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On Symmetric and Skew-Symmetric Operators

Chafiq Benhida[a], Muneo Chō[b], Eungil Ko[c], Ji Eun Lee[d]

[a] *Université de Lille 1, Laboratoire Paul Painlevé*
[b] *Department of Mathematics, Kanagawa University*
[c] *Department of Mathematics, Ewha Womans University*
[d] *Department of Mathematics and Statistics, Sejong University*

Abstract. In this paper we show many spectral properties that are inherited by m-complex symmetric and m-skew complex symmetric operators and give new results or recapture some known ones for complex symmetric operators.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$ and *skew complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$. Many standard operators such as normal operators, algebraic operators of order 2, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, and Volterra integration operators are included in the class of complex symmetric operators. Several authors have studied the structure of complex symmetric operators (see [12]-[14], [18], and [19] for more details). On the other hand, less attention has been paid to skew complex symmetric operators. There are several motivations for such operators. In particular, skew symmetric matrices have many applications in pure mathematics, applied mathematics, and even in engineering disciplines. Real skew symmetric matrices are very important in applications, including function theory, the solution of linear quadratic optimal control problems, robust control problems, model reduction, crack following in anisotropic materials, and others. In view of these applications, it is natural to study skew symmetric operators on the Hilbert space $\mathcal{H}$ (see [22], [26], and [27] for more details). Here we consider larger classes including those operators; namely m-complex and m-skew complex symmetric operators and we show many spectral properties that follow from the so called Jacobson's lemma. This is in particular, applicable to the studied classes.

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Email addresses: benhida@math.univ-lille1.fr (Chafiq Benhida), chiyom01@kanagawa-u.ac.jp (Muneo Chō), eiko@ewha.ac.kr (Eungil Ko), jieunlee7@sejong.ac.kr (Ji Eun Lee)

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space and $D(\lambda, r)$ be the open disc centered at $\lambda \in \mathbb{C}$ and with radius $r > 0$. For an open set U in $\mathbb{C}$, we denote by $\mathcal{O}(U, \mathcal{H})$ and $\mathcal{E}(U, \mathcal{H})$ the Fréchet space of all $\mathcal{H}$-valued analytic functions on U and the Fréchet space of all $\mathcal{H}$-valued C^∞-functions on U, respectively.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property* (SVEP for short) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$, the only analytic solution of the equation $(T - z)f(z) = 0$ is the null function. We use $\sigma_{svep}(T)$ to denote the set of all points where T fails to have the SVEP, and we say that T has the SVEP if T has the SVEP at each $\lambda \in \mathbb{C}$ which means that $\sigma_{svep}(T)$ is the empty set (see [11]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to satisfy *Bishop's property* (β) at $\lambda \in \mathbb{C}$ (resp. $(\beta)_\epsilon$) if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$ and for any sequence (f_n) in $\mathcal{O}(U, \mathcal{H})$ (resp. in $\mathcal{E}(U, \mathcal{H})$) such that whenever, $(T - z)f_n(z) \longrightarrow 0$ in $\mathcal{O}(U, \mathcal{H})$ (resp. in $\mathcal{E}(U, \mathcal{H})$), then $f_n \longrightarrow 0$ in $\mathcal{O}(U, \mathcal{H})$ (resp. in $\mathcal{E}(U, \mathcal{H})$). Let $\sigma_\beta(T)$ (resp. $\sigma_{\beta_\epsilon}(T)$) be the set of all points where T does not have property (β) (resp. $(\beta)_\epsilon$). Then T is said to satisfy Bishop's property (β) (resp. $(\beta)_\epsilon$) precisely when $\sigma_\beta(T) = \emptyset$ (resp. $\sigma_{\beta_\epsilon}(T) = \emptyset$).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *spectral decomposition property* (δ) at λ if there exists an open neighborhood U of λ such that for every finite open cover $\{U_1, \dots, U_n\}$ of $\mathbb{C}$, with $\sigma(T) \setminus U \subseteq U_1$, we have

$$\mathcal{X}_T(\bar{U}_1) + \cdots + \mathcal{X}_T(\bar{U}_n) = \mathcal{H}, \tag{1}$$

where $\mathcal{X}_T(F)$ is the set of elements $x \in \mathcal{H}$ such that the equation $(T - \lambda)f(\lambda) = x$ has a global analytic solution on $\mathbb{C} \setminus F$. Following [25, page 32], $\mathcal{X}_T(F)$ is called the *glocal analytic spectral subspace* associated with F, since the analytic functions in their definition are globally defined on $\mathbb{C} \setminus F$, but will depend on x.

