# ON SUBSTITUTION INVARIANT STURMIAN WORDS: AN APPLICATION OF RAUZY FRACTALS 

V. BERTHÉ, H. EI, S. ITO AND H. RAO


#### Abstract

Sturmian words are infinite words that have exactly $n+1$ factors of length $n$ for every positive integer $n$. A Sturmian word $s_{\alpha, \rho}$ is also defined as a coding over a two-letter alphabet of the orbit of point $\rho$ under the action of the irrational rotation $R_{\alpha}: x \mapsto x+\alpha(\bmod 1)$. A substitution fixes a Sturmian word if and only if it is invertible. The main object of the present paper is to investigate Rauzy fractals associated with two-letter invertible substitutions. As an application, we give an alternative geometric proof of Yasutomi's characterization of all pairs $(\alpha, \rho)$ such that $s_{\alpha, \rho}$ is a fixed point of some non-trivial substitution.


## 1. Introduction

1.1. Sturmian words and substitution invariance. Sturmian words are infinite words over a binary alphabet, say, $\{1,2\}$, that have exactly $n+1$ factors of length $n$ for every positive integer $n$. Sturmian words can also be defined in a constructive way as follows. Let $0<\alpha<1$. Let $\mathbb{T}^{1}=\mathbb{R} / \mathbb{Z}$ denote the one-dimensional torus. The rotation of angle $\alpha$ of $\mathbb{T}^{1}$ is defined by $R_{\alpha}: \mathbb{T}^{1} \rightarrow \mathbb{T}^{1}, x \mapsto x+\alpha$. For a given real number $\alpha$, we introduce the following two partitions of $\mathbb{T}^{1}$ :

$$
\underline{I}_{1}=[0,1-\alpha), \quad \underline{I}_{2}=[1-\alpha, 1) ; \quad \bar{I}_{1}=(0,1-\alpha], \quad \bar{I}_{2}=(1-\alpha, 1] .
$$

Tracing the orbit of $R_{\alpha}^{n}(\rho)$, we define two infinite words for $\rho \in \mathbb{T}^{1}$ :

$$
\begin{aligned}
& \underline{s}_{\alpha, \rho}(n)= \begin{cases}1 & \text { if } R_{\alpha}^{n}(\rho) \in \underline{I}_{1}, \\
2 & \text { if } R_{\alpha}^{n}(\rho) \in \underline{I}_{2},\end{cases} \\
& \bar{s}_{\alpha, \rho}(n)= \begin{cases}1 & \text { if } R_{\alpha}^{n}(\rho) \in \bar{I}_{1}, \\
2 & \text { if } R_{\alpha}^{n}(\rho) \in \bar{I}_{2} .\end{cases}
\end{aligned}
$$

It is well known $([13,25])$ that an infinite word is a Sturmian word if and only if it is equal either to $\bar{s}_{\alpha, \rho}$ or to $\underline{s}_{\alpha, \rho}$ for some irrational number $\alpha$. The word $\underline{s}_{\alpha, \rho}$ is called lower Sturmian word whereas the

Date: July 29, 2006.
1991 Mathematics Subject Classification. Primary 37B10; Secondary 11J70, 68R15.
word $\bar{s}_{\alpha, \rho}$ is called upper Sturmian word. The notation $s_{\alpha, \rho}$ stands in all that follows indifferently for $\bar{s}_{\alpha, \rho}$ or for $\underline{s}_{\alpha, \rho}$ when there is no need to distinguish between the two. A detailed description of Sturmian words can be found in Chapter 2 of [23], see also [28].

Let $\{1,2\}^{*}$ be the free monoid over $\{1,2\}$ endowed with the concatenation operation. A non-erasing homomorphism $\sigma$ of the free monoid $\{1,2\}^{*}$ is called a substitution. An infinite word $s \in\{1,2\}^{\mathbb{N}}$ is a fixed point of the substitution $\sigma$ if $\sigma(s)=s$.

It is well known that the famous Fibonacci word, i.e., the fixed point of the Fibonacci substitution $1 \mapsto 12,2 \mapsto 1$, is a Sturmian word. It is thus natural to ask when a Sturmian word is a fixed point of some non-trivial substitution. More precisely, we want to know

Question 1. For which $\alpha$ and $\rho$ is the Sturmian word $\underline{s}_{\alpha, \rho}$ (resp. $\bar{s}_{\alpha, \rho}$ ) a fixed point of some non-trivial substitution?

By non-trivial substitution, we mean here a substitution that is distinct from the identity. In all that follows, we say that a Sturmian word is substitution invariant if it is a fixed point of a non-trivial substitution.

There is a substantial literature devoted to Question 1. The first step has been made in [14] (Theorem 1 below). When $\rho=\alpha$, we have $\underline{s}_{\alpha, \alpha}=\bar{s}_{\alpha, \alpha}$ since $\alpha$ is an irrational number. We thus denote this word by $s_{\alpha, \alpha}$. It is usually called the characteristic word of $\alpha$. For a number $x$ in a quadratic field, we denote by $x^{\prime}$ the conjugate of $x$ in this field.

Theorem 1 (Crisp et al. [14]). Let $0<\alpha<1$ be an irrational number. Then the following two conditions are equivalent:
(i) the characteristic word $s_{\alpha, \alpha}$ is substitution invariant;
(ii) $\alpha$ is a quadratic irrational with $\alpha^{\prime} \notin[0,1]$.

A quadratic number $\alpha$ with $0<\alpha<1$ and $\alpha^{\prime} \notin[0,1]$ is called a Sturm number according to [2]. Let us note that the simplification of Condition (ii) in Theorem 1 to its present form is due to [2]. Furthermore, the expression of substitutions which fix $s_{\alpha, \alpha}$ can be explicitly obtained from the continued fraction expansion of $\alpha$ (see [14]).

For more results on the homogeneous case (i.e., the case $\rho=\{n \alpha\}$ for $n \in \mathbb{Z}$, where $\{x\}$ stands for the fractional part of $x$ ), see for instance $[8,7,11,16,21,23]$; for results in the non-homogeneous case, see [22, 26, 6]. Some variants of Question 1 are also considered in [27, 10].

Yasutomi has given a complete answer to Question 1 in [35]. Its characterization involves the conjugate of the quadratic real number
$x$ and can be compared to Galois' theorem for simple continued fractions describing numbers having a purely periodic continued fraction expansion.

Theorem 2 (Yasutomi [35]). Let $0<\alpha<1$ and $0 \leq \rho \leq 1$. Then $s_{\alpha, \rho}$ is substitution invariant if and only if the following two conditions are satisfied:
(i) $\alpha$ is an irrational quadratic number and $\rho \in \mathbb{Q}(\alpha)$;
(ii) $\alpha^{\prime}>1,1-\alpha^{\prime} \leq \rho^{\prime} \leq \alpha^{\prime}$ or $\alpha^{\prime}<0, \alpha^{\prime} \leq \rho^{\prime} \leq 1-\alpha^{\prime}$.

Remark 1. Let us note the symmetry between both cases in Assertion (ii) of Theorem 2. Indeed, let $E: 1 \mapsto 2,2 \mapsto 1$ be the substitution exchanging letters; then $\underline{s}_{\alpha, \rho}$ (resp. $\bar{s}_{\alpha, \rho}$ ) is substitution invariant if and only if $\underline{s}_{1-\alpha, 1-\rho}$ (resp. $\bar{s}_{1-\alpha, 1-\rho}$ ) which is equal to $E\left(\underline{s}_{\alpha, \rho}\right)$ (resp. $E\left(\bar{s}_{1-\alpha, 1-\rho}\right)$ ); furthermore, $(\alpha, \rho)$ satisfies $\alpha^{\prime}>1,1-\alpha^{\prime} \leq \rho^{\prime} \leq \alpha^{\prime}$ if and only if $(1-\alpha, 1-\rho)$ satisfies $1-\alpha^{\prime}<0, \alpha^{\prime} \leq 1-\rho^{\prime} \leq 1-\alpha^{\prime}$.

As a corollary of Theorem 2, we easily obtain:
Corollary 1. Let $\alpha$ be a Sturm number. Then
(i) for any $\rho \in \mathbb{Q} \cap(0,1), \underline{s}_{\alpha, \rho}=\bar{s}_{\alpha, \rho}$ is substitution invariant.
(ii) Let $\rho \in[0,1)$. The Sturmian word $\underline{s}_{\alpha,\{n \alpha\}}\left(\right.$ resp. $\left.\bar{s}_{\alpha,\{n \alpha\}}\right)$ is substitution invariant if and only if $n=-1,0,1$. In total we obtain exactly five substitution invariant Sturmian words

$$
\left\{21 s_{\alpha, \alpha}, 12 s_{\alpha, \alpha}, 2 s_{\alpha, \alpha}, 1 s_{\alpha, \alpha}, s_{\alpha, \alpha}\right\}
$$

in the homogeneous case.
Note that (ii) is also proven in [35] and in [16].
Proof. (i) Since $\rho$ is a rational number, we have $\rho^{\prime}=\rho$. Hence condition (ii) of Theorem 2 is fulfilled if $\alpha^{\prime}>1$ or $\alpha^{\prime}<0$.
(ii) Let us first assume that $\alpha^{\prime}>1$. Let $n, p \in \mathbb{Z}$ such that $\rho=$ $\{n \alpha\}=n \alpha-p$. One has $p=[n \alpha]$.

