ON SYMMETRIC FUNCTIONS AND SYMMETRIC FUNCTIONS OF SYMMETRIC FUNCTIONS*

By

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INTRODUCTION

The study of symmetric functions is quite an old one. From the time of Girard (1629) even up to the present day this subject has occupied the attention of many eminent mathematicians. The theory of the roots of algebraic equations in one or more variables has furnished the chief incentive for the development of the theory of symmetric functions. Ingenious methods for computing symmetric functions in terms of what are called *the elementary symmetric functions* have been developed by Hammond, Brioschi, Junker, Dresden and others. Extensive tables of symmetric functions in terms of the elementary symmetric functions may be found in the literature.

Symmetric functions play such a pre-eminent rôle in the mathematical theory of statistics and their computation by direct methods or by general formulas, even when assumptions restricting the groupings of the variates about the various means are made, is so excessively tedious that there has seemed to be need

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of development of the theory of symmetric functions in directions not suggested by the theory of equations. The ingenious methods referred to above are of little or no practical value in statistics; for they express a symmetric function in terms of the elementary symmetric functions whilst here it is necessary to express the symmetric function in terms of what are called the *power sums.* Likewise, and for the same reason, the tables mentioned are of no value to the student of statistics.

Moreover, in the theory of sampling one not only has to deal with symmetric functions of the given variates but with symmetric functions of symmetric functions of the given variates. This then leads to interesting as well as practical developments in the theory of symmetric functions.

In this investigation it is proposed to:

1. Develop symbolic methods which will enable one to express any given symmetric function in terms of the power sums, without knowing the expressions for the symmetric functions of lower weight, and which will also lend themselves readily to the construction of tables;

2. Develop symbolic devices in the more general case of a symmetric function of symmetric functions.

CHAPTER I

Direct Computation

1. Suppose there is given a set of n variates[1] $x_1, \cdot\, x_2, x_3, x_4, \ldots x_n$, no assumptions whatever being made as to their arrangement about the various means. Any rational, integral, algebraic function of these n variates which is unaltered by interchanges or permutations of the variates is called *a symmetric function*. With a few modifications, the usual notation for symmetric functions will be used in this investigation.

The *power sums* $s_1, s_2, s_3, \ldots s_t$:

Let

$$s_1 = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots\cdots + x_n ,$$

$$s_2 = \sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \cdots\cdots + x_n^2 ,$$

$$s_3 = \sum_{i=1}^{n} x_i^3 = x_1^3 + x_2^3 + \cdots\cdots + x_n^3 ,$$

. . . .

. . . .

$$s_t = \sum_{i=1}^{n} x_i^t = x_1^t + x_2^t + \cdots\cdots + x_n^t .$$

Further, let $(a^\alpha b^\beta c^\gamma \cdots)$ represent any symmetric

[1]The variates may be either real or complex numbers.

function of the given variates. In other words, let $(a^{\alpha} b^{\beta} c^{\gamma} \cdots)$ equal the sum of all the terms such as

$$x_1^a x_2^a \cdots x_{\alpha}^a x_{\alpha+1}^b x_{\alpha+2}^b \cdots x_{\alpha+\beta}^b x_{\alpha+\beta+1}^c \cdots x_{\alpha+\beta+\gamma}^c \cdots$$

which can be formed from the n variates, where a, b, c, . . . and α, β, γ, . . . are positive integers and $a > b > c > \cdots > 0$. e. g.

$$(3^2 21) = \sum_{\substack{i=1\\ j=1\\ k=1\\ m=1}}^{n} x_i^3 x_j^3 x_k^2 x_m \,, \quad i \neq j \neq k \neq m\,.$$

DEFINITIONS:

A *partition* of a positive integer t is any set of positive integers whose sum is t. The integers which constitute the partition are called the *parts* of the partition and are enclosed in parentheses ( ). It is desirable to arrange the parts in descending order of magnitude from left to right. Obviously then for any finite positive integer t each partition of t contains a finite number of parts. If there are r parts in the partition of t then the partition is called an *r-part partition of t* or simply an *r-partition of t*. E. G. (33), (321), (3111) are respectively 2-part, 3-part and 4-part partitions of 6. When repeated parts appear in the partition it is customary to write one of the repeated parts with an index corresponding to the number of times that part is repeated. Thus (33) is written (3^2) and (3111) is written (31^3). The number t is called the *weight* of the partition. For a discussion of the formulae for finding the number of partitions of an integer the reader is referred to Whitworth's

"Choice and Chance."[1]

It will now be clear that the notation introduced for the general symmetric function is a partition notation. The *weight* of a symmetric function is the degree in all the variates of any term in the summation. The *order* of a symmetric function is the highest degree in which each variate appears in the summation. For instance, in $\sum x_i^4 x_j^3 x_k^2 =$ (432) the weight is $4+3+2=9$ and the order is 4. It follows that in the partition notation of a symmetric function the weight is given by $a\alpha + b\beta + c\gamma + \ldots$ and the order by a. In the partition notation the power sums become simply (1), (2), (3), ..., (t) respectively.

For the purpose of mathematical statistics, moments rather than the power sums are the important thing. However, the transformation from power sums to moments is so simple that the results of this investigation in terms of power sums may be written in terms of the moments by putting

$$
\begin{aligned}
n\mu'_{1:x} &= s_1 \,, \\
n\mu'_{2:x} &= s_2 \,, \\
n\mu'_{3:x} &= s_3 \,, \\
&\cdots\cdots \\
n\mu'_{t:x} &= s_t \,,
\end{aligned}
$$

where $\mu'_{1:x}$, $\mu'_{2:x}$, $\mu'_{3:x}$, ..., $\mu'_{t:x}$ are the statistical *moments* of the n variates.

2. It is not difficult to express certain symmetric functions in terms of the power sums. Practically all texts in higher algebra devote a section or two to this problem. Most of those which develop general formulae do so by using the properties of the coefficients of an algebraic equation. However, many others have developed general formulae in symmetric functions without

[1] W. A. Whitworth, "Choice and Chance," G. E. Stechert and Co., N. Y., fifth edition, page 100.

making use of the algebraic equation in their derivations. The latter procedure will be followed here in order to emphasize the fact that the interest is not in the theory of equations but in a set of variates such as might appear for instance in a statistical problem. A few of the general formulae of symmetric functions will be developed now by direct computation in order to demonstrate a basic theorem of this work—a theorem which will be stated at the close of this chapter.

Multiplying s_2 and s_1 the result is

$$s_2 s_1 = (x_1^2 + x_2^2 + \cdots + x_n^2)(x_1 + x_2 + \cdots + x_n)$$

$$= (x_1^2 x_2 + x_1^2 x_3 + \cdots + x_{n-1} x_n^2) + (x_1^3 + x_2^3 + \cdots + x_n^3)$$

$$= \sum_{\substack{i=1\\ j=1}}^{n} x_i^2 x_j + \sum_{i=1}^{n} x_i^3, \quad i \neq j$$

$(2)(1) = (21) + (3)$, hence

$(21) = (2)(1) - (3)$

Similarly, if $u \neq v$,

$$s_u s_v = (x_1^u + x_2^u + \cdots + x_n^u)(x_1^v + x_2^v + \cdots + x_n^v)$$

$$= (x_1^u x_2^v + x_1^u x_3^v + \cdots + x_{n-1}^u x_n^v) + (x_1^{u+v} + x_2^{u+v} + \cdots + x_n^{u+v})$$

$$=\sum_{\substack{i=1\\j=1}}^{n} x_i^u x_j^v + \sum_{i=1}^{n} x_i^{u+v} \quad , \ i \neq j \ ,$$

$$= (uv) + (u+v)$$, hence

$$(uv) = (u)(v) - (u-v)$$

However, if $u = v$ a modification is necessary. For then

$$(u)^2 = (x_1^u + x_2^u + \dots + x_n^u)^2$$

$$= (x_1^{2u} + x_2^{2u} + \dots + x_n^{2u}) + (x_1^u x_2^u + \dots + x_{n-1}^u x_n^u)$$

$$= \sum_{i=1}^{n} x_i^{2u} + \sum_{\substack{i=1\\j=1}}^{n} x_i^u x_j^u \ , \ i \neq j \ ,$$

$$= (\overline{2u}) + 2(u^2)$$ and thus

$2!(u^2) = (u)^2 - (\overline{2u})$ where the bar over $2\,u$ indicates ordinary algebraic multiplication of 2 and u, i e. $(\overline{2u}) = s_{2u}$.

If $u \neq v \neq w$, $u+v \neq w$, $u+w \neq v$, $v+w \neq u$, then

$$(u)(v)(w) = (x_1^u + x_2^u + \dots + x_n^u)(x_1^v + x_2^v + \dots + x_n^v)(x_1^w + x_2^w + \dots + x_n^w)$$

$$= (x_1^u x_2^v x_3^w + \dots) + (x_1^{u+v} x_2^w + \dots) + (x_1^{u+v+w} + \dots)$$

$$= \sum_{\substack{i=1\\j=1\\k=1}}^{n} x_i^u x_j^v x_k^w + \sum_{\substack{i=1\\j=1}}^{n} x_i^{u+v} x_j^w + \sum_{\substack{i=1\\j=1}}^{n} x_i^{v+w} x_j^u$$

$$+\sum_{\substack{i=1\\j=1}}^{n} x_i^{u+w} x_j^{v} + \sum_{i=1}^{n} x_i^{u+v+w}, \quad i \neq j \neq k$$

$$= (uvw) + (u+v, w) + (v+w, u) + (u+w, v) + (u+v+w)$$

the commas being used to separate the parts of the partitions. Now applying the result obtained for (uv) to the second, third and fourth terms on the right of this last expression, it becomes, since

$$(u+v, w) = (u+v)(w) - (u+v+w),$$

$$(v+w, u) = (v+w)(u) - (u+v+w),$$

$$(u+w, v) = (u+w)(v) - (u+v+w),$$

$$(u)(v)(w) = (uvw) + (u+v)(w) + (v+w)(u) + (u+w)(v) - 2(u+v+w).$$

Finally

$$(uvw) = (u)(v)(w) - (u+v)(w) - (v+w)(u) - (u+w)(v) + 2(u+v+w)$$

$$= s_u s_v s_w - s_{u+v} s_w - s_{v+w} s_u - s_{u+w} s_v + 2 s_{u+v+w}$$

If $u = v = w$, then a modification is again necessary, and repeating the multiplication with $u = v = w$ it is found that

$$3!(u^3) = (u)^3 - 3(\overline{2u})(u) + 2(\overline{3u})$$
$$= s_u^3 - 3 s_{2u} s_u + 2 s_{3u}.$$

