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ON SYMMETRY OF THE (STRONG) BIRKHOFF–JAMES ORTHOGONALITY IN HILBERT C^*-MODULES

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Abstract. In this note, we prove that the Birkhoff–James orthogonality, as well as the strong Birkhoff–James orthogonality, is a symmetric relation in a full Hilbert $\mathcal{A}$-module V if and only if at least one of the underlying C^*-algebras $\mathcal{A}$ or $\mathbf{K}(V)$ is isomorphic to $\mathbb{C}$.

1. Introduction and preliminaries

Let V be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and let $x, y \in V$. The usual way to define the orthogonality in V is by means of the C^*-valued inner product: we say that x is *orthogonal* to y, and we write $x \perp y$, if $\langle x, y\rangle = 0$. Another concept of orthogonality in a Hilbert C^*-module is the Birkhoff–James orthogonality (see [5], [7]). This concept makes sense in every normed linear space X and, in the case when X is an inner product space, it is equivalent to the usual orthogonality given by the inner product. Recall that, for two elements x and y of a normed linear space X, we say that x *is orthogonal to* y *in the Birkhoff–James sense*; in short, $x \perp_B y$, if

$$\|x\| \leq \|x + \lambda y\|, \quad \forall \lambda \in \mathbb{C}.$$

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Having in mind that in Hilbert C^*-modules the role of scalars is played by the elements of the underlying C^*-algebra, the authors introduced a new concept of orthogonality in [2]; for $x, y \in V$, we say that x is *strongly Birkhoff–James orthogonal* to y; in short, $x \perp_B^s y$, if

$$\|x\| \leq \|x + ya\|, \quad \forall a \in \mathcal{A}.$$

It was shown in [2] that the strong Birkhoff–James orthogonality is stronger than the Birkhoff–James orthogonality, and weaker than the orthogonality with respect to the inner product, that is, $\langle x, y\rangle = 0 \Rightarrow x \perp_B^s y \Rightarrow x \perp_B y$, while the converses do not hold in general. If V is a full Hilbert $\mathcal{A}$-module, then the only case when the orthogonalities $\perp_B^s$ and $\perp_B$ coincide is when $\mathcal{A}$ is isomorphic to $\mathbb{C}$ (see [3, Theorem 3.5]), while orthogonalities $\perp_B^s$ and $\perp$ coincide only when $\mathcal{A}$ or $\mathbf{K}(V)$ is isomorphic to $\mathbb{C}$ (see [3, Theorems 4.7, 4.8]).

Obviously, the orthogonality relation $\perp$ is nondegenerate ($x \perp x$ if and only if $x = 0$); homogenous (if $x \perp y$, then $\lambda x \perp \mu y$, $\forall \lambda, \mu \in \mathbb{C}$); symmetric ($x \perp y$ if and only if $y \perp x$); right-additive (if $x \perp y_1$ and $x \perp y_2$, then $x \perp (y_1 + y_2)$); and left-additive (if $x_1 \perp y$ and $x_2 \perp y$, then $(x_1 + x_2) \perp y$).

In general, the orthogonality relations $\perp_B$ and $\perp_B^s$ are nondegenerate and homogenous, but neither symmetric nor additive (see [2, Remark 2.7(b)] for $\perp_B^s$; the same examples apply for $\perp_B$ because of [3, Proposition 3.1]). In this note, we describe the class of full Hilbert C^*-modules in which the (strong) Birkhoff–James orthogonality is symmetric.

Let us also mention that there are numerous papers about orthogonalities in C^*-algebras and Hilbert C^*-modules, among which considerable attention has been paid to orthogonality preserver problems (see, e.g., [6], [9]).

Before stating our results, let us recall some basic facts about C^*-algebras and Hilbert C^*-modules and introduce our notation.

