ON SUPPORTS OF SEMIGROUPS OF MEASURES

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Let S denote a compact semitopological semigroup (i.e. the multiplication is separately continuous) and $P(S)$ the set of probability measures on S. Then $P(S)$ is a compact semitopological semigroup under convolution and the weak * topology (4). Let Γ be a subsemigroup of $P(S)$ and $S(\Gamma) = \bigcup_{\mu \in \Gamma} \text{supp}\, \mu$ where supp μ is the support of $\mu \in P(S)$. In the case in which S is *commutative* it was shown by Glicksberg in (4) that $S(\Gamma)$ is an algebraic group in S if Γ is an algebraic group. For a general semigroup S, Pym (7) considered $\Gamma = \{\eta\}$, η being an idempotent, and established that $S(\Gamma)$ is a topologically simple subsemigroup of S, i.e. every ideal of $S(\Gamma)$ is dense in $S(\Gamma)$. In this note we prove that if Γ is a simple subsemigroup of $P(S)$ (a semigroup is simple if it contains no proper ideal) which contains an idempotent then $S(\Gamma)$ is a topologically simple subsemigroup of S. We also give an example to show that our conclusion (hence also Pym's) is best possible in the sense that $S(\Gamma)$ is not simple in general.

Next, if S is *jointly* continuous and supp $\Gamma = \bar{S}(\Gamma)$ (the bar denotes closure), we can then obtain that supp Γ must be simple if Γ is assumed simple but not necessarily containing an idempotent. In other words, (Theorem 4) for a jointly continuous semigroup S, the support of a simple subsemigroup of $P(S)$ is a compact simple subsemigroup of S. This fills a lacuna in the literature. In fact it was first stated by Lin as a corollary of the following (5, Theorem 3): " If Σ is a closed subsemigroup of $P(S)$, then the minimal ideal $K(\text{supp}\, \Sigma)$ of supp Σ is the support of the minimal ideal $K(\Sigma)$ of Σ." Unfortunately the proof of this statement given in (5) contains a gap since the set

$$\{\mu \in \Sigma: \ \text{supp}\, \mu \cap K(\text{supp}\, \Sigma) \neq \varnothing\}$$

is not shown to be non-empty although essential use is made of the fact that it is an ideal of Σ. Here we give a direct proof of the " corollary " and deduce the " theorem ". (We note that Theorem 3 of (5) was used to derive several more results in (5), quoted in (8), and employed in the proof of Theorem 2 of (2).)

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Proposition 1. *Suppose S is a compact semitopological semigroup. If Γ is a simple subsemigroup of $P(S)$ and Γ contains an idempotent η, then $S(\Gamma)$ is a topologically simple subsemigroup of S.*

Proof. Let I be an ideal of $S(\Gamma)$ and let $\Omega = \{\mu \in \Gamma\colon \operatorname{supp} \mu \cap I \neq \emptyset\}$. It is easily seen that Ω is non-empty and an ideal of Γ by virtue of (**4**, Lemma 4.1). This gives $\Omega = \Gamma$, whence $\operatorname{supp} \eta \cap I \neq \emptyset$. Since $\operatorname{supp} \eta$ is topologically simple, it follows that $\operatorname{supp} \eta \subset \bar{I}$. Hence for any measure $\mu \in \Gamma = \Gamma\eta\Gamma$, we have $\operatorname{supp} \mu = \overline{\operatorname{supp} \alpha \operatorname{supp} \eta \operatorname{supp} \beta} \subset \bar{I}$ for measures $\alpha, \beta \in \Gamma$. Consequently $S(\Gamma) \subset \bar{I}$, i.e. $S(\Gamma)$ is topologically simple.

