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ON SUBORDINATED DISTRIBUTIONS AND GENERALIZED RENEWAL MEASURES

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Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with partial sums $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$. We investigate the behaviour of $\sum_{n=0}^\infty a_n P(S_n \in x + A)$ as $x \to \pm\infty$, where $(a_n)_{n\in\mathbb{N}_0}$ is a sequence of nonnegative numbers and $A \subset \mathbb{R}$ is a fixed Borel set.

1. Introduction. Various problems in probability theory lead to questions on the asymptotic behaviour of

$$(1) \qquad \sum_{n=0}^{\infty} a_n P(S_n \in x + A), \quad \text{as } x \to \pm\infty,$$

where $(a_n)_{n\in\mathbb{N}_0}$ is some sequence of nonnegative numbers, $(S_n)_{n\in\mathbb{N}_0}$ is the sequence of partial sums, $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$, of another sequence $(X_k)_{k\in\mathbb{N}}$ of i.i.d. random variables, and $A \subset \mathbb{R}$ is a fixed Borel set such as $[0,1]$ or $[0,\infty)$. Examples of such convolution series are subordinated distributions ($\sum_{n=0}^\infty a_n = 1$) which arise as distributions of random sums, and harmonic and ordinary renewal measures ($a_0 = 0$, $a_n = 1/n$ for all $n \in \mathbb{N}$ in the first, $a_n = 1$ for all $n \in \mathbb{N}_0$ in the second case). These examples are in turn essential for the analysis of the large time behaviour of diverse applied models such as branching and queueing processes, they are also of interest in connection with representation theorems such as the Lévy representation of infinitely divisible distributions.

A traditional approach to such problems is via regular variation: If the underlying random variables are nonnegative we can use Laplace transforms and the related Abelian and Tauberian theorems [see, e.g., Stam (1973) in the context of subordination and Feller (1971, XIV.3) in connection with renewal theory; Embrechts, Maejima, and Omey (1984) is a recent treatment of generalized renewal measures along these lines].

The approach of the present paper is based on the Wiener–Lévy–Gel'fand theorem and has occasionally been called the Banach algebra method. In Grübel (1983) we gave a new variant of this method for the special case of lattice distributions, showing that by using the appropriate Banach algebras of sequences, arbitrarily fine expansions are possible under certain assumptions on the higher-order differences of $(P(X_1 = n))_{n\in\mathbb{N}}$. Here we give a corresponding treatment of nonlattice distributions. We restrict ourselves to an analogue of first-order differences and obtain a number of theorems which perhaps are described best as next-term results. To explain this let us consider a special case in more detail.

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Let $Z(t)_{t\geq 0}$ be an age-dependent branching process with lifetime distribution μ and $m < 1$, where m denotes the mean number of offspring [see Athreya and Ney (1972) for the definitions]. It is known that if μ is subexponential, i.e., if

$$\lim_{x\to\infty} (\mu * \mu((x,\infty))/\mu(x,\infty))) = 2,$$

where $*$ denotes convolution, then

$$\lim_{t\to\infty} \frac{EZ(t)}{\mu((t,\infty))} = \frac{1}{1-m}$$

[Athreya and Ney (1972, VI, Theorem 3B(ii))]. Now we have

$$EZ(t) = (1-m)\sum_{k=1}^{\infty} m^{k-1}\mu^{*k}((t,\infty)),$$

which displays this situation as a special case of (1), and it follows from the results to be given in Section 3 below that if μ has finite second moment and a density f which is of bounded variation and satisfies

$$f(x) = O(x^{-4}), \qquad \sup_{y\geq x} f(y) = O(f(2x)), \qquad V_x^{x+1}f = o(f(x)),$$

where $V_x^{x+1}f$ denotes the variation of f on the interval $(x, x+1]$, then

$$\lim_{t\to\infty} \frac{EZ(t) - \dfrac{1}{1-m}\mu((t,\infty))}{f(t)} = \frac{2m\kappa}{(1-m)^2},$$

where κ denotes the first moment of μ.

Banach algebra methods have been applied in the study of subordinated distributions with exponentially decreasing weights by Chover, Ney, and Wainger (1973) and in ordinary renewal theory by Borovkov (1964), Essén (1973), Rogozin (1976b), and others. Especially in renewal theory much attention has been paid to expansions giving the respective next term: Consider the renewal function U,

$$U(x) = \sum_{n=0}^{\infty} P(S_n \leq x) = \sum_{n=0}^{\infty} \mu^{*n}((-\infty, x]),$$

where μ denotes the distribution of X_1. Assume that μ is nonlattice and has finite positive mean m_1. A first refinement of the elementary renewal theorem,

$$U(x) = \frac{x}{m_1} + o(x),$$

has been given by Smith (1964): If $m_2 = EX_1^2 < \infty$, then

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} + o(1).$$

This has in turn been refined by Stone (1965): If μ satisfies Cramér's condition, then

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{1}{m_1^2}\int_x^{\infty}\int_y^{\infty} \mu((t,\infty))\, dt\, dy + \varepsilon(x),$$

where $\varepsilon(x) = o(x^{1-\gamma}\log x)$ if $E|X_1^\gamma| < \infty$ ($\gamma \in \mathbb{N}$). Recently Carlsson (1983) has given the next term: if μ satisfies Cramér's condition and has finite third moment m_3, then

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{1}{m_1^2}\int_x^\infty \int_y^\infty \mu((t,\infty))\,dt\,dy + \frac{m_2}{m_1^3}\int_x^\infty \mu((t,\infty))\,dt + \varepsilon(x),$$

where $\varepsilon(x) = o(x^{-\gamma}\log x)$ if $E|X_1^\gamma| < \infty$ ($\gamma \geq 3$, $\gamma \in \mathbb{N}$).

Roughly, U has been expanded down to the order of the tails of the underlying lifetime distribution μ. We will obtain an expansion with error term of magnitude $\mu([x, x+1])$: If some convolution power of μ has nonvanishing absolutely continuous part and if μ has finite fourth moment, then

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{1}{m_1^2}\int_x^\infty \int_y^\infty \mu((t,\infty))\,dt\,dy + \frac{m_2}{m_1^3}\int_x^\infty \mu((t,\infty))\,dt + \frac{4m_1m_3 - 9m_2^2}{12m_1^4}\mu((x,\infty)) + \varepsilon(x),$$

where $\varepsilon(x) = o(x^{-\gamma})$ if $\mu([x, x+1]) = o(x^{-\gamma})$ ($\gamma \geq 5$, not necessarily an integer).

Beyond adding new terms our method and its extensions to higher-order differences may be regarded as an effective tool for obtaining arbitrarily fine expansions in this situation and many others.

The paper is organized as follows.

Section 2 contains a Wiener–Lévy–Gel'fand result which is basic for everything which follows. In the third section we apply it to the situation (1) in the case of exponentially decreasing a_n's, on using these results we then obtain information relating the asymptotic behaviour of infinitely divisible distributions and their Lévy measures in Section 4. In Section 5 we consider harmonic renewal measures; Section 6 then gives applications to ordinary renewal measures. The last section contains some concluding remarks, indicating extensions and related results. Some lemmas of a more technical nature are deferred to the Appendix.

2. A Wiener–Lévy–Gel'fand result. We first introduce some notation.

Let $\mathscr{M}$ denote the space of all complex-valued measures on the Borel subsets $\mathscr{B}$ of the real line. For any $\mu \in \mathscr{M}$ let $\hat{\mu}$ denote the corresponding Fourier transform, $|\mu|$ the corresponding total variation measure, and $\|\mu\|_{TV} = |\mu|(\mathbb{R})$ its total variation norm. Further let l denote the Lebesgue measure and δ_x the distribution concentrated on x ($x \in \mathbb{R}$). We write $\mu \ll l$ if $\mu \in \mathscr{M}$ is absolutely continuous. For any $\mathscr{M}_1 \subset \mathscr{M}$ put $\hat{\mathscr{M}}_1 = \{\hat{\mu}: \mu \in \mathscr{M}_1\}$, $\mathscr{M}_1^a = \{\mu \in \mathscr{M}_1: \mu \ll l\}$. If $\mu \in \mathscr{M}$, $A \in \mathscr{B}$, then $\mu|_A$ denotes the element of $\mathscr{M}$ with $\mu|_A(B) = \mu(A \cap B)$ for all $B \in \mathscr{B}$. Finally,

$$U_\rho(z) = \{z' \in \mathbb{C}: |z - z'| < \rho\}, \qquad \overline{U_\rho(z)} = \{z' \in \mathbb{C}: |z - z'| \leq \rho\}$$

for all $z \in \mathbb{C}$, $\rho > 0$. We define an operator Σ: $D(\Sigma) \to \mathscr{M}$ as follows,

$$D(\Sigma) = \left\{\mu \in \mathscr{M}\colon \int |F_\mu|\,dl < \infty\right\},$$

where

$$F_\mu(x) = \mu(\mathbb{R}) I_{[0,\infty)}(x) - \mu((-\infty, x]),$$

$$\Sigma\mu(A) = \int_A F_\mu\,dl, \quad \text{for all } A \in \mathscr{B}.$$

Let $\mathscr{D}$ denote the space of all $\mu \in \mathscr{M}$ representable in the form $\mu = \alpha\delta_0 + \Sigma\nu$ with some $\alpha \in \mathbb{C}$, $\nu \in D(\Sigma)$; define Δ: $\mathscr{D} \to D(\Sigma)$ by the requirements $\Delta\mu(\mathbb{R}) = 0$, $\mu = \mu(\{0\})\delta_0 + \Sigma(\Delta\mu)$. In the first lemma of the appendix some simple computation rules for Σ and Δ are listed; we will use these throughout the paper without further comment. Let Δ^n, Σ^n denote the nth iterates of Δ and Σ, respectively.