In like manner, if $u \neq v \neq w \neq z$, $u+v \neq w$, etc., $u+v+w \neq z$, etc., then

$$(uvwz) = (u)(v)(w)(z) - (u)(v)(w+z) - (u)(w)(v+z)$$

$$-(u)(z)(v+w) - (v)(w)(u+z) - (v)(z)(u+w)$$

$$-(w)(z)(u+v) + 2(u)(v+w+z) + 2(v)(u+w+z)$$

$$+2(w)(u+v+z) + 2(z)(u+v+w) + (u+v)(w+z)$$

$$+(u+v)(w+z) + (u+w)(v+z) + (u+z)(v+w) - 6(u+v+w+z).$$

If $u = v = w = z$, then

$$4!(u^4) = (u)^4 - 6(u)^2(\overline{2u}) + 8(u)(\overline{3u}) + 3(\overline{2u})^2 - 6(\overline{4u})$$

$$= s_u^4 - 6s_u^2\, s_{2u} + 8s_u s_{3u} + 3s_{2u}^2 - 6s_{4u}$$

Similar modifications are necessary when some but not all of the parts of the partition are equal. For example,

$$(u)^2(v) = (x_1^u + x_2^u + \cdots + x_n^u)^2(x_1^v + x_2^v + \cdots + x_n^v)$$

$$= \sum_{i=1}^{n} x_i^{2u+v} + \sum_{\substack{i=1\\j=1}}^{n} x_i^{2u} x_j^{v} + \sum_{\substack{i=1\\j=1}}^{n} x_i^{u+v} x_j^{u} + 2\sum_{\substack{i=1\\j=1\\k=1}}^{n} x_i^u x_j^u x^v,$$

$$i \neq j \neq k,$$

$$= (\overline{2u}+v) + (\overline{2u}, v) + 2(u+v, u) + 2(u^2 v)$$

$$= (\overline{2u})(v) + 2(u)(u+v) - 4(\overline{2u}+v) + 2(u^2 v)$$

hence

$$2!(u^2v) = (u)^2(v) - (\overline{2u})(v) - 2(u)(u+v) + 2(\overline{2u}+v)$$

$$= s_u^2 s_v - s_{2u} s_v - 2s_u s_{u+v} + 2 s_{2u+v}$$

3. Proceeding after the above fashion, any symmetric function whatever can be expressed in terms of the power sums. However, the process becomes increasingly cumbersome and the general formula is of no practical value for the purpose of computation. Moreover, it is necessary to use a continuous process, that is, to work from the simpler symmetric functions of small weight to the more complex symmetric functions of greater weight.

A special case may be worth mentioning to illustrate still better the carrying out of the direct process in the general case.

$$(u)^t = (x_1^u + x_2^u + \cdots\cdots + x_n^u)^t$$

Applying the multinomial theorem and assuming that the law holds for $t-1$ and that the symmetric functions of weight less than t are known and transposing all the terms of the right member except the term involving (u^t), it is found that

$$t!(u^t) = \sum(-1)^{v+t} \frac{t!\,(u)^{a_1}(\overline{2u})^{a_2}(\overline{3u})^{a_3}\cdots(\overline{tu})^{a_t}}{1^{a_1}2^{a_2}3^{a_3}\cdots t^{a_t}\cdot a_1!\,a_2!\,a_3!\cdots a_t!}$$

where $a_1, a_2, a_3, \ldots\ldots, a_t$ are either positive integers or zeros such that $a_1 + a_2 + a_3 + \ldots + a_t = v$ and $a_1 + 2a_2 + 3a_3 + \cdots\cdot + ta_t = t$.

In particular, if $u = 1$, then

$$t!(1^t) = \sum (-1)^{v+t} \frac{t!(1)^{a_1}(2)^{a_2}(3)^{a_3}\ldots(t)^{a_t}}{1^{a_1}2^{a_2}3^{a_3}\ldots t^{a_t}\cdot a_1!\,a_2!\,a_3!\cdots a_t!}$$

This last result may be expressed very conveniently in determinant form. Starting with the results obtained in article 2, it is seen that

$$1!(1) = s_1 ,$$

$$2!(1^2) = \begin{vmatrix} s_1 & 1 \\ s_2 & s_1 \end{vmatrix}$$

$$3!(1^3) = \begin{vmatrix} s_1 & 1 & 0 \\ s_2 & s_1 & 2 \\ s_3 & s_2 & s_1 \end{vmatrix} ,$$

$$4!(1^4) = \begin{vmatrix} s_1 & 1 & 0 & 0 \\ s_2 & s_1 & 2 & 0 \\ s_3 & s_2 & s_1 & 3 \\ s_4 & s_3 & s_2 & s_1 \end{vmatrix} ,$$

. . . .

. . . .

$$t!\,(1^t) = \begin{vmatrix} s_1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ s_2 & s_1 & 2 & 0 & \cdot & \cdot & \cdot & & \cdot & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \cdot & \cdot & & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ s_{t-2} & \cdot & \cdot & \cdot & s_3 & s_2 & s_1 & t-2 & & 0 \\ s_{t-1} & s_{t-2} & \cdot & \cdot & \cdot & s_3 & s_2 & s_1 & & t-1 \\ s_t & s_{t-1} & s_{t-2} & \cdot & \cdot & \cdot & s_3 & s_2 & & s_1 \end{vmatrix}$$

To establish this general law it is sufficient to note that the development of this determinant gives as a general term

$$(-1)^{v+t} \frac{t!\, s_1^{a_1}\, s_2^{a_2}\, s_3^{a_3} \cdot \cdot \cdot \cdot s_t^{a_t}}{1^{a_1}\, 2^{a_2}\, 3^{a_3} \cdot \cdot \cdot \cdot t^{a_t} \cdot a_1!\, a_2!\, a_3! \cdots a_t!}$$

where a_1, a_2, a_3, ... , a_t are positive integers or zeros which satisfy the conditions $a_1 + a_2 + a_3 + \cdots + a_t = v$ and $a_1 + 2a_2 + 3a_3 + 4a_4 + \cdots\ ta_t = t$.

Hence the determinant is equal to

$$\sum (-1)^{v+t} \frac{t!\, s_1^{a_1}\, s_2^{a_2} \cdot \cdot \cdot \cdot s_t^{a_t}}{1^{a_1} 2^{a_2} \cdot \cdot \cdot \cdot t^{a_t} \cdot a_1!\, a_2! \cdot \cdot \cdot \cdot a_t!}$$

where, as before, the summation is over all the different terms it is possible to obtain by assigning a_1, a_2, ... , a_t all positive integral values or zeros which satisfy the conditions

$$a_1 + a_2 + \cdots\cdots + a_t = v\,,$$
$$a_1 + 2a_2 + \cdots\cdots + ta_t = t\,.$$

4. This chapter will be concluded here with the statement of a very important theorem which may now be written and which will serve as a basis for the developments in the chapters to follow.

Basic Theorem:

Any symmetric function (defined in article 1) may be expressed as a rational, integral, algebraic function of the power sums.

Further, each term in the expression for the symmetric function in terms of the power sums is of the same weight as the symmetric function itself. Hence a term which does not arise from a partition of the weight of the symmetric function cannot appear in the expression in terms of the power sums.

CHAPTER II

A Differential Operator Method of Computing Symmetric Functions in Terms of the Power Sums

5. Consider a symmetric function $(a^\alpha b^\beta c^\gamma \ldots)$ of weight w of the variates $x_1, x_2, \ldots, x_n$. By the theorem demonstrated in chapter I and stated at the close thereof it is possible to write

$$(a^\alpha b^\beta c^\gamma \ldots) = f(s_1, s_2, \cdots, s_w)$$

where f stands for a rational, integral, algebraic function of the power sums $s_1, s_2, \ldots, s_w$, and where each term in f is of total weight w, i. e. *isobaric*.

In the preceding chapter the direct method of computing a symmetric function in terms of the power sums has been illustrated. But that method has two major disadvantages. In the first place, it is necessary to know the expressions in terms of the power sums of the symmetric functions of lower weight; and in the second place, it becomes altogether impractical for anything but the simplest cases. It is proposed to develop a method which will have neither of these disadvantages—in other words, to develop a method which will express any given symmetric function directly in terms of the power sums without knowing the expressions for the symmetric functions of lower weight, and which will not become too unwieldy. In addition, the method ought to lend itself readily to the construction of tables of symmetric functions in terms of the power sums.

The method developed here will be a differential operator method. It may be stated at the outset that many schemes for determining differential operators which will do the work are possible. The writer has investigated a number of them. The operators developed here are given because they seem to satisfy best the demands just imposed on the method of computation. In fact, their simplicity and the directness with which they produce results indicate that they are the simplest differential operators that can be developed for the problem.

6. Suppose now that a new variate $x_{n+1} = k$ is introduced. What effect will it have on $(a^{\alpha} b^{\beta} c^{\gamma} \ldots)$ and on f? First consider $(a^{\alpha} b^{\beta} c^{\gamma} \ldots)$. Since all the variates enter the symmetric function in exactly the same way, new terms involving k in all the ways in which the other variates appear will be introduced. For example, if the original set of variates is x_1, x_2, x_3, x_4 and the original symmetric function $(32) = \sum x_i^3 x_j^2$, $i \neq j$, then this symmetric function is made up of the terms

$$\begin{array}{cccc} x_1^3 x_2^2 & x_2^3 x_1^2 & x_3^3 x_1^2 & x_4^3 x_1^2 \\ x_1^3 x_3^2 & x_2^3 x_3^2 & x_3^3 x_2^2 & x_4^3 x_2^2 \\ x_1^3 x_4^2 & x_2^3 x_4^2 & x_3^3 x_4^2 & x_4^3 x_3^2 \end{array}$$

Introducing a new variate $x_5 = k$, produces the new terms

$$\begin{array}{cccc} x_1^3 k^2 & x_2^3 k^2 & x_3^3 k^2 & x_4^3 k^2 \\ k^3 x_1^2 & k^3 x_2^2 & k^3 x_3^2 & k^3 x_4^2 \end{array}$$

or that is, produces $\sum k^3 x_i^2$ and $\sum x_i^3 k^2$. And since k is a constant with respect to the summation, these summations may be written $k^3 \sum x_i^2$ and $k^2 \sum x_i^3$, $i = 1, 2, 3, 4$.

