ON SYMMETRIC-TENSOR-VALUED ISOTROPIC FUNCTIONS OF TWO SYMMETRIC TENSORS*

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1. Introduction. We show that any symmetric-tensor-[1]valued isotropic polynomial function $\mathbf{T}(\mathbf{A}, \mathbf{B})$ of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ is expressible as

$$\begin{aligned}\mathbf{T}(\mathbf{A}, \mathbf{B}) &\\ = (h_0 + h_1 I_{10} + h_2 I_{10}^2)\mathbf{I} &+ (h_3 + h_4 I_{10})\mathbf{A} + (h_5 + h_6 I_{10})\mathbf{B} \\ &+ (h_7 + h_8 I_{10})\mathbf{A}^2 + (h_9 + h_{10} I_{10})(\mathbf{AB} + \mathbf{BA}) + (h_{11} + h_{12} I_{10})\mathbf{B}^2 \\ &+ (h_{13} + h_{14} I_{10})(\mathbf{A}^2\mathbf{B} + \mathbf{BA}^2) + (h_{15} + h_{16} I_{10})(\mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}) + h_{17}(\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2) \quad (1.1)\end{aligned}$$

where $h_0, \cdots, h_{17}$ are polynomials in the isotropic invariants $I_1, \cdots, I_9$ defined by

$$I_1, \cdots, I_9 = \text{tr } \mathbf{A}, \text{tr } \mathbf{B}, \text{tr } \mathbf{A}^2, \text{tr } \mathbf{AB}, \text{tr } \mathbf{B}^2, \text{tr } \mathbf{A}^3, \text{tr } \mathbf{A}^2\mathbf{B}, \text{tr } \mathbf{AB}^2, \text{tr } \mathbf{B}^3 \quad (1.2)$$

and where

$$I_{10} = \text{tr } \mathbf{A}^2\mathbf{B}^2. \quad (1.3)$$

It has been shown by Rivlin [1] that any symmetric-tensor-valued isotropic polynomial function of the symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ is expressible as

$$\begin{aligned}\mathbf{T}(\mathbf{A}, \mathbf{B}) = \gamma_0\mathbf{I} &+ \gamma_1\mathbf{A} + \gamma_2\mathbf{B} + \gamma_3\mathbf{A}^2 + \gamma_4(\mathbf{AB} + \mathbf{BA}) \\ &+ \gamma_5\mathbf{B}^2 + \gamma_6(\mathbf{A}^2\mathbf{B} + \mathbf{BA}^2) + \gamma_7(\mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}) + \gamma_8(\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2) \quad (1.4)\end{aligned}$$

where the γ_k are polynomials in the isotropic invariants $I_1, \cdots, I_{10}$ defined by (1.2) and (1.3). There are a number of redundant terms in the expression (1.4). In Sec. 2 we outline the procedures employed to generate the matrix identities which enable us to eliminate these redundant terms and thus to proceed from the expression (1.4) for $\mathbf{T}(\mathbf{A}, \mathbf{B})$ to that defined by (1.1), $\cdots$, (1.3). In Sec. 3 we show that there are no redundant terms in the expression for $\mathbf{T}(\mathbf{A}, \mathbf{B})$ given by (1.1) and hence no further simplification of the expression for $\mathbf{T}(\mathbf{A}, \mathbf{B})$ is possible.

2. Reduction procedure. We may also write the expression (1.1) in the form

$$\mathbf{T}(\mathbf{A}, \mathbf{B}) = \sum_{i,j,k} a_{ijk}\mathbf{H}_{ijk} \quad (2.1)$$

where the a_{ijk} are constants and where $\mathbf{H}_{ijk}$ $(k = 1, 2, \cdots)$ denote the matrices of degree i, j in $\mathbf{A}, \mathbf{B}$ which appear in the expansion of (1.1); for example,

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[1] "Tensor" means three-dimensional second-order tensor.

$$\mathbf{H}_{00k} = \{\mathbf{I}\}, \qquad \mathbf{H}_{10k} = \{\mathbf{A}, \mathbf{I} \operatorname{tr} \mathbf{A}\},$$
$$\mathbf{H}_{11k} = \{\mathbf{I} \operatorname{tr} \mathbf{AB}, \mathbf{I} \operatorname{tr} \mathbf{A} \operatorname{tr} \mathbf{B}, \mathbf{A} \operatorname{tr} \mathbf{B}, \mathbf{B} \operatorname{tr} \mathbf{A}, \mathbf{AB} + \mathbf{BA}\}, \cdots . \tag{2.2}$$

We now outline the procedures used to generate the matrix identities which enable us to reduce (1.4) to (1.1).