We call a monotone decreasing function τ: $\mathbb{R}_+ \to \mathbb{R}_+$ dominatedly varying (τ is a DVF for short) if

$$C(\tau) = \sup_{x \geq 0} \frac{\tau(x)}{\tau(2x)} < \infty.$$

In dependence on some DVF τ we introduce subspaces $\mathscr{M}(\tau)$ and $\mathscr{D}(\tau)$ of $\mathscr{M}$:

$$\mathscr{M}(\tau) = \{\mu \in \mathscr{M}\colon |\mu|((x, x+1]) = o(\tau(x))\},$$

$$\mathscr{D}(\tau) = \{\mu \in \mathscr{D}\colon \Delta\mu \in \mathscr{M}(\tau)\}$$

[here and in the following $o(\tau(x))$ refers to $x \to \infty$ if not qualified otherwise].

THEOREM 1. *Let τ be a DVF with $\tau(x) = O(x^{-2})$. Suppose $\mu_1 \in \mathscr{D}(\tau)$, $\mu_2 \in \mathscr{M}$, and Ψ: $G \to \mathbb{C}$ are such that $G \subset \mathbb{C}$ is open and contains the closure of $\hat{\mu}_1(\mathbb{R})$, Ψ is analytic on G, and*

$$\hat{\mu}_2(\theta) = \Psi(\hat{\mu}_1(\theta)), \quad \textit{for all } \theta \in \mathbb{R}.$$

Then $\mu_2 \in \mathscr{D}(\tau)$.

In the first step of the proof we introduce a norm $\|\cdot\|_0^\tau$ on $\mathscr{D}(\tau)$ such that $(\mathscr{D}(\tau), \|\cdot\|_0^\tau, *)$ is a Banach algebra. In this step we will use the fact that $(\mathscr{M}, \|\cdot\|_{\mathrm{TV}}, *)$ and $(\mathscr{M}(\tau), \|\cdot\|^\tau, *)$, where

$$\|\mu\|^\tau = C(\tau)\left(\|\mu\|_{\mathrm{TV}} + \sup_{x \geq 0} \tau(x)^{-1}|\mu|((x, x+1])\right), \quad \text{for all } \mu \in \mathscr{M}(\tau),$$

are Banach algebras [see Gel'fand, Raikow, and Schilow (1964) and Rogozin (1976a)]; clearly all convolution algebras are commutative and δ_0 is the unit element. In the second step we characterize its maximal ideals. In the third step we complete the proof of the theorem by using a standard procedure from the general theory of Banach algebras.

PROOF OF THEOREM 1. (i) We define a norm on $\mathscr{D}(\tau)$ by

$$\|\mu\|_0^\tau = C_1(\tau)\Big(\|\mu\|_{\mathrm{TV}} + \|\Delta\mu\|_{\mathrm{TV}} + \sup_{x\ge 0} \tau(x)^{-1}|\Delta\mu|((x, x+1])\Big),$$

where $C_1(\tau)$ is as in Lemma A.3. With the help of this lemma we obtain convolution closure of $\mathscr{D}(\tau)$ and also the norm inequality,

$$\|\mu_1 * \mu_2\|_0^\tau \le \|\mu_1\|_0^\tau \|\mu_2\|_0^\tau, \quad \text{for all } \mu_1, \mu_2 \in \mathscr{D}(\tau).$$

To prove completeness, consider a Cauchy sequence $(\mu_n)_{n\in\mathbb{N}}$. We may assume $\mu_n(\{0\}) = 0$ for all $n \in \mathbb{N}$. Then there exist $\mu \in \mathscr{M}$, $\nu \in \mathscr{M}(\tau)$ such that

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathrm{TV}} = 0, \qquad \lim_{n\to\infty} \|\Delta\mu_n - \nu\|^\tau = 0,$$

and it remains to show $\mu = \Sigma\nu$.

We choose a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that $\|\mu_{n_k} - \mu_{n_{k+1}}\|_0^\tau < 2^{-k}$ for all $k \in \mathbb{N}$. Then F,

$$F(t) = \sum_{k=0}^{\infty} \big|\Delta\mu_{n_k}((-\infty, t]) - \Delta\mu_{n_{k+1}}((-\infty, t])\big|,$$

is l-integrable, and since $\Delta\mu_{n_k}((-\infty, t]) \to \nu((-\infty, t])$ for all $t \in \mathbb{R}$ we obtain

$$\lim_{k\to\infty} \int \big|\Delta\mu_{n_k}((-\infty, t]) - \nu((-\infty, t])\big|\, dt = 0$$

by dominated convergence. This means $\lim_{k\to\infty}\|\mu_{n_k} - \Sigma\nu\|_{\mathrm{TV}} = 0$, so $\mu = \Sigma\nu$ as required.

(ii) For any $\mu \in \mathscr{D}$ let $\|\mu\|_0 = \|\mu\|_{\mathrm{TV}} + \|\Delta\mu\|_{\mathrm{TV}}$ which is easily seen to be a norm on $\mathscr{D}$. It is also simple to prove closure of $\mathscr{D}$ with respect to convolution and the corresponding norm inequality, completeness follows by the same arguments as in the case $(\mathscr{D}(\tau), \|\cdot\|_0^\tau)$. So $(\mathscr{D}, \|\cdot\|_0, *)$ is a Banach algebra.

Let I be a maximal ideal in $\mathscr{D}(\tau)$, let $\psi\colon \mathscr{D}(\tau) \to \mathbb{C}$ denote the corresponding (multiplicative and continuous) homomorphism, so $I = \psi^{-1}(\{0\})$ [see, e.g., Rudin (1974, 11.5 Theorem)]. Then

$$|\psi(\delta_x * \mu)| = |\psi(\mu)|, \quad \text{for all } \mu \in \mathscr{D}^a(\tau), \quad x \in \mathbb{R}, \tag{2}$$

since otherwise $n \to \psi(\delta_{nx} * \mu)$ would increase exponentially for some $\mu \in \mathscr{D}^a(\tau)$, $x \in \mathbb{R}$, which contradicts Lemma A.2(i) because of

$$|\psi(\delta_{nx} * \mu)| \le \|\delta_{nx} * \mu\|_0^\tau.$$

Let $\mathscr{K}$ denote the set of those $\mathscr{M}$-elements which have compact support. Then

$$\mathscr{K} \cap \mathscr{D}(\tau) \text{ is dense in } \mathscr{D}(\tau). \tag{3}$$

Given any $\mu \in \mathscr{D}(\tau)$ an approximating sequence is $(\mu_n)_{n\in\mathbb{N}}$ with

$$\mu_n = \mu(\{0\})\delta_0 + \Sigma\nu_n, \qquad \nu_n = \Delta\mu((-\infty, -n))\delta_{-n} + \Delta\mu|_{[-n,n]} + \Delta\mu((n, \infty))\,\delta_0$$

[use $\tau(x) = O(x^{-2})$].

By adapting an argument from Chover et al. (1973, page 281) we show

$$|\psi(\mu)| \le C\|\mu\|_0, \quad \text{for all } \mu \in \mathscr{D}(\tau) \tag{4}$$

with a suitable constant C. We have

$$\|\mu\|_0^\tau \le C\|\mu\|_0, \quad \text{for all } \mu \in \mathscr{D}(\tau) \text{ with support in } [0,1],$$

with a suitable constant C. Because of (3) we can find a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\tau) \cap \mathscr{K}$ such that $(-n, n]$ contains the support of μ_n and $\mu_n \to \mu$ in $\mathscr{D}(\tau)$. We may assume $\mu(\{0\}) = \mu_n(\{0\}) = 0$. Then we obtain on using (2)

$$\begin{aligned}
|\psi(\mu)| &= \lim_{n\to\infty} \left|\psi\left(\sum_{k=-n}^{n-1} \delta_k * \delta_{-k} * \mu_n|_{(k,k+1]}\right)\right| \\
&\le \limsup_{n\to\infty} \sum_{k=-n}^{n-1} \left|\psi\left(\delta_{-k} * \mu_n|_{(k,k+1]}\right)\right| \\
&\le \limsup_{n\to\infty} \sum_{k=-n}^{n-1} \left\|\delta_{-k} * \mu_n|_{(k,k+1]}\right\|_0^\tau \\
&\le C \limsup_{n\to\infty} \sum_{k=-n}^{n-1} \left\|\delta_{-k} * \mu_n|_{(k,k+1]}\right\|_0 \\
&= C \limsup_{n\to\infty} \sum_{k=-n}^{n-1} \left\|\mu_n|_{(k,k+1]}\right\|_0,
\end{aligned}$$

the last step following with the translation invariance of $\|\cdot\|_0$ on $\mathscr{D}^a$, i.e., $\|\delta_\alpha * \mu\|_0 = \|\mu\|_0$ for all $\mu \in \mathscr{D}^a$, $\alpha \in \mathbb{R}$. We have

$$\Delta(\mu_n|_{(k,k+1]}) = -(\Delta\mu_n)((k,\infty))\delta_k + (\Delta\mu_n)|_{(k,k+1]} + \Delta\mu_n((k+1,\infty))\delta_{k+1}$$

and

$$|(\Delta\mu_n)((k,\infty))| \le \int_k^{k+1} |\Delta\mu_n((x,\infty))|\,dx + |\Delta\mu_n|((k,k+1]),$$

which gives

$$\sum_{k=-n}^{n-1} \left\|\mu_n|_{(k,k+1]}\right\|_0 \le 3\|\mu_n\|_0,$$

and so (4) follows.