For $n=-1,0,1$, we have $\rho=1-\alpha, 0, \alpha$, respectively, so that $\rho^{\prime}=1-$ $\alpha^{\prime}, 0, \alpha^{\prime}$. Hence $\rho^{\prime} \in\left[1-\alpha^{\prime}, \alpha^{\prime}\right]$. Therefore $\bar{s}_{\alpha, \rho}$ and $\underline{s}_{\alpha, \rho}$ are substitution invariant.

For $n \geq 2, \rho^{\prime}=n \alpha^{\prime}-p>\alpha^{\prime}$ since $p=[n \alpha] \leq n-1$; for $n \leq-2$, one has $p=[n \alpha]>n \alpha-1 \geq n-1$. Hence $p+1 \geq n+1$ and $\rho^{\prime}=n \alpha^{\prime}-p<1-\alpha^{\prime}$. Therefore, $\bar{s}_{\alpha, \rho}$ and $\underline{s}_{\alpha, \rho}$ are not substitution invariant.

We deduce the case $\alpha^{\prime}<0$ by applying Remark 1 .
1.2. Invertible substitutions. Let $\sigma$ be a substitution over $\{1,2\}$ and let $M_{\sigma}=\left(m_{i j}\right)$ be its incidence matrix, where $m_{i j}$ counts the number of occurrences of the letter $i$ in $\sigma(j)$. We assume that $\operatorname{det} M_{\sigma}=$
$\pm 1$ (the substitution is said to be unimodular) and $M_{\sigma}$ is primitive ( $M_{\sigma}^{n}$ has only positive entries for some non-negative integer $n$ ).

A substitution is said to be invertible if it is an automorphism of the free group generated by the alphabet $\{1,2\}$. Note that if $\sigma$ is an invertible substitution, then its incidence matrix is unimodular.

Theorem 3 (Wen-Wen [34]). Every invertible substitution over $\{1,2\}$ is a composition of the following three invertible substitutions:

$$
\begin{equation*}
1 \mapsto 2,2 \mapsto 1 ; 1 \mapsto 12,2 \mapsto 1 ; \quad 1 \mapsto 21,2 \mapsto 1 \tag{1}
\end{equation*}
$$

Question 1 is related to invertible substitutions according to the following well-known result (see for instance [23]).

Theorem 4. A word is a Sturmian substitution invariant word if and only if it is a fixed point of some primitive and invertible substitution.

Let us illustrate the main idea of the proof of Theorem 2 in [35]. According to the three substitutions in Theorem 3, S. Ito and S. Yasutomi [21] define three transformations from $[0,1]^{2}$ to $[0,1]^{2}$, namely:

$$
\begin{gathered}
T_{1}(\alpha, \rho)=\left(\frac{\alpha}{1+\rho}, \frac{\rho}{1+\alpha}\right), \quad T_{2}(\alpha, \rho)=\left(\frac{1}{2-\alpha}, \frac{\rho}{2-\alpha}\right), \\
T_{3}(\alpha, \rho)=(1-\alpha, 1-\rho) .
\end{gathered}
$$

Then it is proven that a Sturmian word $s_{\alpha, \rho}$ is substitution invariant if and only if there exists a sequence $S_{1}, \ldots, S_{n}$ with $S_{i} \in\left\{T_{1}, T_{2}, T_{3}\right\}$ such that $(\alpha, \rho)=S_{1} \circ \cdots \circ S_{n}(\alpha, \rho)$. Since there are three transformations, the task of determining such $(\alpha, \rho)$ is tedious. Yasutomi's original proof of Theorem 2 in [35] is somewhat technical and lengthy.

Since Theorem 2 is a key elementary result, it is worth giving a proof that is more transparent and accessible. Let us note that a geometric proof based on the use of cut-and-project schemes has also been given in [4]. The proof we present here is based on Rauzy fractals.
1.3. Rauzy fractals. Rauzy fractals (first introduced in [30] in the Tribonacci case) are compact attractors of a graph-directed iterated function system associated with primitive substitutions with some prescribed algebraic properties. For more details, see for instance Chap. 7 in [28]. Rauzy fractals have numerous applications in number theory, ergodic theory, dynamical systems, fractal geometry and tiling theory (see for instance $[3,18,19,20,30,32]$, and Chap. 7 in [28]). The main purpose of the present paper is to describe a new application of Rauzy fractals to Sturmian words and more precisely, to study Rauzy fractals associated with invertibe two-letter substitutions according to [15].

Let us first describe an intuitive approach to Rauzy fractals for twoletter substitutions. We give a more formal definition in Section 2. Let $\sigma$ be a primitive and unimodular substitution over $\{1,2\}$. If $\sigma$ does not admit a fixed point, that is, if the image of 1 (resp. 2) begins with 2 (resp. 1), then $\sigma^{2}$ admits a fixed point. Otherwise, a fixed point of $\sigma$ is still a fixed point of $\sigma^{2}$. Let $s=s_{0} s_{1} s_{2} \ldots$ be a fixed point of $\sigma^{2}$. Let $(1-\alpha, \alpha)$ be the eigenvector with positive entries of $M_{\sigma}$ corresponding to the Perron-Frobenius eigenvalue. We shall call $\alpha$ the characteristic length of the matrix $M_{\sigma}$ or of the substitution $\sigma$, according to the context.

We define an oriented walk on the real line as follows. We start from the origin; in the $n$-th step, if $s_{n-1}=1$, we move to the right side by $\alpha$; if $s_{n-1}=2$, we move to the left side by $1-\alpha$. Taking the closure of the orbit of the origin under this transformation, we obtain

$$
X=\text { closure }\left\{\left|s_{0} s_{1} \ldots s_{k-1}\right|_{1} \cdot \alpha+\left|s_{0} s_{1} \ldots s_{k-1}\right|_{2} \cdot(\alpha-1) ; k \geq 0\right\}
$$

where $\left|s_{0} s_{1} \ldots s_{n-1}\right|_{j}$ stands for the number of occurrences of the letter $j$ in the word $s_{0} s_{1} \ldots s_{n-1}$. Furthermore, we define

$$
\begin{array}{cc}
X_{1}=\text { closure }\left\{\left|s_{0} s_{1} \ldots s_{k-1}\right|_{1} \cdot \alpha+\right. & \left|s_{0} s_{1} \ldots s_{k-1}\right|_{2} \cdot(\alpha-1) ; \\
& \left.k \geq 0, s_{k}=1\right\} \\
X_{2}=\text { closure }\left\{\left|s_{0} s_{1} \ldots s_{k-1}\right|_{1} \cdot \alpha+\right. & \left|s_{0} s_{1} \ldots s_{k-1}\right|_{2} \cdot(\alpha-1)  \tag{2}\\
& \left.k \geq 0, s_{k}=2\right\}
\end{array}
$$

The Rauzy fractals of $\sigma$ are defined as the set $X=X_{1} \cup X_{2}, X_{1}, X_{2}$ in (2). (To be more precise, we shall see in Section 2 that $X, X_{1}, X_{2}$ are an affine image of the Rauzy fractals.)

A central property for our study is that the fixed points of an invertible substitution are Sturmian (see Theorem 4), and hence the associated Rauzy fractals are intervals.

Theorem 5 ([12]). Let $\sigma$ be a primitive unimodular substitution over $\{1,2\}$. Then the Rauzy fractals $X_{1}, X_{2}$ and $X_{1} \cup X_{2}$ are intervals if and only if $\sigma$ is invertible.

A simple proof of this result is given in Section 2.4. Let us note that we only use here in the present paper the following easy implication: the Rauzy fractals of an invertible substitution are intervals.

Let us give a sketch of our proof of Theorem 2. By Theorem 4 and Theorem 5, if a Sturmian word is substitution invariant, then it is a fixed point of some primitive substitution with connected Rauzy fractals.

Let $\sigma$ be an invertible substitution with characteristic length $\alpha$. Then $\alpha$ is a Sturm number, and the Rauzy fractals $X_{1}, X_{2}$ are intervals with
length $1-\alpha$ and $\alpha$, respectively. Suppose $s=\bar{s}_{\alpha, \rho}$ or $s=\underline{s}_{\alpha, \rho}$ is a fixed point of $\sigma^{2}$. (According to Proposition 1 below, we can indifferently consider any of these two words.) One checks that $\rho=1-\alpha-h$, where $\{h\}=X_{1} \cap X_{2}$.

Let $V^{\prime}$ be the line $y=\frac{1-\alpha^{\prime}}{\alpha^{\prime}} x$, where $\alpha^{\prime}$ is the algebraic conjugate of $\alpha$. A broken line in $\mathbb{R}^{2}$, the so-called stepped surface, is associated with line $V^{\prime}$, defined as a discretization of $V^{\prime}$ (see Figure 3).

The sets $X_{1}, X_{2}$ have a self-similar structure: indeed they satisfy a set equation which is controlled by the stepped surface of $V^{\prime}$ (see Lemma 4 and Theorem 6). Hence, by connectedness and self-similarity of Rauzy fractals, we express the intersection $X_{1} \cap X_{2}$ in terms of the stepped surface (see Theorem 8).

Then we show that the stepped surface is associated with the rotation $R_{\gamma}$ with $\gamma=\frac{\alpha^{\prime}-1}{2 \alpha^{\prime}-1}$, which may be considered as the dual rotation of $R_{\alpha}$. An arithmetic characterization of the stepped surface is obtained (see Theorem 10). This allows us to get an algebraic description of the intersection set $X_{1} \cap X_{2}$ for an invertible substitution $\sigma$, which yields a proof of Theorem 2 .