(i) Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ denote 3×3 skew-symmetric matrices. Then we have [2] the identity

$$\boldsymbol{\alpha\beta\gamma} - \boldsymbol{\gamma\beta\alpha} + \boldsymbol{\beta\gamma\alpha} - \boldsymbol{\alpha\gamma\beta} + \boldsymbol{\gamma\alpha\beta} - \boldsymbol{\beta\alpha\gamma} = 2\,\mathbf{I} \operatorname{tr} \boldsymbol{\alpha\beta\gamma}. \tag{2.3}$$

Substitution of

$$\boldsymbol{\alpha} = \mathbf{A}^2\mathbf{B} - \mathbf{BA}^2, \ \boldsymbol{\beta} = \mathbf{B}^2\mathbf{A} - \mathbf{AB}^2, \boldsymbol{\gamma} = \mathbf{AB} - \mathbf{BA} \tag{2.4}$$

into (2.3) will yield, upon application of various of the matrix identities given in [1], a matrix identity of the form

$$(\operatorname{tr} \mathbf{A}^2\mathbf{B}^2)(\mathbf{A}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{A}^2) = \sum_p \alpha_p \mathbf{H}_{44p} \,. \tag{2.5}$$

(ii) Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote skew-symmetric 3×3 matrices and let $\mathbf{c}$ denote a symmetric 3×3 matrix. Then we have [2] the identity[2]

$$\boldsymbol{\alpha\beta}\mathbf{c} + \mathbf{c}\boldsymbol{\beta\alpha} + \boldsymbol{\alpha}\mathbf{c}\boldsymbol{\beta} + \boldsymbol{\beta}\mathbf{c}\boldsymbol{\alpha} + \mathbf{c}\boldsymbol{\alpha\beta} + \boldsymbol{\beta\alpha}\mathbf{c} = (\boldsymbol{\alpha\beta} + \boldsymbol{\beta\alpha}) \operatorname{tr} \mathbf{c}$$
$$+ \mathbf{c} \operatorname{tr} \boldsymbol{\alpha\beta} + \mathbf{I}\,(2 \operatorname{tr} \boldsymbol{\alpha\beta}\mathbf{c} - \operatorname{tr} \mathbf{c} \operatorname{tr} \boldsymbol{\alpha\beta}). \tag{2.6}$$

We substitute

$$\boldsymbol{\alpha} = \boldsymbol{\beta} = (\mathbf{A}^2\mathbf{B} - \mathbf{BA}^2), \quad \mathbf{c} = \mathbf{AB}^2 + \mathbf{B}^2\mathbf{A} \tag{2.7}$$

into (2.6). The resulting identity may be reduced, upon application of (2.5) and identities found in [1], to a matrix identity of the form

$$(\operatorname{tr} \mathbf{A}^2\mathbf{B}^2)^2\mathbf{A} = \sum_p \beta_p \mathbf{H}_{54p} \,. \tag{2.8}$$

Interchanging **A** and **B** in (2.8) yields

$$(\operatorname{tr} \mathbf{A}^2\mathbf{B}^2)\mathbf{B} = \sum_p \gamma_p \mathbf{H}_{45p} \,. \tag{2.9}$$

(iii) We add the two identities obtained by multiplying the identity resulting from substitution of (2.7) into (2.6) on the right by **A** and on the left by **A**. Upon application of identities (2.5), (2.8), and identities appearing in [1], we obtain

$$(\operatorname{tr} \mathbf{A}^2\mathbf{B}^2)^2\mathbf{A}^2 = \sum_p \delta_p \mathbf{H}_{64p} \,. \tag{2.10}$$

Interchanging **A** and **B** in (2.10) yields

$$(\operatorname{tr} \mathbf{A}^2\mathbf{B}^2)^2\mathbf{B}^2 = \sum_p \epsilon_p \mathbf{H}_{46p} \,. \tag{2.11}$$

(iv) We add the two identities obtained by multiplying the identity resulting from substitution of (2.7) into (2.6) on the right by **B** and on the left by **B**. The resulting

