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ON $\mathscr{A}$-SUBMODULES FOR REFLEXIVE OPERATOR ALGEBRAS

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ABSTRACT. In [**2**] the authors described all weakly closed $\mathscr{A}$-submodules of $L(H)$ for a nest algebra $\mathscr{A}$ in terms of order homomorphisms of Lat $\mathscr{A}$. In this paper we prove that for any reflexive algebra $\mathscr{A}$ which is σ-weakly generated by rank-one operators in $\mathscr{A}$, every σ-weakly closed $\mathscr{A}$-submodule can be characterized by an order homomorphism of Lat $\mathscr{A}$. In the case when $\mathscr{A}$ is a reflexive algebra with a completely distributive subspace lattice and $\mathscr{M}$ is a σ-weakly closed ideal of $\mathscr{A}$, we obtain necessary and sufficient conditions for the commutant of $\mathscr{A}$ modulo $\mathscr{M}$ to be equal to AlgLat $\mathscr{M}$.

Let H be a complex Hilbert space, and let $L(H)$ be the set of all bounded linear operators on H. The terminology and notation of this paper concerning nest algebras and reflexive subspaces of $L(H)$ may be found in [**3**].

Let $\mathscr{A}$ be a reflexive subalgebra of $L(H)$. Suppose that $E \mapsto \tilde{E}$ is an order homomorphism of Lat $\mathscr{A}$ into itself (i.e. $E \leq F$ implies $\tilde{E} \leq \tilde{F}$), where Lat $\mathscr{A}$ is the set of all invariant projections for $\mathscr{A}$. Then the set $\mathscr{M} = \{T \in L(H) \colon (I - \tilde{E})TE = 0$ for all $E \in \operatorname{Lat}\mathscr{A}\}$ is clearly a weakly closed $\mathscr{A}$-submodule of $L(H)$.

J. A. Erdos and S. C. Power in [**2**] proved that any weakly closed $\mathscr{A}$-submodule of $L(H)$ for a nest algebra $\mathscr{A}$ is of the above form. Here we prove that this is also true for any reflexive algebra $\mathscr{A}$ which is σ-weakly generated by rank-one operators in $\mathscr{A}$.

The following result is due to J. Kraus and D. R. Larson [**3**].

THEOREM 1. *Let $\mathscr{A}$ be a unital σ-weakly closed algebra which is σ-weakly generated by rank-one operators in $\mathscr{A}$. Then every σ-weakly closed left or right module of $\mathscr{A}$ is reflexive.*

THEOREM 2. *Let $\mathscr{A}$ be as in the above theorem, and let $\mathscr{M}$ be a σ-weakly closed $\mathscr{A}$-submodule of $L(H)$. Then $\mathscr{M}$ has the form*

$$\mathscr{M} = \{T \in L(H) \colon (I - \tilde{E})TE = 0 \text{ for all } E \in \operatorname{Lat}\mathscr{A}\},$$

where $E \mapsto \tilde{E}$ is some order homomorphism of Lat $\mathscr{A}$ *into itself.*

PROOF. For any $E \in \operatorname{Lat}\mathscr{A}$, let $\tilde{E}$ be the orthogonal projection onto $[\mathscr{M}EH] = \bigvee\{\operatorname{ran}(XE) \colon X \in \mathscr{M}\}$. Since $\mathscr{M}$ is an $\mathscr{A}$-submodule, $\tilde{E}$ is invariant under $\mathscr{A}$ and clearly $E \mapsto \tilde{E}$ is an order homomorphism. Let $\mathscr{N} = \{T \in L(H) \colon (I - \tilde{E})TE = 0$ for all $E \in \operatorname{Lat}\mathscr{A}\}$. It is obvious that $\mathscr{N} \supseteq \mathscr{M}$. Conversely, if $T \in \mathscr{N}$, then $(I - \tilde{E})TE = 0$, so $[TEH] \subseteq [\tilde{E}H] = [\mathscr{M}EH]$ for any $E \in \operatorname{Lat}\mathscr{A}$. Now for any