Since $\mathscr{D}(\tau)$ is dense in $(\mathscr{D}, \|\cdot\|_0)$ (4) enables us to extend ψ to the whole of $\mathscr{D}$. Let I' denote the corresponding null space. Then I' is a maximal ideal in $\mathscr{D}$ and $I = I' \cap \mathscr{D}(\tau)$.

Now let I be a maximal ideal in $\mathscr{D}$. Using arguments from the proof of Theorem 1 in Yosida (1978, XI.16) we show that

$$\text{(5)} \qquad \begin{aligned} &\text{either } I \supset \mathscr{D}_a = \{\mu \in \mathscr{D}: \mu \ll l,\ \Delta\mu \ll l\} \\ &\text{or } I = I(\theta_0, \mathscr{D}) = \{\mu \in \mathscr{D}: \hat{\mu}(\theta_0) = 0\}, \quad \text{for some } \theta_0 \in \mathbb{R}. \end{aligned}$$

Let ψ again denote the corresponding homomorphism and assume $\psi(\mu_0) \neq 0$ for some $\mu_0 \in \mathscr{D}_a$. We define $F: \mathbb{R} \to \mathscr{D}$, $h: \mathbb{R} \to \mathbb{C}$ by

$$F(x) = \delta_x * \mu_0, \qquad h(x) = \psi(\mu_0)^{-1}\psi(F(x)).$$

Because of $\mu_0 \in \mathscr{D}_a$, F is continuous and so is h. The multiplicativity of ψ entails $h(x+y) = h(x)h(y)$; h is bounded because of $\|F(x)\|_0 = \|\mu_0\|_0$ for all $x \in \mathbb{R}$. Evidently $h \not\equiv 0$, so for some $\theta_0 \in \mathbb{R}$

$$h(x) = \exp(i\theta_0 x), \quad \text{for all } x \in \mathbb{R}.$$

Now take an arbitrary $\mu_1 \in \mathscr{D} \cap \mathscr{K}$. According to Theorem 3.3.2 in Hille and Phillips (1957) there exists an element μ_2 of $\mathscr{D}$ such that

$$\phi(\mu_2) = \int \phi(F(x))\mu_1(dx),$$

for all continuous linear functionals ϕ: $\mathscr{D} \to \mathbb{C}$. For a fixed $A \in \mathscr{B}$, $\mu \to \mu(A)$ is such a mapping—this identifies μ_2 as $\mu_0 * \mu_1$. ψ is also such a mapping so

$$\psi(\mu_2) = \int h(x)\psi(\mu_0)\mu_1(dx) = \psi(\mu_0)\hat{\mu}_1(\theta_0).$$

This implies $\psi(\mu_1) = \hat{\mu}_1(\theta_0)$ for all $\mu_1 \in \mathscr{D} \cap \mathscr{K}$ and since $\mathscr{D} \cap \mathscr{K}$ is dense in $\mathscr{D}$ this extends to the whole of $\mathscr{D}$ which completes the proof of (5).

We show next

$$(6) \qquad I \supset \mathscr{D}_a \Rightarrow I = \mathscr{D}^a \left(= \{\mu \in \mathscr{D}: \mu(\{0\}) = 0\}\right).$$

Suppose $I \supset \mathscr{D}_a$ and $\mu \in I$ with $\alpha = \mu(\{0\}) \neq 0$. We then have $\mu^{*2} \in I$. Since

$$\mu^{*2} = \alpha^2\delta_0 + 2\alpha\Sigma\Delta\mu + \Sigma(\Delta\mu * \Sigma\Delta\mu)$$

and $\Sigma(\Delta\mu * \Sigma\Delta\mu) \in \mathscr{D}_a$ we obtain

$$\alpha^2\delta_0 + 2\alpha\Sigma\Delta\mu \in I.$$

But then

$$\delta_0 = -\frac{1}{\alpha}(\alpha\delta_0 + 2\Sigma\Delta\mu - 2\mu) \in I,$$

which is impossible. From $I \subset \mathscr{D}^a$, $I = \mathscr{D}^a$ follows since I is a maximal ideal in $\mathscr{D}$ and $\mathscr{D}^a$ is an ideal in $\mathscr{D}$.

We may summarize this part of the proof as follows: The space of maximal ideals of $\mathscr{D}(\tau)$ is $\{I(\theta_0, \mathscr{D}(\tau))$: $\theta_0 \in \mathbb{R} \cup \{\infty\}\}$.

(iii) Let μ_1, μ_2, G and Ψ satisfy the assumptions of the theorem. If $z \notin \overline{\hat{\mu}_1(\mathbb{R})}$ then $\mu_1 - z\delta_0$ is not contained in any maximal ideal of $\mathscr{D}(\tau)$ which implies the existence of a convolution inverse $(\mu_1 - z\delta_0)^{*(-1)}$ in $\mathscr{D}(\tau)$. Integrating the $\mathscr{D}(\tau)$-valued function $z \to (1/2\pi i)\Psi(z)(\mu_1 - z\delta_0)^{*(-1)}$ over a contour $\Gamma \subset G - \overline{\hat{\mu}_1(\mathbb{R})}$ which has index 1(0) with respect to any point of $\overline{\hat{\mu}_1(\mathbb{R})}(G^c)$ we obtain an element of $\mathscr{D}(\tau)$ with Fourier transform $\Psi \circ \hat{\mu}_1$ [see, e.g., Hille and Phillips (1957, 5.2) for a detailed presentation; a similar argument has also been used by Chover et al. (1973, page 264)]. □

In Sections 5 and 6 we will need an analogous result on the spaces $\mathscr{M}(\tau)$. This result follows from the characterization of the maximal ideals of these spaces given by Rogozin and Sgibnev (1980), Gel'fand et al. (1964, Section 30), and step (iii) of the above proof.

THEOREM 2. *Let τ be a DVF. Suppose $\mu_1 \in \mathscr{M}(\tau)$, $\mu_2 \in \mathscr{M}$, and $\Psi: G \to \mathbb{C}$ are such that $G \subset \mathbb{C}$ is open and contains $\hat{\mu}_1(\mathbb{R})$, Ψ is analytic on G,*

$$\hat{\mu}_2(\theta) = \Psi(\hat{\mu}_1(\theta)), \quad \text{for all } \theta \in \mathbb{R},$$

and

$$z \in G, \quad \text{for all } z \in \mathbb{C} \text{ with } \mu_1 - z\delta_0 \in I \text{ for some } I, \quad I \supset \mathscr{M}^a, \quad I \text{ maximal ideal in } \mathscr{M}. \tag{7}$$

Then $\mu_2 \in \mathscr{M}(\tau)$.

Condition (7) is satisfied:

—if $1 \in G$ and $\mu_1 = \delta_0 + \mu$ with some $\mu \in \mathscr{M}^a$,

—if $U_1(1) \subset G$ and $\mu_1 = \delta_0 - \mu$ with some probability μ, a convolution power of which has nonvanishing absolutely continuous part.

The first case is obvious. Here is an elementary argument for the second: Let $(\cdot)_{\text{abs}}, (\cdot)_{\text{sing}}$ denote absolutely continuous and singular parts, respectively. Then $\mu - z\delta_0 \in I$ implies $\mu^{*n} - z^n\delta_0 \in I$ which because of $(\mu^{*n})_{\text{abs}} \in I$ gives

$$\delta_0 - z^{-n}(\mu^{*n})_{\text{sing}} \in I.$$

So if $|z| \geq 1$, I (which is closed) would contain a sequence which tends to δ_0.

The final result of this section gives another property of the spaces $\mathscr{D}(\tau)$, $\mathscr{M}(\tau)$, its proof follows from Lemma A.2(i) and an argument on page 263 in Gel'fand et al. (1964).

LEMMA 3. *Let τ be a DVF, let $a, b, c, d \in \mathbb{R}$ be given with $a < b < c < d$. Then there exists an absolutely continuous $\mu \in \mathscr{D}(\tau) \cap \mathscr{M}(\tau) \cap D(\Sigma)$ such that $\hat{\mu}(\theta) \in [0,1]$ for all $\theta \in \mathbb{R}$ and $\hat{\mu}(\theta) = 1$ on $[b,c]$, $\hat{\mu}(\theta) = 0$ on $[a,d]^c$.*

3. Subordinators with exponentially decreasing weights. Throughout this section we assume that $a_n = o(\exp(-\alpha n))$ for some $\alpha > 0$, and we put $m_k = \sum_{n=1}^{\infty} n^k a_n$ $(k = 1, 2)$.

