Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XI (2012), 999-1007

On surfaces of general type with $q = 5$

MARGARIDA MENDES LOPES, RITA PARDINI AND GIAN PIETRO PIROLA

Abstract. We prove that a complex surface S with irregularity $q(S) = 5$ that has no irrational pencil of genus > 1 has geometric genus $p_g(S) \geq 8$. As a consequence, we are able to classify minimal surfaces S of general type with $q(S) = 5$ and $p_g(S) < 8$. This result is a negative answer, for $q = 5$, to the question asked in [13] of the existence of surfaces of general type with irregularity q that have no irrational pencil of genus > 1 and with the lowest possible geometric genus $p_g = 2q - 3$ (examples are known to exist only for $q = 3, 4$).

Mathematics Subject Classification (2010): 14J29.

1. Introduction

Let S be a smooth complex projective surface with irregularity $q(S) := h^0(\Omega^1_S) \geq 3$. The existence of a fibration $f : S \to B$ with B a smooth curve of genus $b > 1$ ("an irrational pencil of genus $b > 1$") gives much geometrical information on S (*cf.* the survey [14]). However, surfaces with an irrational pencil of genus $b > 1$ can hardly be regarded as "general" among the irregular surfaces of general type: for instance, for $b < q(S)$ the Albanese variety of such a surface S is not simple.

By the classical Castelnuovo-De Franchis theorem (*cf.* [6, Proposition X.9]), if S has no irrational pencil of genus > 1 then the inequality $p_g(S) \geq 2q(S) - 3$ holds, where $p_g(S) := h^0(K_S)$ is, as usual, the geometric genus. This fundamental inequality has been recently generalized in [17] to Kähler varieties of arbitrary dimension.

The surfaces of general type S for which the equality $p_g(S) = 2q(S) - 3$ holds are studied in [13]. There those with an irrational pencil of genus > 1 are classified and the inequality $K_S^2 \geq 7\chi(S) - 1$ is proven for S minimal. However, the question of the existence of surfaces with $p_g(S) = 2q(S) - 3$ having no irrational pencil of

The first author is a member of the Center for Mathematical Analysis, Geometry and Dynamical Systems (IST/UTL). The second and the third author are members of GNSAGA–INdAM. This research was partially supported by FCT (Portugal) through program POCTI/FEDER and Project PTDC/MAT/099275/2008 and by MIUR (Italy) through project PRIN 2007 *"Spazi di moduli e teorie di Lie"*.

Received February 8, 2011; accepted in revised form May 2, 2011.

genus $b > 1$ is wide open. At present, the state of the art is as follows:

- for $q = 3$, the only such surfaces are (the minimal desingularization of) a theta divisor in a principally polarized Abelian threefold ([11, 18]);
- for $q = 4$, a family of examples is constructed in [19];
- for $q \geq 5$, no example is known.

One is led to conjecture that for $q > 4$ there are no surfaces with $p_g = 2q - 3$ that have no irrational pencil. In this note we settle the case $q = 5$:

Theorem 1.1. *Let S be a smooth projective complex surface with $q(S) = 5$ that has no irrational pencils of genus > 1. Then*:

$$p_g(S) \geq 8.$$

As a consequence we obtain the following classification theorem:

Theorem 1.2. *Let S be a minimal complex surface of general type with $q(S) = 5$ and $p_g(S) \leq 7$. Then either*:

(i) $p_g(S) = 6$, $K_S^2 = 16$ *and S is the product of a curve of genus 2 and a curve of genus 3; or*
(ii) $p_g(S) = 7$, $K_S^2 = 24$ *and $S = (C \times F)/\mathbb{Z}_2$, where C is a curve of genus 7 with a free $\mathbb{Z}_2$-action, F is a curve of genus 2 with a $\mathbb{Z}_2$-action such that $F/\mathbb{Z}_2$ has genus 1 and $\mathbb{Z}_2$ acts diagonally on $C \times F$. The map $f\colon S \to C/\mathbb{Z}_2$ induced by the projection $C \times F \to C$ is an irrational pencil of genus 4 with general fibre F of genus 2.*

The idea of the proof of Theorem 1.1 is to obtain contradictory upper and lower bounds for K_S^2 under the assumption that $p_g(S) < 8$ and S is minimal.