A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra with the norm satisfying the C^*-condition $\|a^*a\| = \|a\|^2$. A positive element of a C^*-algebra $\mathcal{A}$ is a self-adjoint element whose spectrum is contained in $[0, \infty)$. If $a \in \mathcal{A}$ is positive, then we write $a \geq 0$. A partial order may be introduced on the set of self-adjoint elements of a C^*-algebra $\mathcal{A}$: if a and b are self-adjoint elements of $\mathcal{A}$ such that $a - b \geq 0$, then we write $a \geq b$ or $b \leq a$. If $a \geq 0$, then there exists a unique positive $b \in \mathcal{A}$ such that $a = b^2$; such an element b, denoted by $a^{\frac{1}{2}}$, is called the *positive square root* of a. An element $p \in \mathcal{A}$ is called a *projection* if $p = p^* = p^2$. A projection p is minimal if there is not a nonzero projection $q \in \mathcal{A}$, $q \neq p$, such that $q \leq p$. A projection $p \in \mathcal{A}$ for which $p\mathcal{A}p = \mathbb{C}p$ is minimal, but the converse does not hold in general.

A linear functional φ of $\mathcal{A}$ is *positive* if $\varphi(a) \geq 0$ for every positive element $a \in \mathcal{A}$. A *state* is a positive linear functional whose norm is equal to one.

A representation of $\mathcal{A}$ in a complex Hilbert space H is a $*$-homomorphism of $\mathcal{A}$ into the C^*-algebra $\mathbf{B}(H)$ of all bounded linear operators acting on H. Any C^*-algebra has a faithful (i.e., injective) representation.

A (*right*) *Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$* (or a (*right*) *Hilbert $\mathcal{A}$-module*) is a linear space which is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner-product $\langle \cdot, \cdot \rangle : V \times V \to \mathcal{A}$ that is sesquilinear, positive definite, and respects the module action; that is,

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in V$, $\alpha, \beta \in \mathbb{C}$,
(2) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in V$, $a \in \mathcal{A}$,
(3) $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in V$,
(4) $\langle x, x\rangle \geq 0$ for $x \in V$; if $\langle x, x\rangle = 0$, then $x = 0$,

and such that V is complete with respect to the norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, $x \in V$. By $\langle V, V\rangle$ we denote the closure of the span of $\{\langle x, y\rangle : x, y \in V\}$. We say that a Hilbert $\mathcal{A}$-module V is *full* if $\langle V, V\rangle = \mathcal{A}$.

Every Hilbert space is a Hilbert $\mathbb{C}$-module. Also, every C^*-algebra $\mathcal{A}$ can be regarded as a Hilbert C^*-module over itself with the inner product $\langle a, b\rangle := a^*b$, and the corresponding norm is just the norm on $\mathcal{A}$ because of the C^*-condition.

In a Hilbert $\mathcal{A}$-module V, we have the following version of the Cauchy–Schwarz inequality:

$$\left|\varphi\big(\langle x, y\rangle\big)\right|^2 \leq \varphi\big(\langle x, x\rangle\big)\varphi\big(\langle y, y\rangle\big), \quad \forall x, y \in V,$$

where φ is a positive linear functional of $\mathcal{A}$.

A mapping $T : V \to V$ on a Hilbert $\mathcal{A}$-module V is called *adjointable* if there exists a mapping $T^* : V \to V$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in V$. Every adjointable operator T is a bounded and $\mathcal{A}$-linear mapping. The set $\mathbf{B}(V)$ of all adjointable mappings acting on V is a C^*-algebra.

For every $x, y \in V$ we define $\theta_{x,y} : V \to V$ by $\theta_{x,y}(z) = x\langle y, z\rangle$. It is easy to see that all $\theta_{x,y}$ are adjointable and that $\theta^*_{x,y} = \theta_{y,x}$. By $\mathbf{K}(V)$ we denote the C^*-algebra spanned by $\{\theta_{x,y} : x, y \in V\}$. Every right Hilbert $\mathcal{A}$-module V may be regarded as a left Hilbert $\mathbf{K}(V)$-module with the inner product $[x, y] := \theta_{x,y}$ for $x, y \in V$. Thus it holds that $\|[x, x]\| = \|\theta_{x,x}\| = \|x\|^2$ for all $x \in V$. For details about C^*-algebras and Hilbert C^*-modules we refer the reader to [8] and [10].

2. Results

Let us first state some known results from [1], [2], [3], and [4] that we shall use in our proofs. (Observe that in [2], instead of the symbols $\perp_B^s$ and $\perp_B$ we used $\perp_*$ and $\perp$, respectively.)