THEOREM 4. *Let τ be a DVF with $\tau(x) = O(x^{-2})$, suppose μ has a density f of bounded variation satisfying $V_x^{x+1} f = o(\tau(x))$, and let $\nu = \sum_{n=1}^{\infty} a_n \mu^{*n}$. Then ν has a density g satisfying $V_x^{x+1} g = o(\tau(x))$. If moreover $\tau(x) = O(x^{-3})$ and μ has finite first moment, then*

$$g(x) = m_1 f(x) + o(\tau(x)).$$

PROOF. Put $\Psi(z) = \sum_{n=1}^{\infty} a_n z^n$, then $\nu(\theta) = \Psi(\hat{\mu}(\theta))$ for all $\theta \in \mathbb{R}$. Since $\hat{\mu}(\mathbb{R}) \subset \overline{U_1(0)}$ and Ψ is analytic on $U_{1+\varepsilon}(0)$ for some $\varepsilon > 0$ Theorem 1 implies $\nu \in \mathscr{D}(\tau)$, which is the first part of the assertion. Further we have

$$\Psi(z) = \Psi(1) + (z-1)\Psi'(1) + (z-1)^2\Phi(z),$$

for some Φ analytic on $U_{1+\varepsilon}(0)$, which gives

$$\hat{\nu}(\theta) = \hat{\nu}(0) - \Psi'(1) + \Psi'(1)\hat{\mu}(\theta) + \left[\widehat{\Sigma\mu}(\theta)^2 i\theta\Phi(\hat{\mu}(\theta))\right] i\theta.$$

We have $\widehat{\Sigma\mu}(\theta)^2 i\theta \in \hat{\mathscr{D}}(\tau)$ by Lemma A.4 and $\Phi(\hat{\mu}(\theta)) \in \hat{\mathscr{D}}(\tau)$ by Theorem 1, so the term in square brackets is in $\hat{\mathscr{D}}(\tau)$. Since the corresponding measure has no point mass in 0 its product with $i\theta$ is in $\hat{\mathscr{M}}(\tau)$. So $\nu - \Psi'(1)\mu \in \mathscr{M}(\tau)$, which means

$$\int_x^{x+1} |g(t) - \Psi'(1)f(t)|\, dt = o(\tau(x)).$$

We know that $V_x^{x+1}g = o(\tau(x))$ from the first part of the proof, $V_x^{x+1}f = o(\tau(x))$ by assumption. Putting this together we get the second assertion of the theorem. □

The next result gives an expansion of the distribution function of ν.

THEOREM 5. *Let τ be a DVF with $\tau(x) = O(x^{-3})$, let μ, ν, and f be as in Theorem 4. Assume that μ has finite first moment κ. Then*

$$\nu((x,\infty)) = (2m_1 - m_2)\mu((x,\infty)) + \tfrac{1}{2}(m_2 - m_1)\mu * \mu((x,\infty)) + o(\tau(x)).$$

If moreover $\tau(x) = O(x^{-4})$ and μ has finite second moment, then

$$\nu((x,\infty)) = m_1\mu((x,\infty)) + (m_2 - m_1)\kappa f(x) + o(\tau(x)).$$

PROOF. With Ψ as in the proof of Theorem 4 and a suitable Φ, analytic on $U_{1+\varepsilon}(0)$ for some $\varepsilon > 0$, we have

$$\widehat{\Sigma\nu}(\theta) - (\Psi'(1) - \Psi''(1))\widehat{\Sigma\mu}(\theta) - \tfrac{1}{2}\Psi''(1)\widehat{\Sigma(\mu*\mu)}(\theta) = \left[\widehat{\Sigma\mu}(\theta)^3 i\theta\Phi(\hat{\mu}(\theta))\right] i\theta,$$

from which we deduce as in that proof

$$\sigma_1 = \Sigma\nu - (\Psi'(1) - \Psi''(1))\Sigma\mu - \tfrac{1}{2}\Psi''(1)\Sigma(\mu * \mu) \in \mathscr{M}(\tau).$$

Using Theorem 4 we obtain $\sigma_1 \in \mathscr{D}(\tau)$, so $\Delta\sigma_1((x,\infty))$ behaves as $o(\tau(x))$. This gives the first equation. The second will follow from the first once we have shown $\Delta\sigma_2((x,\infty)) = o(\tau(x))$, where $\sigma_2 = \Sigma(\mu * \mu) - 2\Sigma\mu - 2\kappa\mu$, which in turn will follow from $\sigma_2 \in \mathscr{D}(\tau) \cap \mathscr{M}(\tau)$. The first set contains σ_2 because of $\mu * \mu - 2\mu \in \mathscr{M}(\tau)$, which we know from Theorem 4. Since μ has finite second moment we have $\Sigma\mu \in D(\Sigma)$ and using Lemma A.5 we see that

$$\Delta^3(\Sigma^2\mu * \Sigma^2\mu) \in \mathscr{M}(\tau).$$

This measure differs from σ_2 only by a multiple of δ_0. □

If f itself is "nearly" dominatedly varying, we have the following result on asymptotic equality.

COROLLARY 6. *Let μ be a probability measure with finite first moment κ and a density f of bounded variation which satisfies*

$$\sup_{y\ge x} f(y) = O(f(2x)), \qquad V_x^{x+1}f = o(f(x)), \qquad f(x) = O(x^{-3}).$$

Then

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = m_1.$$

If moreover $f(x) = O(x^{-4})$ *and the second moment of* μ *is finite, then*

$$\lim_{x\to\infty} \frac{\nu((x,\infty)) - m_1\mu((x,\infty))}{f(x)} = (m_2 - m_1)\kappa.$$

PROOF. Apply Theorem 4 and Theorem 5 with $\tau(x) = \sup_{y\geq x} f(y)$. □

4. Infinitely divisible distributions. Let P be an infinitely divisible (i.d.) distribution with Lévy measure ν; then for some $\alpha \in \mathbb{R}$, $\sigma^2 \geq 0$ we have

$$\hat{P}(\theta) = \exp\left(i\alpha\theta - \frac{1}{2}\sigma^2\theta^2 + \int\left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2}\right)\nu(dx)\right), \quad \text{for all } \theta \in \mathbb{R}.$$

It is well known [see, e.g., Feller (1971, page 572)] that the tails of P and ν are asymptotically equal, i.e.,

$$\lim_{x\to\infty} \frac{P((x,\infty))}{\nu((x,\infty))} = 1, \tag{8}$$

if $\nu((x,\infty))$ is regularly varying. In the one-sided case $P([0,\infty)) = 1$, Embrechts, Goldie and Veraverbeke (1979) obtained a necessary and sufficient condition for (8); see Grübel (1984, Section 4.2) for a partial generalization. As another example of how our method leads to next-term results we have the following theorem.

THEOREM 7. *Let* P *be i.d. with Lévy measure* ν *and let* τ *be a DVF with* $\tau(x) = O(x^{-4})$. *Suppose* $\int x^2\nu(dx) < \infty$ *and assume that the restriction of* ν *to some* $[-C, C]^c$, $C > 0$, *is absolutely continuous and has a density* f *of bounded variation satisfying* $V_x^{x+1} f = o(\tau(x))$. *Then*

$$P((x,\infty)) = \nu((x,\infty)) + \kappa f(x) + o(\tau(x)),$$

where κ *denotes the first moment of* P.

PROOF. If ν has compact support we can continue P to an entire function which implies that P has exponentially decreasing tails [Lukacs (1970, Theorem 7.2.1)], Lemma A.2(i) then gives the assertion.

Assume $c = \nu([-C, C]^c) > 0$ and define $\mu = (1/c)\nu|_{[-C,C]^c}$. Let P_1 denote the i.d. distribution with transform

$$\hat{P}_1(\theta) = \exp\left(i\theta\alpha_1 - \frac{1}{2}\sigma^2\theta^2 + \int\left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2}\right)\nu_1(dx)\right),$$

where

$$\alpha_1 = \alpha - \int_{[-C,C]^c} x/(1+x^2)\nu(dx)$$

and $\nu_1 = \nu|_{[-C,C]}$. Further let

$$P_2 = e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \mu^{*k},$$

then $P = P_1 * P_2$. From the results of Section 3 we know that $P_2 \in \mathscr{D}(\tau)$ and

$$\begin{aligned} P_2((x,\infty)) &= \nu((x,\infty)) + \kappa_2 f(x) + o(\tau(x)), \\ \Delta P_2((x,\infty)) &= f(x) + o(\tau(x)), \end{aligned} \tag{9}$$

where κ_2 denotes the first moment of P_2.

Now consider $\sigma = \Sigma P - \Sigma P_2 - \kappa_1 P_2$, where κ_1 denotes the first moment of P_1. P_1 has exponentially decreasing tails, so $P_1 \in D(\Sigma)$, $\Sigma P_1 \in D(\Sigma)$, and ΣP_1, $\Sigma\Sigma P_1$ are in $\mathscr{M}(\tau)$. We have $P - P_2 = \Sigma P_1 * \Delta P_2 + e^{-c} P_1 - e^{-c}\delta_0$ which implies $\sigma \in \mathscr{D}(\tau)$. Similarly we can show that $\sigma = \Sigma\Sigma P_1 * \Delta P_2 + e^{-c}\Sigma P_1 - \kappa_1 e^{-c}\delta_0$ which gives $\sigma \in \mathscr{M}(\tau)$. So $\Delta\sigma((x,\infty)) = o(\tau(x))$, that is

$$P((x,\infty)) = P_2((x,\infty)) + \kappa_1 \Delta P_2((x,\infty)) + o(\tau(x)),$$

which together with (9) and $\kappa = \kappa_1 + \kappa_2$ completes the proof of the theorem. □

Theorem 1 is used in the above proof in case of (1) with weights decreasing faster than exponential. The results of Section 2 are not needed in full generality since completeness and norm-inequality of the respective algebras suffice. A similar situation may arise in the case of exponentially decreasing weights; see Omey and Willekens (1984) which contains some related results in the case of measures concentrated on the half-line $[0,\infty)$ and Chover et al. (1973, page 267, Remark 2).

5. Harmonic renewal measures. In this section we consider the special case $a_0 = 0$, $a_n = 1/n$ $(n \in \mathbb{N})$ of (1). We then obtain the harmonic renewal measure $\nu_h = \sum_{n=1}^{\infty}(1/n)\mu^{*n}$. Such measures have been investigated by Greenwood, Omey and Teugels (1982) and Grübel (1986). In Grübel (1986) $\mathscr{M}(\tau)$-spaces have been used to estimate ν_h down to the order of $\mu((x,\infty))$, on using the spaces $\mathscr{D}(\tau)$ and Theorem 1 we obtain an expansion with error term of magnitude $\mu([x, x+1)]$. Let l_h^+ denote the measure on $(0,\infty)$ with l-density $(1/x) \wedge 1$.

