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On Submanifolds of $N(k)$-Quasi Einstein Manifolds with a Type of Semi-Symmetric Metric Connection

İnan Ünal[1]

[1]*Department of Computer Engineering, Faculty of Engineering, Munzur University, Tunceli, Turkey*

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Abstract

In this study, we consider the $N(k)-$quasi Einstein manifolds with respect to a type of semi-symmetric metric connection. We suppose that the generator of $N(k)-$quasi-Einstein manifolds is parallel with respect to semi-symmetric metric connection and we classify such manifolds. In addition, we consider the submanifolds of a $N(k)-$quasi Einstein manifold and we obtain some conditions on the totally geodesic and the totally umbilic submanifolds. Finally, we consider a para-Kenmotsu space form as an example of $N(k)-$quasi-Einstein manifolds.

1. Introduction

An Einstein manifold is a Riemannian manifold (M,g) satisfying Einstein fields equation. We determine such manifold by $Ric = \lambda g$, for the Ricci curvature Ric of M non-zero constant λ. In differential geometry, there are many kind of manifolds which satisfy this relation. Einstein manifolds are widely studied by researchers from mathematics and physics. A well known generalization of Einstein manifolds is the notion of quasi-Einstein manifolds defined by Chaki in [5]. Similar to Einstein manifolds , quasi-Einstein manifolds are also occur in the solutions of Einstein field equations. In this manner, quasi-Einstein manifolds have some applications in the general relativity. An example is Robertson-Walker space times [8]. A quasi-Einstein manifold is a Riemannian manifold (M,g) which has the following relation on the Ricci tensor of M;

$$Ric(\Omega_1,\Omega_2) = ag(\Omega_1,\Omega_2) + b\eta(\Omega_1)\eta(\Omega_2) \tag{1.1}$$

for some smooth functions a and b, arbitrary vector fields $\Omega_1,\Omega_2 \in \Gamma(TM)$, where η is a non-zero $1-$form on M such that $g(\Omega_1,\xi) = \eta(\Omega_1)$, $\eta(\xi) = 1$ for a vector field $\xi \in \Gamma(TM)$. We call η by associated $1-$ form and ξ by the generator of the manifold. If a $(2m+1)$-dimensional Riemannian manifold M has an almost contact metric structure (ϕ,ξ,η,g) and Ricci tensor satisfies (1.1) then M is called by an $\eta-$Einstein manifold [1]. So, an $\eta-$Einstein manifold is an example of quasi-Einstein manifolds. Also, a generalized Sasakian space form is a quasi-Einstein manifold [6].

k-nullity distribution of a quasi Einstein manifold is defined as

$$N(k): p \longrightarrow N_p(k) = \left[\Omega_3 \in \Gamma(T_pM) : Rim(\Omega_1,\Omega_2)\Omega_3 = k\left\{g(\Omega_2,\Omega_3)\Omega_1 - g(\Omega_1,\Omega_3)\Omega_2\right\}\right], \tag{1.2}$$

for any Ω_1, $\Omega_2 \in \Gamma(T_pM)$ and $k \in \mathbb{R}$, where Rim is the Riemannian curvature tensor of M. If the generator vector field ξ belongs to $k-$nullity distribution then M is called $N(k)$-quasi Einstein manifold $(NK(QE)_m)$ [5]. A quasi Einstein manifold is an $NK(QE)_m$ manifold if it is conformally flat [15]. In 2004 De and Ghosh [7] prove the existence of $NK(QE)_m$ manifolds and presented some results. In 2008 Özgür [3] examined $NK(QE)_m$ manifolds under some certain curvature conditions. Yıldız et al. [4] worked on $NK(QE)_m$ manifolds with some semi-symmetry conditions and gave examples. The Riemannian geometry of $N(k)-$ quasi-Einstein manifolds have been studied by many researchers in [3, 6, 10, 12, 16].

In this work, we consider a $NK(QE)_m$ manifold admitting a type of semi-symmetric metric connection (SSMC) and we obtain some results on the submanifolds of such manifolds. Also, we present a classification of $NK(QE)_m$ manifold admitting SSMC. We proved some theorems on the totally geodesic and totally umbilical submanifolds. Finally, we consider a para-Kenmotsu space form as an example.