For fixed q and p_g, by Noether's formula giving an upper bound for K^2 is the same as giving a lower bound for the topological Euler characteristic c_2. More precisely, it is the same as giving a lower bound for $h^{1,1}$, the only Hodge number which is not determined by p_g and q. In our situation, the upper bound follows directly from the result of [9] that if S is a surface of general type with $q = 5$, having no irrational pencils, then $h^{1,1} \geq 11 + t$, where t is bigger or equal to the number of curves contracted by the Albanese map.

If the canonical system $|K_S|$ has no fixed components, one can apply the results of [2] to get a lower bound for K_S^2 which is enough to rule out this possibility. Hence the bulk of the proof consists in obtaining a lower bound for K_S^2 under the assumption that $|K_S|$ has a fixed part $Z > 0$. This is done in Section 2, where we improve by 1 in the case $Z > 0$ a well known inequality for surfaces with birational bicanonical map due to Debarre (*cf.* Corollary 2.7). The proof is based on a subtle numerical analysis of the intersection properties of the fixed and moving part of $|K_S|$ that is, we believe, of independent interest.

It would be possible to generalize Theorem 1.1 for $q \geq 6$, if a good lower bound for $h^{1,1}(S)$ could be established. Unfortunately it is very difficult to extend the methods of [9] for $q \geq 6$. Recently, a lower bound on $h^{1,1}$ has been obtained in [12] by completely different methods, but it is not strong enough for our purposes.

Notation and conventions: a *surface* is a smooth complex projective surface. We use the standard notation for the invariants of a surface S: $p_g(S) := h^0(\omega_S) = h^2(\mathcal{O}_S)$ is the *geometric genus*, $q(S) := h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$ is the *irregularity* and $\chi(S) := p_g(S) - q(S) + 1$ is the *Euler–Poincaré* characteristic.

An *irrational pencil of genus* b of a surface S is a fibration $f : S \to B$, where B is a smooth curve of genus $b > 0$.

We use $\equiv$ to denote linear equivalence and $\sim$ to denote numerical equivalence of divisors.

An effective divisor D on a smooth surface is k*-connected* if for every decomposition $D = A + B$, with $A, B > 0$ one has $AB \geq k$. (Recall that on a minimal surface of general type every n-canonical divisor is 1-connected and, unless $n = 2$ and $K_S^2 = 1$, it is also 2-connected (*cf.* [3])).

ACKNOWLEDGEMENTS. We wish to thank Letterio Gatto and Stavros Papadakis for very helpful conversations.

2. Reider divisors

Let S be a surface and let M be a nef and big divisor on S such that $M^2 \geq 5$. By Reider's theorem, if a point P of S is a base point of $|K_S + M|$, then there is an effective divisor E passing through P such that either:

- $E^2 = -1$, $ME = 0$ or
- $E^2 = 0$, $ME = 1$.

This suggests the following definition:

Definition 2.1. Let M be a nef and big divisor on a surface S. An effective divisor E such that $E^2 = k$ and $EM = s$ is called a (k, s) divisor of M.

By [8, (0.13)], the $(-1, 0)$ divisors and the $(0, 1)$ divisors are 1-connected. In addition, if E is a $(-1, 0)$ divisor, using the index theorem one shows that the intersection form on the components of E is negative definite. In particular, there exist only finitely many $(-1, 0)$ divisors of M on S.

Lemma 2.2. *Let M be a nef divisor with $M^2 \geq 5$ on a surface S. Then*:

(i) *if E is a reducible $(0, 1)$ divisor E of M, and $0 < C < E$ then $C^2 < 0$;*
(ii) *if E_1, E_2 are two distinct $(0, 1)$ divisors of M, then $E_1E_2 = 0$ and E_1 and E_2 are disjoint.*

Proof. Let E, C be as in (i). The index theorem gives $C^2 < 0$ if $MC = 0$ and $C^2 \leq 0$ if $MC = 1$. Assume that $C^2 = 0$. Then $EC = (E - C)C > 0$, since E is 1-connected, and therefore $(E + C)^2 \geq 2$. Since $M^2 \geq 5$ and $M(C + E) = 2$ we have a contradiction to the index theorem. Hence $C^2 < 0$.