THEOREM 8. *Let τ be a DVF with $\tau(x) = O(x^{-3})$. Suppose that some convolution power of μ has nonvanishing absolutely continuous component and that μ has finite second and positive first moment m_1. Then $\mu([x, x+1]) = o(\tau(x))$ implies*

$$\left|\nu_h - l_h^+ + \frac{1}{m_1}\Sigma\mu\right|([x, x+1]) = o(\tau(x)).$$

PROOF. Let ν_e be the measure on $(0,\infty)$ with l-density $(1/x)(1 - e^{-x})$ and put $\nu_1 = \nu_h - \nu_e$. Let log: $G \to \mathbb{C}$, $G = \{z \in \mathbb{C}\colon \operatorname{Re}(z) > 0 \vee \operatorname{Im}(z) \neq 0\}$, denote the principal branch of the logarithm. Then, under the assumptions of the theorem, $\nu_1 \in \mathscr{M}$ and $\hat{\nu}_1(\theta) = -\log(1 - \hat{\mu}(\theta) + \widehat{\Sigma\mu}(\theta))$ [see Grübel (1986)] and we have to show $\nu_1 + (1/m_1)\Sigma\mu \in \mathscr{M}(\tau)$. Since $m_1 = \widehat{\Sigma\mu}(0) > 0$ we can find $\alpha > 0$ such that $\mathrm{Re}(\widehat{\Sigma\mu}(\theta)) \geq 1/2m_1 > 0$ if $|\theta| \leq 4\alpha$. Because of Lemma 3 we can find some $\mu_1 \in \mathscr{M}(\tau) \cap D(\Sigma)$, depending on α, such that

$$\hat{\mu}_1(\mathbb{R}) \subset [0,1], \qquad \hat{\mu}_1(\theta) = 1, \quad \text{if } |\theta| \leq \alpha, \qquad \hat{\mu}_1(\theta) = 0, \quad \text{if } |\theta| \geq 2\alpha,$$

and $\Sigma\mu_1 \in \mathscr{M}(\tau)$. We consider separately $\gamma_1 = (\nu_1 + (1/m_1)\Sigma\mu) * \mu_1$ and $\gamma_2 = (\nu_1 + (1/m_1)\Sigma\mu) * (\delta_0 - \mu_1)$.

Choose $\mu_2 \in \mathscr{M}(\tau) \cap \mathscr{D}(\tau)$ with

$$\hat{\mu}_2(\mathbb{R}) \subset [0,1], \qquad \hat{\mu}_2(\theta) = 1, \quad \text{if } |\theta| \leq 2\alpha \quad \text{and} \quad \hat{\mu}_2(\theta) = 0, \quad \text{if } |\theta| \geq 4\alpha,$$

and put

$$\mu_3 = \delta_0 - \mu_2 + \mu_2 * \Sigma\mu$$

and

$$\mu_4 = \delta_0 - \mu_2 + \mu_2 * \mu_e,$$

where μ_e denotes the exponential distribution with parameter 1. Then

$$\hat{\gamma}_1(\theta) = \left(-\log \hat{\mu}_3(\theta) + \frac{1}{m_1}\hat{\mu}_3(\theta) + \log \hat{\mu}_4(\theta)\right)\hat{\mu}_1(\theta).$$

For some function Φ analytic on G and some constant C we have

$$\log \hat{\mu}_3(\theta) - \frac{1}{m_1}\hat{\mu}_3(\theta) = C + (i\theta)^2\widehat{\Sigma\mu_3}(\theta)^2\Phi(\hat{\mu}_3(\theta)).$$

Since $\hat{\mu}_3(\theta)$ is on the line connecting 1 and $\widehat{\Sigma\mu}(\theta)$ we have $\hat{\mu}_3(\theta) \in G$ on $|\theta| \leq 4\alpha$ by the choice of α; if $|\theta| \geq 4\alpha$ we have $\hat{\mu}_3(\theta) = 1$. So Theorem 1 applies and on using Lemma A.4 we obtain $(i\theta)^2\widehat{\Sigma\mu_3}(\theta)^2\Phi(\hat{\mu}_3(\theta)) \in \hat{\mathscr{M}}(\tau)$. Further we get $\log \hat{\mu}_4(\theta) \in \hat{\mathscr{M}}(\tau)$ from Theorem 2, so $\gamma_1 \in \mathscr{M}(\tau)$ follows.

In order to handle γ_2 we first choose some $\mu_5 \in \mathscr{M}(\tau)$ with

$$\hat{\mu}_5(\mathbb{R}) \subset [0,1], \qquad \hat{\mu}_5(\theta) = 1, \quad \text{on } |\theta| \leq \alpha/2, \qquad \hat{\mu}_5(\theta) = 0, \quad \text{on } |\theta| \geq \alpha,$$

and put

$$\mu_6 = \delta_0 - \mu + \mu_5 * \mu.$$

Then

$$\hat{\gamma}_2(\theta) = (\hat{\mu}_1(\theta) - 1)\log \hat{\mu}_6(\theta) + i\theta\widehat{\Sigma\mu_1}(\theta)\log\left(1 + \widehat{\Sigma\mu_5}(\theta)\right) + \frac{1}{m_1}\widehat{\Sigma\mu_1}(\theta)(1 - \hat{\mu}(\theta)).$$

Theorem 2 and the remarks following it give $\log \hat{\mu}_6(\theta) \in \hat{\mathscr{M}}(\tau)$. Since $\mathrm{Re}(\widehat{\Sigma\mu_5}(\theta)) = 0$ Theorem 1 gives $\log(1 + \widehat{\Sigma\mu_5}(\theta)) \in \hat{\mathscr{D}}(\tau)$, also $\Sigma\mu_1 \in \mathscr{D}(\tau)$. From this we obtain $\gamma_2 \in \mathscr{M}(\tau)$ which completes the proof of the theorem. □

The main result of Embrechts et al. (1984) gives conditions for

$$\nu_h([x, x+h]) \sim l_h^+([x, x+h]).$$

These also apply to the more general situation where the sequence $(a_n)_{n \in \mathbb{N}_0}$ from (1) is only assumed to be regularly varying with index -1. However, for the asymptotic behaviour of the differences $\nu_h([x, x+h]) - l_h^+([x, x+h])$, the precise form of the coefficients a_n is decisive—this may readily be seen on using the results of Embrechts et al. (1984).

6. Renewal measures. The special case of (1) which has most extensively been dealt with is the one with $a_n = 1$ for all $n \in \mathbb{N}_0$. We then obtain the (ordinary) renewal measure $\nu = \sum_{n=0}^{\infty} \mu^{*n}$. In this situation an expansion of $\nu([x, x+1])$ with error term of magnitude $\mu([x, x+1])$ has already been given by Essén (1973). Our first result complements Theorem 3.2 of Essén (1973) with respect to the conditions on the reference function τ (see also 7.4 below). The second part of our result makes more concrete a remark following this theorem. Let l^+ denote the restriction of l to $(0, \infty)$.

THEOREM 9. *Let τ be a DVF with $\tau(x) = O(x^{-3})$. Suppose that some convolution power of μ has nonvanishing absolutely continuous component and that μ has finite second moment m_2 and positive first moment m_1. Then $\mu([x, x+1]) = o(\tau(x))$ implies*

$$\left| \nu - \frac{1}{m_1} l^+ - \frac{2}{m_1^2} \Sigma\Sigma\mu + \frac{1}{m_1^3} \Sigma\Sigma\mu * \Sigma\mu \right| ([x, x+1]) = o(\tau(x)).$$

If moreover μ has finite third moment and $\tau(x) = O(x^{-4})$, then $\mu([x, x+1]) = o(\tau(x))$ implies

$$\left| \nu - \frac{1}{m_1} l^+ - \frac{1}{m_1^2} \Sigma\Sigma\mu + \frac{m_2}{m_1^3} \Sigma\mu \right| ([x, x+1]) = o(\tau(x)).$$

PROOF. It follows from the calculations in Section 3 of Carlsson (1983) that

$$\left(\nu - \frac{1}{m_1} l^+ \right)^{\wedge} (\theta) = \frac{\widehat{\Sigma\Sigma\mu}(\theta)}{m_1 \widehat{\Sigma\mu}(\theta)}.$$

We are thus led to expand $\Psi(z) = 1/z$ about $\widehat{\Sigma\mu}(0) = m_1$ which gives

$$\left(\nu - \frac{1}{m_1} l^+ - \frac{2}{m_1^2} \Sigma\Sigma\mu + \frac{1}{m_1^3} \Sigma\Sigma\mu * \Sigma\mu \right)^{\wedge} (\theta) = \frac{\widehat{\Sigma\Sigma\mu}(\theta) \left(\widehat{\Sigma\mu}(\theta) - m_1 \right)^2}{m_1^3 \widehat{\Sigma\mu}(\theta)}.$$

On using the same arguments as in the proof of Theorem 8 we will show that the right side of this equation is in $\mathscr{M}(\tau)$.