Email address and ORCID number**:** **inanunal@munzur.edu.tr, https:// orcid.org/0000-0003-1318-9685 (İ. Ünal)**

2. N(k)-quasi Einstein manifolds with a type of semi-symmetric metric connection

In the Riemannian geometry, we know that the Levi-Civita connection have no torsion and it is a metric connection. Also, there are many type of connections which has torsion and not symmetric. One of them is a semi-symmetric metric connection (SSMC) . In the [17] Yano defined a type of SSMC. Murathan and Özgür [3] studied Riemannian manifolds with this connection under some semi-symmetry conditions. The authors consider the parallel unit vector field with respect to the Levi-Civita connection. In this section, we consider a $NK(QE)_m$ manifold with the parallel vector field ξ with respect to SSMC. We present some results related to SSMC.
Let M be an $m-$dimensional $NK(QE)_m$ manifold and define a map on M by

$$\overset{\approx}{\nabla}_{\Omega_1}\Omega_2 = \widetilde{\nabla}_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1,\Omega_2)\xi \tag{2.1}$$

for all $\Omega_1,\Omega_2 \in \Gamma(TM)$, where $\widetilde{\nabla}$ is the Levi-Civita connection (LCC) on M. The map $\overset{\approx}{\nabla}$ on M defines a semi-symmetric metric connection [17]. The Riemannian curvature of M with respect to $\overset{\approx}{\nabla}$ was obtained in [17] as;

$$\begin{aligned}\overset{\approx}{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = \widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &- \omega(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) + \omega(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4) \\ &- g(\Omega_2,\Omega_3)\omega(\Omega_1,\Omega_4) + g(\Omega_1,\Omega_3)\omega(\Omega_2,\Omega_4)\end{aligned} \tag{2.2}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4 \in \Gamma(TM)$, where ω is defined as

$$\omega(\Omega_1,\Omega_2) = (\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2 - \eta(\Omega_1)\eta(\Omega_2) + \frac{1}{2}g(\Omega_1,\Omega_2).$$

From (2.1) we obtain

$$\overset{\approx}{\nabla}_{\Omega_1}\xi = \widetilde{\nabla}_{\Omega_1}\xi + \Omega_1 - \eta(\Omega_1)\xi.$$

Suppose that $\overset{\approx}{\nabla}_{\Omega_1}\xi = 0$. Then, we recall ξ by parallel vector field with respect to SSMC. Thus, we get

$$\widetilde{\nabla}_{\Omega_1}\xi = -\Omega_1 + \eta(\Omega_1)\xi. \tag{2.3}$$

On the other hand, we have

$$(\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2 = \widetilde{\nabla}_{\Omega_1}\eta(\Omega_2) - \eta(\widetilde{\nabla}_{\Omega_1}\Omega_2).$$

Since, $\widetilde{\nabla}$ is a metric connection i.e $(\widetilde{\nabla}_{\Omega_1}g)(\Omega_2,\Omega_3) = g(\widetilde{\nabla}_{\Omega_1}\Omega_2,\Omega_3) + g(\Omega_3,\widetilde{\nabla}_{\Omega_1}\Omega_2)$, from (2.3) we get

$$(\tilde{\nabla}_{\Omega_1}\eta)\Omega_2 = -g(\Omega_1,\Omega_2) + \eta(\Omega_1)\eta(\Omega_2).$$

Thus, we obtain $\omega(\Omega_1,\Omega_2) = -\frac{1}{2}g(\Omega_1,\Omega_2)$ and so from (2.2), we get

$$\overset{\approx}{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = \widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) + g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4). \tag{2.4}$$

In [2] it was proved that in a $NK(QE)_m$ manifold $k = \frac{a+b}{m-1}$. Thus, from (1.2) , we obtain

$$\overset{\approx}{R}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = (\frac{a+b}{m-1} + 1)[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)] \tag{2.5}$$

Finally, we state that

Theorem 2.1. *Let M be a $NK(QE)_m$ manifold with respect to a SSMC $\overset{\approx}{\nabla}$ and ξ be a parallel vector field with respect to $\overset{\approx}{\nabla}$. We have following classifications;*

- *If $a+b = 1-m$ then M is locally isometric to $m-$dimensional Euclidean space $\mathbb{E}^m$,*
- *If $a+b > 1-m$ then M is locally isometric to $m-$dimensional sphere $S^m(\frac{a+b}{m-1}+1)$,*
- *If $a+b < 1-m$ then M is locally isometric to $m-$dimensional hyperbolic space $H^n(\frac{a+b}{m-1}+1)$.*