This paper is organized as follows. We first review in Section 2 some basic facts on Rauzy fractals. We then discuss in Section 2.4 the connectedness of Rauzy fractals for a two-letter alphabet. Theorem 5 is proven in this section. In Section 3, we study set equations of Rauzy fractals, especially in the invertible case. The intersection set $X_{1} \cap X_{2}$ for invertible substitutions is determined in Section 4. In Section 5, an algebraic characterization of the stepped surface is given. A proof of Theorem 2 is given in Section 6.

## 2. RAUZY FRACTALS

In this section we review some basic facts on Rauzy fractals. We present here all definitions that apply to a two-letter alphabet, which is sufficient for our purpose. Note that the notation, which is adapted from [18], is slightly different from [3].
2.1. Sturm numbers. Let $\sigma$ be a primitive unimodular substitution over $\{1,2\}$. Let $\beta$ be the maximal eigenvalue of its incidence matrix $M_{\sigma}$. Its algebraic conjugate $\beta^{\prime}$ is also an eigenvalue of $M_{\sigma}$. By the Perron-Frobenius' theorem, we have $\beta>1$. Now $\beta \beta^{\prime}=\operatorname{det} M_{\sigma}= \pm 1$ implies $\left|\beta^{\prime}\right|<1$. Therefore $\beta$ is a Pisot number and the substitution $\sigma$ is said to be of Pisot type.

It is well-known that the densities of letters exist in fixed points of primitive substitutions (see [29]). Furthermore, the vector of densities
of the letters 1 and 2 denoted by $(1-\alpha, \alpha)$, with $0 \leq \alpha \leq 1$, is easily proven to be an expanding eigenvector, i.e., an eigenvector associated with the expanding eigenvalue $\beta$. Let us recall that $\alpha$ is called the characteristic length of $M_{\sigma}$. The characteristic length $\alpha$ is (irrational) quadratic; the vector $\left(1-\alpha^{\prime}, \alpha^{\prime}\right)$ is an eigenvector associated with the eigenvalue $\beta^{\prime}$. Still by Perron-Frobenius' theorem, coordinates $1-\alpha^{\prime}$, $\alpha^{\prime}$ cannot both be positive, hence $\alpha^{\prime}\left(1-\alpha^{\prime}\right) \leq 0$, which implies that $\left.\alpha^{\prime} \notin\right] 0,1[$. Hence $\alpha$ is a Sturm number.

Conversely, any Sturm number is the characteristic length of a primitive unimodular matrix $M$ of size $2 \times 2$. Indeed, if $\alpha$ is a Sturm number, then $s_{\alpha, \alpha}$ is a fixed point of an invertible primitive substitution $\sigma$ following Theorem 1, and hence $\alpha$ is the characteristic length of $M_{\sigma}$. We thus have proven the lemma below.

Lemma 1. A number $\alpha \in(0,1)$ is a Sturm number if and only if there exists a $2 \times 2$ primitive unimodular matrix $M$ with non-negative integral entries such that $(1-\alpha, \alpha)$ is an expanding eigenvector of $M$. Consequently, if the Sturmian word $s_{\alpha, \rho}$ is substitution invariant, then this implies that $\alpha$ is a Sturm number.

Example 1. Let $\sigma$ be the substitution $1 \mapsto 121,2 \mapsto 12$, i.e., the square of the Fibonacci substitution. This substitution admits as a unique fixed point the Fibonacci word $s_{\alpha, \alpha}$, with $\alpha=\frac{3-\sqrt{5}}{2}$, whose first terms are

$$
121121211211212112121
$$

One has $M_{\sigma}=\left[\begin{array}{ll}2 & 1 \\ 1 & 1\end{array}\right], \beta=\frac{3+\sqrt{5}}{2}$, and $\beta^{\prime}=\frac{3-\sqrt{5}}{2}=\alpha=\frac{1}{\beta}>0$.
We will also need the following lemma.
Lemma 2 ([24, 34, 8]). Let $\sigma$ be a non-trivial substitution over $\{1,2\}$. The following three conditions are equivalent:
(i) $\sigma$ is primitive invertible;
(ii) for any Sturmian word $s, \sigma(s)$ is still a Sturmian word;
(iii) there exists a Sturmian word s such that $\sigma(s)$ is a Sturmian word.

The equivalence between (i) and (ii) is due to [24] and [34], the equivalence with (iii) is proven in [8]. For more details, see [23].
2.2. Upper and lower Sturmian sequences. In this subsection, we show that $\underline{s}_{\alpha, \rho}$ is substitution invariant if and only if $\bar{s}_{\alpha, \rho}$ is also substitution invariant.

Proposition 1. Let $0<\alpha<1$ be an irrational number and $0 \leq \rho \leq 1$. Then $\underline{s}_{\alpha, \rho}$ is substitution invariant if and only $\bar{s}_{\alpha, \rho}$ is also substitution invariant.

Proof. Suppose $\underline{s}_{\alpha, \rho}=s_{0} s_{1} s_{2} \ldots$ is a fixed point of the non-trivial substitution $\sigma$. According to Lemma 2, $\sigma$ is primitive invertible. By primitivity, one has

$$
\begin{equation*}
\left|\sigma^{2}(1)\right| \geq 2 \text { and }\left|\sigma^{2}(2)\right| \geq 2 \tag{3}
\end{equation*}
$$

Note that $\underline{s}_{\alpha, \rho}=s_{0} s_{1} s_{2} \ldots$ is also a fixed point of the non-trivial substitution $\sigma^{2}$. Let us prove that $\bar{s}_{\alpha, \rho}$ is a fixed point of $\sigma^{2}$.

Let us assume that $\underline{s}_{\alpha, \rho} \neq \bar{s}_{\alpha, \rho}$ (otherwise, there is nothing to prove). One has either

$$
\begin{align*}
& \underline{s}_{\alpha, \rho}=s_{0} \ldots s_{n-1} 21 s_{n+2} \cdots=s_{0} \ldots s_{n-1} 21 s_{\alpha, \alpha}  \tag{4}\\
& \bar{s}_{\alpha, \rho}=s_{0} \ldots s_{n-1} 12 s_{n+2} \cdots=s_{0} \ldots s_{n-1} 12 s_{\alpha, \alpha}
\end{align*}
$$

or

$$
\begin{align*}
& \underline{s}_{\alpha, \rho}=1 s_{\alpha, \alpha},  \tag{5}\\
& \bar{s}_{\alpha, \rho}=2 s_{\alpha, \alpha} .
\end{align*}
$$

Let $s=\underline{s}_{\alpha, \rho}$ and $s^{\prime}=\bar{s}_{\alpha, \rho}$. We assume that we are in case (4); case (5) can be handled in the same way.

It is shown in [33] (as a consequence of Theorem 3) that if $\tau$ is an invertible substitution over a two-letter alphabet, then there exist two words $u$ and $v$ such that either $\tau(12)=u 12 v, \tau(21)=u 21 v$, or $\tau(12)=u 21 v, \tau(21)=u 12 v$. By applying twice this result, one deduces that there exist a finite word $w$ and an infinite word $t$, such that

$$
\begin{equation*}
\sigma^{2}(s)=w 21 t, \quad \sigma^{2}\left(s^{\prime}\right)=w 12 t \tag{6}
\end{equation*}
$$

One first deduces that $t=s_{\alpha, \alpha}$. Indeed, $12 t$ and $21 t$ are two Sturmian words with the same angle $\alpha$. Second, we deduce from $\sigma^{2}\left(s_{0} s_{1} \ldots s_{n-1}\right)=$ $s_{0} s_{1} \ldots s_{n-1} u=w$ and (3) that $w$ and $s_{0} s_{1} \ldots s_{n-1}$ are equal to the empty word. Again by (4) and (6), we have

$$
\sigma^{2}(s)=21 t, \quad \sigma^{2}\left(s^{\prime}\right)=12 t=s^{\prime}
$$

Hence $s^{\prime}$ is a fixed point of $\sigma^{2}$.
2.3. Definition of Rauzy fractals. Let $\vec{e}_{1}, \vec{e}_{2}$ be the canonical basis of $\mathbb{R}^{2}$. Let $f:\{1,2\}^{*} \rightarrow \mathbb{Z}^{2}$ be the Parikh map, also called abelianization homomorphism, defined by $f(w)=|w|_{1} \vec{e}_{1}+|w|_{2} \vec{e}_{2}$, where $|w|_{i}$ denotes the number of occurrences of the letter $i$ in $w$.

Let $V$ be the expanding eigenspace of the matrix $M_{\sigma}$ corresponding to the eigenvalue $\beta$, and $V^{\prime}$ the contracting eigenspace corresponding to $\beta^{\prime}$. The expanding subspace is generated by the vector $\vec{v}=(1-$ $\alpha, \alpha)$, therefore the contracting subspace is generated by the vector
$\vec{v}^{\prime}=\left(1-\alpha^{\prime}, \alpha^{\prime}\right)$. Then $V \oplus V^{\prime}=\mathbb{R}^{2}$ is a direct sum decomposition of $\mathbb{R}^{2}$. According to this direct sum, two natural projections are defined:

$$
\pi: \mathbb{R}^{2} \rightarrow V^{\prime} \quad \text { and } \quad \pi^{\prime}: \mathbb{R}^{2} \rightarrow V
$$

We define the Rauzy fractal associated with $\sigma$ as the closure of the projection according to $\pi$ of the vertices of the broken line (illustrated in Figure 1) obtained by applying map $f$ to the prefixes of a given fixed point of $\sigma^{2}$. (We recall that $\sigma^{2}$ always admits a fixed point since we work on a two-letter alphabet.)