If moreover in equation (1) the glocal subspaces are closed, then T is said to be decomposable at λ.

The δ-spectrum $\sigma_\delta(T)$ and the decomposability spectrum $\sigma_{dec}(T)$ are defined in a similar way.

It is well-known that

Decomposable $\Rightarrow$ Bishop's property $(\beta) \Rightarrow$ SVEP.

In general, the converse implications do not hold (see [25] for more details).

The properties (β) and (δ) are known to be dual to each other in the sense that $\sigma_\delta(T) = \sigma_\beta(T^*)$. An operator is decomposable at λ if it has both (β) and (δ) at λ. Thus $\sigma_{dec}(T) = \sigma_\delta(T) \cup \sigma_\beta(T)$. We refer to [1, 2, 6, 7, 25] for further details on local spectral theory.

3. Some Useful Spectral Properties

In this section, we study an antilinear operator which is the only type of nonlinear operators that are important in quantum mechanics. We state the basic definition and properties of such operators. An operator T on $\mathcal{H}$ is *antilinear* if for all $x, y \in \mathcal{H}$

$$T(\alpha x + \beta y) = \overline{\alpha} Tx + \overline{\beta} Ty$$

holds for all $\alpha, \beta \in \mathbb{C}$.

Lemma 3.1. *Let B and C be antilinear operators on $\mathcal{H}$. Then the following properties hold;*

- *BC and CB are linear operators.*
- *$\gamma B + \delta C$ is an antilinear operator for any $\gamma, \delta \in \mathbb{C}$.*
- *If D is a linear operator, then BD, DB, CD, and DC are antilinear operators.*

- *If B^{-1} exists, then B^{-1} is an antilinear operator.*

The following definition comes from [3, Page 639], [31, Page 51], and [29, Page 259]; for an antilinear operator T, a *Hermitian adjoint* operator of T on $\mathcal{H}$ is an antilinear operator $T^\dagger : \mathcal{H} \to \mathcal{H}$ with the property;

$$\langle Tx, y\rangle = \overline{\langle x, T^\dagger y\rangle} \tag{2}$$

for all $x, y \in \mathcal{H}$. If an antilinear operator T is bounded, then, by the Riesz representation theorem, the Hermitian adjoint of T exists and is unique ([10, Page 90]). For antilinear operators T and R, we get immediately from (2) that $(T^\dagger)^\dagger = T$, $(T + R)^\dagger = T^\dagger + R^\dagger$ and $(TR)^\dagger = R^\dagger T^\dagger$.

Let's start by the following result which is a slight variation of Jacobson's lemma.

Proposition 3.2. *Let B and C be two antilinear bounded operators on a Hilbert space $\mathcal{H}$. Then BC and CB are in $\mathcal{L}(\mathcal{H})$ and*

$$I - CB \quad \text{is invertible} \iff I - BC \quad \text{is invertible.} \tag{3}$$

3.1. Global Spectral Properties

If T is a bounded linear operator on $\mathcal{H}$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_r(T)$, and $\sigma_c(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, the residual spectrum, and continuous spectrum of T, respectively. As more or less direct consequences of the last proposition we have the following ones;

Proposition 3.3. *Let B and C be two antilinear bounded operators on a Hilbert space $\mathcal{H}$. Then the following statements hold;*

- $\sigma(BC) \setminus \{0\} = \sigma(CB)^* \setminus \{0\}$
- $\sigma_p(BC) \setminus \{0\} = \sigma_p(CB)^* \setminus \{0\}$
- $\sigma_{ap}(BC) \setminus \{0\} = \sigma_{ap}(CB)^* \setminus \{0\}$
- $\sigma_r(BC) \setminus \{0\} = \sigma_r(CB)^* \setminus \{0\}$
- $\sigma_c(BC) \setminus \{0\} = \sigma_c(CB)^* \setminus \{0\}$

where $E^* := \{\bar{\lambda} : \lambda \in E\}$ for $E \subset \mathbb{C}$.

The result above is valid for various distinguished part of the spectrum as it has been illustrated for bounded linear operators in the previous works (see [4], [5] and the reference therein).