Hence $\sum_{\substack{i=1\\j=1}}^{4} x_i^3 x_j^2$ becomes $\sum_{\substack{i=1\\j=1}}^{4} x_i^3 x_j^2 + k^3 \sum_{i=1}^{4} x_i^2 + k^2 \sum_{i=1}^{4} x_i^3$, $i \neq j$,

i. e., (32) becomes (32) $+ k^3(2) + k^2(3)$.

Similarly k must enter $(a^{\alpha} b^{\beta} c^{\gamma} \ldots)$ just as every other variate does. As a result new terms are produced and $(a^{\alpha} b^{\beta} c^{\gamma} \ldots)$ becomes $(a^{\alpha} b^{\beta} c^{\gamma} \ldots)$

$$+ k^a (a^{\alpha-1} b^{\beta} c^{\gamma} \ldots) + k^b (a^{\alpha} b^{\beta-1} c^{\gamma} \ldots)$$

$$+ k^c (a^{\alpha} b^{\beta} c^{\gamma-1} \ldots) + \ldots \ldots$$

Next find what happens to $f(s_1, s_2, \ldots, s_w)$ when the new variate $x_{n+1} = k$ is introduced. From the definition of the power sums it follows that

$$\begin{array}{lll} s_1 & \text{becomes} & s_1 + k, \\ s_2 & \text{becomes} & s_2 + k^2, \\ s_3 & \text{becomes} & s_3 + k^3, \\ \ldots & & \\ \ldots & & \\ s_t & \text{becomes} & s_t + k^t, \\ \ldots & & \\ \ldots & & \\ s_w & \text{becomes} & s_w + k^w. \end{array}$$

Hence $f(s_1, s_2, \ldots, s_w)$ becomes

$$f(s_1 + k, s_2 + k^2, \ldots, s_w + k^w).$$

Taylor's series for several variables is

$$\begin{aligned} f(x+h, y+k, z+m, \ldots) = {} & f(x, y, z, \ldots) \\ & + (h\partial/\partial x + k\partial/\partial y + m\partial/\partial z + \ldots) f \\ & + (h\partial/\partial x + k\partial/\partial y + m\partial/\partial z + \ldots)^2 \frac{f}{2!} \\ & + (h\partial/\partial x + k\partial/\partial y + m\partial/\partial z + \ldots)^3 \frac{f}{3!} \\ & + \ldots \ldots \end{aligned}$$

where the multiplication of operators is algebraic.

Applying Taylor's series to the function under consideration, the result is

$$\begin{aligned} f(s_1+k, s_2+k^2, \cdots, s_w+k^w) &= f(s_1, s_2, \cdots, s_w) \\ &+ (k\partial/\partial s_1 + k\partial/\partial s_2 + \cdots + k^w\partial/\partial s_w) f \\ &+ (k\partial/\partial s_1 + k\partial/\partial s_2 + \cdots + k^w\partial/\partial s_w)^2 \frac{f}{2!} \\ &+ (k\partial/\partial s_1 + k\partial/\partial s_2 + \cdots + k^w\partial/\partial s_w)^3 \frac{f}{3!} \\ &\cdots\cdots\cdots \\ &+ (k\partial/\partial s_1 + k\partial/\partial s_2 + \cdots + k^w\partial/\partial s_w)^w \frac{f}{w!}, \end{aligned}$$

all other terms being identically zero.

Now let

$$d_1 = \partial/\partial s_1, \quad d_2 = \partial/\partial s_2, \cdots,$$
$$d_v = \partial/\partial s_v, \cdots, \quad v = 1, 2, 3, \cdots, w.$$

Then $d_1^2 = (\partial/\partial s_1)(\partial/\partial s_1) = \partial^2/\partial s_1^2$ and similarly $d_v^u = \partial^u/\partial s_v^u$

It is now possible to write

$$\begin{aligned} f(s_1+k, s_2+k^2, \cdots, s_w+k^w) &= f \\ &+ (kd_1 + k^2d_2 + k^3d_3 + \cdots + k^wd_w) f \\ &+ (kd_1 + k^2d_2 + k^3d_3 + \cdots + k^wd_w)^2 \frac{f}{2!} \\ &+ (kd_1 + k^2d_2 + k^3d_3 + \cdots + k^wd_w)^3 \frac{f}{3!} \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &+ (kd_1 + k^2d_2 + k^3d_3 + \cdots + k^wd_w)^w \frac{f}{w!} \end{aligned}$$

Multiplying out and collecting coefficients of powers of k, this becomes

$$f(s_1+k,\ s_2+k^2,\cdots,s_w+k^w)=(1+kD_1+k^2D_2+k^3D_3+\cdots+k^wD_w)f,$$

all other terms vanishing, where

$$(1)\quad \left[\begin{array}{l}
D_1=d_1,\\
2!D_2=d_1^2+2d_2,\\
3!D_3=d_1^3+6d_1d_2+6d_3,\\
4!D_4=d_1^4+12d_1^2d_2+24d_1d_3+12d_2^2+24d_4,\\
5!D_5=d_1^5+20d_1^3d_2+60d_1^2d_3+60d_1d_2^2+120d_1d_4+120d_2d_3+120d_5,\\
6!D_6=d_1^6+30d_1^4d_2+120d_1^3d_3+180d_1^2d_2^2+360d_1^2d_4\\
\qquad\qquad +720d_1d_2d_3+720d_1d_5+720d_2d_4+120d_2^3+360d_3^2+720d_6,\\
\text{etc.}
\end{array}\right.$$

Applying the multinomial theorem and then picking out the coefficient of k^t, the general term in this coefficient is found to be of the form

$$\frac{d_a^A\, d_b^B\, d_c^C\cdots}{A!\ B!\ C!\cdots}$$

where a, b, c, . . . and A, B, C, . . . are positive integers which satisfy the condition $aA+bB+cC+\cdots=t$.

Hence

$$t!\,D_t = \sum \frac{t!\, d_a^A d_b^B d_c^C}{A!\, B!\, C!} \qquad \text{where } aA + bB + cC + \cdots = t;$$

i. e. the sum of all the different terms which can be formed by assigning to $a, b, c, \ldots\ldots, A, B, C. \ldots$ all positive integral values which satisfy the condition $aA + bB + cC + \cdots = t$.

From the above relations it follows also that

$$(2)\quad \left[\begin{array}{l} d_1 = D_1, \\ 2d_2 = -(D_1^2 - 2D_2), \\ 3d_3 = (D_1^3 - 3D_1D_2 + 3D_3), \\ 4d_4 = -(D_1^4 - 4D_1^2D_2 + 2D_2^2 + 4D_1D_3 - 4D_4), \\ 5d_5 = (D_1^5 - 5D_1^3D_2 + 5D_1^2D_3 + 5D_1D_2^2 - 5D_2D_3 - 5D_1D_4 + 5D_5). \\ 6d_6 = -(D_1^6 - 6D_1^4D_2 + 6D_1^3D_3 - 6D_1^2D_4 + 9D_1^2D_2^2 - 12D_1D_2D_3 \\ \qquad + 6D_1D_5 + 6D_2D_4 - 2D_2^3 + 3D_3^2 - 6D_6). \\ td_t = (-1)^{t+1} \sum (-1)^{t+v} \dfrac{(v-1)!\, t\; D_a^A D_b^B D_c^C \cdots}{A!\; B!\; C! \cdots\cdots} \end{array}\right.$$

where $a, b, c, \ldots ; A, B, C, \ldots$ are positive integers and where the summation is over all the different terms which it is possible to obtain by assigning positive integral values to $a, b, c, \ldots ; A, B, C, \ldots$ which satisfy the conditions $A + B + C + \ldots = v$, $aA + bB + cC + \cdots = t$.

7. Now since $(a^\alpha b^\beta c^\gamma \ldots) = f$, therefore replacing f by $(a^\alpha b^\beta c^\gamma \ldots)$ the effect of the introduction of

the new variate $x_{n+1} = k$ may be written

$$(1+kD_1+k^2D_2+\cdots+k^wD_w)(a^\alpha b^\beta c^\gamma\cdots) = (a^\alpha b^\beta c^\gamma\cdots)$$
$$+k^a(a^{\alpha-1}b^\beta c^\gamma\cdots)+k^b(a^\alpha b^{\beta-1}c\cdots)+k^c(a^\alpha b^\beta c^{\gamma-1}\cdots)$$
$$+\cdots\cdots$$

Equating coefficients of equal powers of k, it is obvious that

$$(3)\quad \left[\begin{array}{l} D_a(a^\alpha b^\beta c^\gamma\cdots) = (a^{\alpha-1}b^\beta c^\gamma\cdots), \\ D_b(a^\alpha b^\beta c^\gamma\cdots) = (a^\alpha b^{\beta-1}c^\gamma\cdots), \\ D_c(a^\alpha b^\beta c^\gamma\cdots) = (a^\alpha b^\beta c^{\gamma-1}\cdots), \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ D_a^\alpha D_b^\beta D_c^\gamma\cdots(a^\alpha b^\beta c^\gamma\cdots) = 1, \quad \text{and also that} \\ D_r(a^\alpha b^\beta c^\gamma\cdots) = 0 \text{ if } r \text{ is not among } a, b, c; \\ \cdots\cdots \end{array}\right.$$

The relations between d and D given above enable one to express $(a^\alpha b^\beta c^\gamma\ldots)$ in terms of the power sums.