Let take an orthonormal basis of M as $\{E_1,E_2,...,E_{m-1},E_m = \xi\}$. Then with taking sum over $1 \leq i \leq m$ in (2.4) we obtain

$$\sum_{i=1}^{m}\overset{\approx}{Rim}(\Omega_1,E_i,E_i,\Omega_4) = \sum_{i=1}^{m}\{\widetilde{Rim}(\Omega_1,E_i,E_i,\Omega_4) + g(E_i,E_i)g(\Omega_1,\Omega_4) - g(\Omega_1,E_i)g(E_i,\Omega_4)\}$$

and so, we get

$$\overset{\approx}{Ric}(\Omega_1,\Omega_4) = \widetilde{Ric}(\Omega_1,\Omega_4) + (m-1)g(\Omega_1,\Omega_4)$$

for all $\Omega_1,\Omega_2 \in \Gamma(TM)$. Then from (1.1), we obtain

$$\overset{\approx}{Ric}(\Omega_1,\Omega_4) = (a+(m-1))g(\Omega_1,\Omega_4) + bm\eta(\Omega_1)\eta(\Omega_2)$$

Finally, we conclude that;

Theorem 2.2. *Let M be an $NK(QE)_m$ manifold with respect to a LCC $\tilde{\nabla}$ and ξ be a parallel vector field with respect to SSMC $\overset{\approx}{\nabla}$. Then M is an $NK(QE)_m$ manifold with respect to $\overset{\approx}{\nabla}$.*

3. Submanifolds of N(k)-quasi Einstein manifolds with a type of semi-symmetric metric connection

Let M be an $m-$dimensional $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$and N be an $n-$dimensional submanifold of M. Suppose that the generator vector field ξ tangent to N. Thus, we have two subbundles of TM as TN and $TN^{\perp}$ such that $TM = TN \oplus TN^{\perp}$.The subbundles TN and $TN^{\perp}$ are called tangent bundle and normal bundle of N, respectively. Let recall some classical equations from the submanifold theory. For details we refer to reader [1].
The Gauss equation is given by

$$\widetilde{\nabla}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \sigma(\Omega_1,\Omega_2)$$

for all $\Omega_1,\Omega_2 \in \Gamma(TN)$, where $\sigma(\Omega_1,\Omega_2)$ denote the second fundamental form, and $\widetilde{\nabla},\nabla$ are the Levi-Civita connections on M and N, respectively.
The Weingarten equation is

$$\widetilde{\nabla}_{\Omega_1}W = -A_W\Omega_1 + \nabla^{\perp}_{\Omega_1}W$$

for all $\Omega_1 \in \Gamma(TN)$ and $W \in \Gamma(TN^{\perp})$, where A_W is the shape operator related to W, $\nabla^{\perp}$ is the induced normal connection on the normal bundle $TN^{\perp}$. Consider the definition of SSMC $\overline{\overline{\nabla}}$ and using the Gauss equation, we get

$$\overline{\overline{\nabla}}_{\Omega_1}\Omega_2 = \nabla_{\Omega_1}\Omega_2 + \eta(\Omega_2)\Omega_1 - g(\Omega_1,\Omega_2)\xi + \sigma(\Omega_1,\Omega_2). \tag{3.1}$$

Suppose that ξ is parallel with respect to $\overline{\overline{\nabla}}$, then we obtain

$$\nabla_{\Omega_1}\xi = -\Omega_1 + \eta(\Omega_1)\xi - \sigma(\Omega_1,\xi).$$

Hence, we provide the following lemma.

Lemma 3.1. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\tilde{\nabla}$, N be a submanifold of M, and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. Then, we get*

$$\nabla_{\Omega_1}\xi = -\Omega_1 + \eta(\Omega_1)\xi\ ,\ \ \sigma(\Omega_1,\xi) = 0$$

for all $\Omega_1 \in \Gamma(TN)$, where $\xi \in \Gamma(TN)$.