Figure 1. The broken line.
More precisely, let $s=\left(s_{k}\right)_{k \geq 0}$ be a fixed point of $\sigma^{2}$. We first define

$$
Y=\left\{f\left(s_{0} \ldots s_{k-1}\right) ; k \geq 0\right\}
$$

where the notation $s_{0} \ldots s_{k-1}$ stands for the empty word when $k=0$. We then divide $Y$ into two parts:

$$
Y_{1}=\left\{f\left(s_{0} \ldots s_{k-1}\right) ; s_{k}=1\right\}, Y_{2}=\left\{f\left(s_{0} \ldots s_{k-1}\right) ; s_{k}=2\right\}
$$

Projecting $Y_{1}, Y_{2}$ onto the contracting eigenspace $V^{\prime}$ and taking the closures, we get

$$
\vec{X}_{1}=\overline{\pi\left(Y_{1}\right)}, \quad \vec{X}_{2}=\overline{\pi\left(Y_{2}\right)}
$$

We call $\vec{X}_{1}$ and $\vec{X}_{2}$ the Rauzy fractals of the substitution $\sigma$. It is shown in [18] that the Rauzy fractals are independent of the choice of the fixed point in the definition.

Clearly, Rauzy fractals $\vec{X}_{1}$ and $\vec{X}_{2}$ are one-dimensional objects. One has

$$
\begin{equation*}
\vec{e}_{1}=-\frac{\alpha^{\prime}}{\alpha-\alpha^{\prime}} \vec{v}+\frac{\alpha}{\alpha-\alpha^{\prime}} \vec{v}^{\prime} \text { and } \vec{e}_{2}=\frac{1-\alpha^{\prime}}{\alpha-\alpha^{\prime}} \vec{v}+\frac{\alpha-1}{\alpha-\alpha^{\prime}} \vec{v}^{\prime} . \tag{7}
\end{equation*}
$$

Hence an easy computation shows that

$$
X_{1}=\phi\left(\vec{X}_{1}\right), \quad X_{2}=\phi\left(\vec{X}_{2}\right)
$$

where $X_{1}, X_{2}$ are defined in (2) and $\phi$ is the linear map defined by

$$
\begin{equation*}
\phi: V^{\prime} \rightarrow \mathbb{R}, \phi\left(\frac{x \vec{v}^{\prime}}{\alpha-\alpha^{\prime}}\right)=x \tag{8}
\end{equation*}
$$

By abuse of language, we also call $X, X_{1}$ and $X_{2}$ the Rauzy fractals of the substitution $\sigma$.

Barge and Diamond showed in [5] that every Pisot substitution over a two-letter alphabet satisfies a certain combinatorial condition, called the strong coincidence condition. Thanks to this, one can show that

Lemma 3 ([18]). Let $\sigma$ be a primitive Pisot substitution over two letters. Then

$$
\mu\left(X_{1}\right)=1-\alpha, \quad \mu\left(X_{2}\right)=\alpha
$$

where $\mu$ is the Lebesgue measure and $\alpha$ is the characteristic length of $M_{\sigma}$.
2.4. Connectedness of Rauzy fractals. It is generally hard to decide whether Rauzy fractals are connected (see for instance [1, 12]). However, in the two-letter case we have a complete characterization given by Theorem 5. We provide an elementary proof of this folklore result.

## Proof of Theorem 5.

Let $\sigma$ be a primitive invertible substitution. Let $s$ be a fixed point of $\sigma^{2}$. By Theorem $4, s$ is a Sturmian word. Indeed, if $s^{\prime}$ is any Sturmian word with the same initial letter as $s$, then the sequence of Sturmian words (according to Lemma 2) $\left(\sigma^{2 n}\left(s^{\prime}\right)\right)_{n \geq 1}$ converges to $s$. Hence $s$ has at most $n+1$ factors of length $n$. Since $\sigma$ is both unimodular and primitive, we infer that the density of the letter 1 in $s$ is irrational, which implies that $s$ is aperiodic and thus, a Sturmian word.

Let $\alpha, \rho$ such that $s=s_{\alpha, \rho}$ (which means indifferently either $\underline{s}_{\alpha, \rho}$ or $\left.\bar{s}_{\alpha, \rho}\right)$. Let us first prove that the points $f\left(s_{0} \cdots s_{k-1}\right)$, for $k \in \mathbb{N}$, stay at a bounded distance of the line $V$; more precisely, they stay between the lines $y=\frac{\alpha}{1-\alpha} x+\frac{\rho-1}{1-\alpha}$ and $y=\frac{\alpha}{1-\alpha} x+\frac{\rho}{1-\alpha}$, which directly
implies that $\mu\left(X_{1} \cup X_{2}\right) \leq 1$. Indeed, the broken line defined by the vertices $f\left(s_{0} \cdots s_{k-1}\right)$, for $k \in \mathbb{N}$, is a cutting sequence (see for instance [23]), that is, it corresponds to the approximation of the line $y=$ $\frac{\alpha}{1-\alpha} x+\frac{\rho}{1-\alpha}-1$ by the broken line with integer vertices obtained by progressing by unit segments, either up or to the right, always going in the direction of the line, and starting from the origin point $(0,0)$ : one first notes that $s_{0}=1$ if and only if $\frac{\rho}{1-\alpha}-1<0$; furthermore, if $\alpha<1 / 2$ (resp. $\alpha>1 / 2$ ), the vertex $f\left(s_{0} \cdots s_{k-1}\right)$ is below (resp. above) the line $y=\frac{\alpha}{1-\alpha} x+\frac{\rho}{1-\alpha}-1$ if and only if $s_{k}=2$ (resp. $s_{k}=1$ ).

Moreover, by (7), $\phi \circ \pi \circ f(1)=\alpha$, and $\phi \circ \pi \circ f(2)=\alpha-1$. This implies that

$$
\begin{cases}\phi \circ \pi \circ f\left(s_{0} \cdots s_{k}\right)=\phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right)+\alpha & \text { when } s_{k}=1  \tag{9}\\ \phi \circ \pi \circ f\left(s_{0} \cdots s_{k}\right)=\phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right)+\alpha-1 & \text { when } s_{k}=2 .\end{cases}
$$

Hence

$$
\begin{equation*}
\forall k \in \mathbb{N}, \phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right) \equiv k \alpha \bmod 1 \tag{10}
\end{equation*}
$$

By irrationality of $\alpha$, we deduce that Rauzy fractals are intervals.
Conversely, let $\sigma$ be a primitive unimodular substitution over $\{1,2\}$. We first assume that the Rauzy fractals of $\sigma$, namely $X_{1}, X_{2}$, and $X=X_{1} \cup X_{2}$, are intervals. Let $s=\left(s_{k}\right)_{k \geq 0}$ be a fixed point of $\sigma^{2}$ which defines $X_{1}$ and $X_{2}$. Let $\alpha$ stand for the characteristic length of $M_{\sigma}$. Equations (9) and (10) still hold.

According to Lemma 3, $\mu\left(X_{1} \cap X_{2}\right)=0, \mu\left(X_{1}\right)=1-\alpha$ and $\mu\left(X_{2}\right)=$ $\alpha$. Furthermore, $X_{1}+\alpha \subset X=X_{1} \cup X_{2}$, by (9). Hence there exists $h \in \mathbb{R}$ such that $X_{1}=[-1+\alpha+h, h]$ and $X_{2}=[h, h+\alpha]$.

If the sequence $\left(\phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right)\right)_{k \geq 1}$ never takes as value one of the endpoints of $X_{1}$ and $X_{2}$, then one has according to (9)
$\left\{\begin{array}{l}\forall k \in \mathbb{N}, s_{k}=1 \text { if and only if } \phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right) \in(-1+\alpha+h, h) \\ \forall k \in \mathbb{N}, s_{k}=2 \text { if and only if } \phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right) \in(h, h+\alpha) .\end{array}\right.$
We deduce from (10) that $s=\bar{s}_{\alpha, 1-\alpha-h}=\underline{s}_{\alpha, 1-\alpha-h}$.
If there exists $k \geq 1$ such that $\phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right)=-1+\alpha+h$, then $s=\underline{s}_{\alpha, 1-\alpha-h}$. Similarly, if there exists $k \geq 1$ such that $\phi \circ \pi \circ$ $f\left(s_{0} \cdots s_{k-1}\right)=h+\alpha$, then $s=\bar{s}_{\alpha, 1-\alpha-h}$.

We assume now that there exists $k \geq 1$ such that $\phi \circ \pi \circ f\left(s_{0} \cdots s_{k-1}\right)=$ $h$. If $s_{k}=1$, then $\phi \circ \pi \circ f\left(s_{0} \cdots s_{k}\right)=h+\alpha$, and $s=\bar{s}_{\alpha, 1-\alpha-h}$. If $s_{k}=2$, then $\phi \circ \pi \circ f\left(s_{0} \cdots s_{k}\right)=h+\alpha-1$, and $s=\underline{s}_{\alpha, 1-\alpha-h}$.