Lemma 2.1. *Let V be a Hilbert $\mathcal{A}$-module. Then the following statements hold for every $x, y \in V$:*

(1) *$x \perp_B y$ if and only if there is a state φ of $\mathcal{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle) = 0$;*
(2) *$x \perp_B^s y$ if and only if $x \perp_B ya$ for all $a \in \mathcal{A}$, that is, if and only if $x \perp_B^s ya$ for all $a \in \mathcal{A}$;*
(3) *$x \perp_B^s y$ if and only if $x \perp_B y\langle y, x\rangle$;*
(4) *$x \perp_B^s y$ if and only if there is a state φ of $\mathcal{A}$ such that $\varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle\langle y, x\rangle) = 0$;*
(5) *$x \perp_B y$ if and only if $\langle x, x\rangle \perp_B \langle x, y\rangle$ if and only if $\langle x, x\rangle \perp_B \langle y, x\rangle$;*
(6) *$x \perp_B^s y$ if and only if $\langle x, x\rangle \perp_B^s \langle x, y\rangle$;*
(7) *if $\langle x, y\rangle \geq 0$ then $x \perp_B y \Leftrightarrow x \perp_B^s y$;*
(8) *$x \perp_B^s (\|x\|^2 x - x\langle x, x\rangle)$.*

In the first result we obtain a necessary condition on an element $x \in V$ which has the symmetry property.

Theorem 2.2. *Let V be a Hilbert $\mathcal{A}$-module, and let $x \in V \setminus \{0\}$ be such that one of the following conditions holds:*

(a) *for every $y \in V$ such that $x \perp_B^s y$, it holds that $y \perp_B^s x$;*
(b) *for every $y \in V$ such that $x \perp_B y$, it holds that $y \perp_B x$.*

Then $\langle x, x\rangle$ is a scalar multiple of a minimal projection in $\mathcal{A}$.

Proof. Without loss of generality we may assume that $\|x\| = 1$.

Suppose that (a) holds. By Lemma 2.1(8), for every $x \in V$, it holds that $x \perp_B^s (x - x\langle x, x\rangle)$, and, by Lemma 2.1(2), $x \perp_B^s (x\langle x, x\rangle - x\langle x, x\rangle^2)$; that is, $x \perp_B^s x(\langle x, x\rangle - \langle x, x\rangle^2)$. By symmetry, $x(\langle x, x\rangle - \langle x, x\rangle^2) \perp_B^s x$, and again by Lemma 2.1(2), $x(\langle x, x\rangle - \langle x, x\rangle^2) \perp_B^s x(\langle x, x\rangle - \langle x, x\rangle^2)$. From the nondegeneracy of $\perp_B^s$, it follows that $x(\langle x, x\rangle - \langle x, x\rangle^2) = 0$, from which $\langle x, x\rangle = \langle x, x\rangle^2$; that is, $\langle x, x\rangle$ is a projection.

Let us show that the projection $p = \langle x, x\rangle$ is minimal. Let $q \in \mathcal{A}$ be a projection such that $0 \le q \le p$, $q \ne p$. Let $\pi : \mathcal{A} \to \mathbf{B}(H)$ be a faithful representation of $\mathcal{A}$ in a Hilbert space H. Then $\pi(p)$ and $\pi(q)$ are projections such that $0 \le \pi(q) \le \pi(p)$ and $\pi(q) \ne \pi(p)$. Therefore, there is a unit vector $\xi \in H$ such that $\pi(p)\xi = \xi$ and $\pi(q)\xi = 0$. Then

$$\big\|\pi(p) + \lambda\pi(q)\big\| \ge \big\|\big(\pi(p) + \lambda\pi(q)\big)\xi\big\| = \|\xi\| = 1 = \big\|\pi(p)\big\|$$

for all $\lambda \in \mathbb{C}$. Since π is isometric, we have $\|p + \lambda q\| \ge \|p\|$ for all $\lambda \in \mathbb{C}$; that is, $p \perp_B q$, which can be written as $p \perp_B q\langle q, p\rangle$ and then, by Lemma 2.1(3), $p \perp_B^s q$. Since $q = pq = \langle x, xq\rangle$, we have $\langle x, x\rangle \perp_B^s \langle x, xq\rangle$, and so Lemma 2.1(6) implies $x \perp_B^s xq$. By the symmetry assumption, we have $xq \perp_B^s x$; this implies $xq \perp_B^s xq$, and so $xq = 0$. Then $q = \langle x, xq\rangle = 0$. This proves that p is minimal.