Notice also that if B and C are antilinear, then C is naturally a mapping of various objects related to BC into those related to CB. For example,

$$C : \ker(BC - \lambda)^p \longrightarrow \ker(CB - \bar{\lambda})^p.$$

3.2. Spectral Picture and Weyl theorem

Let T be an operator in $\mathcal{L}(\mathcal{H})$. Recall that T is said to be a *Fredholm* operator if $\mathrm{Im}(T)$ is closed and if $\dim(\ker(T))$ and $\mathrm{codim}(\mathrm{Im}(T))$ are finite. *Left Fredholm* operators (resp. *Right Fredholm* operators) are given by operators T with closed range and such that $\dim(\ker(T))$ is finite (resp. $\mathrm{codim}(\mathrm{Im}(T))$ is finite). An operator T is said to be *semi-Fredholm* if it is either left Fredholm or right Fredholm. Thus T is semi-Fredholm if $\mathrm{Im}(T)$ is closed and if ($\dim(\ker(T))$ or $\mathrm{codim}(\mathrm{Im}(T))$ is finite. The *index* of a semi Fredholm operator T is defined to be $\mathrm{ind}(T) = \dim(\ker(T)) - \mathrm{codim}(\mathrm{Im}(T))$.

Denote by $\mathcal{F}$ (resp. $\mathcal{SF}$, $\mathcal{LF}$, and $\mathcal{RF}$) the family all Fredholm (resp. semi-Fredholm, left Fredholm, and right Fredholm) operators. The essential spectrum of T is $\sigma_e(T) = \{\lambda \in \mathbb{C} \,:\, T - \lambda \notin \mathcal{F}\}$. The left essential spectrum $\sigma_{le}(T)$ and the right essential spectrum $\sigma_{re}(T)$ are defined similarly. For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$ is called the *ascent* of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $\mathrm{Im}(T^q) = \mathrm{Im}(T^{q+1})$ is called the *descent* of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. We define Weyl spectrum, $\sigma_w(T)$ and Browder spectrum, $\sigma_b(T)$, by

$$\begin{aligned}\sigma_w(T) &= \bigcap_{\{K \text{ is compact}\}} \sigma(T + K)\\ &= \{\lambda \in \mathbb{C} \,:\, T - \lambda \text{ is not Fredholm with index zero}\}\end{aligned}$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm of finite ascent and descent}\}.$$

It is well-known that the mapping $ind : \mathcal{S}F \longrightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ defined by

$$T \longmapsto \mathrm{ind}(T) = \dim(\ker(T)) - \mathrm{codim}(\mathrm{Im}(T))$$

is continuous. We usually call a *hole* in $\sigma_e(T)$ any bounded component of $\mathbb{C} \setminus \sigma_e(T)$ and a *pseudo hole* in $\sigma_e(T)$ a component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or $\sigma_e(T) \setminus \sigma_{re}(T)$. The *spectral picture* $SP(T)$ of an operator was introduced by C. Pearcy in [28] as the collection of holes and pseudo holes and the associated Fredholm indices.

From the discussion made above; we know that for every B and C antilinear bounded operators on $\mathcal{H}$ and every $\lambda \neq 0$, $\mathrm{Im}(BC - \lambda)$ is closed if and only if $\mathrm{Im}(CB - \bar{\lambda})$ is closed and $\dim(\ker(BC - \lambda)) = \dim(\ker(CB - \bar{\lambda}))$. Thus one easily concludes that the spectral picture of BC is linked (beside 0 eventually) to the spectral picture of CB in the following way;

$$(\mathfrak{h}, \mathrm{ind}(\mathfrak{h})) \in SP(BC) \Longleftrightarrow (\mathfrak{h}^*, \mathrm{ind}(\mathfrak{h})) \in SP(CB) \text{ where } \mathfrak{h}^* := \{\bar{\lambda} | \ \lambda \in \mathfrak{h}\}.$$

We notice that it has been shown in [5] that this holds in the linear case (B and C are in $\mathcal{L}(\mathcal{H})$). See also in [15] the discussion about 0 which could be in a pseudohole of BC and not in a pseudohole of CB.

An operator T in $\mathcal{L}(\mathcal{H})$ is said to satisfy *Weyl's theorem* if

$$\sigma_w(T) = \sigma(T) \setminus \pi_{00}(T)$$

where

$$\pi_{00}(T) = \{\lambda \in iso(\sigma(T)) : 0 < \dim(\ker(T - \lambda)) < \infty\}$$

and $iso(E)$ is the set of all isolated points of E. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.

