol{T} \subseteq K$, and A_K denotes all functions in A which vanish on K. ψ is called the *minimum function* for T. The minimum function is undefined if $\ker \phi = 0$.

For each $n \geqq 1$, ϕ induces a (contractive) homomorphism ϕ_n of the Banach algebra $A \otimes M_n (\subseteq C(\boldsymbol{T}) \otimes M_n)$, whose kernel is $\psi \cdot A_K \otimes M_n$: T is called *maximal* if the canonical homomorphism of $A \otimes M_n / \psi \cdot A_K \otimes M_n$ induced by ϕ_n is isometric, for every $n \geqq 1$. It can be seen that this usage of the term is in harmony with the foregoing.

Let ψ be an inner function such that $Z_\psi \cap \boldsymbol{T}$ is of measure zero. Then S_ψ is an example of a maximal operator which has ψ as its minimum function. Moreover, S_ψ is irreducible and $S_\psi{}^* S_\psi - S_\psi S_\psi{}^*$ has finite rank.

THEOREM 6. *Let T_1 and T_2 be irreducible operators such that both commutators $T_i{}^* T_i - T_i T_i{}^*$ are compact. Assume that both operators possess minimum functions and are maximal. Then T_1 and T_2 are unitarily equivalent if, and only if, their minimum functions are proportional.*

Our third application is to the Volterra operator V, defined on $L^2(0, 1)$ by $Vf(x) = \int_0^x f(t)dt$, $f \in L^2(0, 1)$. V is well known to be compact and irreducible. If $C_0, C_1, \cdots, C_k$ are $n \times n$ matrices, then $p(z) = C_0 + C_1 z + \cdots + C_k z^k$ defines an M_n-valued polynomial; we define $p(T)$ for an operator $T \in L(H)$ by $p(T) = C_0 \otimes I + C_1 \otimes T + \cdots + C_k \otimes T^k$, regarded as an operator on $\boldsymbol{C}^n \otimes H$.

THEOREM 7. *Let T be an irreducible operator for which $T^*T - TT^*$ is compact. Suppose $\|p(T)\| = \|p(V)\|$ for every matrix-valued polynomial p. Then T is unitarily equivalent to V.*

The norm condition $\|p(T)\| = \|p(V)\|$ is assumed to hold for every M_n-valued polynomial p, and every $n \geqq 1$. It can be shown that the condition for $n = 1$ already implies $\|p(T)\| \leqq \|p(V)\|$ for $n > 1$, but we do not know if the opposite inequality is also redundant for $n > 1$.

Two counterexamples are noteworthy. First, there exist (three-dimensional) operators S and T for which $\|p(S)\| = \|p(T)\|$ holds for all scalar-valued polynomials p, but not for all matrix-valued polynomials; put differently, not every isometric representation of a subalgebra of a C^*-algebra is completely contractive, even when the subalgebra is singly-generated. Second, while a characterization like Theorem 7 holds for operators other than V, it is of limited generality. For example, if ψ_1 and ψ_2 are inner functions such that Z_{ψ_1} and Z_{ψ_2} both contain the entire unit circle, then S_{ψ_i} is irreducible and $S_{\psi_i}^*S_{\psi_i} - S_{\psi_i}S_{\psi_i}^*$ is compact, $\|p(S_{\psi_1})\| = \|p(S_{\psi_2})\|$ is valid for every matrix-valued polynomial p, but these two operators are not unitarily equivalent if ψ_1 and ψ_2 are not proportional. The second statement of Theorem 3 explains why the proof of Theorem 7 breaks down for these examples.

Finally, we remark that Theorems 6 and 7 remain valid when the hypothesis $T^*T - TT^*$ compact is replaced with the weaker condition: the commutator ideal in $C^*(T)$ is a minimal (closed, two-sided) ideal. Full details and further developments will appear elsewhere.

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