Also, we know that

$$(\widetilde{\nabla}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = \nabla^{\perp}_{\Omega_1}(\sigma(\Omega_1,\Omega_2)) - \sigma(\nabla_{\Omega_1}\Omega_2,\Omega_3) - \sigma(\Omega_2,\nabla_{\Omega_1}\Omega_3) \tag{3.2}$$

for all $\Omega_1,\Omega_2,\Omega_3 \in \Gamma(TN)$ [1].

Definition 3.2. *Let M be an $NK(QE)_m$ manifold and N be submanifold of M. If the covariant derivation of the second fundamental form vanishes, then N is called parallel submanifold [1].*

Theorem 3.3. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. If N is parallel submanifold with respect to LCC $\widetilde{\nabla}$ then it is not parallel submanifold with respect to SSMC $\overline{\overline{\nabla}}$.*

Proof. From the definition of SSMC $\overline{\overline{\nabla}}$, we have

$$\begin{aligned}(\overline{\overline{\nabla}}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = {} & \widetilde{\nabla}_{\Omega_1}\sigma(\Omega_1,\Omega_2) - \sigma(\widetilde{\nabla}_{\Omega_1}\Omega_2,\Omega_3) - \eta(\Omega_2)\sigma(\Omega_1,\Omega_3) - g(\Omega_1,\Omega_2)\sigma(\xi,Z)\\ & - \sigma(\Omega_2,\tilde{\nabla}_{\Omega_1}\Omega_3) - \eta(\Omega_3)\sigma(\Omega_1,\Omega_2) - g(\Omega_1,\Omega_3)\sigma(\Omega_2,\xi).\end{aligned}$$

Since ξ is parallel with respect to SSMC $\overline{\overline{\nabla}}$, by using Lemma 3.1 we obtain

$$(\overline{\overline{\nabla}}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = \nabla^{\perp}_{\Omega_1}(\sigma(\Omega_1,\Omega_2)) - \sigma(\nabla_{\Omega_1}\Omega_2,\Omega_3) - \sigma(\Omega_2,\nabla_{\Omega_1}\Omega_3) - \eta(\Omega_2)\sigma(\Omega_1,\Omega_3) - \eta(\Omega_3)\sigma(\Omega_1,\Omega_2).$$

Suppose that, N is parallel with respect to LCC $\widetilde{\nabla}$. Then, from (3.2) we have

$$(\overline{\overline{\nabla}}_{\Omega_1}\sigma)(\Omega_2,\Omega_3) = -\eta(\Omega_2)\sigma(\Omega_1,\Omega_3) - \eta(\Omega_3)\sigma(\Omega_1,\Omega_2).$$

Thus N is not parallel with respect to SSMC $\overline{\overline{\nabla}}$. □

We also state following result.

Corollary 3.4. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. If N is parallel with respect to SSMC $\overline{\overline{\nabla}}$ then it is not parallel with respect to LCC $\widetilde{\nabla}$.*

The Codazzi equation for N is given by

$$\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) + g(\sigma(\Omega_1,\Omega_3),\sigma(\Omega_2,\Omega_4)) - g(\sigma(\Omega_2,\Omega_3),\sigma(\Omega_1,\Omega_4)) \tag{3.3}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4 \in \Gamma(TN)$, where $\widetilde{Rim}$ is the Riemannian curvature tensor of M and Rim is the Riemannian curvature tensor of N [1].
Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$ and N be a submanifold of M. From (2.4) and (3.2), we get

$$\begin{aligned}\overline{\overline{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) + g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4)) - g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4)\\ &+ g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4).\end{aligned}$$

Thus , by using (2.5) we obtain

$$Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = \frac{a+b}{m-1}[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) + g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)] - g(\sigma(\Omega_1,\Omega_3),\sigma(\Omega_2,\Omega_4)) + g(\sigma(\Omega_2,\Omega_3),\sigma(\Omega_1,\Omega_4)]$$

Finally, we state the following theorem.

Theorem 3.5. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. If N is totally geodesic , then N is an $NK(QE)_m$ manifold with $k = \frac{a+b}{m-1}$.*

On the other hand if N is totally umbilical, i.e. $\sigma(\Omega_1,\Omega_2) = Hg(\Omega_1,\Omega_2)$, then we get

$$Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) = (\frac{a+b}{m-1} + g(H,H))[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) + g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)].$$

where H is the mean curvature of N. Therefore we can state following theorem.

