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ON SYNCHRONIZED SEQUENCES AND THEIR SEPARATORS *

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Abstract. We introduce the notion of a k-synchronized sequence, where k is an integer larger than 1. Roughly speaking, a sequence of natural numbers is said to be k-synchronized if its graph is represented, in base k, by a right synchronized rational relation. This is an intermediate notion between k-automatic and k-regular sequences. Indeed, we show that the class of k-automatic sequences is equal to the class of bounded k-synchronized sequences and that the class of k-synchronized sequences is strictly contained in that of k-regular sequences. Moreover, we show that equality of factors in a k-synchronized sequence is represented, in base k, by a right synchronized rational relation. This result allows us to prove that the separator sequence of a k-synchronized sequence is a k-synchronized sequence, too. This generalizes a previous result of Garel, concerning k-regularity of the separator sequences of sequences generated by iterating a uniform circular morphism.

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Introduction

The notion of a sequence computed by a finite automaton with output function, receiving in input the expansion of natural numbers in a given base, was introduced by Cobham [4]. He proved that sequences obtained in this way are exactly the

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* *This paper is dedicated to Aldo de Luca in occasion of its 60th birthday. Without his guidance, it would not have been possible for us to undertake scientific research.*

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images, by a letter-to-letter morphism, of the fixed points of uniform morphisms. This kind of sequences are known as uniform tag sequences or automatic sequences. A fundamental result of Christol *et al.* [3] gives an arithmetical characterization of automatic sequences (see also [2]).

Allouche and Shallit [1] propose a generalization of the notion of automatic sequence, introducing regular sequences. When considering sequences with values in the ring of relative integers, regular sequences can be generated by finite automata with multiplicity in $\mathbb{Z}$ reading the expansion of natural numbers in a given base.

In this paper we consider an intermediate notion between those of automatic and regular sequences. More precisely, we consider sequences whose graph is represented, in a fixed base k, by a right synchronized rational relation.

We recall that the addition of natural numbers and the natural ordering of $\mathbb{N}$ are typical examples of relations which are represented, in any base $k \geq 2$, by right synchronized rational relations. We shall say, briefly, that they are *k-synchronized relations*. A sequence whose graph is a k-synchronized relation will be called a *k-synchronized sequence*.

In Section 2, we shall prove that the class of k-synchronized sequences is strictly included in the class of k-regular sequences and that a sequence of integers is k-automatic if and only if it is k-synchronized and bounded. Moreover, any k-synchronized sequence grows at most linearly.

We also show that the class of k-synchronized sequences is closed for composition and sum as well as for the operation consisting in multiplying by a non-negative rational scalar and then truncating to an integer.

In Section 3, we study factors of k-synchronized sequences. We shall prove that if $u = (u_n)_{n=0}^{\infty}$ is a k-synchronized sequence, then the triples (i, j, h) such that

$$u_i u_{i+1} \dots u_{i+h-1} = u_j u_{j+1} \dots u_{j+h-1}$$

constitute a k-synchronized relation.

Garel [9] studied the separators of sequences generated by iteration of uniform morphisms. The n-th separator of a sequence $u = (u_n)_{n=0}^{\infty}$ is the shortest factor of u, if any exists, having its leftmost occurrence starting "at position n". In other terms, the n-th separator is the shortest "new" factor appearing in the sequence after the first n elements.

The notion of separator is deeply related to the string matching problem. Moreover, it seems to be very similar to some concepts concerning the automaticity of unary languages [10]. Indeed, the (deterministic) automaticity $\mathcal{A}_u(n)$ of the sequence u is the minimal number of states of a deterministic finite state automaton with output on the one-letter alphabet $\{1\}$ outputting u_m on input 1^m for all $m \leq n$. As proved in [10], one has $\mathcal{A}_u(n) = n - t + 2$, where t is the length of the shortest factor of u having its leftmost occurrence ending "at position n".

Notice that, unless u is ultimately periodic, the n-th separator exists for all natural n [9], so that one can consider the sequence of the lengths of the separators, called *separator sequence*.

Here, we shall prove that the separator sequence of a k-synchronized sequence is k-synchronized, too. We notice that this result contains, as a particular case, a result proved by Garel [9] by means of complex combinatorial techniques stating that the separator sequences of sequences generated by iteration of circular uniform morphisms of modulus k are k-regular.

1. Preliminaries

Let A be a non-empty set of symbols or *alphabet*. We shall denote by A^* the free monoid generated by A. The elements of A are usually called *letters* and those of A^* *words*. The *length* of a word $w \in A^*$ will be denoted by $|w|$. The *empty word* will be denoted by ϵ.

An *infinite word* on the alphabet A is any unending sequence of letters

$$u = u_0 u_1 \cdots u_n \cdots ,$$

$u_n \in A$, $n \in \mathbb{N}$. For any pair of integers i, j such that $0 \leq i \leq j$, we shall denote by $u[i,j]$ the word

$$u[i,j] = u_i u_{i+1} \cdots u_j.$$

The words $u[i,j]$ are said to be *factors* of the infinite word u.

In this paper, we shall often identify a sequence of natural numbers $(u_n)_{n=0}^{\infty}$ with the infinite word $u_0 u_1 \cdots u_n \cdots$ on the alphabet $A = \mathbb{N}$. Thus, we shall consider factors of a numeric sequence.