Choose $\alpha > 0$ with $\mathrm{Re}(\widehat{\Sigma\mu}(\theta)) \geq \frac{1}{2} m_1$ on $|\theta| \leq 4\alpha$ and then $\mu_1 \in \mathscr{D}(\tau) \cap \mathscr{M}(\tau)$ with

$$\hat{\mu}_1(\mathbb{R}) \subset [0, 1], \qquad \hat{\mu}_1(\theta) = 1, \text{ if } |\theta| \leq \alpha, \qquad \hat{\mu}_1(\theta) = 0, \text{ if } |\theta| \geq 2\alpha.$$

We then have

$$\frac{\widehat{\Sigma\Sigma\mu}(\theta)\left(\widehat{\Sigma\mu}(\theta) - m_1\right)^2}{\widehat{\Sigma\mu}(\theta)} = \gamma_1(\theta) - \gamma_2(\theta),$$

with

$$\gamma_1(\theta) = i\theta\left[\left(\widehat{\Sigma\Sigma\mu}(\theta)^3 i\theta\right)\left(\frac{\hat{\mu}_1(\theta)}{\widehat{\Sigma\mu}(\theta)}\right)\right]$$

and

$$\gamma_2(\theta) = \left[i\theta\left(\widehat{\Sigma\Sigma\mu}(\theta)^2 i\theta\right)\left(\widehat{\Sigma\Sigma\mu}(\theta) i\theta\right)\right]\left[\frac{1 - \hat{\mu}_1(\theta)}{1 - \hat{\mu}(\theta)}\right],$$

where the brackets are meant to indicate that because of Lemma A.4 it remains to prove

$$\frac{\hat{\mu}_1(\theta)}{\widehat{\Sigma\mu}(\theta)} \in \hat{\mathscr{D}}(\tau), \qquad \frac{1 - \hat{\mu}_1(\theta)}{1 - \hat{\mu}(\theta)} \in \hat{\mathscr{M}}(\tau).$$

Since $z \to 1/z$ is analytic on G, the domain of analyticity of the principal branch of the logarithm, these relations follow with exactly the same constructions as in the proof of Theorem 8 (use μ_2, μ_3 on the first and μ_5, μ_6 on the second relation).

The second part of the theorem will follow from the first part and

$$\Sigma\Sigma\mu * \Sigma\mu = m_1\Sigma\Sigma\mu + m_2\Sigma\mu + \gamma,$$

for some $\gamma \in \mathscr{M}(\tau)$. Since μ has finite third moment now we may apply Σ to $\Sigma\Sigma\mu$ again and then use Lemma A.5 as done at the end of the proof of Theorem 5. □

Our final result deals with the renewal function. The relevance of the different expansions may perhaps be understood best by applying it to a special class of measures such as mixtures of Pareto distributions and a distribution with support bounded from above.

THEOREM 10. *Let τ be a DVF with $\tau(x) = O(x^{-3})$. Suppose that some convolution power of μ has nonvanishing absolutely continuous component and that μ has finite second moment m_2 and positive first moment m_1. Put $U(x) = \nu((-\infty, x])$ for all $x \in \mathbb{R}$. Then $\mu([x, x+1]) = o(\tau(x))$ implies*

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{3}{m_1^2}\Sigma\Sigma\mu((x,\infty)) + \frac{3}{m_1^3}\Sigma\Sigma\mu * \Sigma\mu((x,\infty))$$
$$- \frac{1}{m_1^4}\Sigma\Sigma\mu * \Sigma\mu * \Sigma\mu((x,\infty)) + o(\tau(x)).$$

If moreover μ has finite third moment m_3 and $\tau(x) = O(x^{-4})$, then

$\mu([x, x+1]) = o(\tau(x))$ *implies*

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{2}{m_1^2}\Sigma\Sigma\mu(x,\infty)) + \frac{1}{m_1^3}\Sigma\Sigma\mu * \Sigma\mu((x,\infty)) - \frac{3m_2^2}{4m_1^4}\mu((x,\infty)) + o(\tau(x)).$$

If moreover μ *has finite fourth moment and* $\tau(x) = O(x^{-5})$, *then* $\mu([x, x+1]) = o(\tau(x))$ *implies*

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{1}{m_1^2}\Sigma\Sigma\mu((x,\infty)) + \frac{m_2}{m_1^3}\Sigma\mu((x,\infty)) + \frac{4m_1m_3 - 9m_2^2}{12m_1^4}\mu((x,\infty)) + o(\tau(x)).$$

PROOF. Put

$$\nu_1 = \nu - \frac{1}{m_1}l^+ - \frac{1}{m_1^2}\Sigma\Sigma\mu + \frac{1}{m_1^3}\Sigma\Sigma\mu * (\Sigma\mu - m_1\delta_0) - \frac{1}{m_1^4}\Sigma\Sigma\mu * (\Sigma\mu - m_1\delta_0)^{*2}.$$

We know from the preceding theorem that

$$\nu - \frac{1}{m_1}l^+ - \frac{1}{m_1^2}\Sigma\Sigma\mu + \frac{1}{m_1^3}\Sigma\Sigma\mu * (\Sigma\mu - m_1\delta_0) \in \mathscr{M}(\tau),$$

which together with Lemma A.4 gives $\nu_1 \in \mathscr{M}(\tau)$. Further we have

$$\widehat{\Sigma\nu_1}(\theta) = -\frac{\widehat{\Sigma\Sigma\mu}(\theta)^2\left(\widehat{\Sigma\mu}(\theta) - m_1\right)^2}{m_1^4\widehat{\Sigma\mu}(\theta)};$$

using the same construction as in the proof of Theorem 9 we see that $\Sigma\nu_1 \in \mathscr{M}(\tau)$. Hence $\nu_1((x,\infty)) = o(\tau(x))$, which is the first assertion of the theorem.

Lemma A.5 gives $\Delta(\Sigma^3\mu * \Sigma^3\mu) = \Sigma\mu_1$ for some $\mu_1 \in \mathscr{D}(\tau)$, so we obtain

$$\Delta^3(\Sigma^2\mu * \Sigma^3\mu * \Sigma^3\mu) \in \mathscr{M}(\tau),$$

with Lemma A.4. This means $\Sigma\sigma \in \mathscr{M}(\tau)$ where

$$\sigma = \Sigma\Sigma\mu * (\Sigma\mu - m_1\delta_0)^{*2} - m_2(\Sigma\mu - m_1\delta_0)^{*2} + \tfrac{1}{4}m_2^2\mu.$$

We also have $\sigma \in \mathscr{M}(\tau)$, so $\sigma((x,\infty)) = o(\tau(x))$, which gives

$$\Sigma\Sigma\mu * (\Sigma\mu - m_1\delta_0)^{*2}((x,\infty)) = m_2(\Sigma\mu - m_1\delta_0)^{*2}((x,\infty)) - \tfrac{1}{4}m_2^2\mu((x,\infty)) + o(\tau(x)).$$

Similarly we obtain

$$\Sigma\mu * \Sigma\mu((x,\infty)) = 2m_1\Sigma\mu((x,\infty)) + m_2\mu((x,\infty)) + o(\tau(x)),$$

and on inserting we get the second part of the theorem. The same procedure, using additionally Lemma A.6, also gives the third formula of the theorem. □

7. Concluding remarks. It should be clear that, e.g., Theorem 7 admits a corollary on asymptotic equality similar to Theorem 5, that the harmonic renewal function can be analyzed in the same way as the renewal function in Theorem 10, etc. In this section we indicate some less obvious extensions, some of these are carried out in my habilitation thesis [Grübel (1984)], on which the present paper is based.

7.1. *The limit* $x \to -\infty$. In Sections 3 and 4 it is enough to switch to μ_-, $\mu_-(A) = \mu(\{x\colon -x \in A\})$, apply the $x \to \infty$ result and then transform back in order to obtain the corresponding $x \to -\infty$ results. This simple argument fails in Sections 5 and 6 since the assumptions on μ are no longer symmetric. It is convenient to introduce $\mathscr{D}_-(\tau) = \{\mu\colon \mu_- \in \mathscr{D}(\tau)\}$ and similarly $\mathscr{M}_-(\tau)$, the results of Section 2 also hold for these spaces. Replacing $\mathscr{D}$, $\mathscr{M}$ by $\mathscr{D}_-$, $\mathscr{M}_-$ we obtain the desired result (the expansion in Theorem 8, e.g., is stated such as to remain valid if $x \to -\infty$).

7.2. *Big-O results.* The analogue of Theorem 1 for spaces such as

$$\mathscr{D}^O(\tau) = \{\mu \in \mathscr{D}\colon |\Delta\mu|((x, x+1]) = O(\tau(x))\}$$

holds if $\tau(x) = O(x^{-2})$ is replaced by $\tau(x) = o(x^{-2})$. The only point in the proof to be observed is that measures with compact support are no longer dense in $\mathscr{D}^O(\tau)$. We may however approximate any $\mu \in \mathscr{D}^O(\tau)$ by a sequence $(\mu_n)_{n \in \mathbb{N}} \subset \mathscr{D}$ of measures with compact support in the sense of

$$\left\|(\mu - \mu_n)^{*2}\right\|_0^\tau \to 0 \quad \text{as } n \to \infty,$$

because of the multiplicativity of the homomorphisms this suffices (the details are somewhat laborious). This together with the corresponding result on $\mathscr{M}$-spaces [Rogozin (1976a) and Rogozin and Sgibnev (1980)] leads to O-versions of the theorems in Sections 3–6.

7.3. *Noninfinitesimal differences.* Let $\Delta_\varepsilon\colon \mathscr{M} \to \mathscr{M}$ ($\varepsilon > 0$) be defined by

$$\Delta_\varepsilon\mu(A) = \mu(A) - \mu(A + \varepsilon), \quad \text{for all } A \in \mathscr{B},$$

and put

$$\mathscr{D}_\varepsilon(\tau) = \{\mu \in \mathscr{M}\colon \Delta_\varepsilon\mu \in \mathscr{M}(\tau)\}.$$

Using such spaces of measures characterized by the asymptotic behaviour of their "noninfinitesimal differences" we obtain weaker conclusions under weaker assumptions. A typical result is the following which should be compared with Theorem 7: Let P be infinitely divisible with Lévy measure ν, let τ be a DVF with $\tau(x) = O(x^{-4})$, $\varepsilon > 0$. Suppose $\int x^2\nu(dx) < \infty$ and $\nu|_{[-C, C]^c} \in \mathscr{D}_\varepsilon(\tau)$ for some $C > 0$. Then

$$\int_{(x, x+\varepsilon)} P((t, \infty))\, dt = \int_{(x, x+\varepsilon)} \nu((t, \infty))\, dt + \kappa\nu((x, x+\varepsilon]) + o(\tau(x)),$$

where κ denotes the first moment of P.