One particular case is worthy of mention. If 1 is not among $a, b, c, \ldots$ then $D_1(a^\alpha b^\beta c^\gamma\ldots) = 0$ and hence $d_1 f = 0$ and therefore also $d_1^2 f = d_1^3 f = \cdots = d_1^w f = 0$. In this case the operator relations may be written simply

$$(1')\quad \left|\begin{array}{l} D_1 = 0, \\ D_2 = d_2, \\ D_3 = d_3, \end{array}\right.$$

$$\left\lfloor\begin{aligned} 2!D_4 &= d_2^2 + 2d_4, \\ D_5 &= d_2 d_3 + d_5, \\ 3!D_6 &= d_2^3 + 6d_2 d_4 + 3d_3^2 + 6d_6, \\ &\text{etc.} \end{aligned}\right.$$

and

$$(2') \quad \left\lceil\begin{aligned} d_1 &= 0, \\ d_2 &= D_2, \\ d_3 &= D_3, \\ 2!d_4 &= 2D_4 - D_2^2, \\ d_5 &= D_5 - D_2 D_3, \\ 3!d_6 &= 6D_6 - 3D_3^2 + 2D_2^3 - 6D_2 D_4, \\ &\text{etc.} \end{aligned}\right.$$

Hence when 1 is not among $a, b, c, \ldots$ then s_1 cannot appear in the expression of $(a^\alpha b^\beta c^\gamma \ldots)$ in terms of the power sums, i. e. all the coefficients of terms involving s_1 vanish identically. But it must not be assumed that if $s_1 = 0$ then $d_1 f = 0$. Ordinarily this will not be true. It is necessary to find $\partial f/\partial s_1$ and in it set $s_1 = 0$. In statistics $s_1 = 0$ corresponds to the case where the variates are grouped about their arithmetic mean, i. e. so that $M_x = 0$.

8. The application of these operators d and D to the computation of a symmetric function in terms of the power sums will now be demonstrated. After that their use in the construction of tables will be considered.

Suppose it is desired to express (3^2) in terms of the power sums. The only terms which may appear are given by the partitions of 6. There are eleven partitions of 6. Hence let

$$(3^2) = a_1 s_1^6 + a_2 s_1^4 s_2 + a_3 s_1^3 s_3 + a_4 s_1^2 s_2^2 + a_5 s_1^2 s_4 + a_6 s_1 s_5$$

$$+ a_7 s_1 s_2 s_3 + a_8 s_2^3 + a_9 s_3^2 + a_{10} s_2 s_4 + a_{11} s_6 .$$

Since (3^2) does not contain 1 as a part, $D_1 = d_1 = 0$ and s_1 cannot appear on the right side of the above equation, i. e.

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0.$$

Now operate on the left side of the equation with D_2 and on the right with d_2.

$$D_2 (3^2) = 0,$$

$$d_2 f = 3 a_8 s_2^2 + a_{10} s_4 ,$$

hence $0 = 3 a_8 s_2^2 + a_{10} s_4$ and therefore $a_8 = a_{10} = 0$. Operating on the left with D_3 and on the right with d_3 gives $a_9 = \frac{1}{2}$ since $D_3 (3^2) = (3)$ and $d_3 f = 2 a_9 s_3$, i. e. $s_3 = 2 a_9 s_3$. Operating on the left with $6 D_6$ and on the right with $d_2^3 + 6 d_2 d_4 + 3 d_3^2 + 6 d_6$ gives $0 = 6 a_9 + 6 a_{11}$ and thus $a_{11} = -\frac{1}{2}$. Hence

$$(3^2) = (s_3^2 - s_6)/2.$$

Similarly let

$$(31^2) = a_1 s_1^5 + a_2 s_1^3 s_2 + a_3 s_1^2 s_3 + a_4 s_1 s_2^2 + a_5 s_1 s_4 + a_6 s_2 s_3 + a_7 s_5 .$$

Operate on the right with d_1^2 and on the left with D_1^2.

This gives

$$s_3 = 20\, a_1 s_1^3 + 6\, a_2 s_1 s_2 + 2 a_3 s_3$$

hence

$$a_1 = a_2 = 0, \quad a_3 = \tfrac{1}{2}$$

Operate on the right with $2d_2$ and on the left with $(D_1^2 - 2D_2)$. Then $-s_3 = 4a_4 s_1 s_2 + 2a_6 s_3$ and $a_4 = 0$, $a_6 = -\frac{1}{2}$

Operate on the right with $4d_4$ and on the left with $-(D_1^4 - 4D_1^2 D_2 + 2D_2^2 + 4D_1 D_3 - 4D_4)$. Then

$$-4s_1 = 4a_5 s_1 \,, \quad a_5 = -1$$

Similarly, operating on the right with $5\, d_5$ and on the left with its equivalent in terms of D, the result is $5 = 5a_7$, $a_7 = 1$. Hence

$$(31^2) = (s_1^2 s_3 - 2s_1 s_4 - s_2 s_3 + 2s_5)/2 \,.$$

In the case of (3^2) the operations on the left were performed with D_1, D_2, D_3 and $6D_6$, and on the right with their equivalent expressions in terms of d_1, d_2, d_3, d_4, d_5, d_6, with $D_1 = d_1 = 0$. In the case of (31^2) the operations on the right were performed with d_1^2, $2d_2$, $4d_4$ and $5d_5$, and on the left with their equivalent expressions in terms of D_1, D_2, D_3, D_4, D_5. Obviously it is immaterial from a theoretical point of view which procedure is followed. For practical purposes it will usually be found that the procedure followed in the case of (31^2) is preferable.

9. The application of the operators to the construction of tables of symmetric functions in terms of the power sums will now be illustrated.

Weight 1:

1. $(1) = s_1$.

Weight 2:

1. $(2) = s_2$.

2. $(1^2) = a_1 s_1^2 + a_2 s_2$.

$D_1(1^2) = d_1(a_1 s_1^2 + a_2 s_2)$, $a_1 = 1/2$.

$2D_2(1^2) = (d_1^2 + 2d_2)(a_1 s_1^2 + a_2 s_2)$, $a_2 = -a_1 = -1/2$.

$(1^2) = (s_1^2 - s_2)/2$.

Weight 3:

For all the symmetric functions of weight 3 f will be of the form

$$f = a_1 s_1^3 + a_2 s_2 s_1 + a_3 s_3 .$$

$$d_1 f = 3a_1 s_1^2 + a_2 s_2 .$$

$$(d_1^3 + 6d_1 d_2 + 6d_3) f = 6(a_1 + a_2 + a_3).$$

1. $(3) = s_3$.

2. $(21) = s_2 s_1 - s_3$, since $D_1(21) = (2) = s_2$; therefore $a_1 = 0$, $a_2 = 1$; $6D_3(21) = 0$ and hence $a_3 = -a_2 = -1$.

3. $(1^3) = (s_1^3 - 3s_2 s_1 + 2s_3)/6$ since $D_1(1^3) = (1^2)$ and $(1^2) = (s_1^2 - s_2)/2$;

therefore $a_1 = 1/6,\ a_2 = -1/2;$

$6D_3(1^3) = 0$ hence $a_3 = -a_1 - a_2 = 1/3.$

Weight 4:

For all the symmetric functions of weight 4 f will have the form

$$f = a_1 s_1^4 + a_2 s_1^2 s_2 + a_3 s_1 s_3 + a_4 s_2^2 + a_5 s_4.$$

$$d_1 f = 4a_1 s_1^3 + 2a_2 s_1 s_2 + a_3 s_3.$$

$$(d_1^2 + 2d_2) f = 2(6a_1 + a_2) s_1^2 + 2(a_2 + 2a_4) s_2.$$

$$(d_1^4 + 12 d_1^2 d_2 + 24 d_1 d_3 + 12 d_2^2 + 24 d_4) f = 24(a_1 + a_2 + a_3 + a_4 + a_5).$$

1. $(4) = s_4$

2. $(2^2) = (s_2^2 - s_4)/2$ since $D_1(2^2) = 0,$

$a_1 = a_2 = a_3 = 0;\quad 2D_2(2^2) = 2(2) = 2s_2,$

$a_4 = 1/2;\ 24D_4(2^2) = 0,\quad a_5 = -1/2.$

3. $(31) = s_3 s_1 - s_4$ since $D_1(31) = (3) = s_3,$

$a_1 = a_2 = 0,\ a_3 = 1.\quad 2D_2(31) = 0,$

$a_4 = 0;\ 24D_4(31) = 0,\quad a_5 = -1.$

4. $(21^2) = (s_1^2 s_2 - 2s_1 s_3 - s_2^2 + 2s_4)/2$ since

$D_1(21^2) = (21) = s_2 s_1 - s_3,\ a_1 = 0,\ a_2 = 1/2,\ a_3 = 1;$

$$2D_2(21^2) = 2(1^2) = (s_1^2 - s_2), 2a_4 + a_2 = -1/2,$$

$$a_4 = -1/2\,;\; 24D_4(21^2) = 0,\;\; a_5 = 1.$$

5. $(1^4) = (s_1^4 - 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 - 6s_4)/24,$

since $\quad D_1(1^4) = (1^3) = (s_1^3 - 3s_2 s_1 + 2s_3)/6$

$$a_1 = 1/24,\; a_2 = -1/4,\; a_3 = 1/3\,;\; 2D_2(1^4) = 0,$$

$$2a_4 = -a_2,\;\; a_4 = 1/8\,;\; 24D_4(1^4) = 0,\;\; a_5 = -1/4.$$

Weight 5:

$$f = a_1 s_1^5 + a_2 s_1^3 s_2 + a_3 s_1^2 s_3 + a_4 s_1 s_2^2 + a_5 s_1 s_4 + a_5 s_2 s_3 + a_7 s_5.$$

$$d_1 f = 5a_1 s_1^4 + 3a_2 s_1^2 s_2 + 2a_3 s_1 s_3 + a_4 s_2^2 + a_5 s_4.$$

$$(d_1^2 + 2d_2) f = 2(10a_1 + a_2) s_1^3 + 2(3a_2 + 2a_4) s_1 s_2$$

$$+ 2(a_3 + a_6) s_3.$$

$$(d_1^5 + 20d_1^3 d_2 + 60d_1^2 d_3 + 60d_1 d_2^2 + 120d_1 d_4 + 120d_2 d_3$$

$$+ 120d_5) f = 120(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7).$$

1. $(5) = s_5$

2. $(32) = s_3 s_2 - s_5,\quad$ since $D_1(32) = 0,$

$a_1=a_2=a_3=a_4=a_5=0$; $2D_2(32)=2(3)$,

$a_3+a_6=1$, $a_6=1$; $120d_5(32)=0$, $a_7=-a_6=-1$.