Next we prove (ii). We have:

$$M^2 \geq 5, \quad M(E_1 + E_2) = 2, \quad M(E_1 - E_2) = 0,$$

hence by the index theorem we obtain:

$$2E_1E_2 = (E_1 + E_2)^2 \leq 0, \quad -2E_1E_2 = (E_1 - E_2)^2 \leq 0.$$

So $E_1E_2 = 0$. By 1-connectedness of E_1, E_2 we conclude that neither divisor is contained in the other. Then we can write $E_1 = A + B$, $E_2 = A + C$ where $A \geq 0$, $B, C > 0$ and B and C have no common components.

Since M is nef and $ME_i = 1$, we have $1 \geq MB(= MC)$ and so $B^2 \leq 0$, $C^2 \leq 0$. Then, since $0 = (E_1 - E_2)^2 = (B - C)^2$, we conclude that $B^2 = C^2 = BC = 0$. Hence by (i) $B = E_1$ and $C = E_2$, namely $A = 0$ and E_1 and E_2 are disjoint. □

Lemma 2.3. *Let S be a surface and let M be a nef and big divisor such that the linear system $|M|$ has no fixed components. Let E be a* $(0, 1)$ *divisor of M and let C be the only irreducible component of E such that $MC = 1$. Then either $|M|$ has a base point on C or C is a smooth rational curve.*

Proof. Suppose $|M|$ has no base points on C. Then, since $MC = 1$ the restriction map $H^0(M) \to H^0(C, M|_C)$ has image of dimension at least 2. It follows that C is a smooth rational curve. □

Proposition 2.4. *Let X be a non ruled surface and let M be a divisor of X such that*:

- $M^2 \geq 5$,
- *the linear system $|M|$ has no fixed components and maps X onto a surface.*

Let C be an irreducible curve contained in the fixed locus of $|K_X + M|$. Then either:

(i) *C is contained in a* $(-1, 0)$ *divisor of M, $MC = 0$ and $C^2 < 0$;*
or
(ii) *C is contained in a* $(0, 1)$ *divisor of M, $MC \leq 1$ and $C^2 \leq 0$.*

Proof. Let $P \in C$ be a point. By Reider's theorem, there is a $(-1, 0)$ divisor or a $(0, 1)$ divisor of M passing through P.

Assume for contradiction that C is not a component of any $(-1, 0)$ or $(0, 1)$ divisor of M. Since there are only finitely many distinct $(-1, 0)$ divisors of M in S, we can assume that there is a $(0, 1)$ divisor passing through a general point P of C. It follows that there are infinitely many $(0, 1)$ divisors on S. Recall that two distinct

(0, 1) divisors are disjoint by Lemma 2.2. Thus, since $|M|$ has a finite number of base points, by Lemma 2.3 X is ruled, against the assumptions.

So C is contained in a $(-1, 0)$ divisor or a (0, 1) divisor E of M. In the first case, M being nef implies that $MC = 0$ and so $C^2 < 0$ by the index theorem. In the second case, again by nefness $MC \leq 1$ and again by the index theorem $C^2 \leq 0$. □

Lemma 2.5. *Let S be a surface and let M be a nef and big divisor of S and let E be a* (0, 1) *divisor of M. If L is a divisor such that* $(M - L)^2 > 0$ *and* $M(M - L) > 0$*, then* $EL \leq 0$.

Proof. Write $\gamma := M(M - L)$. Then $M(\gamma E - (M - L)) = 0$. Since $(M - L)^2 > 0$ and $E^2 = 0$, $\gamma(M - L) \not\sim E$. Thus, by the index theorem $0 > (\gamma E - (M - L))^2 = -2\gamma E(M - L) + (M - L)^2$.