Example 2. In the preceding proposition, $S(\Gamma)$ need not be simple. For instance, take the semigroup S given in (**1**, IV.7.1), i.e. $S = I \times I \times I$ with the usual topology, where $I = [0, 1]$. Consider the separately continuous function $f\colon I \times I \to I$ defined by $f(x, z) = 2xz/(x^2+z^2)$ for x, z not both zero, and $f(0, 0) = 0$. Then S endowed with the multiplication

$$(x, y, z)(x', y', z') = (x, f(x, z'), z')$$

is a compact semitopological semigroup. Its minimal ideal

$$K(S) = \{(x, f(x, z), z)\colon x, z \in I\}$$

is not closed since $(0, 1, 0) \in \bar{K}(S) \backslash K(S)$. Now for any $a \in K(S)$ and $\mu \in P(S)$ we obtain the relation $\delta(a)\mu\delta(a) = \delta(a)$ ($\delta(a)$ denotes the unit point mass at a), since

$$\delta(a)\mu\delta(a)(f) = \int f(ata)d\mu(t) = \int f(a)d\mu(t) = f(a) = \delta(a)(f)$$

for all $f \in C(S)$. It follows that the measure $\eta = \mu\delta(a)\mu$ is idempotent. If $\Gamma = \{\eta\}$ and μ is a measure with S as its support (e.g. the restriction to S of the Lebesgue measure on $\mathbf{R}^3$), then $S(\Gamma) = \operatorname{supp} \eta = \overline{SaS} = \bar{K}(S)$ which is not simple.

Example 3. With the hypotheses of Proposition 1, $\operatorname{supp} \Gamma$ may not be topologically simple. For instance, take the one-point compactification $S = \mathbf{R} \cup \{\infty\}$ of $\mathbf{R}$ which is the additive group of real numbers with the operation extended by $x + \infty = \infty + x = \infty$ for $x \in S$. It follows that S is a compact semitopological semigroup (**1**, IV.1.1.1.1(a)). Let $\Gamma = \{\delta(x)\colon x \in \mathbf{R}\}$. Then $S(\Gamma) = \mathbf{R}$ and so $\operatorname{supp} \Gamma = S$ which is clearly not topologically simple.

In the remainder of this paper, let S be a compact jointly continuous semigroup. We have the next simple lemma which extends Lemma 1.3.11 of (**6**).

Lemma 4. *Suppose A is a subsemigroup of S. Then A is topologically simple if and only if $\bar{A}$ is simple.*

Proof. Let $\bar{A}$ be simple and take an ideal I of A. It is obvious that $\bar{I}$ is an ideal of $\bar{A}$ and so $\bar{I} = \bar{A} \supset A$, i.e. A is topologically simple. To show the converse, we model on the proof of (**1**, II.3.6). In fact for any $a \in A$ we see that $\bar{A}a\bar{A} \cap A \neq \emptyset$ is an ideal of A and so $\bar{A}a\bar{A} \supset A$. Whence $\bar{A}a\bar{A} = \bar{A}$. Now we check that $\bar{A}x\bar{A} = \bar{A}$ for $x \in \bar{A}$. Clearly there exists a net (x_α) in A such that $x_\alpha \to x$ and $\bar{A}x_\alpha\bar{A} = \bar{A}$. Let $b \in \bar{A}$ and we obtain two nets (c_α) and (d_α) in $\bar{A}$

such that $b = c_\alpha x_\alpha d_\alpha$. By passing to subnets if necessary, we have $c_\alpha \to c$ and $d_\alpha \to d$, giving $b = cxd \in \bar{A}x\bar{A}$. It follows that $\bar{A} \subset \bar{A}x\bar{A}$. Thus $\bar{A}$ is simple.

Theorem 5. (1) *If* Γ *is a simple subsemigroup of* $P(S)$, *then* supp Γ *is a compact simple subsemigroup of* S.

(2) *If* Σ *is a closed subsemigroup of* $P(S)$, *then* $\operatorname{supp} K(\Sigma) = K(\operatorname{supp} \Sigma)$.

Proof. (1) Because $\operatorname{supp} \bar{\Gamma} = \operatorname{supp} \Gamma$ (cf. (3), p. 55), we consider the compact semigroup $\bar{\Gamma}$ which contains an idempotent η (6, Theorem 1.1.10). Moreover, $\bar{\Gamma}$ is simple by Lemma 4. Then it follows from Proposition 1 that $S(\bar{\Gamma})$ is topologically simple and so $\operatorname{supp} \bar{\Gamma}$ is simple by Lemma 4 again. That supp Γ is simple follows.

(2) Clearly $\operatorname{supp} K(\Sigma)$ is an ideal of the semigroup supp Σ and hence contains $K(\operatorname{supp} \Sigma)$. The result is now immediate.

Remark. In the theorem above, it is clear that $S(\Gamma)$ is a topologically simple semigroup, but we have been unable to determine whether this semigroup is always simple.

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