Proposition 3.4. *Let B and C be two antilinear bounded operators on a Hilbert space $\mathcal{H}$. If $0 \in \pi_{00}(BC) \cap \pi_{00}(CB)$ or $0 \notin \pi_{00}(CB) \cup \pi_{00}(BC)$ then;*
(i) *BC satisfies Weyl's theorem if and only if CB satisfies Weyl's theorem.*
(ii) *BC satisfies Browder's theorem if and only if CB satisfies Browder's theorem.*

Proof. It's easy to see that we have $\pi_{00}(BC) \setminus \{0\} = \pi_{00}(CB) \setminus \{0\}$, $\sigma_b(BC) \setminus \{0\} = \sigma_b(CB) \setminus \{0\}$, and $\sigma_w(BC) \setminus \{0\} = \sigma_w(CB) \setminus \{0\}$. Then the result follows.

3.3. Local Spectral Properties

Notice that for the local part of the spectrum we have

Proposition 3.5. *Let B and C be antilinear bounded operators on a Hilbert space $\mathcal{H}$. Then BC and CB are in $\mathcal{L}(\mathcal{H})$ and*

- $\sigma_{svep}(BC) = \sigma_{svep}(CB)^*$
- $\sigma_{\beta}(BC) = \sigma_{\beta}(CB)^*$
- $\sigma_{\beta_\epsilon}(BC) = \sigma_{\beta_\epsilon}(CB)^*$
- $\sigma_{\delta}(BC) = \sigma_{\delta}(CB)^*$
- $\sigma_{dec}(BC) = \sigma_{dec}(CB)^*$.

Proof. We only show the second statement for (β). Assume that $\bar{\lambda}_0 \notin \sigma_\beta(CB)$ and so CB has (β) at $\bar{\lambda}_0$. Let U be an open subset of $D(\lambda_0, r)$, with $r > 0$ given by the definition of property (β) at $\bar{\lambda}_0$ and let $f_n : U \to \mathcal{H}$ be a sequence of analytic functions such that $(BC - \lambda)f_n(\lambda) \longrightarrow 0$ in the topology of $\mathcal{O}(U, \mathcal{H})$ which means

$$\lim_{n\to\infty} \|(BC - \lambda)f_n(\lambda)\|_K = 0$$

for every compact set K in U, where $\|f(\lambda)\|_K := \sup_{\lambda\in K} \|f(\lambda)\|$ for an $\mathcal{H}$-valued function f. Since B and C are antilinear, it follows that

$$\lim_{n\to\infty} \|(CB - \overline{\lambda})Cf_n(\lambda)\|_K = 0$$

for every compact subset K of U or equivalently, $\lim_{n\to\infty} \|(CB - \lambda)Cf_n(\overline{\lambda})\|_{K^*} = 0$ for every compact K^* of U^*. Moreover, since CB has (β) at $\bar{\lambda}_0$, it implies that $\lim_{n\to\infty} \|Cf_n(\overline{\lambda})\|_{K^*} = 0$ for every compact K^* of U^* and so $\lim_{n\to\infty} \|Cf_n(\lambda)\|_K = 0$ for every compact K of U. Therefore, $\lim_{n\to\infty} \|BCf_n(\lambda)\|_K = 0$ and consequently $\lim_{n\to\infty} \|\lambda f_n(\lambda)\|_K = 0$ for every compact K of U. Since 0 has (β), it follows that $\lim_{n\to\infty} \|f_n(\lambda)\|_K = 0$ on U. So BC has (β) at λ_0 meaning that $\lambda_0 \notin \sigma_\beta(BC)$. The other inclusion is obtained by symmetry.

We may observe that we "lost" 0 by passing from local to global spectra and a natural question arises about when we get the coincidence of the spectrum. In the following proposition, we give some conditions that ensure that. But of course, there are others that are not in our scope. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$ where T^* denotes the adjoint of T. Using the definition of the Hermitian adjoint of an antilinear operator as (2), we can define a normal antilinear operator as follows; an antilinear bounded operator A on $\mathcal{H}$ is called *normal* if A and $A^\dagger$ commute where $A^\dagger$ satisfies (2) (see [30, Section 4.1, Page 27]).

Proposition 3.6. *Let B and C be antilinear bounded operators on a Hilbert space $\mathcal{H}$. Then $\sigma(BC) = \sigma(CB)^*$ in the following cases;*

1. *C and B are injective.*
2. *C and $C^\dagger$ are injective.*
3. *C or B is injective with dense range.*
4. *C and B are not injective.*
5. *C and $C^\dagger$ are not injective.*
6. *C or B is normal.*

Proof. 1. Assume that C is injective. In this case, we prove that $\sigma(BC) \subset \sigma(CB)^*$. Indeed, what's left is to show that if $0 \in \sigma(BC)$, then $0 \in \sigma(CB)^*$ from Proposition 3.3. Suppose that $0 \in \sigma(BC)$. We have two possible cases:

(i) BC is not one to one or (ii) BC is not onto.