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$x \in H$, denote by E the orthogonal projection onto $[\mathscr{A}x]$. Then $E \in \operatorname{Lat}\mathscr{A}$ and so $Tx \in T[EH] \subseteq \tilde{E}H = [\mathscr{M}EH] = [\mathscr{M}[\mathscr{A}x]] = [\mathscr{M}x]$. By the definition of reflexive subspace of $L(H)$, we know that $T \in \mathscr{M}$ since $\mathscr{M}$ is reflexive by Theorem 1. It follows that $\mathscr{M} \supseteq \mathscr{N}$ and so $\mathscr{M} = \mathscr{N}$. The proof is complete.

It is known from [**4**] that a commutative subspace lattice $\mathscr{L}$ is completely distributive iff the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$ is σ-weakly dense in $\operatorname{Alg}\mathscr{L}$. Thus we have the following result:

COROLLARY 3. *If $\mathscr{L}$ is a commutative and completely distributive subspace lattice, then every σ-weakly closed* $\operatorname{Alg}\mathscr{L}$*-submodule is of the form*

$$\{T \in L(H): (I - \tilde{E})TE = 0 \text{ for all } E \in \mathscr{L}\},$$

where $E \mapsto \tilde{E}$ is an order homomorphism of $\mathscr{L}$ into itself.

COROLLARY 4. *Let $\mathscr{A}$ and $\mathscr{M}$ be as in Theorem* 2. *Then $\mathscr{M}$ is an ideal of $\mathscr{A}$ if and only if $\tilde{E} \leq E$ for every $E \in \operatorname{Lat}\mathscr{A}$.*

LEMMA 5. *Let $\mathscr{A}$ and $\mathscr{M}$ be as in Theorem* 2. *Then $\mathscr{M}$ is an algebra if and only if $\tilde{\tilde{E}} \leq \tilde{E}$ for every $E \in \operatorname{Lat}\mathscr{A}$.*

PROOF. Clearly if $\mathscr{M}$ is an algebra, then for any $E \in \operatorname{Lat}\mathscr{A}$

$$\tilde{\tilde{E}}H = [\mathscr{M}[\mathscr{M}EH]] = [\mathscr{M}\mathscr{M}EH] \subseteq [\mathscr{M}EH] = \tilde{E}H$$

so $\tilde{\tilde{E}} \leq \tilde{E}$.

For the converse, if $S, T \in \mathscr{M}$, since

$$STE = S\tilde{E}TE = \tilde{\tilde{E}}S\tilde{E}TE = \tilde{\tilde{E}}STE$$

we have $(I - \tilde{E})STE = 0$, and so $ST \in \mathscr{M}$ which shows that $\mathscr{M}$ is an algebra.

LEMMA 6. *Let $\mathscr{A}$ and $\mathscr{M}$ be as in Theorem* 2. *Then $P \in \operatorname{Lat}\mathscr{M}$ if and only if there exists some $E \in \operatorname{Lat}\mathscr{A}$ such that*

$$\tilde{E} \leq P \leq E.$$

PROOF. If $\tilde{E} \leq P \leq E$ for some $E \in \operatorname{Lat}\mathscr{A}$ and $T \in \mathscr{M}$, then $TP = TEP = \tilde{E}TEP = P\tilde{E}TEP = PTP$. Hence $P \in \operatorname{Lat}\mathscr{M}$.

Conversely if $P \in \operatorname{Lat}\mathscr{M}$, let E be the projection onto $[\mathscr{A}PH]$. Then $E \in \operatorname{Lat}\mathscr{A}$, $E \geq P$ and

$$\tilde{E}H = [\mathscr{M}[\mathscr{A}PH]] = [\mathscr{M}\mathscr{A}PH] \subseteq [\mathscr{M}PH] \subseteq PH.$$

Thus $\tilde{E} \leq P \leq E$. The proof is complete.