7.4. *Other reference functions.* DVF's can exhibit oscillatory behaviour, e.g., for every $\alpha > \beta > 0$ it is possible to find a DVF τ such that

$$\tau(x) \geq x^{-\beta}, \quad \text{for all } x \text{ in some unbounded set,}$$
$$\tau(x) \leq x^{-\alpha}, \quad \text{for all } x \text{ in some unbounded set.}$$

Also there does not exist a constant γ such that $\tau_1(x) = O(x^{\gamma}\tau(x))$ for all integrable DVF's τ—this is the reason for the additional upper bounds on τ in our results. The situation is easier if τ is assumed to be regularly varying (such functions are asymptotically equal to DVF's) since $\tau_1(x) = O(x\tau(x))$ then. We could have worked with an extra condition such as $\tau_1(x) = O(x\tau(x))$ [this leads to a conceptual simplification, see Essén (1973)] which would exclude heavy oscillating reference functions but include more slowly decreasing ones.

7.5. *Higher-order $\mathscr{D}$-spaces and Wiener–Hopf factors.* Perhaps the most substantial extension of our method would be the introduction of higher-order $\mathscr{D}$-spaces. In principle this should lead to arbitrarily fine expansions (under more and more restrictive conditions on μ, of course). A particular case in which the results then possibly might be of interest is that of the distribution μ^+ of the first positive sum S_N, $N = \inf\{n \in \mathbb{N}\colon S_n > 0\}$ [$(S_n)_{n \in \mathbb{N}_0}$ still denotes the sequence of partial sums of an i.i.d. sequence with distribution μ]. Then the harmonic renewal measure corresponding to μ^+ is the restriction to $(0, \infty)$ of the harmonic renewal measure of μ. Results relating the behaviour of a probability to that of its harmonic renewal measure can be obtained with the methods of Section 3, so putting this together with the step described in Section 5 we get expansions of μ^+ in terms of μ. The results obtainable along these lines at the present stage, i.e., with first-order $\mathscr{D}$-spaces, are much easier to come by directly, see e.g., Grübel (1985). Higher-order $\mathscr{D}$-spaces however should lead to expansions no longer attainable by elementary means.

APPENDIX

Here we collect some lemmas of a more technical nature. The first lemma lists some properties of Σ and Δ, we omit the simple proofs. Lemma A.2 gives some elementary properties of dominatedly varying functions, all others are concerned with estimating the total variation of convolution products.

LEMMA A.1. (i) *Σ and Δ are linear operators.*

(ii) *If $\mu_1 \in D(\Sigma)$ satisfies $\mu_1(\mathbb{R}) = 0$, then for any $\mu_2 \in \mathscr{M}$ we have $\mu_2 * \mu_1 \in D(\Sigma)$, and $\Sigma(\mu_2 * \mu_1) = \mu_2 * \Sigma\mu_1$.*

(iii) $\Delta(\mu_1 * \mu_2) = \mu_1(\{0\})\Delta\mu_2 + \mu_2 * \Delta\mu_1$ *for all* $\mu_1, \mu_2 \in \mathscr{D}$.

(iv) $\Sigma\Delta\mu = \mu - \mu(\{0\})\delta_0$ *for all* $\mu \in \mathscr{D}$, $\Delta\Sigma\mu = \mu - \mu(\mathbb{R})\delta_0$ *for all* $\mu \in D(\Sigma)$.

(v) $\widehat{\Sigma\mu}(\theta) = [\hat{\mu}(\theta) - \hat{\mu}(0)]/i\theta$ *for all* $\mu \in D(\Sigma)$, $\widehat{\Delta\mu}(\theta) = i\theta(\hat{\mu}(\theta) - \hat{\mu}(\infty))$ *for all* $\mu \in \mathscr{D}$, *where* $\hat{\mu}(\infty) = \lim_{\theta \to \infty}\hat{\mu}(\theta)$ $(= \mu(\{0\}))$.

If τ is a DVF with $\tau(x) = O(x^{-n-1})$, let $\tau_0 = \tau$, $\tau_k = \int_x^\infty \tau_{k-1}(t)\,dt$ for $k = 1, \dots, n$.

LEMMA A.2. *Let τ be a DVF.*

(i) *There exist $k \in \mathbb{N}$, $C > 0$ such that $\tau(x) \geq Cx^{-k}$ for all $x \geq 1$. Moreover,*

$$\sup_{y \geq 0} \frac{\tau(y)}{\tau(y+x)} = O(x^k), \quad \textit{for some } k \in \mathbb{N}.$$

(ii) *Assume $\tau(x) = O(x^{-n-1})$ for some $n \in \mathbb{N}$. Then $\tau_1, \dots, \tau_n$ are DVF's again and for $k = 1, \dots, n$,*

$$\tau_k(x)\tau_{n+1-k}(x) = O(\tau(x)).$$

PROOF. (i) See Feller (1971, page 289) for the first part. Further, by definition of $C(\tau)$,

$$\tau(x+y) \leq C(\tau)^{F(x)}\tau(y), \quad \text{for all } y \geq 1$$

with $F(x) = [\log_2 x] + 2$.

(ii) $\tau_1(x) \leq C(\tau)\int_x^\infty \tau(2t)\,dt = \frac{1}{2}C(\tau)\int_{2x}^\infty \tau(t)\,dt = \frac{1}{2}C(\tau)\tau_1(2x)$, so τ_1 is a DVF again. This gives the first part of the statement. To prove the second part we note that

$$\tau_{k+1}(x) = O\left(\tau_k(x)^{1-1/(n+1-k)}\right), \qquad k = 0, \dots, n-1,$$

since $\tau_k(x) \leq Cx^{-n-1+k}$ for all $x \geq 0$ with some suitable C and then, with $F(x) = \tau_k(x)^{-1(n+1-k)}$,

$$\tau_{k+1}(x) \leq \int_x^{F(x)} \tau_k(x)\,dt + C\int_{F(x)}^\infty t^{-n-1+k}\,dt$$
$$\leq F(x)\tau_k(x) + \frac{C}{n-k}F(x)^{-n+k}.$$

On using an induction argument we obtain

$$\tau_k(x) = O\left(\tau(x)^{1-k/(n+1)}\right), \qquad k = 1, \dots, n,$$

which gives the assertion. □

We write $\Sigma(D(\Sigma))$ for the range of Σ. This set is easily seen to be closed with respect to convolution.

LEMMA A.3. *Let τ be a DVF with $\tau(x) = O(x^{-2})$. For any $\mu \in \Sigma(D(\Sigma))$, $x \geq 0$, put*

$$C(\mu, x) = \sup_{y \geq x} \tau(y)^{-1}|\Delta\mu|((y, y+1]).$$

Then there exists a constant $C_1(\tau) \geq 1$ such that

$$C(\mu_1 * \mu_2, x) \leq C_1(\tau)\left[C\left(\mu_1, \frac{x}{2}\right)\|\mu_2\|_{\mathrm{TV}} + C\left(\mu_2, \frac{x}{2}\right)\|\mu_1\|_{\mathrm{TV}} + C\left(\mu_1, \frac{x}{2}\right)C\left(\mu_2, \frac{x}{2}\right)\right],$$

for all $\mu_1, \mu_2 \in \Sigma(D(\Sigma))$, $x \geq 0$.

PROOF. Suppose $\mu_i = \Sigma\nu_i$ with $\nu_i(\mathbb{R}) = 0$ for $i = 1,2$. Let $n \in \mathbb{N}$, $0 \le x = x_0 < \cdots < x_n = x + 1$ be given. Then for any $k \in \{1, \dots, n\}$

$$\begin{aligned}(\Delta(\mu_1 * \mu_2))((x_{k-1}, x_k]) &= \nu_1 * \mu_2((x_{k-1}, x_k]) \\ &= \int_{(-\infty, x/2]} \nu_1((x_{k-1} - y, x_k - y])\mu_2(dy) \\ &\quad + \int_{(x/2, \infty)} \nu_1((x_{k-1} - y, x_k - y])\mu_2(dy).\end{aligned}$$

We use partial integration and $\nu_2(\mathbb{R}) = 0$ to transform the second term:

$$\begin{aligned}&\int_{(x/2,\infty)} \nu_1((x_{k-1} - y, x_k - y])\nu_2((y, \infty))\, dy \\ &\quad = -\mu_1\left(\left(x_{k-1} - \frac{x}{2}, x_k - \frac{x}{2}\right]\right)\nu_2\left(\left(\frac{x}{2}, \infty\right)\right) \\ &\qquad + \int_{(x/2,\infty)} \mu_1((x_{k-1} - y, x_k - y])\nu_2(dy).\end{aligned}$$

Taking absolute values, summing over k and then taking suprema with respect to partitions of $(x, x + 1]$ we obtain

$$\begin{aligned}|\Delta(\mu_1 * \mu_2)|((x, x + 1]) &\le \int_{(-\infty, x/2]} |\nu_1|((x - y, x + 1 - y])|\mu_2|(dy) \\ &\quad + \int_{(x/2,\infty)} |\mu_1|((x - y, x + 1 - y])|\nu_2|(dy) \\ &\quad + |\mu_1|\left(\left(\frac{x}{2}, \frac{x}{2} + 1\right]\right)\left|\nu_2\left(\left(\frac{x}{2}, \infty\right)\right)\right|.\end{aligned}$$