3. $(41)=s_4s_1-s_5$, since $D_1(41)=(4)$,

$a_1=a_2=a_3=a_4=0$, $a_5=1$; $2D_2(41)=0$, $a_6=0$;

$120d_5(41)=0$, $a_7=-1$.

4. $(2^21)=(s_2^2s_1-s_4s_1-2s_3s_2+2s_5)/2$, since

$D_1(2^21)=(2^2)$, $a_5=-1/2$, $a_1=a_2=a_3=0$,

$a_4=1/2$; $2D_2(2^21)=2(21)$, $a_6=-1$;

$120D_5(2^21)=0$, $a_7=1$.

5. $(31^2)=(s_3s_1^2-2s_4s_1-s_3s_2+2s_5)/2$, since

$D_1(31^2)=(31)$, $a_1=a_2=0$, $a_3=1/2$, $a_4=0$, $a_5=-1$;

$2D_2(31^2)=0$, $a_6=-a_3=-1/2$; $120D_5(31^2)=0$, $a_7=1$

6. $(21^3)=(s_2s_1^3-3s_3s_1^2-3s_2^2s_1+6s_4s_1+5s_3s_2-4s_5)/6$,

since $D_1(21^3)=(21^2)$, $a_1=0$, $a_2=1/6$,

$a_3=-1/2$, $a_4=-1/2$, $a_5=1$; $2D_2(21^3)=2(1^3)$,

$a_6=5/6$; $120D_5(21^3)=0$, $a_7=-2/3$.

7. $(1^5)=(s_1^5-10s_2s_1^3+20s_3s_1^2+15s_2^2s_1-30s_4s_1-20s_3s_2$

$+24s_5)/120$, since $D_1(1^5)=(1^4)$, $a_1=1/120$, $a_2=-1/12$, $a_3=1/6$,

$a_4=1/8$, $a_5=-1/4$; $2D_2(1^5)=0$, $a_6=-a_3=-1/6$; $120D_5(1^5)=0$, $a_7=1/5$.

Weight 6:

$$f = a_1 s_1^6 + a_2 s_2 s_1^4 + a_3 s_3 s_1^3 + a_4 s_2^2 s_1^2 + a_5 s_4 s_1^2$$
$$+ a_6 s_3 s_2 s_1 + a_7 s_5 s_1 + a_8 s_4 s_2 + a_9 s_2^3 + a_{10} s_3^2 + a_{11} s_6 \,.$$

$$d_1 f = 6a_1 s_1^5 + 4a_2 s_2 s_1^3 + 3a_3 s_3 s_1^2$$
$$+ 2a_4 s_2^2 s_1 + 2a_5 s_4 s_1 + a_6 s_3 s_2 + a_7 s_5 \,.$$

$$(d_1^2 + 2d_2) f = 2(15a_1 + a_2) s_1^4 + 2(6a_2 + 2a_4) s_2 s_1^2$$
$$+ 2(3a_3 + a_6) s_3 s_1 + 2(a_4 + 3a_9) s_2^2 + 2(a_5 + a_8) s_4$$

$$(d_1^3 + 6d_1 d_2 + 6d_3) f = 6(20a_1 + 4a_2 + a_3) s_1^3$$
$$+ 6(4a_2 + 4a_4 + a_7) s_2 s_1 + 6(a_3 + a_6 + 2a_{10}) s_3 \,.$$

$$(d_1^6 + 30d_1^4 d_2 + 120d_1^3 d_3 + 180d_1^2 d_2^2 + 360 d_1^2 d_4 + 720 d_1 d_2 d_3$$
$$+ 720 d_1 d_5 + 720 d_2 d_4 + 120 d_2^3 + 360 d_3^2 + 720 d_6) f$$
$$= 720(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11}).$$

1. $(6) = s_6 \,.$

2. $(3^2) = (s_3^2 - s_6)/2$, since operating on this symmetric function with D_1 . 2 D_2 . 6 D_3 . 720 D_6 and comparing coefficients of the symmetric functions thus obtained with the result of the operations on f above, gives

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$$
$$a_{10} = 1/2 \,, \quad a_{11} = -1/2 \,.$$

3. $(2^3) = (s_2^3 - 3s_4 s_2 + 2s_6)/6$. For operating with D_1 and comparing coefficients of $D_1 (2^3) = 0$ with $d_1 f$ above gives $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$.

Similarly, operating with 2 D_2 gives $a_8 = 1/6$, $a_9 = -1/2$. Operating with $6D_3$ gives $a_{10} = 0$. Operating with 720 D_6 gives $a_{11} = 1/3$.

4. $(42) = s_4 s_2 - s_6$.

5. $(51) = s_5 s_1 - s_6$.

6. $(321) = s_3 s_2 s_1 - s_5 s_1 - s_4 s_2 - s_3^2 + 2s_6$.

7. $(41^2) = (s_4 s_1^2 - 2s_5 s_1 - s_4 s_2 + 2s_6)/2$.

8. $(2^2 1^2) = (s_2^2 s_1^2 - s_4 s_1^2 - 4s_3 s_2 s_1 + 4 s_5 s_1$

$$+5s_4 s_2 - s_2^3 + 2s_3^2 - 6s_6)/4.$$

9. $(31^3) = (s_3 s_1^3 - 3s_4 s_1^2 - 3s_3 s_2 s_1$

$$+6s_5 s_1 + 3s_4 s_2 + 2s_3^2 - 6s_6)/6.$$

10. $(21^4) = (s_2 s_1^4 - 4s_3 s_1^3 - 6s_2^2 s_1^2 + 12 s_4 s_1^2 + 20 s_3 s_2 s_1$

$$-16 s_5 s_1 - 18 s_4 s_2 + 3s_2^3 + 8s_3^2 + 16 s_6)/24.$$

11. $(1^6) = (s_1^6 - 15 s_2 s_1^4 + 40 s_3 s_1^3 + 45 s_2^2 s_1^2 - 90 s_4 s_1^2 - 120 s_3 s_2 s_1$

$$+144 s_5 s_1 + 90 s_4 s_2 - 15 s_2^3 + 40 s_3^2 - 120 s_6)/720.$$

Note that only the four operator relations given above have been used in finding the expressions for all eleven symmetric functions of weight 6.

CHAPTER III

SYMMETRIC FUNCTIONS OF SYMMETRIC FUNCTIONS. A PROBLEM IN SAMPLING

10. Consider again the n variates $x_1, x_2, \cdots, x_n$. Let $s_{1:x}, s_{2:x}, s_{3:x}, \cdots\cdots, s_{t:x}$ denote the power sums, the x subscript being introduced here to keep in the foreground the fact that the summation is with respect to x. Now raise each variate to the power m, where m is a positive integer. Thus a new set of variates is produced, viz. $x_1^m, x_2^m, \cdots, x_n^m$. Suppose now that samples, each containing r variates, $(r \leqq n)$, are drawn in all possible ways from these n new variates. Obviously there will be ${}_nC_r$ samples. Denote[1] them as follows:

$$z_1 = x_1^m + x_2^m + \cdots\cdots + x_r^m = \sum^{r:1} x^m,$$

$$z_2 = x_2^m + x_3^m + \cdots\cdots + x_{r+1}^m = \sum^{r:2} x^m,$$

$$z_3 = x_3^m + x_4^m + \cdots\cdots + x_{r+2}^m = \sum^{r:3} x^m,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$z_{{}_nC_r} = x_{n-r+1}^m + \cdots \quad \cdot + x_n^m = \sum^{r:{}_nC_r} x^m,$$

[1]Notation suggested by Editorial, *Annals of Mathematical Statistics*, 1 (1930), page 100.

where $z_i = \sum^{r:i} x^m$ is the sum of the r variates appearing in the i'th sample.

Further, let

$$s_{1:z} = \sum_{i=1}^{{}_nC_r} z_i\,,$$

$$s_{2:z} = \sum_{i=1}^{{}_nC_r} z_i^2\,,$$

$$s_{3:z} = \sum_{i=1}^{{}_nC_r} z_i^3\,,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$s_{t:z} = \sum_{i=1}^{{}_nC_r} z_i^t$$

represent the power sums with respect to z.

Now, since each z_i is a symmetric function of certain of the $x_1^m, x_2^m, \cdot \cdot \cdot \cdot \cdot, x_n^m$, any symmetric function of the z_i is a symmetric function of symmetric functions. The situation here is then considerably more complex than in the preceding chapters. The problem now is to express any symmetric function of the z_i in terms of the power sums with respect to x. It is not difficult to imagine how much more complicated and tedious the direct computation is here than in the problem already dealt with. But these symmetric functions, particularly the power sums with respect to z, play such an important rôle in the theory of sampling that it is now proposed to develop a differential operator method for expressing symmetric functions of the z, in terms of the power sums with respect to x.

On account of the presence here of symmetric functions of both x and z it is necessary to modify the notation of the preceding chapters. Let $(a^{\alpha} b^{\beta} c^{\gamma} \cdot \cdot)_x$ be the general symmetric function with respect to x and $(a^{\alpha} b^{\beta} c^{\gamma} \cdot \cdot)_z$ the same general symmetric function with respect to z. Under this notation the power sums with respect to x may be written $(1)_x$, $(2)_x$, . . ., $(t)_x$, and the power sums with respect to z become $(1)_z$, $(2)_z$, . . ., $(t)_z$.