So $E(M - L) > 0$, and therefore $EL \leq 0$. □

Proposition 2.6. *Let S be a smooth minimal surface of general type and let M be a divisor such that*

- $Z := K_S - M > 0$;
- *the linear system* $|M|$ *has no fixed components and maps S onto a surface.*

Then the following hold:

(i) *if* $M^2 \geq 5 + KZ$, *then* $h^0(2M) < h^0(K_S + M)$;
(ii) *if* $M^2 \geq 5$, $(M - Z)^2 > 0$ *and* $M(M - Z) > 0$, *then there are no* (0, 1) *divisors of M. Furthermore* $h^0(2M) < h^0(K_S + M)$ *and every irreducible fixed component C of* $|K_S + M|$ *satisfies* $MC = 0$.

Proof. We observe first of all that $h^0(2M) = h^0(K_S + M)$ if and only if Z is the fixed part of $|K_S + M|$.

(i) Assume for contradiction that $h^0(2M) = h^0(K_S + M)$. Let C be an irreducible component of Z. By Proposition 2.4, $C^2 \leq 0$ and $MC \leq 1$. Now

$$-2 \leq C^2 + KC \leq C^2 + KZ,$$

and hence $C^2 \geq -2 - KZ$. It follows

$$(M - C)^2 = M^2 - 2MC + C^2 \geq M^2 - 2 - 2 - KZ = M^2 - 4 - KZ > 0.$$

In addition, we have:

$$M(M - C) = (M - C)^2 + C(M - C) \geq (M - C)^2 - C^2 \geq (M - C)^2 > 0.$$

Since $MZ \geq 2$ by the 2-connectedness of canonical divisors, there is at least a component D of Z such that $MD > 0$. By Proposition 2.4, we have $MD = 1$ and D is contained in a (0, 1) divisor E of M. Then Lemma 2.5 gives $EC \leq 0$ for all the components of Z, and so $EZ \leq 0$.

But now since $ME = 1$ and $E^2 = 0$ we obtain that $KE = 1 + EZ \leq 1$. On the other hand, $K_S E$ is > 0 by the index theorem and it is even by the adjunction formula, hence we have a contradiction.

(ii) Let E be a (0, 1) divisor of M. Then we have $EZ \leq 0$ by Lemma 2.5 and we get a contradiction as above. So there are no (0, 1) divisors of M on S. Hence by Proposition 2.4 every irreducible fixed curve of $|K_S + M|$ satisfies $MC = 0$. Since $MZ \geq 2$ by the 2-connectedness of the canonical divisors, not every component of Z can be a fixed component of $|K_S + M|$ and therefore $h^0(K_S + M) > h^0(2M)$. □

As a consequence, we obtain the following refinement of [10, Theorem 3.2 and Remark 3.3]:

Corollary 2.7. *Let S be a minimal surface of general type whose canonical map is not composed with a pencil. Denote by M the moving part and by Z the fixed part of* $|K_S|$. *If* $Z > 0$ *and* $M^2 \geq 5 + K_S Z$, *then*

$$K_S^2 + \chi(S) = h^0(K_S + M) + K_S Z + MZ/2 \geq h^0(2M) + K_S Z + MZ/2 + 1.$$

Furthermore, if $h^0(K_S + M) = h^0(2M) + 1$ *then* $|K_S + M|$ *has base points and there is a* $(-1, 0)$ *divisor or a* $(0, 1)$ *divisor E of M such that* $EZ \geq 1$.

Proof. Since M is nef and big, by Kawamata-Viehweg vanishing $h^0(K_S + M) = \chi(K_S + M)$, hence the equality follows by the Riemann-Roch theorem whilst the inequality is Proposition 2.6, (i).