In the first case (i), there exists $x \in \mathcal{H} \setminus \{0\}$ such that $BCx = 0$. Applying C we obtain $CBCx = 0$. Then either $Cx = 0$ or $Cx \neq 0$. Since C is injective and $x \neq 0$, it follows that $Cx \neq 0$. Thus, CB is not injective.

For the second case (ii), suppose that CB is surjective which implies that C is surjective. Since C is already injective, it becomes invertible. Now, $BC = C^{-1}(CB)C$ and then is surjective which contradicts our assumption.

If B is injective, we obtain the other inclusion $\sigma(CB)^* \subset \sigma(BC)$.

2. Since B and C are antilinear, it follows that BC is linear. Then (2) gives that

$$\langle x, (BC)^* y\rangle = \langle (BC)x, y\rangle = \overline{\langle Cx, B^\dagger y\rangle} = \langle x, C^\dagger B^\dagger y\rangle$$

for all $x, y \in \mathcal{H}$. This means that $(BC)^* = C^\dagger B^\dagger$ and so $(CB)^* = B^\dagger C^\dagger$. Hence the result is obtained by passing to the adjoint and Hermitian adjoint by the proof of 1.

3. If T is an antilinear, then (2) implies that $\langle Tx, y\rangle = 0$ if and only if $\overline{\langle x, T^\dagger y\rangle} = 0$. Therefore $x \in \ker(T)$ if and only if $x \in \mathrm{Im}(T^\dagger)^\perp$. This means that $\ker(T) = \{0\}$ if and only if $\overline{\mathrm{Im}(T^\dagger)} = \mathcal{H}$. Hence T is injective if and only if $T^\dagger$ has dense range. So 3. is equivalent to 2.

For 4., we have $\sigma(CB)^* \setminus \{0\} = \sigma(BC) \setminus \{0\}$.

If C is not injective, then there exists $x \neq 0$ in $\mathcal{H}$ such that $Cx = 0$. This implies $BCx = 0$ and consequently $0 \in \sigma(BC)$. By using the adjoints and symmetry, we have $0 \in \sigma(CB)^*$.

5. By the same reason, 5. is equivalent to 4.

6. If C is normal, then $\|Cx\| = \|C^\dagger x\|$. Then, both C and $C^\dagger$ are injective or not injective and the result follows from the previous cases 2. and 5.

4. Applications

4.1. When C is a conjugation

Recall that C is a conjugation on $\mathcal{H}$ if $C : \mathcal{H} \longrightarrow \mathcal{H}$ is an antilinear operator that satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. In the following result, we provide the adjoint of conjugation operators.

Theorem 4.1. *Let C be a conjugation on $\mathcal{H}$. Then the Hermitian adjoint of C is the conjugation C, i.e., $C^\dagger = C$. Conversely, assume that C is antilinear with $C^2 = I$. If $C^\dagger = C$, then C is a conjugation on $\mathcal{H}$.*

Proof. Since C is antilinear, it follows from (2) that

$$\langle Cx, y\rangle = \overline{\langle x, C^\dagger y\rangle} \tag{4}$$

for all $x, y \in \mathcal{H}$. On the other hand, since C is conjugation, it follows that

$$\langle Cx, y\rangle = \langle Cy, x\rangle \tag{5}$$

for all $x, y \in \mathcal{H}$. Equations (4) and (5) imply

$$\overline{\langle x, Cy\rangle} = \langle Cy, x\rangle = \langle Cx, y\rangle = \overline{\langle x, C^\dagger y\rangle}$$

for all $x, y \in \mathcal{H}$. Hence we conclude that $C = C^\dagger$.

Conversely, suppose that C is antilinear with $C^2 = I$. If $C^\dagger = C$, then (2) implies that

$$\langle Cx, y\rangle = \overline{\langle x, C^\dagger y\rangle} = \overline{\langle x, Cy\rangle} = \langle Cy, x\rangle$$

for all $x, y \in \mathcal{H}$. Hence C is a conjugation on $\mathcal{H}$.