11. Case $m=1$.

Consider first of all the case of samples when $m=1$. In developing an operator method for expressing $(a^{\alpha} b^{\beta} c^{\gamma} \cdot \cdot)_z$ in terms of the power sums with respect to x it will not be necessary to deal with this general case. For the operators developed in chapter II will express $(a^{\alpha} b^{\beta} c^{\gamma} \cdot \cdot \cdot)_z$ in terms of the power sums with respect to z. Hence all that is required is an operator method for expressing the power sums with respect to z in terms of the power sums with respect to x.

That it is possible to express the power sums with respect to z in terms of the power sums with respect to x can be demonstrated by direct methods. Recall the theorem stated at the close of chapter I and note also that in any power sum with respect to z each term is a symmetric function (a power sum in fact) of certain of the x_1, x_2, . . ., x_n. Each x enters exactly the same as every other x and the power sum with respect to x is unaltered by interchanges or permutations of x_1, x_2, . . ., x_n. Hence the symmetric function with respect to z is also a symmetric function with respect to x and therefore can be expressed as a rational, integral, algebraic function of the power sums with respect to x. Moreover, as before, each term in the rational, integral, algebraic function of the power sums with respect to x will be of total weight w if the symmetric function of the z_i is of weight w; that is, the symmetric function is of the same weight in x as it is in z. This last conclusion follows directly from the definition of the z_i.

Although the problem here is more complicated than that in chapter II, nevertheless the approach to the problem in that case suggests a beginning here. Let

$$(w)_z = f(s_{1:x}, s_{2:x}, \cdots, s_{w:x}),$$

where f is a rational, integral, algebraic function of the power sums with respect to x. Since $(w)_z$ is of weight w, no power sum of weight greater than w can appear in f, i. e. no power sum higher than $s_{w:x}$.

Introducing a new variate $x_{n+1} = k$, as before, changes $f(s_{1:x}, s_{2:x}, \ldots, s_{w:x})$ into $f(s_{1:x} + k, s_{2:x} + k^2, \ldots, s_{w:x} + k^w)$. But it has already been shown that this new f may be written

$$f(s_{1:x} + k, s_{2:x} + k^2, \cdots, s_{w:x} + k^w) = (1 + kD_1 + k^2 D_2 + \cdots + k^w D_w) f$$

where, if $d_v = \partial / \partial s_{v:x}$, the relations between D and d are given by (1) and (2) of chapter II.

What is the effect of the new variate $x_{n+1} = k$ on $(w)_z$? If no further assumptions are made then obviously there will now be ${}_{n+1}C_r$ samples. The introduction of new samples complicates things and no operator relations are obtained. It would seem desirable to preserve the number of samples. This may be done by making suitable assumptions. Just as the new variate is arbitrarily introduced, so its behaviour in the sampling process may be arbitrarily determined in any way that will bring results. With this in mind, select any one of the original variates, say x_i. Let $qx_i = k = x_{n+1}$. Now assume that $k = qx_i$ is so related with x_i that in the sampling process every sample which contains x_i also contains qx_i, i. e. contains $(q+1)x_i$. In other words, in order to keep the number of samples the same, x_i and qx_i are always taken together in the samples.

Now each variate appears in $(1)_z$ exactly ${}_{n-1}C_{r-1}$

times. Hence $(q+1)x_i$ appears ${}_{n-1}C_{r-1}$ times in the new $(1)_z$. Therefore the new $(1)_z$ is equal to the original $(1)_z$ increased by $qx_i \cdot {}_{n-1}C_{r-1} = k \cdot {}_{n-1}C_{r-1}$. Similarly $(2)_z$ becomes $(2)_z + 2k(1)_{z'} + k^2 \cdot {}_{n-1}C_{r-1}$ where the prime above z indicates here, and in what follows, that $(t)_{z'}$ is obtaind from $(t)_z$ by replacing n and r by $n-1$ and $r-1$ respectively in the expression for $(t)_z$ in terms of the power sums with respect to x. For example, since $(1)_z = {}_{n-1}C_{r-1} \cdot s_{1:x}$, then

$$(1)_{z'} = {}_{n-2}C_{r-2} \cdot s_{1:x} \, .$$

Applying the multinomial theorem to the samples, the effect of the new variate may be written

$(1)_z$ becomes $(1)_z + k \cdot {}_{n-1}C_{r-1}$,

$(2)_z$ becomes $(2)_z + 2k(1)_{z'} + k^2 \cdot {}_{n-1}C_{r-1}$,

$(3)_z$ becomes $(3)_z + 3k(2)_{z'} + 3k^2(1)_{z'} + k^3 \cdot {}_{n-1}C_{r-1}$,

. . . .

$$\begin{aligned}(w)_z \text{ becomes } (w)_z &+ {}_wC_1 \cdot k(w-1)_{z'} + {}_wC_2 \cdot k^2(w-2)_{z'} \\ &+ \cdots \\ &+ {}_wC_v \cdot k^v (w-v)_{z'} + \cdots \\ &+ \cdots \\ &+ {}_wC_{w-1} \cdot k^{w-1}(1)_{z'} + k^w \cdot {}_{n-1}C_{r-1} \, .\end{aligned}$$

Now since $(w)_z = f$, therefore

$$
\begin{aligned}
(1+kD_1+k^2D_2+\cdots+k^wD_w)(w)_z = (w)_z &+ {}_wC_1\cdot k(w-1)_{z'} \\
&+ {}_wC_2\cdot k^2(w-2)_{z'} + {}_wC_3\cdot k^3(w-3)_{z'} \\
&+ \cdots \\
&+ {}_wC_v\cdot k^v(w-v)_{z'} + \cdots \\
&+ \cdots \\
&+ k^w\cdot {}_{n-1}C_{r-1}
\end{aligned}
$$

Equating coefficients of equal powers of k it follows that:

$$
\begin{aligned}
D_1(w)_z &= {}_wC_1\cdot(w-1)_{z'}, \\
D_2(w)_z &= {}_wC_2\cdot(w-2)_{z'}, \\
D_3(w)_z &= {}_wC_3\cdot(w-3)_{z'}, \\
&\cdots \\
D_v(w)_z &= {}_wC_v\cdot(w-v)_{z'}, \\
&\cdots \\
D_{w-1}(w)_z &= {}_wC_{w-1}\cdot(1)_{z'}, \\
D_w(w)_z &= {}_{n-1}C_{r-1}, \\
D_u(w)_z &= 0 \quad \text{if } u>w.
\end{aligned}
$$

12. Before proceeding to the application of these operators it ought to be remarked that other sets of differential operators

can be developed. For instance, it is possible to develop a complete set of differential operator relations by adding k to each of the given variates. But the operators thus obtained are very cumbersome in comparison with those developed above. The statement made with respect to the operators developed in chapter II may be repeated here. There is every reason to believe that the differential operators developed here are the simplest that can be obtained for the problem.

13. The use of the operators developed in this chapter will now be illustrated by computing a few power sums with respect to z in terms of the power sums with respect to x.

1. Let $(1)_z = a_1 s_{1:x}$. Then

$$D_1 (1)_z = d_1 a_1 s_{1:x} \quad \text{, that is}$$

$$_{n-1}C_{r-1} = a_1 \,. \qquad \text{Hence}$$

$$(1)_z = {}_{n-1}C_{r-1} \; s_{1:x} \,.$$

2. Let

$$(2)_z = f = a_1 s^2_{1:x} + a_2 s_{2:x} \,.$$

$$D_1 (2)_z = d_1 f ,$$

$$2 (1)_{z'} = d_1 f ,$$

$$2 \cdot {}_{n-2}C_{r-2} \cdot s_{1:x} = 2 a_1 s_{1:x} , \quad a_1 = {}_{n-2}C_{r-2} \,.$$

$$2!\, D_2 (2)_z = (d_1^2 + 2 d_2) f ,$$

$$2 \; {}_{n-1}C_{r-1} = 2(a_1 + a_2), \quad a_2 = {}_{n-1}C_{r-1} - {}_{n-2}C_{r-2} \,,$$

$$(2)_z = {}_{n-2}C_{r-2} \; s^2_{1:x} + ({}_{n-1}C_{r-1} - {}_{n-2}C_{r-2}) \, s_{2:x} \,.$$

3. Let

$$(3)_z = f = a_1 s_{1:x}^3 + a_2 s_{1:x} s_{2:x} + a_3 s_{3:x}.$$

$$D_1 (3)_z = d_1 f,$$

$$3(2)_{z'} = d_1 f,$$

$$3 \cdot {}_{n-3}C_{r-3} \cdot s_{1:x}^2 + 3({}_{n-2}C_{r-2} - {}_{n-3}C_{r-3}) s_{2:x}$$

$= 3a_1 s_{1:x}^2 + a_2 s_{2:x}$, hence $a_1 = {}_{n-3}C_{r-3}$,

$$a_2 = 3({}_{n-2}C_{r-2} - {}_{n-3}C_{r-3}).$$

$$3!\, D_3 (3)_z = (d_1^3 + 6 d_1 d_2 + 6 d_3) f,$$

$$6 \cdot {}_{n-1}C_{r-1} = 6(a_1 + a_2 + a_3),$$

$$a_3 = {}_{n-1}C_{r-1} - 3 \cdot {}_{n-2}C_{r-2} + 2 \cdot {}_{n-3}C_{r-3}.$$

$$(3)_z = {}_{n-3}C_{r-3} \cdot s_{1:x}^3 + 3({}_{n-2}C_{r-2} - {}_{n-3}C_{r-3}) s_{1:x} s_{2:x}$$

$$+({}_{n-1}C_{r-1} - 3 \cdot {}_{n-2}C_{r-2} + 2 \cdot {}_{n-3}C_{r-3}) s_{3:x}.$$