Suppose that (b) holds. Again, $x \perp_B^s (x - x\langle x, x\rangle)$, and therefore $x \perp_B^s (x\langle x, x\rangle - x\langle x, x\rangle^2)$. Then we have $x \perp_B (x\langle x, x\rangle - x\langle x, x\rangle^2)$, and by the symmetry assumption, $(x\langle x, x\rangle - x\langle x, x\rangle^2) \perp_B x$. Since $\langle x\langle x, x\rangle - x\langle x, x\rangle^2, x\rangle = \langle x, x\rangle^2 - \langle x, x\rangle^3 \ge 0$, by Lemma 2.1(7), it follows that $(x\langle x, x\rangle - x\langle x, x\rangle^2) \perp_B^s x$. Then, as before, it follows that $p := \langle x, x\rangle$ is a projection.

To show that p is minimal, suppose that $q \in \mathcal{A}$ is a projection such that $0 \le q \le p$, $q \ne p$. As before, we conclude that $x \perp_B^s xq$. Then $x \perp_B xq$ and, by the symmetry assumption, we have $xq \perp_B x$. Since $\langle xq, x\rangle = qp = q \ge 0$, we conclude that $xq \perp_B^s x$, from which, as before, $q = 0$. ☐

The converse of the previous theorem does not hold, as the following example shows.

Example 2.3. Let $V = \mathcal{A} = C([0, 1] \cup [2, 3])$ be the C^*-algebra of all continuous complex-valued functions on $[0, 1] \cup [2, 3]$ regarded as a Hilbert C^*-module over itself. Let $x \in \mathcal{A}$ be defined as

$$x(t) = \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \in [2, 3]. \end{cases}$$

Then $\langle x, x\rangle = x$, and this is a minimal projection in $\mathcal{A}$. Let

$$y(t) = \begin{cases} t & \text{if } x \in [0,1], \\ 0 & \text{if } x \in [2,3]. \end{cases}$$

Then $x \perp_B^s y$, since, for every $a \in \mathcal{A}$, it holds that

$$\|x + ya\| \geq \big|x(0) + y(0)a(0)\big| = 1 = \|x\|.$$

However, $y \not\perp_B^s x$, since $y \perp_B^s x$ would imply $y \perp_B^s xy = y$, and then $y = 0$. Since $\langle x, y\rangle \geq 0$, by Lemma 2.1(7), we deduce that $x \perp_B y$, but $y \not\perp_B x$.

The following result is a kind of converse of Theorem 2.2.

Proposition 2.4. *Let V be a Hilbert $\mathcal{A}$-module, and let $x \in V$ be such that $\langle x, x\rangle \mathcal{A} \langle x, x\rangle = \mathbb{C}\langle x, x\rangle$.*

(a) *For every $y \in V$ such that $x \perp_B^s y$, it holds that $\langle x, y\rangle = 0$.*
(b) *For every $y \in V$ such that $x \perp_B y$, it holds that $\langle x, x\rangle\langle y, x\rangle = 0$.*

Proof. If $x = 0$, then the statements are trivial, so suppose that $x \neq 0$. Without loss of generality we may assume that $\|x\| = 1$. Denote $p = \langle x, x\rangle$. Since $\langle x, x\rangle$ is a projection, we have $x = x\langle x, x\rangle$.

(a) If $x \perp_B^s y$, then, by Lemma 2.1(6) and (2), $\langle x, x\rangle \perp_B^s \langle x, y\rangle$, and therefore $\langle x, x\rangle \perp_B^s \langle x, y\rangle\langle y, x\rangle$. Since

$$\langle x, y\rangle\langle y, x\rangle = \langle x, x\rangle\langle x, y\rangle\langle y, x\rangle\langle x, x\rangle = \lambda\langle x, x\rangle,$$

for some $\lambda \in \mathbb{C}$, we have $\langle x, x\rangle \perp_B^s \lambda\langle x, x\rangle$, from which it follows that $\lambda = 0$ and then $\langle x, y\rangle = 0$.