For the second assertion it suffices to notice that $h^0(K_S + M) = h^0(2M) + 1$ means that the image of the restriction map $H^0(K_S + M) \to H^0(Z, (K_S + M)|_Z)$ is 1-dimensional. Since $(K_S + M)Z \geq 2$, the system $|K_S + M|$ has necessarily base points. Thus there is a $(-1, 0)$ divisor or a $(0, 1)$ divisor E of M. By adjunction $K_S E - E^2$ is even and so necessarily $EZ \geq 1$. □

3. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Let $a \colon S \to A$ be the Albanese map of S. Notice that by the classification of surfaces the assumptions that $q(S) = 5$ and S has no irrational pencil of genus > 1 imply that S is of general type and a is generically finite onto its image. Without loss of generality we may assume that S is minimal. By [5], an irregular surface of general type having no irrational pencils of genus > 1 satisfies $p_g \geq 2q - 3$. We assume for contradiction that $p_g(S) = 7 = 2q(S) - 3$, so that $\chi(S) = 3$. We denote by $\varphi_K \colon S \to \mathbb{P}^6$ the canonical map and by Σ the canonical image. Since $q(S) > 2$, Σ is a surface by [20].

We denote by t the rank of the cokernel of the map $a^* \colon \mathrm{NS}(A) \to \mathrm{NS}(S)$. Note that t is bigger than or equal to the number of irreducible curves contracted by the Albanese map.

Denote as usual by $b_i(S)$ the i-th Betti number and by $c_2(S)$ the second Chern class of S. By [9, Theorem 1,(3)], we have $b_2(S) \geq 31 + t$, namely $c_2(S) \geq 13 + t$. By Noether's formula this is equivalent to:

$$K_S^2 \leq 23 - t. \tag{3.1}$$

Denote by $\mathbb{G}$ the Grassmannian of 2-planes of $H^0(\Omega_S^1)^\vee$ and by $\mathbb{G}^\vee$ the Grassmannian of 2-planes in $H^0(\Omega_S^1)$. By the Castelnuovo–De Franchis theorem, the kernel of the map $\rho\colon \bigwedge^2 H^0(\Omega_S^1) \to H^0(K_S)$ does not contain any nonzero simple tensor. Hence ρ induces a morphism $\mathbb{G}^\vee \to \mathbb{P}(H^0(K_S))$ which is finite onto its image. Since $\dim \mathbb{G}^\vee = 6$, it follows that $\ker \rho$ has dimension 3, ρ is surjective and it induces a finite map $\mathbb{G}^\vee \to \mathbb{P}(H^0(K_S))$. As a consequence, we have the following facts:

(a) the surface S is generalized Lagrangian, namely there exist independent 1-forms $\eta_1, \dots \eta_4 \in H^0(\Omega_S^1)$ such that $\eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4 = 0$. In addition, we may assume that $\eta_1 \wedge \eta_2$ is a general 2-form of S. In that case, the fixed part of the linear system $\mathbb{P}(\wedge^2 V)$, where $V =< \eta_1, \dots \eta_4 >$, coincides with the fixed part of the canonical divisor (*cf.* [15, Section 3]) .
(b) the canonical image Σ is contained in the intersection of $\mathbb{G}$ with the codimension 3 subspace $T = \mathbb{P}(\operatorname{Im} \rho^\vee) \subset \mathbb{P}^9 = \mathbb{P}(\bigwedge^2 H^0(\Omega_S^1))$ (where $\rho^\vee$ is the transpose of ρ),
(c) since $\mathbb{G}^\vee$ is the dual variety of $\mathbb{G}$, the space T is not contained in an hyperplane tangent to $\mathbb{G}$, hence $Y := \mathbb{G} \cap T$ is a smooth threefold.

Using the Lefschetz hyperplane section theorem we see that $\operatorname{Pic}(Y)$ is generated by the class of a hyperplane. Then Σ is the scheme theoretic intersection of Y with a hypersurface of degree $m \geq 2$ of $\mathbb{P}^6$. Thus, since $\mathbb{G}$ has degree 5 (*cf.* [16, Corollary 1.11]), it follows that $\deg \Sigma = 5m$ and by [16, Proposition 1.9] we have $\omega_\Sigma = \mathcal{O}_\Sigma(m-2)$. By [13, Theorem 1.2], the degree d of φ_K is different from 2. Since $K_S^2 \leq 23$ by (3.1), the inequality $K_S^2 \geq d \deg \Sigma = 5dm$ gives $d = 1$, namely φ_K is birational onto its image. So we have $m \geq 3$, since $\omega_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}}(-5)$ (*cf.* [16, Proposition 1.9]) and Σ is of general type.