We thus have proved that $s$ is a Sturmian word. According to Lemma 2 , this implies that $\sigma^{2}$, and thus $\sigma$, are invertible.

We deduce from the previous proof the following:
Corollary 2. Let $\sigma$ be a primitive invertible substitution. Then there exists $h \in \mathbb{Z}$ such that the Rauzy fractals satisfy

$$
X_{1}=[-1+\alpha+h, h], \quad X_{2}=[h, \alpha+h],
$$

where $\alpha$ is the characteristic length of $\sigma$. Furthermore, if $\bar{s}_{\alpha, \rho}$ or $\underline{s}_{\alpha, \rho}$ is a fixed point point of $\sigma^{2}$, then $\rho=1-\alpha-h$.
Example 2. Let us continue Example 1. One has $X_{1}=[-\alpha, 1-2 \alpha]$, $X_{2}=[1-2 \alpha, 1-\alpha], h=1-2 \alpha$.

## 3. Self-Similarity of Rauzy fractals

In this section, we discuss the self-similar structure of Rauzy fractals $X_{1}$ and $X_{2}$, while paying special attention to the case $\sigma$ invertible. The stepped surface is shown to play an important role.
3.1. Set equations of Rauzy fractals. Let $\sigma$ be a primitive substitution over $\{1,2\}$ and let $\beta$ be the Perron-Frobenius eigenvalue of $M_{\sigma}$.

It is well-known $([3],[32],[18])$ that $\vec{X}_{1}$ and $\vec{X}_{2}$, and thus $X_{1}$ and $X_{2}$, have a self-similar structure, i.e., both $\frac{1}{\beta^{\prime}} X_{1}$ and $\frac{1}{\beta^{\prime}} X_{2}$ are unions of translated copies of $X_{1}$ and $X_{2}$. (We recall that $\left|\beta^{\prime}\right|<1$.) In order to describe the corresponding set equations, we introduce the following notation: let $D_{1}$ (resp. $D_{2}$ ) be the set of these $(a, i) \in \mathbb{R} \times\{1,2\}$ such that $X_{i}+a \subset \frac{1}{\beta^{\prime}} X_{1}$ (resp. $X_{i}+a \subset \frac{1}{\beta^{\prime}} X_{2}$ ), that is,

$$
\frac{1}{\beta^{\prime}} X_{1}=\bigcup_{(a, i) \in D_{1}} X_{i}+a, \quad \frac{1}{\beta^{\prime}} X_{2}=\bigcup_{(b, i) \in D_{2}} X_{i}+b
$$

For the explicit form of $D_{1}, D_{2}$, we refer to [3] for the general case, and to Section 3.4, in the present case. To give an intuitive flavour of the explicit form, let us just note that any vertex $f\left(s_{0} \cdots s_{k-1}\right)$ of the broken line has form $f\left(\sigma\left(s_{0} \cdots s_{q-1}\right)\right)+f(p)$, for a prefix $p$ of $\sigma\left(s_{q}\right)$. Its projection yields the multiplication by $1 / \beta^{\prime}$, and thus belongs to $X_{s_{k}} / \beta^{\prime}$. The first part $f\left(\sigma\left(s_{0} \cdots s_{q-1}\right)\right)$ contributes by projection to an interval $X_{s_{q}}$ and $f(p)$ induces a translation of this interval.
Example 3. We continue Example 2. One checks that

$$
\begin{gathered}
\frac{X_{1}}{\beta^{\prime}}=[-1,1 / \alpha-2]=[-1,1-\alpha]=\left(X_{1}+\alpha-1\right)+X_{1}+X_{2} \\
\frac{X_{2}}{\beta^{\prime}}=[1 / \alpha-2,1 / \alpha-1]=[1-\alpha, 2-\alpha]=\left(X_{1}+1\right)+\left(X_{2}+1\right) .
\end{gathered}
$$

One has $D_{1}=\{(\alpha-1,1),(0,1),(0,2)\}$ and $D_{2}=\{(1,1),(1,2)\}$.
3.2. The stepped surface. Recall that $V^{\prime}$ is the contracting eigenline of $M_{\sigma}$. We denote the upper closed half-plane delimited by $V^{\prime}$ as $\left(V^{\prime}\right)^{+}$, and the lower open half-plane delimited by $V^{\prime}$ as $\left(V^{\prime}\right)^{-}$. We define

$$
S=\left\{\left[z, i^{*}\right] ; \quad z \in \mathbb{Z}^{2}, z \in\left(V^{\prime}\right)^{+} \text {and } z-\vec{e}_{i} \in\left(V^{\prime}\right)^{-}\right\},
$$

where the notation $\left[z, i^{*}\right]$, for $z \in \mathbb{Z}^{2}$ and $i^{*} \in\left\{1^{*}, 2^{*}\right\}$, endows the point $z$ in $\mathbb{Z}^{2}$ with color $i^{*}=1^{*}, 2^{*}$. Intuitively $S$ consists of the collection of these colored points $\left[z, i^{*}\right]$ which are close to the contracting eigenline $V^{\prime}$.

We now define $\overline{\left[z, 1^{*}\right]}$ (resp. $\overline{\left[z, 2^{*}\right]}$ ) as the closed line segment from $z$ to $z+\vec{e}_{2}$ (resp. to $z+\vec{e}_{1}$ ) (see Figure 2). Then the stepped surface $\bar{S}$ of $V^{\prime}$ is defined as the broken line consisting of the following segments

$$
\bar{S}=\bigcup_{\left[z, i^{*}\right] \in S} \overline{\left[z, i^{*}\right]} .
$$

It is easily seen to be connected. A piece of a stepped surface is depicted in Figure 3 for the example of Example 1. By abuse of language, the formal set $S$ will also be called the stepped surface of $V^{\prime}$.


Figure 2. The segments $\overline{\left[0,1^{*}\right]}$ and $\overline{\left[0,2^{*}\right]}$.


Figure 3. A piece of the stepped surface for $1 \mapsto 121$, $2 \mapsto 12$.

It turns out that the set equations of the Rauzy fractals are controlled by the stepped surface. An explicit expression of sets $D_{1}$ and $D_{2}$ is given in [3], from which one immediately deduces the following facts:

Lemma 4 ( $[3,18])$. Using the notation above:
i) for any $(a, i) \in D_{1} \cup D_{2}$, there exists an element $\left[z, i^{*}\right] \in S$ such that $\phi \circ \pi(z)=a$;
ii) $(0,1),(0,2) \in D_{1} \cup D_{2}$;
iii) $\left(n_{i j}\right)_{1 \leq i, j \leq 2}={ }^{t} M_{\sigma}$, where $n_{i j}$ counts the number of elements $(a, i)$ in the set $D_{j}$.
3.3. Tiling associated with the stepped surface. Projecting the stepped surface $\bar{S}$ onto $V^{\prime}$, we first obtain a tiling $\mathcal{J}^{\prime}$ of $V^{\prime}$ :

$$
\mathcal{J}^{\prime}=\left\{\pi\left(\overline{\left[z, i^{*}\right]}\right) ;\left[z, i^{*}\right] \in S\right\}
$$

Applying the linear transformation $\phi$ (see (8)), we then get a tiling $\mathcal{J}$ of the real line:

$$
\mathcal{J}=\left\{\phi \circ \pi\left(\overline{\left[z, i^{*}\right]}\right) ;\left[z, i^{*}\right] \in S\right\} .
$$

Tiling $\mathcal{J}$ is a tiling with two prototiles. Indeed

$$
\mathcal{J}=\left\{\phi \circ \pi(z)+J_{i} ;\left[z, i^{*}\right] \in S\right\}
$$

where

$$
J_{1}=\phi \circ \pi \overline{\left[0,1^{*}\right]}=[-1+\alpha, 0], J_{2}=\phi \circ \pi \overline{\left[0,2^{*}\right]}=[0, \alpha] .
$$

We label the tiles of $\mathcal{J}$ on the right side of the origin by the sequence $T_{0}, T_{1}, T_{2}, \ldots$, where $T_{n+1}$ is the rightside neighbour of $T_{n}$. Likewise we label the tiles of $\mathcal{J}$ on the left side of the origin by $T_{-1}, T_{-2}, \ldots$ One has $\mathcal{J}=\left\{T_{k} ; k \in \mathbb{Z}\right\}$. We furthermore define the two-sided sequence $\left(g_{k}\right)_{k \in \mathbb{Z}}$ as the sequence of left endpoints of tiles $T_{k}$ (one has $g_{0}=0$ ). An arithmetic description of the sequence $\left(g_{k}\right)_{k \in \mathbb{Z}}$ is given in Section 5.

Example 4. We continue Example 3. One has $g_{-2}=2(\alpha-1), g_{-1}=$ $\alpha-1, g_{0}=0, g_{1}=\alpha, g_{2}=1$.
3.4. Set equations of connected Rauzy fractals. According to Corollary 2 , if $\sigma$ is a primitive invertible substitution, then there exists a real number $h$ such that $X_{1}=[-1+\alpha+h, h], X_{2}=[h, h+\alpha]$, that is,

$$
X_{1}=J_{1}+h, \quad X_{2}=J_{2}+h,
$$

where $J_{1}=[-1+\alpha, 0]$ and $J_{2}=[0, \alpha]$ are the two prototiles of tiling $\mathcal{J}$.