[2] It has been noted by the referee that the identities (2.3) and (2.6) are special cases of an identity given by Spencer and Rivlin [3 (see equation (4.13) on page 55)].

identity may be reduced, upon application of (2.5), (2.8), (2.9) and identities appearing in [1], to a matrix identity of the form

$$(\text{tr } \mathbf{A}^2\mathbf{B}^2)^2(\mathbf{AB} + \mathbf{BA}) = \sum_p \lambda_p \mathbf{H}_{55p} . \tag{2.12}$$

(v) We add the two identities obtained by multiplying the identity resulting from substitution of (2.7) into (2.6) on the right by $\mathbf{B}^2$ and on the left by $\mathbf{B}^2$. The resulting identity may be reduced, upon application of (2.5), (2.8), $\cdots$ and identities appearing in [1], to a matrix identity of the form

$$(\text{tr } \mathbf{A}^2\mathbf{B}^2)^2(\mathbf{AB}^2 + \mathbf{B}^2\mathbf{A}) = \sum_p \mu_p \mathbf{H}_{56p} . \tag{2.13}$$

Interchanging **A** and **B** in (2.13) yields

$$(\text{tr } \mathbf{A}^2\mathbf{B}^2)^2(\mathbf{A}^2\mathbf{B} + \mathbf{BA}^2) = \sum_p \nu_p \mathbf{H}_{65p} . \tag{2.14}$$

(vi) We multiply the identity obtained by substituting (2.4) into (2.3) on the left by $\mathbf{A}^2\mathbf{B}^2$ and then take the trace of the resulting identity. This yields, upon application of identities given in [1],

$$(\text{tr } \mathbf{A}^2\mathbf{B}^2)^3 = \beta_0 + \beta_1 \text{ tr } \mathbf{A}^2\mathbf{B}^2 + \beta_2 (\text{tr } \mathbf{A}^2\mathbf{B}^2)^2 \tag{2.15}$$

where $\beta_0, \cdots, \beta_2$ are polynomials in the invariants $I_1, \cdots, I_9$ defined by (1.2).

With the aid of the identities (2.5), (2.8), $\cdots$, (2.15), we readily see that the expression (1.4) for $\mathbf{T}(\mathbf{A}, \mathbf{B})$ reduces to the expression for $\mathbf{T}(\mathbf{A}, \mathbf{B})$ defined by (1.1), (1.2) and (1.3). We note that in (2.5), (2.8), $\cdots$, (2.14), the $\alpha_p, \cdots, \nu_p$ are constants and the $\mathbf{H}_{ijp}$ are matrices defined as in (2.1) and (2.2).

3. Irreducibility of (1.1). Let g_{mn} denote the number of linearly independent symmetric-tensor-valued isotropic polynomial functions of degree m, n in **A**, **B**. Let p_{mn} denote the number of monomial terms of degree m, n in **A**, **B** appearing in the expression (1.1). We shall see below that $p_{mn} = g_{mn}$ for all m, n. Suppose that there are still redundant terms of degree m, n in **A**, **B** present in the expression (1.1). If we were to eliminate these, the resulting expression $\bar{\mathbf{T}}(\mathbf{A}, \mathbf{B})$ would contain $\bar{p}_{mn} < g_{mn}$ monomial terms of degree m, n in **A**, **B**. This would mean that not every symmetric-tensor-valued isotropic polynomial function of degree m, n in **A**, **B** would be expressible in the form $\bar{\mathbf{T}}(\mathbf{A}, \mathbf{B})$ and hence would also not be expressible in the form (1.1). However, we have shown above that every symmetric-tensor-valued isotropic polynomial function of **A**, **B** is expressible in the form (1.1). We conclude that by showing $p_{mn} = g_{mn}$ for all m and n we have verified that there are no redundant terms in the expression (1.1). Hence no further simplification of the expression for $\mathbf{T}(\mathbf{A}, \mathbf{B})$ is possible. We now proceed to show that $p_{mn} = g_{mn}$.