(b) Suppose $x \perp_B y$. By Lemma 2.1(5), it follows that $\langle x, x\rangle \perp_B \langle y, x\rangle$ and then $\langle x, x\rangle^2 \perp_B \langle x, x\rangle\langle y, x\rangle$; that is, $\langle x, x\rangle \perp_B \langle x, x\rangle\langle y, x\rangle$. Since

$$\langle x, x\rangle\langle y, x\rangle = \langle x, x\rangle\langle y, x\rangle\langle x, x\rangle = \lambda\langle x, x\rangle,$$

for some $\lambda \in \mathbb{C}$, we conclude that $\lambda = 0$ and $\langle x, x\rangle\langle y, x\rangle = 0$. □

Remark 2.5. Let $\mathcal{A}$ be a C^*-algebra such that there is $p \in \mathcal{A} \setminus \{0\}$ satisfying $p\mathcal{A}p = \mathbb{C}p$. (As an example, one can take a C^*-algebra $\mathcal{A}$ of all compact operators on some Hilbert space and any one-dimensional projection $p \in \mathcal{A}$.) Let V be a full Hilbert $\mathcal{A}$-module. Let $y \in V$ be such that $yp \neq 0$ (such an element exists since V is a full Hilbert $\mathcal{A}$-module). Let $x = yp$. Then it holds that

$$\langle x, x\rangle = \langle yp, yp\rangle = p\langle y, y\rangle p \in p\mathcal{A}p,$$

and so $\langle x, x\rangle = \lambda p$ for some $\lambda > 0$. Thus we have

$$\langle x, x\rangle \mathcal{A} \langle x, x\rangle = \lambda^2(p\mathcal{A}p) = \lambda^2(\mathbb{C}p) = \mathbb{C}\langle x, x\rangle,$$

and so x satisfies the assumption of Proposition 2.4.

Let us now state our main result.

Theorem 2.6. *Let V be a full Hilbert $\mathcal{A}$-module. The following statements are equivalent:*

(a) *$\perp_B$ is a symmetric relation;*

(b) $\perp_B^s$ *is a symmetric relation;*
(c) $\perp_B^s$ *coincides with the inner product orthogonality;*
(d) $\mathcal{A}$ *or* $\mathbf{K}(V)$ *is isomorphic to* $\mathbb{C}$.

Proof. By [3, Theorems 4.7, 4.8], we know that (c)$\Leftrightarrow$(d).

It is obvious that (c)$\Rightarrow$(b).

If (d) holds, then V is an inner product space with the norm $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$ or $\|x\| = [x,x]^{\frac{1}{2}}$, depending on whether $\mathcal{A}$ or $\mathbf{K}(V)$ is isomorphic to $\mathbb{C}$. If $\mathcal{A}$ is isomorphic to $\mathbb{C}$, then it holds that $x \perp_B y$ precisely when $\langle x,y\rangle = 0$, while in the case when $\mathbf{K}(V)$ is isomorphic to $\mathbb{C}$, we have $x \perp_B y$ if and only if $[x,y] = 0$. Note that, in both cases, $\perp_B$ is a symmetric relation; that is, (a) holds.

Let us prove (b)$\Rightarrow$(c). First, observe that it follows from Theorem 2.2 that $\langle v,v\rangle$ is a scalar multiple of a minimal projection for every $v \in V$, and so

$$v\langle v,v\rangle = \|v\|^2 v, \quad \forall v \in V. \tag{2.1}$$

Let $x,y \in V$ be such that $x \perp_B^s y$. If $y = 0$, then $\langle x,y\rangle = 0$. Suppose that $y \neq 0$. Without loss of generality we may assume that $\|y\| = 1$. Then $x \perp_B^s y\langle y,x\rangle$, and so, by symmetry, $y\langle y,x\rangle \perp_B^s x$. Then, by Lemma 2.1(6), it holds that $\langle y\langle y,x\rangle, y\langle y,x\rangle\rangle \perp_B^s \langle y\langle y,x\rangle, x\rangle$. By using (2.1) we get

$$\big\langle y\langle y,x\rangle, y\langle y,x\rangle\big\rangle = \langle x,y\rangle\langle y,y\rangle\langle y,x\rangle = \big\langle x, y\langle y,y\rangle\big\rangle\langle y,x\rangle = \langle x,y\rangle\langle y,x\rangle,$$

and so $\langle x,y\rangle\langle y,x\rangle \perp_B^s \langle x,y\rangle\langle y,x\rangle$. Therefore, $\langle x,y\rangle\langle y,x\rangle = 0$, and so $\langle x,y\rangle = 0$. This proves our statement.