Let $(a, i) \in D_{1}$. There exists an element $\left[z, i^{*}\right] \in S$ such that $\phi \circ$ $\pi(z)=a$ by Lemma 4 . Let $k \in \mathbb{Z}$ such that $\phi \circ \pi \overline{\left[z, i^{*}\right]}=T_{k}$; then

$$
X_{i}+a=J_{i}+h+a=T_{k}+h .
$$

We thus can introduce two subsets $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ of $\mathcal{J}$ such that

$$
\frac{X_{1}}{\beta^{\prime}}=\left(\bigcup_{T \in \mathcal{D}_{1}} T\right)+h, \quad \frac{X_{2}}{\beta^{\prime}}=\left(\bigcup_{T \in \mathcal{D}_{2}} T\right)+h
$$

On the one hand, the tiles in $\mathcal{D}_{1} \cup \mathcal{D}_{2}$ do not overlap according to [5] and [3]. On the other hand, these tiles must form a connected patch of $\mathcal{J}$ since $X_{1}, X_{2}, X_{1} \cup X_{2}$ are intervals according to Theorem 5. Hence we have proven that

Theorem 6. Let $X_{1}=[-1+\alpha+h, h], X_{2}=[h, h+\alpha]$ be the Rauzy fractals of the primitive invertible substitution $\sigma$. Then

$$
\frac{X_{1}}{\beta^{\prime}}=\left(\bigcup_{T \in \mathcal{D}_{1}} T\right)+h, \quad \frac{X_{2}}{\beta^{\prime}}=\left(\bigcup_{T \in \mathcal{D}_{2}} T\right)+h
$$

where $\mathcal{D}_{1}, \mathcal{D}_{2}$ and $\mathcal{D}_{1} \cup \mathcal{D}_{2}$ are connected patches of the tiling $\mathcal{J}$.
Example 5. We continue Example 4. One has $\mathcal{D}_{1}=\left\{T_{-2}, T_{-1}, T_{0}\right\}$, $\mathcal{D}_{2}=\left\{T_{1}, T_{2}\right\}, \frac{X_{1}}{\beta^{\prime}}=h+T_{-2}+T_{-1}+T_{0}, \frac{X_{2}}{\beta^{\prime}}=h+T_{1}+T_{2}$.

## 4. Invertible substitutions with a given incidence matrix

In this section, we give a more detailed description of the Rauzy fractals of invertible substitutions with a given incidence matrix.
4.1. A list of invertible substitutions with a given incidence matrix. Let $M=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ be a primitive unimodular matrix. A very interesting result on invertible substitutions is given in [31]:
Theorem 7 (Séébold [31]). Let $M=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ be a primitive unimodular matrix with non-negative entries. The number of invertible substitutions with incidence matrix $M$ is equal to $a+b+c+d-1$.

Let $\sigma$ be an invertible substitution with incidence matrix $M_{\sigma}=M$. According to Lemma $4,(0,1),(0,2) \in D_{1} \cup D_{2}$, hence we have

$$
\begin{equation*}
T_{-1}, T_{0} \in \mathcal{D}_{1} \cup \mathcal{D}_{2} \tag{11}
\end{equation*}
$$

By Lemma 4 iii), we have

$$
\begin{equation*}
\text { Card } \mathcal{D}_{1}=\operatorname{Card} D_{1}=a+b, \quad \text { Card } \mathcal{D}_{2}=\operatorname{card} D_{2}=c+d \tag{12}
\end{equation*}
$$

Let us assume that the determinant of $M$ is equal to 1 . (We will not need to subsequently consider the case $\operatorname{det}(M)=-1$, but a similar study can be conducted.) In this case, $1 / \beta^{\prime}=\beta>0$ so that $\frac{X_{1}}{\beta^{\prime}}$ is on the left side of $\frac{X_{2}}{\beta^{\prime}}$. Hence by Theorem 6 , the patch $\mathcal{D}_{1}$ is on the left
side of $\mathcal{D}_{2}$. By Theorem 6, (11), and (12), we infer that there exists $k$ with $1 \leq k \leq a+b+c+d-1$ such that

$$
\begin{align*}
& \mathcal{D}_{1}=\left\{T_{-k}, T_{-k+1}, \ldots, T_{-k+a+b-1}\right\} \\
& \mathcal{D}_{2}=\left\{T_{-k+a+b}, T_{-k+a+b+1}, \ldots, T_{-k+a+c+b+d-1}\right\} \tag{13}
\end{align*}
$$

Hence there are at most $a+b+c+d-1$ invertible substitutions with incidence matrix $M$, and their set equations are deduced from (13). On the other hand, Theorem 7 asserts that there are exactly $a+b+c+d-1$ such substitutions. Since the set equations for different substitutions are distinct, we conclude that there is a one-to-one correspondence between the invertible substitutions with incidence matrix $M$ and the set equations determined by (13). We denote these substitutions by $\sigma_{k}, 1 \leq k \leq a+b+c+d-1$.
4.2. Intersection point of Rauzy fractals. For each of the substitutions $\sigma_{k}$ defined in the previous section, there exists $\rho_{k}$ such that $s_{\alpha, \rho_{k}}$ (which means indifferently either $\bar{s}_{\alpha, \rho_{k}}$ or $\underline{s}_{\alpha, \rho_{k}}$ ) is a fixed point of $\sigma_{k}^{2}$ according to the proof of Proposition 1 .

Let $1 \leq k \leq a+b+c+d-1$. Let $X_{1}=\left[-1+\alpha+h_{k}, h_{k}\right], \quad X_{2}=$ [ $h_{k}, \alpha+h_{k}$ ] be the Rauzy fractals of $\sigma_{k}$. One has $\rho_{k}=1-\alpha-h_{k}$ according to Corollary 2. Below we use the connectedness and the selfsimilarity of Rauzy fractals to determine $h_{k}$ and thus $\rho_{k}$. Let us recall that $\left(g_{k}\right)_{k \in \mathbb{Z}}$ stands for the sequence of left endpoints of tiles $T_{k}$ in $\mathcal{J}$.

Theorem 8. Let $M$ be a $2 \times 2$ primitive matrix with non-negative entries such that $\operatorname{det} M=1$. Let $\sigma_{k}, 1 \leq k \leq a+b+c+d-1$, be the invertible substitutions with incidence matrix $M$, and let $X_{1}=$ $\left[-1+\alpha+h_{k}, h_{k}\right], \quad X_{2}=\left[h_{k}, \alpha+h_{k}\right]$ be the Rauzy fractals of $\sigma_{k}^{2}$. Let $\beta$ be the maximal eigenvalue of $M$. Then

$$
h_{k}=\frac{g_{-k+a+b}}{\beta-1} .
$$

Proof. On the one hand, $\frac{X_{1}}{\beta^{\prime}} \cap \frac{X_{2}}{\beta^{\prime}}=\left\{\left(\beta^{\prime}\right)^{-1} h_{k}\right\}=\left\{\beta h_{k}\right\}$. On the other hand, this intersection point is the left endpoint of the interval $\cup\left\{T+h_{k} ; T \in \mathcal{D}_{2}\right\}$, i.e., the left endpoint of $T_{-k+a+b}+h_{k}$. So we get $g_{-k+a+b}+h_{k}=\beta h_{k}$, and $h_{k}=\frac{g_{-k+a+b}}{\beta-1}$.
Theorem 9. Let $M$ be a $2 \times 2$ primitive matrix with non-negative entries such that $\operatorname{det} M=1$. Let $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{a+b+c+d-1}$ be the invertible substitutions with incidence matrix $M$. Let $G:=\left\{g_{k} ; k \in \mathbb{Z}\right\}$. Then the Sturmian word $s_{\alpha, \rho}$ is a fixed point of the substitution $\sigma_{k}^{2}$ if and only if

$$
0 \leq \rho \leq 1 \text { and }(\rho+\alpha-1) \in \frac{G}{1-\beta}
$$

Proof. By Theorem 8, one has $h_{a+b+c+d-1}<\cdots<h_{2}<h_{1}$. Hence a real number $h$ belongs to the set $\left\{h_{1}, h_{2}, \ldots, h_{a+b+c+d-1}\right\}$ if and only if

$$
\begin{equation*}
h \in \frac{G}{\beta-1} \text { and } h_{a+b+c+d-1} \leq h \leq h_{1} . \tag{14}
\end{equation*}
$$

The values $h_{1}$ and $h_{a+b+c+d-1}$ remain to be determined. For the substitution $\sigma_{1}$, the set $\mathcal{D}_{1}$ is equal to $\left\{T_{-1}, T_{0}, \ldots, T_{a+b-2}\right\}$. By Lemma 4 iii), the numbers of tiles in $\mathcal{D}_{1}$ of length $1-\alpha$ and $\alpha$ are $a$ and $b$, respectively. Since $\left|T_{-1}\right|=1-\alpha$, we have

$$
g_{a+b-1}=(a-1)(1-\alpha)+b \alpha=(\beta-1)(1-\alpha) .
$$

Here we use the equality $a(1-\alpha)+b \alpha=\beta(1-\alpha)$, which follows from the fact that $(1-\alpha, \alpha)$ is an expanding eigenvector of $M$. Therefore $h_{1}=1-\alpha$. A similar argument shows that $h_{a+b+c+d-1}=-\alpha$.