Corollary 4.2. ([13]) *Let B and C be conjugations on $\mathcal{H}$. Then BC and CB are unitary.*

Proof. If B and C are conjugations on $\mathcal{H}$, then (2) implies that $(BC)^* = C^\dagger B^\dagger$ and $(CB)^* = B^\dagger C^\dagger$ as in the proof of Proposition 3.6. By Theorem 4.1, we know that $(BC)^* = CB$ and $(CB)^* = BC$. Hence BC and CB are unitary (see also [13, Lemma 1]).

Using Proposition 3.5, we obtain the following theorem;

Theorem 4.3. *Let T be in $\mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then*

- $\sigma_{svep}(CTC) = \sigma_{svep}(T)^*$
- $\sigma_{\beta}(CTC) = \sigma_{\beta}(T)^*$
- $\sigma_{\beta_\epsilon}(CTC) = \sigma_{\beta_\epsilon}(T)^*$
- $\sigma_{\delta}(CTC) = \sigma_{\delta}(T)^*$
- $\sigma_{dec}(CTC) = \sigma_{dec}(T)^*$.

Proof. A straight forward application of Proposition 3.5, with $B = TC$, implies the same local spectral properties of $CB = CTC$ and $BC = TCC = T$. Observe that up to now, we are using the only fact that $C^2 = I$.

Now, applying Proposition 3.6, we have the following results (see also [20, Lemma 3.21]).

Theorem 4.4. *Let T be in $\mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then*

$$\sigma_{\bullet}(CTC) = \sigma_{\bullet}(T)^*$$

when $\sigma_{\bullet} \in \{\sigma, \sigma_p, \sigma_{ap}, \sigma_c, \sigma_r, \sigma_{su}, \sigma_e, \sigma_w, \dots\}$.

Proof. If T is a bounded linear operator and C is a conjugation on $\mathcal{H}$, then TC is antilinear and C is clearly normal from Theorem 4.1. Hence $\sigma(CTC) = \sigma(T)^*$ from 6. of Proposition 3.6. Using also that $C^2 = I$, we obtain similarly the remain cases.

Theorem 4.5. *Let T be in $\mathcal{L}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then T satisfies Weyl's (or Browder's) theorem if and only if CTC satisfies Weyl's (or Browder's) theorem.*

5. Helton Classes

Let A and B be two given operators in $\mathcal{L}(\mathcal{H})$. Recall the definition of the usual derivation operator $\delta_{A,B}(X)$ given by

$$\delta_{A,B}(X) = AX - XB \quad \text{for} \quad X \in \mathcal{L}(\mathcal{H}).$$

For every positive integer k, we have

$$\delta^k_{A,B}(X) = \delta_{A,B}(\delta^{k-1}_{A,B}(X)) \quad \text{for} \quad X \in \mathcal{L}(\mathcal{H}).$$

Definition 5.1. *Let A and B be in $\mathcal{L}(\mathcal{H})$. An operator B is said to be in $Helton_k(A)$ if $\delta^k_{A,B}(I) = 0$.*

The following result can be found in [23, Section 3.6].

Theorem 5.2. *Let A and B be in $\mathcal{L}(\mathcal{H})$. If B is in $Helton_k(A)$ then $\sigma_p(B) \subset \sigma_p(A)$, $\sigma_{ap}(B) \subset \sigma_{ap}(A)$, and $\sigma_{su}(A) \subset \sigma_{su}(B)$. In particular, $\sigma(A) \subset \sigma(B)$ when A has the SVEP. Moreover, if A and B^* have the SVEP, then $\sigma(A) = \sigma(B)$.*

Proof. For the first inclusion, if $\lambda \in \sigma_p(B)$, then there exists a nonzero $x \in \mathcal{H}$ such that $(B - \lambda)x = 0$. Since $B - \mu \in \text{Helton}_k(A - \mu)$ for any μ, it follows that $(A - \lambda)^k x = 0$. If $\lambda \notin \sigma_p(A)$, then $A - \lambda$ is one-to-one. Hence $x = 0$ which is a contradiction for a nonzero x. So $\lambda \in \sigma_p(A)$.

For the second inclusion, if $\lambda \in \sigma_{ap}(B)$, then there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that $\lim_{n\to\infty} \|(B - \lambda)x_n\| = 0$. Since $B - \lambda \in \text{Helton}_k(A - \lambda)$, it follows that $\lim_{n\to\infty} \|(A - \lambda)^k x_n\| = 0$. If $\lambda \notin \sigma_{ap}(A)$, then $A - \lambda$ is bounded below. Hence $\lim_{n\to\infty} \|x_n\| = 0$. This is a contradiction for $\|x_n\| = 1$. Hence $\lambda \in \sigma_{ap}(A)$.