Write $|K_S| = |M| + Z$, where Z is the fixed part and M is the moving part. If $Z = 0$, then in view of (a) we have $K_S^2 \geq 8\chi = 24$ by [2, Theorem 1.2]. This would contradict (3.1), hence $Z > 0$.

Since $m > 2$, every quadric that contains Σ must contain Y. Recall that Y is obtained from $\mathbb{G}$ by intersecting with 3 independent linear sections. Denote by R the homogeneous coordinate ring of $\mathbb{G}$. Since R is Cohen–Macaulay and Y has codimension 3 in $\mathbb{G}$, these 3 linear sections form an R-regular sequence. As a consequence (*cf.* [7, Proposition 1.1.5]) the (vector) dimension of the space of quadrics of $\mathbb{P}^6$ containing Y is the same as the (vector) dimension of the space of quadrics of $\mathbb{P}^9$ containing $\mathbb{G}$. Since the latter dimension is 5 (*cf.* [16, Proposition 1.2]), it follows that:

$$h^0(2M) \geq h^0(\mathcal{O}_{\mathbb{P}^6}(2)) - 5 = 23.$$

Then by (3.1) and Corollary 2.7 we have:

$$26-t \geq K_S^2+\chi(S) = h^0(K_S+M)+K_S Z+MZ/2 \geq 23+K_S Z+MZ/2+1. \quad (3.2)$$

So $K_S Z+MZ/2 \leq 2-t$. Recall that $MZ \geq 2$ by the 2-connectedness of canonical divisors.

Assume $K_S Z = 0$. Then every component of Z is an irreducible smooth rational curve with self-intersection -2 and as such it is contracted by the Albanese map. Since $K_S Z+MZ/2 \leq 2-t$, the only possibility is $t = 1$ and $MZ = 2$. Hence $Z = rA$, where A is a -2-curve. Since $MZ = 2$ and $K_S Z = 0$, we have $Z^2 = -2$ and so $r = 1$. Hence Z is a -2-cycle of type A_1; in particular it is reduced and, in the terminology of [2], it is contracted by any subspace $V \subseteq H^0(\Omega^1_S)$. Then, again by (a) and [2, Theorem 1.2], we get $K^2 \geq 8\chi = 24$, a contradiction.

So $K_S Z > 0$. Then by (3.2) necessarily $K_S Z = 1$, $MZ = 2$ (yielding $Z^2 = -1$) and $h^0(K_S + M) = 24 \leq h^0(2M) + 1$. As we have already remarked, the canonical image Σ has degree ≥ 15. Therefore $M^2 \geq 15 > 5 + K_S Z = 6$ and, by Corollary 2.7, there is a $(-1, 0)$ or a $(0, 1)$ divisor E of M. Since the hypotheses of Proposition 2.6, (ii) are satisfied, E must be a $(-1, 0)$ divisor of M.

Then $M(E+Z) = 2$ and so by the algebraic index theorem $M^2(E+Z)^2-4 \leq 0$, yielding $(E+Z)^2 \leq 0$. Since $(E+Z)^2 = -2+2EZ$ and, by Corollary 2.7, $EZ \geq 1$, the only possibility is $EZ = 1$ and $(E+Z)^2 = 0$. In this case $K_S(E+Z) = 2$ and this is impossible by the proof of [2, Proposition 8.2], which shows that a minimal irregular surface with $q \geq 4$, having no irrational pencils of genus > 1, cannot have effective divisors of arithmetic genus 2 and self-intersection 0. □

Proof of Theorem 1.2. By [5], a surface of general type S with $q(S) = 5$ has $p_g(S) \geq 6$ and, in addition, if $p_g(S) = 6$ then S is the product of a curve of genus C and a curve of genus 3. Now statement (ii) is a consequence of Theorem 1.1 and [13, Theorem 1.1]. □

References

[1] M. A. BARJA, *Numerical bounds of canonical varieties*, Osaka J. Math. **37** (2000), 701–718.