It may be shown from group-theoretic considerations that the number g_{mn} of linearly independent symmetric-tensor-valued isotropic polynomial functions of degree m and n respectively in the symmetric tensors **A** and **B** is given by the coefficient of $a^m b^n$ in the expansion of the function

$$G(a, b) = (2\pi)^{-1} \int_0^{2\pi} (e^{2i\theta} + e^{i\theta} + 2 + e^{-i\theta} + e^{-2i\theta})F(a, \theta)F(b, \theta)(1 - \cos\theta)\, d\theta \tag{3.1}$$

where

$$F(a, \theta) = [(1 - ae^{2i\theta})(1 - ae^{i\theta})(1 - a)^2(1 - ae^{-i\theta})(1 - ae^{-2i\theta})]^{-1}. \tag{3.2}$$

The integral (3.1) may be converted into a contour integral by setting $e^{i\theta} = z$ and evaluated by the method of residues. We obtain, after a lengthy computation,

$$G(a, b) = H(a, b)/K(a, b) \tag{3.3}$$

where

$$\begin{aligned} H(a, b) &= 1 + a + b + a^2 + ab + b^2 + a^2b + ab^2 + 2a^2b^2 + a^3b^2 + a^2b^3 \\ &\quad + a^4b^2 + a^3b^3 + a^2b^4 + a^4b^3 + a^3b^4 + a^4b^4, \\ K(a, b) &= (1 - a)(1 - a^2)(1 - a^3)(1 - b)(1 - b^2)(1 - b^3)(1 - ab)(1 - a^2b)(1 - ab^2). \end{aligned} \tag{3.4}$$

The coefficients h_i appearing in (1.1) are polynomials in $I_1, \cdots, I_9$ and are expressible as

$$h_i = h^{(i)}_{i_1 i_2 \cdots i_9} I_1^{i_1} I_2^{i_2} \cdots I_9^{i_9} \tag{3.5}$$

where the $h^{(i)}_{i_1 i_2 \cdots i_9}$ are constants. We note that

$$(1 + I_1 + I_1^2 + \cdots)(1 + I_2 + I_2^2 + \cdots) \cdots (1 + I_9 + I_9^2 + \cdots) \tag{3.6}$$

is equal to the sum of all of the monomial terms appearing in the expression (3.5) for h_i. The number of monomial terms of degree m, n in **A**, **B** in the expression (3.6) and hence also in (3.5) is given by the coefficient of $a^m b^n$ in the expression obtained from (3.6) by replacing $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9$ by a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3 respectively. This yields

$$\begin{aligned} &(1 + a + a^2 + \cdots)(1 + b + b^2 + \cdots) \cdots (1 + b^3 + b^6 + \cdots) \\ &\quad = [(1 - a)(1 - b)(1 - a^2)(1 - ab)(1 - b^2)(1 - a^3)(1 - a^2b)(1 - ab^2)(1 - b^3)]^{-1} \\ &\quad = [K(a, b)]^{-1} \end{aligned} \tag{3.7}$$

where we have employed formal expansions such as $(1 - a)^{-1} = 1 + a + a^2 + a^3 + \cdots$. We then see that the number of monomial terms of degree m, n in **A**, **B** appearing in the terms

$$h_0\mathbf{I},\ h_1(\mathrm{tr}\ \mathbf{A}^2\mathbf{B}^2)\mathbf{I},\ h_2(\mathrm{tr}\ \mathbf{A}^2\mathbf{B}^2)^2\mathbf{I},\ h_3\mathbf{A},\ h_4(\mathrm{tr}\ \mathbf{A}^2\mathbf{B}^2)\mathbf{A},\ \cdots \tag{3.8}$$

is given by the coefficient of $a^m b^n$ in the expansions of

$$\frac{1}{K(a, b)},\ \frac{a^2b^2}{K(a, b)},\ \frac{a^4b^4}{K(a, b)},\ \frac{a}{K(a, b)},\ \frac{a^3b^2}{K(a, b)},\ \cdots \tag{3.9}$$

respectively. From (1.1) and (3.9), we see that the number p_{mn} of monomial terms of degree m, n in **A**, **B** contained in the expression (1.1) for **T**(**A**, **B**) is given by the coefficient of $a^m b^n$ in the expansion of $G(a, b)$ defined by (3.3) and (3.4). Since this also gives the number of linearly independent symmetric-tensor-valued isotropic polynomial functions of degree m, n in **A**, **B**, we have verified that $p_{mn} = g_{mn}$.

References

[1] R. S. Rivlin, J. Rat. Mech. Anal. **4**, 681 (1955)
[2] G. F. Smith and R. S. Rivlin (to be published)
[3] A. J. M. Spencer and R. S. Rivlin, Arch. Rat. Mech. Anal. **9**, 45 (1962)