The implication (a)$\Rightarrow$(c) is proved in a similar way. First, Theorem 2.2 implies (2.1). Let $x,y \in V \setminus \{0\}$ be such that $x \perp_B^s y$. Again assume that $\|y\| = 1$. Then $x \perp_B y\langle y,x\rangle$, and so, by symmetry, $y\langle y,x\rangle \perp_B x$. Then, by Lemma 2.1(5), it holds that $\langle y\langle y,x\rangle, y\langle y,x\rangle\rangle \perp_B \langle y\langle y,x\rangle, x\rangle$. As before, by using (2.1), we get

$$\big\langle y\langle y,x\rangle, y\langle y,x\rangle\big\rangle = \langle x,y\rangle\langle y,x\rangle,$$

and so we have $\langle x,y\rangle\langle y,x\rangle \perp_B \langle x,y\rangle\langle y,x\rangle$. It follows that $\langle x,y\rangle = 0$. □

Corollary 2.7. *The relation* $\perp_B^s$ *(resp.,* $\perp_B$*) is symmetric in a* C^**-algebra* $\mathcal{A}$ *if and only if* $\mathcal{A} \simeq \mathbb{C}$.

Remark 2.8. It would also be interesting to describe Hilbert C^*-modules in which relations $\perp_B$ or $\perp_B^s$ are left- or right-additive.

This problem is easy to solve in the case of a unital C^*-algebra $\mathcal{A}$ (with the unit e), regarded as a Hilbert C^*-module over itself. Namely, suppose that $a \in \mathcal{A}$ is noninvertible. Then aa^* or a^*a is noninvertible. Assume that $b := aa^*$ is noninvertible. By [2, Remark 2.7(a)], $e \perp_B b$ and $e \perp_B (\|b\|e - b)$, and so, if $\perp_B$ is right-additive, then $e \perp_B \|b\|e$, from which $b = 0$ and then $a = 0$. The same conclusion is obtained in the case when a^*a is noninvertible. This proves that every nonzero element of $\mathcal{A}$ is invertible, and so $\mathcal{A} \simeq \mathbb{C}$.

The same proof works for right-additivity of $\perp_B^s$, since $b \geq 0$ and $\|b\|e - b \geq 0$, and therefore, by Lemma 2.1(7), $e \perp_B b \Leftrightarrow e \perp_B^s b$ and $e \perp_B (\|b\|e - b) \Leftrightarrow e \perp_B^s (\|b\|e - b)$.

Suppose that $\perp_B$ is left-additive. Let $a \in \mathcal{A}$ be positive and noninvertible. Let φ be a state of $\mathcal{A}$ such that $\varphi(a) = 0$. Then $\varphi(\|a\|e - a) = \|a\| = \|\|a\|e - a\|$.

(Indeed, since a is positive and noninvertible, $\|a\|$ belongs to the spectrum of $\|a\|e - a \geq 0$, and so $\|a\| \leq \|\|a\|e - a\|$. On the other hand, $0 \leq \|a\|e - a \leq \|a\|e$, and so $\|\|a\|e - a\| \leq \|a\|$; hence $\|a\| = \|\|a\|e - a\|$.) Further, by [3, Lemma 4.1], $\varphi((\|a\|e - a)^2) = \|\|a\|e - a\|^2$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \big|\varphi\big((\|a\|e - a)a\big)\big|^2 &= \big|\varphi\big((\|a\|a^{\frac{1}{2}} - a^{\frac{3}{2}})a^{\frac{1}{2}}\big)\big|^2 \\ &\leq \big|\varphi\big((\|a\|a^{\frac{1}{2}} - a^{\frac{3}{2}})^2\big)\big|\big|\varphi(a)\big| = 0, \end{aligned}$$

and so $\varphi((\|a\|e - a)a) = 0$. By Lemma 2.1(1), this gives $(\|a\|e - a) \perp_B a$, which, together with $\|a\|e \perp_B a$, by left-additivity gives $a \perp_B a$; that is, $a = 0$. So, $\mathcal{A} \simeq \mathbb{C}$. Since $(\|a\|e - a)a \geq 0$, by Lemma 2.1(7), we have $(\|a\|e - a) \perp_B a \Leftrightarrow (\|a\|e - a) \perp_B^s a$, and so the same proof works for left-additivity of $\perp_B^s$.

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