4. Let

$$(4)_z = f = a_1 s_{1:x}^4 + a_2 s_{1:x}^2 s_{2:x} +$$

$$a_3 s_{1:x} s_{3:x} + a_4 s_{2:x}^2 + a_5 s_{4:x}.$$

$$D_1 (4)_z = d_1 f,$$

$$4(3)_{z'} = d_1 f,$$

$$4[{}_{n-4}C_{r-4} s_{1:x}^3 + 3({}_{n-3}C_{r-3} - {}_{n-4}C_{r-4}) s_{1:x} s_{2:x}$$

$$+({}_{n-2}C_{r-2} - 3\cdot{}_{n-3}C_{r-3} + 2\cdot{}_{n-4}C_{r-4})s_{3:x}]$$

$$= 4a_1 s_{1:x} + 2a_2 s_{1:x} s_{2:x} + a_3 s_{3:x},$$

$$a_1 = {}_{n-4}C_{r-4}, \quad a_2 = 6({}_{n-3}C_{r-3} - {}_{n-4}C_{r-4}),$$

$$a_3 = 4({}_{n-2}C_{r-2} - 3\cdot{}_{n-3}C_{r-3} + 2\cdot{}_{n-4}C_{r-4}).$$

$$2!\,D_2(4) = (d_1^2 + 2d_2)f,$$

$$12(2)_{x'} = (d_1^2 + 2d_2)f,$$

$$12\cdot{}_{n-3}C_{r-3}\cdot s_{1:x}^2 + 12({}_{n-2}C_{r-2} - {}_{n-3}C_{r-3})s_{2:x}$$

$$= 2(6a_1 + a_2)s_{1:x}^2 + 2(a_2 + 2a_4)s_{2:x},$$

$$a_4 = 3({}_{n-2}C_{r-2} - 2\cdot{}_{n-3}C_{r-3} + {}_{n-4}C_{r-4}).$$

$$4!\,D_4(4)_x = (d_1^4 + 12d_1^2 d_2 + 24d_1 d_3 + 12d_2^2 + 24d_4)f,$$

$$24\cdot{}_{n-1}C_{r-1} = 24(a_1 + a_2 + a_3 + a_4 + a_5),$$

$$a_5 = {}_{n-1}C_{r-1} - 7\cdot{}_{n-2}C_{r-2} + 12\cdot{}_{n-3}C_{r-3} - 6\cdot{}_{n-4}C_{r-4}.$$

$$(4)_x = {}_{n-4}C_{r-4}\cdot s_{1:x}^4 + 6({}_{n-3}C_{r-3} - {}_{n-4}C_{r-4})s_{1:x}^2 s_{2:x}$$

$$+4({}_{n-2}C_{r-2} - 3\cdot{}_{n-3}C_{r-3} + 2\cdot{}_{n-4}C_{r-4})s_{1:x} s_{3:x}$$

$$+3({}_{n-2}C_{r-2} - 2\cdot{}_{n-3}C_{r-3} + {}_{n-4}C_{r-4})s_{2:x}^2$$

$$+({}_{n-1}C_{r-1} - 7\cdot{}_{n-2}C_{r-2} + 12\,{}_{n-3}C_{r-3}$$

$$-6\cdot{}_{n-4}C_{r-4})s_{4:x}$$

14. Now consider the case where m is any positive integer. Write $x_i^m = y_i$. The operators developed in this chapter will express any power sum with respect to z in terms of the power sums with respect to y, i. e. in terms of $s_{1:y}$, $s_{2:y}$, $s_{3:y}$, ... But obviously $s_{v:y} = s_{mv:x}$ and hence the operators of this chapter will express any symmetric function which is a power sum with respect to z in terms of power sums with respect to x, viz. in terms of $s_{m:x}$, $s_{2m:x}$, $s_{3m:x}$, . . . where m is a positive integer. Hence the operators developed in chapters II and III will express any symmetric function of z_i, $i = 1, 2, \ldots, {}_nC_r$, $z_i = \sum^{r:i} x^m$, m a positive integer, in terms of power sums with respect to x. In particular

$$(1)_z = {}_{n-1}C_{r-1} \cdot s_{m:x},$$

$$(2)_z = {}_{n-2}C_{r-2} \cdot s_{m:x}^2 + ({}_{n-1}C_{r-1} - {}_{n-2}C_{r-2}) s_{2m:x}$$

$$(3)_z = {}_{n-3}C_{r-3} \cdot s_{m:x}^3 + 3({}_{n-2}C_{r-2} - {}_{n-3}C_{r-3}) s_{m:x} s_{2m:x}$$

$$+ ({}_{n-1}C_{r-1} - 3 \cdot {}_{n-2}C_{r-2} + 2 \cdot {}_{n-3}C_{r-3}) s_{3m:x}$$

15. Consider again the case $m = 1$. $\rho_1 = {}_{n-1}C_{r-1}$, $\rho_2 = {}_{n-2}C_{r-2}$, ..., $\rho_k = {}_{n-k}C_{r-k}$, $k \leq r$. Then[1]

$$s_{1:z} = \rho_1 s_{1:x},$$

$$s_{2:z} = \rho_2 s_{1:x}^2 + (\rho_1 - \rho_2) s_{2:x},$$

$$s_{3:z} = \rho_3 s_{1:x}^3 + 3(\rho_2 - \rho_3) s_{1:x} s_{2:x} + (\rho_1 - 3\rho_2 + 2\rho_3) s_{3:x},$$

$$s_{4:z} = \rho_4 s_{1:x}^4 + 6(\rho_3 - \rho_4) s_{1:x}^2 s_{2:x}$$

[1]Notation suggested in Editorial, *Annals of Mathematical Statistics*, 1 (1930), page 104.

$$+4(\rho_2-3\rho_3+2\rho_4)s_{1:x}s_{3:x}+3(\rho_2-2\rho_3+\rho_4)s_{2:x}^2$$

$$+(\rho_1-7\rho_2+12\rho_3-6\rho_4)s_{4:x},$$

etc.

The question as to whether the coefficients in the above expressions follow any simple law now arises. Instead of $\rho_k = {}_{n-k}C_{r-k}$, $k \leqq r$, write $\rho^k = {}_{n-k}C_{r-k}$, $k \leqq r$. Let

$$P_1(\rho)=\rho,$$

$$P_2(\rho)=\rho-\rho^2,$$

$$P_3(\rho)=\rho-3\rho^2+2\rho^3,$$

$$P_4(\rho)=\rho-7\rho^2+12\rho^3-6\rho^4,$$

etc.

Further, let P_t be the expression obtained from $P_t(\rho)$ by going back to subscripts instead of exponents. Then

$$P_1=\rho_1,$$

$$P_2=\rho_1-\rho_2,$$

$$P_3=\rho_1-3\rho_2+2\rho_3,$$

$$P=\rho_1-7\rho_2+12\rho_3-6\rho_4,$$

etc.

The expressions for $s_{1:x}$, $s_{2:x}$, ... may now be written:

$$s_{1:z} = P_1 s_{1:x},$$

$$s_{2:z} = P_1^2 s_{1:x}^2 + P_2 s_{2:x},$$

$$s_{3:z} = P_1^3 s_{1:x}^3 + 3P_1P_2 s_{1:x} s_{2:x} + P_3 s_{3:x},$$

$$s_{4:z} = P_1^4 s_{1:x}^4 + 6P_1^2P_2 s_{1:x}^2 s_{2:x} + 4P_1P_3 s_{1:x} s_{3:x} + 3P_2^2 s_{2:x}^2 + P_4 s_{4:x},$$

etc.

where, of course, $P_r P_s P_t \cdot \cdot \cdot \cdot$ is to be found by multiplying $P_r(\rho) P_s(\rho) P_t(\rho) \cdot \cdot \cdot \cdot \cdot$ and then changing the exponents in the result into subscripts. e. g. To find P_2^2 first find $P_2^2(\rho) = (\rho - \rho^2)^2 = \rho^2 - 2\rho^3 + \rho^4$ and then change the exponents into subscripts, obtaining $P_2^2 = \rho_2 - 2\rho_3 + \rho_4$.

One further step is necessary in order to emphasize the law for the formation of these expressions for $s_{1:z}$, $s_{2:z}$, $\cdot \cdot \cdot \cdot$

They may be written in the form

$$s_{1:z} = 1!\left(\frac{P_1 s_{1:x}}{1!}\right),$$

$$s_{2:z} = 2!\left(\frac{P_1^2 s_{1:x}^2}{2!} + \frac{P_2 s_{2:x}}{2!}\right),$$

$$s_{3:z} = 3!\left(\frac{P_1^3 s_{1:x}^3}{3!} + \frac{P_1 P_2 s_{1:x} s_{2:x}}{1!\,2!} + \frac{P_3 s_{3:x}}{3!}\right),$$

$$s_{4:z} = 4!\left(\frac{P_1^4 s_{1:x}^4}{4!} + \frac{P_1^2 P_2 s_{1:x}^2 s_{2:x}}{2!\,1!\,2!} + \frac{P_1 P_3 s_{1:x} s_{3:x}}{1!\,3!} + \frac{P_2^2 s_{2:x}^2}{2!(2!)^2} + \frac{P_4 s_{4:x}}{4!}\right)$$

$$s_{t:z} = t!\sum \frac{p_i^I p_j^J p_k^K \cdots s_i^I s_j^J s_k^K \cdots}{(i!)^I (j!)^J (k!)^K \cdots I!\,J!\,K! \cdots}$$

After computing by the direct method the first eight moments, under the assumption that $s_{1:x} = M_x = 0$, an article[1] which appeared in the *Annals of Mathematical Statistics* gives the following law for the formation of the functions $P_t(\rho)$ for $t = 1, 2, \cdots, 8$: If $c_{i:t}$ is the coefficient of ρ^i in the expression for $P_t(\rho)$, then

$$c_{i:t} = i c_{i:t-1} - (i-1) c_{i-1:t-1} \, .$$

This is equivalent to saying that

$$P_t(\rho) = \sum_{m=0}^{t-1} \left[(m+1) c_{m+1:t-1} - m c_{m:t-1} \right] \rho^{m+1} .$$

That this law holds for all values of $t = 1, 2, \ldots$ is now easily established. For if it be assumed that this law holds for the expression for $(t-1)_z$ in terms of the power sums with respect to x, then it holds also for $(t)_z$ because the operators $D_1, D_2, \cdots, D_t$ and the equivalent operators in terms of $d_1, d_2, \cdots, d_t$ will express $(t)_z$ in terms of the power sums with respect to z and of weight less than t. And the coefficients of the terms in the expression for $(t)_z$ are seen to depend only on the coefficients of these power sums of weight less than t. e. g. Suppose the law holds for $t = 1, 2$. Let

$$(3)_z = Q_1 s_{1:x}^3 + Q_2 s_{1:x} s_{2:x} + Q_3 s_{3:x} \, .$$

---

[1] Editorial, *Annals of Mathematical Statistics*, 1 (1930), page 107.