[2] M. A. BARJA, J. C. NARANJO and G. P. PIROLA, *On the topological index of irregular surfaces*, J. Algebraic Geom. **16** (2007), 435–458.

[3] W. BARTH, C. PETERS and A. VAN DE VEN, "Compact Complex Surfaces", Ergebnisse der Mathematik, 3. Folge, Band 4, Springer, Berlin, 1984.

[4] A. BEAUVILLE, *L'application canonique pour les surfaces de type général*, Invent. Math. **55** (1979), 121–140.

[5] A. BEAUVILLE, *L'inégalité $p_g \geq 2q - 4$ pour les surfaces de type général*, appendix to [10].

[6] A. BEAUVILLE, "Complex Algebraic Surfaces", second edition, L.M.S Student Texts 34, Cambridge University Press, Cambridge, 1996.

[7] W. BRUNS and J. HERZOG, "Cohen-Macaulay Rings", revised edition, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1998.

[8] F. CATANESE, C. CILIBERTO and M. MENDES LOPES, *On the classification of irregular surfaces of general type with non birational bicanonical map*, Trans. Amer. Math. Soc. **350** (1998), 275–308.
[9] A. CAUSIN and G. P. PIROLA, *Hermitian matrices and cohomology of Kaehler varieties*, Manuscripta Math. **121** (2006), 157–168.
[10] O. DEBARRE, *Inégalités numériques pour les surfaces de type général*, with an appendix by A. Beauville, Bull. Soc. Math. France **110** (1982), 319–346.
[11] C. D. HACON and R. PARDINI, *Surfaces with $p_g = q = 3$*, Trans. Amer. Math. Soc. **354** (2002), 2631–2638.
[12] R. LAZARSFELD and M. POPA, *Derivative complex BGG correspondence and numerical inequalities for compact Khaeler manifolds*, Invent. Math. **182** (2010), 605–633.
[13] M. MENDES LOPES and R. PARDINI, *On surfaces with $p_g = 2q - 3$*, Adv. Geom. **10** (2010), 549–555.
[14] M. MENDES LOPES and R. PARDINI, *The geography of irregular surfaces*, In: "Current Developments in Algebraic Geometry", Math. Sci. Res. Inst. Publ., Cambridge Univ. Press **59** (2012), 349–378.
[15] M. MENDES LOPES and R. PARDINI, *Severi type inequalities for surfaces with ample canonical class*, Comment. Math. Helv. **86** (2011), 401–414.
[16] S. MUKAI, "Curves and Grassmannians", Algebraic geometry and related topics (Inchon, 1992), 19–40, Conf. Proc. Lecture Notes Algebraic Geom., I, Int. Press, Cambridge, MA, 1993.
[17] G. PARESCHI and M. POPA, *Strong generic vanishing and a higher dimensional Castelnuovo-de Franchis inequality*, Duke Math. J. **150** (2009), 269–28.
[18] G. P. PIROLA, *Algebraic surfaces with $p_g = q = 3$ and no irrational pencils*, Manuscripta Math. **108** (2002), 163–170.
[19] C. SCHOEN, *A family of surfaces constructed from genus 2 curves*, Internat. J. Math. **18** (2007), 585–612.
[20] G. XIAO, *Irregularity of surfaces with a linear pencil*, Duke Math. J. **55** (1987), 596–602.

Departamento de Matemática
Instituto Superior Técnico
Universidade Técnica de Lisboa
Av. Rovisco Pais
1049-001 Lisboa, Portugal
mmlopes@math.ist.utl.pt

Dipartimento di Matematica
Università di Pisa
Largo B. Pontecorvo, 5
56127 Pisa, Italy
pardini@dm.unipi.it

Dipartimento di Matematica
Università di Pavia
Via Ferrata, 1
27100 Pavia, Italy
gianpietro.pirola@unipv.it