For the third inclusion, if $B-\lambda$ is surjective, then $B-\lambda \in \text{Helton}_k(A-\lambda)$ implies $A-\lambda$ is surjective. This means that $\sigma_{su}(A) \subset \sigma_{su}(B)$.

For the last statement, if A has the SVEP and $B \in \text{Helton}_k(A)$ then B has the SVEP from [23, Theorem 3.2.1]. By the above relation and [25], we obtain that $\sigma(A) = \sigma_{su}(A) \subset \sigma_{su}(B) = \sigma(B)$. Since $B \in \text{Helton}_k(A)$, it holds that $A^* \in \text{Helton}_k(B^*)$. If B^* has the SVEP, then A^* has the SVEP from [23]. Thus $\sigma_{ap}(A) = \sigma(A)$ and $\sigma_{ap}(B) = \sigma(B)$ from [25]. Hence $\sigma(B) \subset \sigma(A)$ and so $\sigma(A) = \sigma(B)$.

We also have the following theorem from [23, Theorems 3.2.1 and 3.7.1] or [25].

Theorem 5.3. *Let A and B be in $\mathcal{L}(\mathcal{H})$. If B is in $\text{Helton}_k(A)$, then*

- *A has the SVEP at $\lambda \Longrightarrow B$ has the SVEP at λ.*
- *A has (β) at $\lambda \Longrightarrow B$ has (β) at λ.*
- *A has $(\beta)_\epsilon$ at $\lambda \Longrightarrow B$ has $(\beta)_\epsilon$ at λ.*

Proof. Suppose that A has $(\beta)_\epsilon$ at λ and $f_n : U \to \mathcal{H}$ is a sequence of analytic functions such that $(B-\lambda)f_n(\lambda) \longrightarrow 0$ in $\mathcal{E}(U,\mathcal{H})$. Then we get that

It's obvious that $\delta_{A,B}(X) = \delta_{A-\mu,B-\mu}(X)$ for every $X \in \mathcal{L}(\mathcal{H})$ and every $\mu \in \mathbb{C}$. Thus $\delta^k_{A,B}(X) = \delta^k_{A-\mu,B-\mu}(X)$ for every $X \in \mathcal{L}(\mathcal{H})$, every $\mu \in \mathbb{C}$ and every integer k

Since

$$\begin{aligned}
0 &= \delta^k_{A-\lambda,B-\lambda}(I)f_n(\lambda)\\
&= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(A-\lambda)^j(B-\lambda)^{k-j}f_n(\lambda)\\
&= \sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}(A-\lambda)^j(B-\lambda)^{k-j}f_n(\lambda) + (A-\lambda)^k f_n(\lambda)\\
&= [\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}(A-\lambda)^j(B-\lambda)^{k-j-1}](B-\lambda)f_n(\lambda) + (A-\lambda)^k f_n(\lambda)
\end{aligned}$$

it follows that $(A-\lambda)^k f_n(\lambda) \longrightarrow 0$ in $\mathcal{E}(U,\mathcal{H})$. Moreover, since A has $(\beta)_\epsilon$ at λ, $(A-\lambda)^{k-1}f_n(\lambda) \longrightarrow 0$ in $\mathcal{E}(U,\mathcal{H})$. By induction, $f_n \longrightarrow 0$ in $\mathcal{E}(U,\mathcal{H})$. Hence B has $(\beta)_\epsilon$ at λ. The remaining statements hold by similar arguments.

6. *m*-Complex and *m*-skew Complex Symmetric Operators

Let m be a positive integer. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that $\Delta_m(T) = 0$ where

$$\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C. \tag{6}$$

(See [8] and [9]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-skew complex symmetric operator* if there exists some conjugation C such that $\Gamma_m(T) = 0$ where

$$\Gamma_m(T) := \sum_{j=0}^{m} \binom{m}{j} T^{*j} C T^{m-j} C. \tag{7}$$

For example, if N is a nilpotent of order 2, then N is complex symmetric from [14, Theorem 2]. But if $T = I + N$, then is easy to see that T is a 3-complex symmetric operator which is not 2-complex symmetric.