The following theorems give a description of the commutant $C(\mathscr{A}, \mathscr{M})$ of $\mathscr{A}$ modulo $\mathscr{M}$ for an $\mathscr{A}$-submodule $\mathscr{M}$. Recall that $C(\mathscr{A}, \mathscr{M})$ is defined by

$$C(\mathscr{A}, \mathscr{M}) = \{T \in L(H): TA - AT \in \mathscr{M} \text{ for every } A \in \mathscr{A}\}.$$

THEOREM 7. *Let $\mathscr{A}$ be a reflexive algebra with commutative and completely distributive subspace lattice* $\operatorname{Lat}\mathscr{A}$, *and let $\mathscr{M}$ be a σ-weakly closed $\mathscr{A}$-submodule containing $\mathscr{A}$. Then $C(\mathscr{A}, \mathscr{M}) = \mathscr{M}$.*

PROOF. Obviously $C(\mathscr{A}, \mathscr{M}) \supseteq \mathscr{M}$. For any $T \in C(\mathscr{A}, \mathscr{M})$ and $E \in \operatorname{Lat}\mathscr{A}$, since $\tilde{E} \in \mathscr{A}$, we have $(I - \tilde{E})(T\tilde{E} - \tilde{E}T)E = 0$, so $(I - \tilde{E})TE = 0$ since $\tilde{E} \geq E$, which implies $T \in \mathscr{M}$. The proof is complete.

COROLLARY 8. *Let $\mathscr{A}$ be as in Theorem 7. If $H^1(\mathscr{A}, L(H)) = 0$, then $H^1(\mathscr{A}, \mathscr{M}) = 0$ for any σ-weakly closed $\mathscr{A}$-submodule $\mathscr{M}$ containing $\mathscr{A}$, where $H^1(\mathscr{A}, \mathscr{M})$ is the first Hochschild cohomology space (see* [**1**]) *of $\mathscr{A}$ with coefficients in $\mathscr{M}$.*

REMARK. In the paper [**1**] Christensen proved that if $\mathscr{A}$ is a reflexive algebra with commutative subspace lattice Lat $\mathscr{A}$, then for any σ-weakly closed algebra $\mathscr{M}$ containing $\mathscr{A}$, $H^1(\mathscr{A}, \mathscr{M}) = H^1(\mathscr{A}, L(H))$. In Corollary 8, $\mathscr{M}$ need not be an algebra but we need Lat $\mathscr{A}$ to be completely distributive.

LEMMA 9. *Let $\mathscr{A}$ be as in Theorem 7, and let $\mathscr{M}$ be a σ-weakly closed ideal of $\mathscr{A}$ determined by an order homomorphism $E \mapsto \tilde{E}$. Then $PC(\mathscr{A}, \mathscr{M}) \mid_{PH} \subseteq (P\mathscr{A} \mid_{PH})'$ for any $E \in \operatorname{Lat}\mathscr{A}$, where $P = E - \tilde{E}$.*

PROOF. See the proof of (ii) of Lemma 3.1 in [**2**].

DEFINITION 10. Let $\mathscr{A}$ be a reflexive algebra, and let $\mathscr{M}$ be a σ-weakly closed ideal of $\mathscr{A}$ determined by an order homomorphism $E \mapsto \tilde{E}$. $\mathscr{M}$ is said to have property (*) if $PC(\mathscr{A}, \mathscr{M}) \mid_{PH} = \{\lambda I_{PH}\}$ for any $P = E - \tilde{E} > 0$.

Note that if $\mathscr{A}$ is a nest algebra and $\mathscr{M}$ is a σ-weakly closed ideal of $\mathscr{A}$, then $\mathscr{M}$ has property (*).

THEOREM 11. *Let $\mathscr{A}$ and $\mathscr{M}$ be as in Lemma 9. Then $C(\mathscr{A}, \mathscr{M}) \supseteq \operatorname{AlgLat}\mathscr{M}$. Furthermore, $C(\mathscr{A}, M) = \operatorname{AlgLat}\mathscr{M}$ if and only if $\mathscr{M}$ has property* (*).