Remember now that $\rho_{k}=1-\alpha-h_{k}$. The theorem follows from (14).

## 5. The stepped surface

In this section, we give an arithmetic description of the stepped surface $\bar{S}$. We first define the two-sided word $\left(t_{n}\right)_{n \in \mathbb{Z}}$ as:

$$
\forall n \in \mathbb{Z}, t_{n}= \begin{cases}1, & \text { if }\left|T_{n}\right|=1-\alpha \\ 2, & \text { if }\left|T_{n}\right|=\alpha .\end{cases}
$$

It is well known that Sturmian words can also be described as cutting sequences (see for instance [23]). One checks according to [3] that $\left(t_{n}\right)_{n \in \mathbb{Z}}$ is the upper two-sided cutting sequence of the line $V^{\prime}: y=$ $\frac{1-\alpha^{\prime}}{\alpha^{\prime}} x$. Hence

$$
\begin{equation*}
t_{-1} t_{-2} t_{-3} \cdots=1 s_{\gamma, \gamma}, \quad t_{0} t_{1} t_{2} \cdots=2 s_{\gamma, \gamma}, \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma=\frac{\alpha^{\prime}-1}{2 \alpha^{\prime}-1} . \tag{16}
\end{equation*}
$$

Let $R_{\gamma}: x \mapsto x+\gamma$ be the rotation of angle $\gamma$ of the torus $\mathbb{T}^{1}$. We deduce from (15) that for all positive $k$

$$
\begin{aligned}
& \left|t_{-1} t_{-2} \ldots t_{-k}\right|_{1} \cdot \gamma+\left|t_{-1} t_{-2} \ldots t_{-k}\right|_{2} \cdot(\gamma-1)=R_{\gamma}^{k}(0) \\
& \left|t_{0} t_{1} \ldots t_{k-1}\right|_{1} \cdot \gamma+\left|t_{0} t_{1} \ldots t_{k-1}\right|_{2} \cdot(\gamma-1)=-R_{\gamma}^{-k}(0) .
\end{aligned}
$$

By definition of $\left(g_{k}\right)_{k \in \mathbb{Z}}$, one has for every nonnegative $k$

$$
\begin{gathered}
g_{-k}=\left|t_{-1} t_{-2} \ldots t_{-k}\right|_{1} \cdot(\alpha-1)+\left|t_{-1} t_{-2} \ldots t_{-k}\right|_{2} \cdot(-\alpha), \\
g_{k}=\left|t_{0} t_{1} \ldots t_{k-1}\right|_{1} \cdot(1-\alpha)+\left|t_{0} t_{1} \ldots t_{k-1}\right|_{2} \cdot \alpha .
\end{gathered}
$$

Hence

$$
\begin{equation*}
\forall k \in \mathbb{Z}, \frac{g_{k}^{\prime}}{2 \alpha^{\prime}-1}=R_{\gamma}^{-k}(0) \tag{17}
\end{equation*}
$$

where $g_{k}^{\prime}$ denotes the conjugate of $g_{k}$. This thus provides an arithmetic description of the stepped surface.

Theorem 10. One has

$$
\begin{aligned}
& G=\left\{g \in \mathbb{Z}[\alpha] ; \quad 0 \leq g^{\prime}<2 \alpha^{\prime}-1\right\} \text { when } \alpha^{\prime}>1, \\
& G=\left\{g \in \mathbb{Z}[\alpha] ; \quad 2 \alpha^{\prime}-1<g^{\prime} \leq 0\right\} \text { when } \alpha^{\prime}<0 .
\end{aligned}
$$

Proof. We assume that $\alpha^{\prime}>1$. The case $\alpha^{\prime}<0$ can be handled similarly. Note that

$$
\left\{R_{\gamma}^{k}(0) ; k \in \mathbb{Z}\right\}=\{m \gamma+n ; 0 \leq m \gamma+n<1\} .
$$

This together with (17) imply that

$$
\begin{aligned}
G & =\left\{g ; g^{\prime}=m\left(\alpha^{\prime}-1\right)+n\left(2 \alpha^{\prime}-1\right) ; m, n \in \mathbb{Z}, 0 \leq g^{\prime}<2 \alpha^{\prime}-1\right\} \\
& =\left\{g ; g=m(\alpha-1)+n(2 \alpha-1) ; m, n \in \mathbb{Z}, 0 \leq g^{\prime}<2 \alpha^{\prime}-1\right\} \\
& =\left\{g \in \mathbb{Z}[\alpha] ; 0 \leq g^{\prime}<2 \alpha^{\prime}-1\right\} .
\end{aligned}
$$

Remark 2. For a Sturm number $\alpha$, it is easy to check that $\gamma=\frac{\alpha^{\prime}-1}{2 \alpha^{\prime}-1}$ is also a Sturm number. We say that $\gamma$ is the dual of $\alpha$. One checks that $\gamma$ and $\alpha$ are duals of each other. In some sense, rotation $R_{\gamma}$ is the dual rotation of $R_{\alpha}$.

## 6. Proof of Theorem 2

In this section, we prove Theorem 2.
Theorem 2. (Yasutomi [35].) Let $0<\alpha<1$ and $0 \leq \rho \leq 1$. Then $s_{\alpha, \rho}$ is substitution invariant if and only if the following two conditions are satisfied:
(i) $\alpha$ is an irrational quadratic number and $\rho \in \mathbb{Q}(\alpha)$;
(ii) $\alpha^{\prime}>1,1-\alpha^{\prime} \leq \rho^{\prime} \leq \alpha^{\prime}$ or $\alpha^{\prime}<0, \alpha^{\prime} \leq \rho^{\prime} \leq 1-\alpha^{\prime}$.
6.1. An algebraic lemma. We first need a preliminary lemma.

Lemma 5. Let $\beta$ be a quadratic algebraic unit, and $\alpha$ be an irrational number in $\mathbb{Q}(\beta)$. Then for any $\rho \in \mathbb{Q}(\beta)$, there exists an arbitrary large even number $n$ such that $\rho\left(\beta^{n}-1\right) \in \mathbb{Z}[\alpha]$.

Proof. Let $\mathcal{A}$ stand for the ring of algebraic integers in $\mathbb{Q}(\beta)$. First we claim that for any $\rho \in \mathbb{Q}(\beta)$, there exists an arbitrary large number $n$ such that $\rho\left(\beta^{n}-1\right) \in \mathcal{A}$. Indeed, let $\delta \in \mathcal{A}$ such that $\delta \rho \in \mathcal{A}$. Then at least two terms in the sequence $\left(\delta \rho \beta^{n}\right)_{n \geq 0}$ belong to the same
residue class modulo the principal ideal of $\mathcal{A}$ generated by $\delta$. Hence $\delta \rho\left(\beta^{n_{1}}-\beta^{n_{2}}\right)$ is divisible by $\delta$ in $\mathcal{A}$ for some $n_{1}>n_{2}$. Since $\beta$ is an algebraic unit, $\delta \rho\left(\beta^{n}-1\right)$ is also divisible by $\delta$ for $n=n_{1}-n_{2}$, and $\rho\left(\beta^{n}-1\right) \in \mathcal{A}$. This proves our claim. Note furthermore that obviously we can decide that $n$ is an even number. We thus have proven that for every $N>0$,

$$
\mathbb{Q}(\beta)=\bigcup_{n \geq N} \frac{\mathcal{A}}{\beta^{2 n}-1}
$$

We then prove that there is a rational number $K$ such that $\mathcal{A} \subset$ $K \mathbb{Z}[\alpha]$. Indeed, let $d$ be the square-free integer such that $\mathbb{Q}(\beta)=$ $\mathbb{Q}(\sqrt{d})$. Then there are integers $a, b$ and $c \neq 0$ such that $\alpha=\frac{a+b \sqrt{d}}{c}$. Note that $b \neq 0$ since $\alpha$ is irrational. It is well known that any element in $\mathcal{A}$ must have the form $(m \sqrt{d}+n) / 2$. Since

$$
(m \sqrt{d}+n) / 2=\frac{m c \alpha-m a+n b}{2 b}
$$

is an element of $\frac{\mathbb{Z}[\alpha]}{2 b}$, our assertion is true by taking $K=\frac{1}{2 b}$.
Therefore for any $N>0$, we have

$$
\mathbb{Q}(\beta)=\bigcup_{n \geq N} \frac{\mathcal{A}}{\beta^{2 n}-1} \subset K \bigcup_{n \geq N} \frac{\mathbb{Z}[\alpha]}{\beta^{2 n}-1} \subseteq \mathbb{Q}(\beta)
$$

Multiplying every term of the above formula by $K^{-1}$, we obtain

$$
\mathbb{Q}(\beta)=\bigcup_{n \geq N} \frac{\mathbb{Z}[\alpha]}{\beta^{2 n}-1}
$$

6.2. Proof of Theorem 2. Now we are in a position to prove Theorem 2.

Necessity. Let us suppose that $s_{\alpha, \rho}$ is a fixed point of the non-trivial primitive invertible substitution $\sigma$. Let $\beta$ be the maximal eigenvalue of $M_{\sigma}$. We may assume that $\operatorname{det} M=1$, for otherwise we consider $\sigma^{2}$ instead of $\sigma$.