Theorem 3.6. *Let M be an $NK(QE)_m$ manifold with respect to SSMC $\overline{\overline{\nabla}}$, N be a submanifold of M and ξ be a parallel vector field with respect to SSMC $\overline{\overline{\nabla}}$. If N is totally umbilical , then N is a generalized real space form.*

Example 3.7. *Let M be a $(2m+1)-$dimensional smooth manifold. (ϕ,ξ,η) is called an almost para-contact structure on M such that*

$$\phi^2\Omega = \Omega - \eta(\Omega)\xi,\;\; \phi(\xi) = 0,\quad \eta\circ\phi = 0,\quad \eta(\xi) = 1 \tag{3.4}$$

where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form, and Ω is an arbitrary vector field on M [18]. M is called a para-Kenmotsu (PK) manifold if we have

$$\left(\widetilde{\nabla}_{\Omega_1}\phi\right)\Omega_2 = -g(\phi\Omega_1,\Omega_2)\xi + \eta(\Omega_2)\phi\Omega_1 \tag{3.5}$$

for all $\Omega_1,\Omega_2 \in \Gamma(TM)$ [14]. Thus on M, we have

$$\widetilde{\nabla}_{\Omega_1}\xi = -\phi^2\Omega_1 \tag{3.6}$$

for all $\Omega_1 \in \Gamma(TM)$.
Let $\overline{\overline{\nabla}}$ be a SSMC defined in (2.1) on M. Thus, we get $\overline{\overline{\nabla}}_{\Omega_1}\xi = 0$, i.e ξ is parallel with respect to SSMC $\overline{\overline{\nabla}}$.
The $\phi-$sectional curvature of PK-manifold is defined as the sectional curvature of plane section spanned by Ω_1 and $\phi\Omega_1$, for unit vector field Ω_1. If M has constant $\phi-$sectional curvature c then we have

$$\begin{aligned}\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \left(\frac{c-3}{4}\right)[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)],\\ &+ \left(\frac{c+1}{4}\right)[g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4) - g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4) + 2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4),\\ &+ \eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4) - \eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4) + g(\Omega_1,\Omega_3)\eta(\Omega_2)\eta(\Omega_4) - g(\Omega_2,\Omega_3)\eta(\Omega_1)\eta(\Omega_4)].\end{aligned} \tag{3.7}$$

A PK-manifold M with above curvature relation is called a PK-space form. For details see [13]. The Ricci curvature of a PK-space forms is given by

$$\widetilde{Ric}(\Omega_1,\Omega_2) = (\frac{(m+1)(c+1)}{4} - (m-1))g(\Omega_1,\Omega_2) - \frac{(m+1)(c+1)}{4}\eta(\Omega_1)\eta(\Omega_2). \tag{3.8}$$

This shows M is a quasi-Einstein manifold with $a = \frac{(m+1)(c+1)}{4} - (m-1)$, $b = \frac{(m+1)(c+1)}{4}$. On a PK-manifold we have

$$(\widetilde{\nabla}_{\Omega_1}\eta)\Omega_2 = g(\Omega_1,\Omega_2) - \eta(\Omega_1)\eta(\Omega_2), \tag{3.9}$$

thus we obtain

$$\omega(\Omega_1,\Omega_2) = \frac{3}{2}g(\Omega_1,\Omega_2) - 2\eta(\Omega_1)\eta(\Omega_2). \tag{3.10}$$

By using (2.2), the curvature of a PK-manifold admitting SSMC given in (2.1) is

$$\begin{aligned}\overline{\overline{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) - 3(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4) - g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4),\\ &+ \eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4) - \eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4) + \eta(\Omega_2)\eta(\Omega_4)g(\Omega_1,\Omega_3) - \eta(\Omega_1)\eta(\Omega_4)g(\Omega_2,\Omega_3)).\end{aligned}$$

Also, from (3.7), on a PK-space form we get

$$\begin{aligned}\widetilde{\widetilde{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \left(\frac{c-15}{4}\right)(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4) \\ &+\left(\frac{c-11}{4}\right)\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)+(\eta(\Omega_2)\eta(\Omega_4)g(\Omega_1,\Omega_3) \\ &+\eta(\Omega_1)\eta(\Omega_4)g(\Omega_2,\Omega_3)) \\ &+\left(\frac{c+1}{4}\right)[g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)+2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4)].\end{aligned} \tag{3.11}$$

A generalized para-Sasakian space form (GPSSF) is an almost para-contact metric manifold (M,ϕ,ξ,η,g) with the following curvature relation;

$$\begin{aligned}\widetilde{Rim}(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= F_1[g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)] \\ &+F_2(-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)+g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,\Omega_4)-2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4)) \\ &\times F_3(\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)\eta(\Omega_2)\eta(\Omega_4)-g(\Omega_2,\Omega_3)\eta(\Omega_1)\eta(\Omega_4)).\end{aligned}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4$ vector fields.