The first term is not larger than $\|\mu_2\|_{\mathrm{TV}} C(\mu_1, x/2)\tau(x/2)$. If $y \in (x/2 + j, x/2 + j + 1]$ we have

$$|\mu_1|((x - y, x + 1 - y]) \le |\mu_1|\left(\left(\frac{x}{2} - j - 1, \frac{x}{2} - j + 1\right]\right),$$

so

$$\int_{(x/2,\infty)} |\mu_1|((x - y, x + 1 - y])|\nu_2|(dy) \le 2|\mu_1|(\mathbb{R}) \sup_{y \ge x/2} |\nu_2|((y, y + 1]).$$

This yields the upper bound $2\|\mu_1\|_{\mathrm{TV}} C(\mu_2, x/2)\tau(x/2)$ for the second term. Further we have

$$\begin{aligned}|\mu_1|\left(\left(\frac{x}{2}, \frac{x}{2} + 1\right]\right) &\le C\left(\mu_1, \frac{x}{2}\right)\tau_1\left(\frac{x}{2}\right) \sup_{y \ge 0} \frac{\tau(y)}{\tau(y + 1)}, \\ \left|\nu_2\left(\left(\frac{x}{2}, \infty\right)\right)\right| &\le C\left(\mu_2, \frac{x}{2}\right)\tau_1\left(\frac{x}{2}\right) \sup_{y \ge 0} \frac{\tau(y)}{\tau(y + 1)}.\end{aligned}$$

So putting the pieces together and using Lemma A.2(ii) we obtain the assertion. □

LEMMA A.4. *Let* τ *be a DVF with* $\tau(x) = O(x^{-3})$ *and suppose* $\mu_1, \mu_2 \in D(\Sigma) \cap \mathscr{D}(\tau)$. *Then*

$$\Delta((\Sigma\mu_1) * (\Sigma\mu_2)) \in \mathscr{D}(\tau).$$

PROOF. We may assume $\mu_1(\{0\}) = \mu_2(\{0\}) = 0$. We first prove the lemma under the additional assumption $\mu_1(\mathbb{R}) = 0$. Then $\Delta\Delta(\Sigma\mu_1 * \Sigma\mu_2) = \Delta\mu_1 * \Sigma\mu_2$. Let $0 \le x = x_0 < \cdots < x_n = x + 1$ be given. Then

$$(\Delta\mu_1 * \Sigma\mu_2)((x_{k-1}, x_k]) = \left(\int_{(-\infty, x/2]} + \int_{(x/2,\infty)}\right) \Delta\mu_1((x_{k-1} - y, x_k - y])\Sigma\mu_2(dy).$$

If we take absolute values, sum over k and then take the supremum with respect to partitions of $(x, x+1]$, the first integral leads to

$$\int_{(-\infty, x/2]} |\Delta\mu_1|((x - y, x + 1 - y])|\Sigma\mu_2|(dy),$$

which is $o(\tau(x))$ because of $\Delta\mu_1 \in \mathscr{M}(\tau)$ and $|\Sigma\mu_2|(\mathbb{R}) < \infty$. We integrate twice partially the second term,

$$\begin{aligned}\int_{(x/2,\infty)} &\Delta\mu_1((x_{k-1}, x_k - y])\Sigma\mu_2(dy)\\ &= -\mu_1((x_{k-1} - x/2, x_k - x/2])\mu_2((x/2, \infty))\\ &\quad -\Sigma\mu_1((x_{k-1} - x/2, x_k - x/2])\,\Delta\mu_2((x/2,\infty))\\ &\quad +\int_{(x/2,\infty)} \Sigma\mu_1((x_{k-1} - y, x_k - y])\,\Delta\mu_2(dy).\end{aligned}$$

Because of $\Delta\mu_i \in \mathscr{M}(\tau)$ we have

$$|\mu_1|([x/2, x/2 + 1]) = o(\tau_1(x)), \qquad |\Delta\mu_2|((x/2, \infty)) = o(\tau_1(x)),$$

and

$$|\mu_2((x/2,\infty))| = o(\tau_2(x)), \qquad |\Sigma\mu_1|((x/2, x/2 + 1]) = o(\tau_2(x)),$$

so the first two terms lead to terms which on using Lemma A.2(ii) may be estimated as $o(\tau(x))$.

The last integral will lead to

$$\int_{(x/2,\infty)} |\Sigma\mu_1|((x - y, x + 1 - y])|\Delta\mu_2|(dy).$$

On estimating the integrand on intervals $(x/2 + j, x/2 + j + 1]$, $j \in \mathbb{N}_0$, and using $\Delta\mu_2 \in \mathscr{M}(\tau)$ as in the proof of Lemma A.3 we obtain $o(\tau(x))$-behaviour for this term too. This settles the lemma in the case $\mu_1(\mathbb{R}) = 0$.

In the general case put $\mu_{2+i} = \mu_i - \mu_i(\mathbb{R})\rho$, $i = 1, 2$, where ρ denotes the uniform distribution on $[0, 1]$. Then

$$\Sigma\mu_i = \Sigma\mu_{2+i} + \mu_i(\mathbb{R})\Sigma\rho, \qquad i = 1, 2,$$

so

$$\begin{aligned}\Delta(\Sigma\mu_1 * \Sigma\mu_2) &= \Delta(\Sigma\mu_3 * \Sigma\mu_4) + \mu_1(\mathbb{R})\,\Delta(\Sigma\mu_4 * \Sigma\rho)\\ &\quad +\mu_2(\mathbb{R})\,\Delta(\Sigma\mu_3 * \Sigma\rho) + \mu_1(\mathbb{R})\mu_2(\mathbb{R})\,\Delta(\Sigma\rho * \Sigma\rho).\end{aligned}$$

The first three terms are in $\mathscr{D}(\tau)$ by the first part of the proof, and the last term leads to a measure with compact support. □

The last two lemmas are higher-order analogues of the preceding one; a sketch of how refinements of the above arguments lead to a proof of the second should suffice.

LEMMA A.5. *Let τ be a DVF with $\tau(x) = O(x^{-4})$. Then $\mu_1, \mu_2 \in \mathscr{D}(\tau) \cap D(\Sigma^2)$ implies*

$$\Delta^2(\Sigma^2\mu_1 * \Sigma^2\mu_2) \in \mathscr{D}(\tau).$$

LEMMA A.6. *Let τ be a DVF with $\tau(x) = O(x^{-5})$. Then $\mu_1, \mu_2 \in \mathscr{D}(\tau) \cap D(\Sigma^3)$ implies*

$$\Delta^3(\Sigma^3\mu_1 * \Sigma^3\mu_2) \in \mathscr{D}(\tau).$$

PROOF. We first give the proof under the additional assumptions $\mu_1(\mathbb{R}) = \Sigma\mu_1(\mathbb{R}) = \Sigma^2\mu_1(\mathbb{R}) = 0$. Then we obtain as in the proof of Lemma A.4, integrating by parts four times now,

$$|\Delta^4(\Sigma^3\mu_1 * \Sigma^3\mu_2)|((x, x+1]) \le \sum_{i=1}^{6} I_i,$$

with

$$I_1 = \int_{(-\infty, x/2]} |\Delta\mu_1|((x-y, x+1-y])|\Sigma^3\mu_2|(dy),$$

$$I_2 = \int_{(x/2, \infty)} |\Sigma^3\mu_1|((x-y, x+1-y])|\Delta\mu_2|(dy),$$

$$I_3 = |\mu_1|\left(\left(\frac{x}{2}, \frac{x}{2}+1\right]\right)|\Sigma^2\mu_2|\left(\left(\frac{x}{2}, \infty\right)\right),$$

$$I_4 = |\Sigma\mu_1|\left(\left(\frac{x}{2}, \frac{x}{2}+1\right]\right)|\Sigma\mu_2|\left(\left(\frac{x}{2}, \infty\right)\right),$$

$$I_5 = |\Sigma^2\mu_1|\left(\left(\frac{x}{2}, \frac{x}{2}+1\right]\right)|\mu_2|\left(\left(\frac{x}{2}, \infty\right)\right),$$

$$I_6 = |\Sigma^3\mu_1|\left(\left(\frac{x}{2}, \frac{x}{2}+1\right]\right)|\Delta\mu_2|\left(\left(\frac{x}{2}, \infty\right)\right).$$

$I_1, I_2 = o(\tau(x))$ follows with the now familiar arguments. Further we have $I_{2+i} = o(\tau_i(x)\tau_{5-i}(x))$, $i = 1, \ldots, 4$, so $o(\tau(x))$-behaviour of the remaining terms follows from Lemma A.2(ii).

In the second step we remove the additional assumptions. This is easy once we have shown that there exists a measure μ_3 with compact support such that

$$\mu_3(\mathbb{R}) = \mu_1(\mathbb{R}), \qquad \Sigma\mu_3(\mathbb{R}) = \Sigma\mu_1(\mathbb{R}), \qquad \Sigma^2\mu_3(\mathbb{R}) = \Sigma^2\mu_1(\mathbb{R}).$$

Let ρ denote the uniform distribution on $(0,1)$. Then for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$,

$\mu_3 = \alpha_1\rho + \alpha_2\Sigma\rho + \alpha_3\Sigma^2\rho$ will do since the above conditions lead to a system of linear equations with regular matrix of coefficients. □

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