By Lemma 1, $\alpha$ must be a Sturm number. From Theorem 9, we deduce

$$
1-\alpha-\rho=h \in \frac{G}{\beta-1} \subseteq \frac{\mathbb{Z}[\alpha]}{\beta-1} \subseteq \mathbb{Q}(\beta)
$$

Hence $\rho \in \mathbb{Q}(\beta)=\mathbb{Q}(\alpha)$, so condition (i) is necessary.
Concerning (ii), we need only to consider the case $\alpha^{\prime}>1$ according to Remark 1. Note that $s_{\alpha, \rho}$ is also a fixed point of $\sigma^{n}$, for any $n \geq 1$,
and in particular for any even number $n$; furthermore, substitutions $\sigma^{n}$ share the same stepped surface. Hence

$$
\begin{gathered}
\rho+\alpha-1 \in \frac{G}{1-\beta^{n}}, \\
\rho^{\prime}+\alpha^{\prime}-1 \in \frac{\left\{g^{\prime} ; g \in G\right\}}{1-\left(\beta^{\prime}\right)^{n}} .
\end{gathered}
$$

By Theorem 10, we have

$$
0 \leq \rho^{\prime}+\alpha^{\prime}-1<\frac{2 \alpha^{\prime}-1}{1-\left(\beta^{\prime}\right)^{n}}
$$

Note that the above formula holds for every even number $n$. By letting $n$ tend to infinity, $\left(\beta^{\prime}\right)^{n}$ vanishes, and we conclude that $1-\alpha^{\prime} \leq \rho^{\prime} \leq \alpha^{\prime}$.

Sufficiency. Suppose that $(\alpha, \rho)$ satisfies (i) and (ii). According to Remark 1, we may assume here again that $\alpha^{\prime}>1$, so $\rho^{\prime}+\alpha^{\prime}-$ $1 \in\left[0,2 \alpha^{\prime}-1\right]$. Since $\alpha$ is a Sturm number, there exists a primitive substitution $\sigma$ such that $s_{\alpha, \alpha}$ is a fixed point of $\sigma$ (Theorem 1). Let $\beta$ be the maximal eigenvalue of the incidence matrix $M_{\sigma}$. We may assume that $\operatorname{det} M_{\sigma}=1$, otherwise we consider $\sigma^{2}$ instead of $\sigma$.

Obviously $\alpha \in \mathbb{Q}(\beta)$. Condition (i) implies that $\rho \in \mathbb{Q}(\alpha)=\mathbb{Q}(\beta)$. Hence $\rho+\alpha-1 \in \mathbb{Q}(\beta)$, so by Lemma 5 , there exist an even number $n$ and $g \in \mathbb{Z}[\alpha]$ such that

$$
\rho+\alpha-1=\frac{g}{1-\beta^{n}}
$$

Let us prove that $g$ is actually an element of $G$. Assumptions $\alpha^{\prime}>1$ and $1-\alpha^{\prime} \leq \rho^{\prime} \leq \alpha^{\prime}$ imply that $0 \leq \rho^{\prime}+\alpha^{\prime}-1 \leq 2 \alpha^{\prime}-1$. Now $0<1-\left(\beta^{\prime}\right)^{n}<1$ since $n$ is even. Hence

$$
g^{\prime}=\left(\rho^{\prime}+\alpha^{\prime}-1\right)\left(1-\left(\beta^{\prime}\right)^{n}\right) \in\left[0,2 \alpha^{\prime}-1\right)
$$

so $g \in G$ by Theorem 10. We thus have proven that $\rho+\alpha-1 \in \frac{G}{1-\beta^{n}}$. This together with $0 \leq \rho \leq 1$ implies that $s_{\alpha, \rho}$ is substitution invariant (by Theorem 9).

Acknowledgement The authors thank the anonyous referees for their pertinent comments which have helped us to increase the quality of the exposition, as well as P. Arnoux, J. Berstel, A. Rémondière, J.-i. Tamura and Z.-Y. Wen for many valuable discussions.

## References

[1] Akiyama S. and Gjini N., Connectedness of number theoretic tilings, Archiv. der Math. 82 (2004), 153-163.
[2] Allauzen C., Une caractérisation simple des nombres de Sturm, J. Théor. Nombres Bordeaux 10 (1998), 237-241.
[3] Arnoux P. and Ito S., Pisot substitutions and Rauzy fractals, Bull. Belg. Math. Soc. 8 (2001), 181-207.
[4] Baláži P., Masáková S. and Pelantová E., Complete characterization of substitution invariant Sturmian sequences, Integers: electronic journal of combinatorial number theory 5 (2005), A14.
[5] Barge M. and Diamond B., Coincidence for substitutions of Pisot type, Bull. Soc. Math. France 130 (2002), no. 4, 619-626.
[6] Bernardi D., Guerziz A. and Koskas M., Sturmian Words: description and orbits, preprint.
[7] Berstel J. and Séébold P., A remark on morphic Sturmian words, Theoretical Informatics and Applications 28 (1994), 255-263.
[8] Berstel J. and Séébold P., Morphismes de Sturm, Bull. Belg. Math. Soc. Simon Stevin 1 (1994), 175-189.
[9] Berthé V. and Vuillon L. Tilings and rotations on the torus: a twodimensional generalization of Sturmian sequences, Discrete Math. 223 (2000), 27-53.
[10] Berthé V., Holton C. and Zamboni L.Q., Initial powers of Sturmian words, Acta Arith. 122 (2006), 315-347.
[11] Brown T. C., Descriptions of the characteristic sequence of an irrational, Canad. Math. Bull. 36 (1993), 15-21.
[12] Canterini V., Connectedness of geometric representation of substitutions of Pisot type, Bull. Belg. Math. Soc. Simon Stevin 10 (2003), 77-89.
[13] Coven E. M. and Hedlund G. A., Sequences with minimal block growth, Math. Systems Theory 7 (1973), 138-153.
[14] Crisp D., Moran W., Pollington A., and Shiue P., Substitution invariant cutting sequence, J. Théor. Nombres Bordeaux 5 (1993), 123-137.
[15] Ei H. and Ito S., Decomposition theorem on invertible substitutions, Osaka J. Math. 35 (1998), 821-834.
[16] Fagnot I., A little more about morphic Sturmian words, Theoret. Informatics Appl. 3 (2006), to appear.
[17] Falconer K., Techniques in Fractal Geometry, Oxford University Press, 5th edition, 1979.
[18] Ito S. and Rao H., Atomic surfaces, tilings and coincidence I. Irreducible case, Israel J. Math. 153 (2006), 129-156.
[19] Ito S. and Rao H., Purely periodic $\beta$-expansions with Pisot unit base, Proc. Amer. Math. Soc. 133 (2005), 953-964.
[20] Ito S. and Sano Y., On periodic $\beta$-expansions of Pisot numbers and Rauzy fractals, Osaka J. Math. 38 (2001), 349-368.
[21] Ito S. and Yasutomi S., On continued fractions, substitutions and characteristic sequences $[n x+y]-[(n-1) x+y]$, Japan J. Math. 16 (1990), 287-306.
[22] Komatsu T. and van der Poorten A. J., Substitution invariant Beatty sequences, Japan J. Math., New ser. 22 (1996), 349-354.
[23] Lothaire M., Algebraic combinatorics on words, Cambridge University Press 2002.
[24] Mignosi F. and Séébold P., Morphismes sturmiens et règles de Rauzy, J. Théor. Nombres Bordeaux 5 (1993), 221-233.
[25] Morse M. and Hedlund G. A., Symbolic dynamics II. Sturmian trajectories, Amer. J. Math. 62 (1940), 1-42.
[26] Parvaix B., Propriétés d'invariance des mots sturmiens, J. Théor. Nombres Bordeaux 9 (1997), 351-369.
[27] Parvaix B., Substitution invariant Sturmian bisequences, J. Théor. Nombres Bordeaux 11 (1999), 201-210.
[28] Pytheas Fogg N., Substitutions in Arithmetics, Dynamics and Combinatorics, V. Berthé, S. Ferenczi, C.Mauduit, A. Siegel eds., Lecture Notes in Mathematics 1794, Springer Verlag, 2002.
[29] M. Queffélec, Substitution Dynamical Systems. Spectral Analysis, Lecture Notes in Math. 1294, Springer-Verlag (1987).
[30] Rauzy G., Nombres algebriques et substitutions, Bull. Soc. Math. France 110 (1982), 147-178.
[31] Séébold P., On the conjugation of standard morphisms, Theoret. Comput. Sci. 195 (1998), 91-109.
[32] Sirvent V. and Wang Y., Geometry of Rauzy fractals, Pacific J. Math. 206 (2002), 465-485.
[33] Tan B. and Wen Z.-Y., Invertible substitutions and Sturmian sequences, European J. of Combinatorics 24 (2003), 983-1002.
[34] Wen Z.-X. and Wen Z.-Y., Local isomorphisms of invertible substitutions, C. R. Acad. Sci. Paris 318 Série I (1994), 299-304.
[35] Yasutomi S.-I., On Sturmian sequences which are invariant under some substitutions, in Number theory and its applications (Kyoto, 1997), pp. 347-373, Kluwer Acad. Publ., Dordrecht, 1999.
V. Berthé, LIRMM 161 rue Ada F-34392 Montpellier cedex 5, France
H. Ei, Department of Information and System Engineering, Faculty of Science Engineering, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8851, JAPAN
S. Ito, Department of Information and System Engineering, Kanazawa University, Kanazawa, Japan
H. Rao, Department of Mathematics, Tsinghua University, Beijing, China