Corollary 3.8. *A PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ is a GPSSF with $F_1=\frac{c-15}{4}$, $F_2=-\frac{c-11}{4}$ and $F_3=\frac{c+1}{4}$.*

Let take an orthonormal basis of M by $E_1,E_2,...E_n,E_{m+1}=\phi E_1,...,E_{2m}=\phi E_m,\xi$. By choosing $\Omega_2=\Omega_3=E_i$ and taking sum over i such that $1\le i\le 2m$ in (3.11) then, we obtain

$$\widetilde{\widetilde{Ric}}(\Omega_1,\Omega_2)=(\frac{m(c-15)-2}{2})g(\Omega_1,\Omega_2)+\frac{c-11}{4}(1-2m)\eta(\Omega_1)\eta(\Omega_4).$$

Thus, M is a quasi-Einstein manifold. So, we state;

Corollary 3.9. *A PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ is a quasi-Einstein manifold.*

This is compatible with Theorem 2.2.
Let N be a submanifold of PK-space form M with respect to $\overline{\overline{\nabla}}$. Then, we have

$$\begin{aligned}Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \widetilde{\widetilde{Rim}}(\Omega_1,\Omega_2,\Omega_3,\Omega_4)-g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))+g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4) \\ &-g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)+g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)\end{aligned}$$

and from (3.11) we get

$$\begin{aligned}Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4) &= \left(\frac{c-19}{4}\right)(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4) \\ &+\left(\frac{c-11}{4}\right)(\eta(\Omega_1)\eta(\Omega_3)g(\Omega_2,\Omega_4)-\eta(\Omega_2)\eta(\Omega_3)g(\Omega_1,\Omega_4) \\ &+\eta(\Omega_2)\eta(\Omega_4)g(\Omega_1,\Omega_3)-\eta(\Omega_1)\eta(\Omega_4)g(\Omega_2,\Omega_3)) \\ &+\left(\frac{c+1}{4}\right)(g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,,\Omega_4)-g(\Omega_1,\phi\Omega_3)g(\phi\Omega_2,,\Omega_4)+2g(\Omega_1,\phi\Omega_2)g(\phi\Omega_3,\Omega_4)) \\ &-g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))+g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4)\end{aligned}$$

for all $\Omega_1,\Omega_2,\Omega_3,\Omega_4\in\Gamma(TN)$.
Suppose that ξ is normal to N and N is an anti-invariant submanifold i.e. $\phi\Omega_1\in\Gamma(TN^{\perp})$, for $\Omega_1\in\Gamma(TN)$. Then, we get

$$Rim(\Omega_1,\Omega_2,\Omega_3,\Omega_4)=(\frac{c-19}{4})(g(\Omega_2,\Omega_3)g(\Omega_1,\Omega_4)-g(\Omega_1,\Omega_3)g(\Omega_2,\Omega_4)+g(\sigma(\Omega_1,\Omega_3)\sigma(\Omega_2,\Omega_4))-g(\sigma(\Omega_2,\Omega_3)\sigma(\Omega_1,\Omega_4).$$

Thus, we state following results.

Corollary 3.10. *Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be an anti-invariant submanifold of M with ξ is normal to N. If N is totally geodesic, then N is $N(k)-$manifold.*

Corollary 3.11. *Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be an anti-invariant submanifold of M with ξ is normal to N. If N is totally umbilical, then N is a reel space form.*

Corollary 3.12. *Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be an anti-invariant submanifold of M with ξ is normal to N. If N is totally geodesic. Then N is an Einstein manifold.*

Let M be a PK-space form with respect to SSMC $\overline{\overline{\nabla}}$ and N be a submanifold of M. If ξ is tangent to submanifold N, then Lemma 3.1 is verified. Also, for the same submanifold the Theorem 3.3 is verified.

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