Remark 6.1. *It is easy to see that*

- *$T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator which means that $CTC \in Helton_m(T^*)$.*
- *$T \in \mathcal{L}(\mathcal{H})$ is an m-skew complex symmetric operator which means that $-CTC \in Helton_m(T^*)$.*

Therefore, we have the following.

Theorem 6.2. *Let T be in $\mathcal{L}(\mathcal{H})$. If T is an m-complex symmetric operator, then*

- *T^* has the SVEP at $\lambda \Longrightarrow T$ has the SVEP at $\bar{\lambda}$.*
- *T^* has (β) at $\lambda \Longrightarrow T$ has (β) at $\bar{\lambda}$.*
- *T^* has $(\beta)_\epsilon$ at $\lambda \Longrightarrow T$ has $(\beta)_\epsilon$ at $\bar{\lambda}$.*

Proof. We only prove the third statement.

$$\begin{aligned} T^* \text{ has } (\beta)_\epsilon \text{ at } \lambda &\Longrightarrow CTC \text{ has } (\beta)_\epsilon \text{ at } \lambda \\ &\Longrightarrow T \text{ has } (\beta)_\epsilon \text{ at } \bar{\lambda}. \end{aligned}$$

By using Theorem 5.3 and Theorem 4.3, we obtain the last statement.

In particular, we recapture a part of some known results (see [8, Theorem 4.7]).

Corollary 6.3. *Let T be in $\mathcal{L}(\mathcal{H})$. If T is an m-complex or m-skew complex symmetric operator, then*

$$T^* \text{ has } (\beta) \Longleftrightarrow T \text{ is decomposable}.$$

For example, if T is a nilpotent operator of order $k > 2$, then T^* is nilpotent of order $k > 2$ and so T^* is $(2k-1)$-complex symmetric from [8, Example 3.1]. Moreover, in this case, T^* has (β). Hence T is decomposable from Corollary 6.3.

7. Complex Symmetric and Skew-Complex Symmetric Operators

Notice that, if $m = 1$, the definition given above matches the one of complex symmetric and skew-complex symmetric operators. One could wonder why we are considering this special case separately. There are at least two reasons.

-The first one is:

For an arbitrary conjugation C and an operator T on $\mathcal{H}$, one can write T as a sum of a complex symmetric operator and a skew-complex symmetric operator. Namely, $T = A+B$ where $A = -\frac{1}{2}\Gamma_1(T^*)$ and $B = -\frac{1}{2}\Delta_1(T^*)$ where $A = CA^*C$, $B = -CB^*C$.

-The second one is:

$\text{Helton}_1(A) = \{A\}$. Thus we have the coincidence of many spectra (instead of the inclusion).

Corollary 7.1. *Let T be in $\mathcal{L}(\mathcal{H})$.*

1. *If T is a complex symmetric operator, then*

- *T^* has the SVEP at $\lambda \iff T$ has the SVEP at $\bar{\lambda}$.*
- *T^* has (β) at $\lambda \iff T$ has (β) at $\bar{\lambda}$.*
- *T^* has $(\beta)_\epsilon$ at $\lambda \iff T$ has $(\beta)_\epsilon$ at $\bar{\lambda}$.*

2. *If T is a skew complex symmetric operator, then*
   - *T^* has the SVEP at $\lambda \iff T$ has the SVEP at $-\bar{\lambda}$.*
   - *T^* has (β) at $\lambda \iff T$ has (β) at $-\bar{\lambda}$.*
   - *T^* has $(\beta)_\epsilon$ at $\lambda \iff T$ has $(\beta)_\epsilon$ at $-\bar{\lambda}$.*

Furthermore, we have

Corollary 7.2. *Let T be in $\mathcal{L}(\mathcal{H})$.*

1. *If T is a complex symmetric operator, then*

$$\sigma_\bullet(T^*) = \sigma_\bullet(T)^*.$$

2. *If T is a skew complex symmetric operator, then*

$$\sigma_\bullet(T^*) = -\sigma_\bullet(T)^*$$

when $\sigma_\bullet \in \{\sigma, \sigma_p, \sigma_{ap}, \sigma_c, \sigma_r, \sigma_{su}, \sigma_e, \sigma_w, \dots\}$.

In the following corollary, we also recapture the result in [18, Theorem 4.4].

Corollary 7.3. *Let T be in $\mathcal{L}(\mathcal{H})$. If T is a complex symmetric or a skew complex symmetric operator, then T satisfies Weyl's (or Browder's) theorem if and only if T^* satisfies Weyl's (or Browder's) theorem.*

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