PROOF. Let $\mathscr{M}$ be determined by an order homomorphism $E \mapsto \tilde{E}$. By Corollary 4 we know that $\tilde{E} \le E$ for any $E \in \operatorname{Lat}\mathscr{A}$. First we prove $C(\mathscr{A}, \mathscr{M}) \supseteq \operatorname{AlgLat}\mathscr{M}$. Let $T \in \operatorname{AlgLat}\mathscr{M}$. For any $A \in \mathscr{A}$, we need to prove $TA - AT \in \mathscr{M}$. For all $E \in \operatorname{Lat}\mathscr{A}$ and $G \le E - \tilde{E}$, $\tilde{E} + G$ is invariant for T by Lemma 6. Thus $(E - \tilde{E})T(E - \tilde{E})$ leaves every subprojection of $E - \tilde{E}$ invariant. And so

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E})$$

for some scalar λ_E. Note that T is also in $\mathscr{A}$. Now we have

$$\begin{aligned}(I - \tilde{E})(TA - AT)E &= (I - \tilde{E})TEAE - (I - \tilde{E})AETE \\ &= (E - \tilde{E})T((E - \tilde{E}) + \tilde{E})A((E - \tilde{E}) + \tilde{E}) \\ &\quad - (E - \tilde{E})A((E - \tilde{E}) + \tilde{E})T((E - \tilde{E}) + \tilde{E}) \\ &= (E - \tilde{E})T(E - \tilde{E})A(E - \tilde{E}) - (E - \tilde{E})A(E - \tilde{E})T(E - \tilde{E}) \\ &= \lambda_E(E - \tilde{E})A(E - \tilde{E}) - (E - \tilde{E})A\lambda_E(E - \tilde{E}) \\ &= 0.\end{aligned}$$

Thus $TA - AT \in \mathscr{M}$ for all $A \in \mathscr{A}$, and hence $T \in C(\mathscr{A}, \mathscr{M})$. Now we prove the last statement of the theorem.

(i) Suppose that $C(\mathscr{A}, \mathscr{M}) = \operatorname{AlgLat}\mathscr{M}$. From the proof of the first statement of this theorem, we know that for any $E \in \operatorname{Lat}\mathscr{A}$, and any $T \in C(\mathscr{A}, \mathscr{M}) = \operatorname{AlgLat}\mathscr{M}$,

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E}).$$

(ii) Suppose that $\mathscr{M}$ has property (*). For any $T \in C(\mathscr{A}, \mathscr{M})$ and $E \in \operatorname{Lat}\mathscr{A}$, we need to prove

$$(I - (\tilde{E} + Q))T(\tilde{E} + Q) = 0$$

for any subprojection $Q \le E - \tilde{E}$.

Since $E, \tilde{E} \in \mathscr{A}$, we have

$$(I - \tilde{E})(T\tilde{E} - \tilde{E}T)E = 0,$$
$$(I - \tilde{E})(TE - ET)E = 0.$$

Thus

$$(I - \tilde{E})T\tilde{E} = 0$$

and

$$\begin{aligned}(I - \tilde{E})TE &= (E - \tilde{E})TE \\ &= (E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E}).\end{aligned}$$

Therefore

$$\begin{aligned}&(I - (\tilde{E} + Q))T(\tilde{E} + Q) \\ &\quad = (I - \tilde{E})TQ - QTQ \\ &\quad = \lambda_E(E - \tilde{E})Q - \lambda_E Q(E - \tilde{E})Q = 0\end{aligned}$$

and hence $T \in \operatorname{AlgLat} \mathscr{M}$. The proof is completed.

COROLLARY 12. *Let $\mathscr{A}$ and $\mathscr{M}$ be as in Lemma* 9. *If $(P\mathscr{A}\,|_{PH})' = \{\lambda I_{PH}\}$ for every $P = E - \tilde{E} > 0$, then $C(\mathscr{A}, \mathscr{M}) = \operatorname{AlgLat}\mathscr{M}$.*

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