Operate on the left with D_1 and on the right with d_1.

$$3(2)_{z'} = 3Q_1 s^2_{1:x} + Q_2 s_{2:x}\,, \qquad \text{hence}$$

$$Q_1 = P_1^{\,3}, \quad Q_2 = 3P_1P_2\,.$$

Operate on the left with $6D_3$ and on the right with $(d_1^{\,3} + 6d_1d_2 + 6d_3)$. Then

$$6P_1 = 6(Q_1 + Q_2 + Q_3), \qquad \text{therefore}$$

$$Q_3 = P_1 - Q_1 - Q_2\,,$$

$$= P_1 - P_1^{\,3} - 3P_1P_2$$

But

$$P_1(\rho) - P_1^{\,3}(\rho) - 3P_1(\rho)P_2(\rho) = \rho - \rho^3 - 3\rho(\rho - \rho^2)$$

$$= \rho - 3\rho^2 + 2\rho^3$$

$$= P_3(\rho).$$

Hence

$$Q_3 = P_3$$

16. Consider the functions $P_t(\rho),\ t = 1, 2, \cdots, 10, \cdots$

$$P_1(\rho) = \rho,$$

$$P_2(\rho) = \rho - \rho^2,$$

$$P_3(\rho) = \rho - 3\rho^2 + 2\rho^3;$$

$$P_4(\rho) = \rho - 7\rho^2 + 12\rho^3 - 6\rho^4;$$

$$P_5(\rho) = \rho - 15\rho^2 + 50\rho^3 - 60\rho^4 + 24\rho^5;$$

$$P_6(\rho) = \rho - 31\rho^2 + 180\rho^3 - 390\rho^4 + 360\rho^5 - 120\rho^6,$$

$$P_7(\rho) = \rho - 63\rho^2 + 602\rho^3 - 2100\rho^4$$

$$+ 336\rho^5 - 2520\rho^6 + 720\rho^7,$$

$$P_8(\rho) = \rho - 127\rho^2 + 1932\rho^3 - 10206\rho^4$$

$$+ 25200\rho^5 - 31920\rho^6 + 20160\rho^7 - 5040\rho^8;$$

$$P_9(\rho) = \rho - 255\rho^2 + 6050\rho^3 - 46620\rho^4 + 166824\rho^5$$

$$- 317520\rho^6 + 332640\rho^7 - 181440\rho^8 + 40320\rho^9;$$

$$P_{10}(\rho) = \rho - 511\rho^2 + 18860\rho^3 - 204630\rho^4$$

$$+ 1020600\rho^5 - 2739240\rho^6 + 3329424\rho^7$$

$$- 3780000\rho^8 + 1814400\rho^9 + 362880\rho^{10}.$$

Those who are familiar with the calculus of finite differences will recognize the coefficients in the above expressions, neglecting their signs, as the numbers appearing in the table of values of

$$\Delta^m 1^n = (\Delta^m x^n)_{x=1}$$

If $u(x)$ and $v(x)$ are functions of x then

$$\Delta^n u(x).\, v(x) = v(x)\Delta^n u(x) + {}_nC_1 \cdot \Delta\, v(x)\, \Delta^{n-1} u(x+1)$$

$$+ {}_nC_2 \cdot \Delta^2 v(x) \cdot \Delta^{n-2} u(x+2) + \cdots$$

Now $x^n = x \cdot x^{n-1}$. Hence, letting $v(x) = x$ and $u(x) = x^{n-1}$; $\Delta^m x^n = \Delta^m x \cdot x^{n-1} = x\,\Delta^m x^{n-1}$ $+ m\,\Delta^{m-1}(x+1)^{n-1}$ and all the other terms vanish. Also $(x+1)^{n-1} = E x^{n-1} = (1+\Delta)x^{n-1}$. Therefore

$$\Delta^m x^n = x\Delta^m x^{n-1} + m\,\Delta^{m-1}(1+\Delta)x^{n-1}$$

$$= x\Delta^m x^{n-1} + m(\Delta^m x^{n-1} + \Delta^{m-1} x^{n-1}).$$

It is now possible to write

$$P_t(\rho) = \sum_{m=0}^{t-1} (-1)^m (\Delta^m 1^{t-1}) \cdot \rho^{m+1}.$$

To show that this law is equivalent to the law given above, viz:

$$P_t(\rho) = \sum_{m=0}^{t-1} \Big[(m+1)c_{m+1:\,t-1} - mc_{m:t-1}\Big]\rho^{m+1}$$

assume they are equivalent for $P_t(\rho)$ and show that they are then equivalent for $P_{t+1}(\rho)$ That is, assume

$$\sum_{m=0}^{t-1}(-1)^m\left[(m+1)\Delta^m 1^{t-2}+m\Delta^{m-1}1^{t-2}\right]\rho^{m+1}$$

$$=\sum_{m=0}^{t-1}\left[(m+1)c_{m+1:t-1}-mc_{m:t-1}\right]\rho^{m+1}$$

Then
$(-1)^m\Delta^m 1^{t-2}=c_{m+1:t-1}$ and $(-1)^{m-1}\Delta^{m-1}1^{t-2}$ $c_{m:t-1}$ For $P_{t+1}(\rho)$ the two laws are equivalent if

$$\sum_{m=0}^{t}(-1)^m\left[(m+1)\Delta^m 1^{t-1}+m\Delta^{m-1}1^{t-1}\right]\rho^{m+1}$$

$$=\sum_{m=0}^{t}\left[(m+1)c_{m+1:t}-mc_{m:t}\right]\rho^{m+1}$$

But this is true since if $c_{m+1:t-1}=(-1)^m\Delta^m 1^{t-2}$
then

$$\begin{aligned}c_{m+1:t}&=(m+1)c_{m+1:t-1}-mc_{m:t-1}\\&=(-1)^m\left[(m+1)\Delta^m 1^{t-2}+m\Delta^{m-1}1^{t-2}\right]\\&=(-1)^m\Delta^m 1^{t-1}.\end{aligned}$$

Similarly, since $c_{m:t-1}=(-1)^{m-1}\Delta^{m-1}1^{t-2}$, then

$$c_{m:t}=(-1)^{m-1}\Delta^{m-1}1^{t-1}.$$

17. Since $\Delta^m 1^n=\Delta^m(1+\Delta)0^n$, it is possible to write

$$P_t(\rho) = \sum_{m=0}^{t-1} (-1)^m \Delta^m \, 1^{t-1} \cdot \rho^{m+1}$$
$$= \sum_{m=0}^{t-1} (-1)^m \Delta^m (1+\Delta) x^{t-1} \Big]_{x=0} \rho^{m+1}$$

The latter expression on the right suggests that $P_t(\rho)$ may be expressed as a function of ρ and x, with x set equal to zero for each particular value of t. Suppose that $F_t(x, \rho)_{x=0}$ is such a function. Obviously F can be neither a polynomial in x nor a rational function of any kind in x; for setting x equal to zero would show that F would have the same value for all values of t. The nature of the expression suggests that x enters F only as a variable with respect to which differentiation is to be carried out, x then being set equal to zero. There are two main reasons for this assumption. First of all, since x enters the difference expression only as a variable with respect to which differencing is performed, x being set equal to zero after each differencing, the guess is that x enters F only as a variable with respect to which differentiation is to be carried out, x being set equal to zero after each differentiation. Besides this there is the intimate relation between Δ and d/dx. For instance, $1+\Delta = e^{d/dx}$, $d/dx = \log(1+\Delta)$ and hence Δ^n can be replaced by a function of the n'th degree in d/dx and vice versa. Further, since the difference expression contains Δ^t it is reasonable to try to express F as a function involving d^t/dx^t. Now let $F_t(x,\rho)_{x=0} = (d^t/dx^t).\phi(x,\rho)]_{x=0}$. Since t differentiations, none of which are to give results identically zero, are to be carried out then ϕ cannot be a rational function of x. Also functions which involve the possibility of the derivative being infinite are excluded. Hence try a transcendental function of x and ρ. The exponential function will not satisfy the conditions. Try

$\phi(x,\rho) = \log\ f(x,\rho)$. And again f cannot be a rational function of x. Suppose f is an exponential function of x, say $f(x,\rho) = R(\rho, e^x)$. Then

$$P_t(\rho) = \frac{d^t}{dx^t} \cdot \log\ R(\rho, e^x)\Big]_{x=0}$$

The simplest case would be $R(\rho, e^x) = \rho e^x$. But this does not satisfy $P_1(\rho) = \rho$. Nor does $R(\rho, e^x) = \rho e^x + \rho$; nor $R(\rho, e^x) = \rho e^x - \rho$. But $R(\rho, e^x) = \rho e^x + 1 - \rho$ does satisfy the conditions since it has been shown[1] that

$$\frac{d^t}{dx^t} \cdot \log\ (\rho e^x + 1 - \rho)\Big]_{x=0}$$

satisfies the law $c_{i:t} = i c_{i:t-1} - (i-1) c_{i-1:t-1}$, where $c_{i:t}$ is the coefficient of ρ^i in $P_t(\rho)$.

Hence $P_t(\rho)$ can be written in the three equivalent forms for all values of t:

$$P_t(\rho) = \sum_{m=0}^{t-1} (-1)^m (\Delta^m 1^{t-1}) \cdot \rho^{m+1}$$

$$P_t(\rho) = \sum_{m=0}^{t-1} \Big[(m+1) c_{m+1:t-1} - m c_{m:t-1} \Big] \rho^{m+1}.$$

$$P_t(\rho) = \frac{d^t}{dx^t} \log\ (\rho e^x + 1 - \rho)\Big]_{x=0}$$

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[1]Editorial, *Annals of Mathematical Statistics*, 1 (1930), pages 107, 108. Also see remark on "Sampling Polynomials," page 120.

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