Let A_i, $1 \leq i \leq r$, be r alphabets, $r \geq 2$. By *relation* on the alphabets A_i, $1 \leq i \leq r$, we mean any subset of the direct product

$$M = A_1^* \times A_2^* \times \cdots \times A_r^*,$$

i.e., any element of the monoid $\wp(M)$ of the subsets of M. A relation is *rational* if it belongs to the smallest submonoid of $\wp(M)$ containing the finite parts and closed for the operations of finite union and submonoid generation. A relation $\rho \subseteq M$ is *length-preserving* if for all element $(w_1, \ldots, w_r) \in \rho$ one has $|w_1| = |w_2| = \cdots = |w_r|$.

We recall, as this will be useful later, that if $\rho \subseteq A_1^* \times A_2^*$ is a rational relation, then for all $w_2 \in A_2^*$ the set $\{w_1 \in A_1^* \mid (w_1, w_2) \in \rho\}$ is a regular subset of A_1^*.

Let $\$ \notin \cup_{i=1}^r A_i$ be a new symbol. A relation $\rho \subseteq M$ is said to be *right synchronized rational* if the relation

$$\left\{ (\$^{t-|w_1|} w_1, \ldots, \$^{t-|w_r|} w_r) \mid (w_1, \ldots, w_r) \in \rho,\ t = \max_{1 \leq i \leq r} |w_i| \right\},$$

is a length-preserving rational relation. An equivalent definition, in terms of "letter-to-letter automata with initial function" is given in [8]. In the case of binary relations, it is essentially equivalent to saying that a subset of $A_1 \times A_2$ is a right synchronized rational relation if and only if it can be expressed as a finite union of products RL, with L a length-preserving rational relation and R a rational subset of $A_1 \times \{\epsilon\}$ or of $\{\epsilon\} \times A_2$.

Let $k \geq 2$ be an integer. For any $n \in \mathbb{N}$, we shall denote by $[n]_k$ the standard expansion of n in base k. Thus, $[n]_k$ is a word on the *digit alphabet* $D_k = \{0, 1, \ldots, k-1\}$.

We shall say that a subset σ of $\mathbb{N}^r$ is a *k-synchronized relation* if the relation

$$\{([n_1]_k, \ldots, [n_r]_k) \mid (n_1, \ldots, n_r) \in \sigma\}$$

is a right synchronized rational relation in $D_k^* \times \cdots \times D_k^*$.

By *projection* of a relation $\rho \subseteq A_1^* \times A_2^* \times \cdots \times A_r^*$, $r \geq 3$, we mean any of the r relations

$$\{(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_r) \mid \exists w_i \in A_i^*,\ (w_1, \ldots, w_r) \in \rho\}.$$

It is well known that the class of right synchronized rational relations is closed for Boolean operations, Cartesian product, projection, and permutation of coordinates [6], properly contains the class of recognizable relations and is properly contained in the class of deterministic rational relations (see, *e.g.* [8]).

Thus, by the definition of k-synchronized relation, we derive immediately the following closure properties.

Proposition 1.1. *The class of k-synchronized relations is closed for Boolean operations, Cartesian product, projection, and permutation of coordinates.*

Example 1.2. The relation $\Delta_{\mathbb{N}} = \{(n, n) \mid n \in \mathbb{N}\}$ is obviously a k-synchronized relation, for any $k \geq 2$.

It is well known that the sum of natural numbers, *i.e.*, the relation

$$\rho_1 = \{(m, n, m+n) \mid m, n \in \mathbb{N}\}$$

is a k-synchronized relation, for all $k \geq 2$. To prove this result, which belongs to folklore, one can easily adapt the proof of the right subsequentiality of this relation given in [5] (see also [7]). Also the natural "order relation"

$$\rho_2 = \{(m, n) \mid m, n \in \mathbb{N},\ m \leq n\}$$

is a k-synchronized relation for all $k \geq 2$. Indeed, it is the projection of ρ_1 on the first and third coordinates, so that the conclusion follows from Proposition 1.1. Clearly, also the "strict order"

$$\rho_2' = \{(m, n) \mid m, n \in \mathbb{N},\ m < n\} = \rho_2 \setminus \Delta_{\mathbb{N}}$$

is a k-synchronized relation for all $k \geq 2$.

Now we introduce the main object of this paper.

Definition 1.3. A sequence of natural numbers $u = (u_n)_{n=0}^{\infty}$ will be called a *k-synchronized sequence* if its graph $G_u = \{(n, u_n) \mid n \in \mathbb{N}\}$ is a k-synchronized relation.

We recall the notions of k-automatic [4] and k-regular [3] sequences. The *k-kernel* of a sequence $u = (u_n)_{n=0}^{\infty}$ is the set of the subsequences

$$(u_{i+k^j n})_{n=0}^{\infty}, \quad j \in \mathbb{N},\ 0 \le i < k^j.$$

Definition 1.4. A sequence $u = (u_n)_{n=0}^{\infty}$ is said to be a *k-automatic sequence* (or *uniform tag sequence* of *modulus* k) if its k-kernel is a finite set. A sequence of integers $u = (u_n)_{n=0}^{\infty}$ is said to be *k-regular* if its k-kernel generates a $\mathbb{Z}$-module of finite type.

Cobham [4] proved that a sequence u on an alphabet A is k-automatic if and only if there exist a sequence v on a finite alphabet B and morphisms $g : B^* \to B^*$ and $h : B^* \to A^*$ such that

$$|g(b)| = k, |h(b)| = 1 \text{ for all } b \in B, \quad v = g(v), \quad u = h(v),$$

where g and h are extended to infinite words by continuity. Thus, in particular, infinite words generated by iteration of a uniform morphism of modulus k are k-automatic. A different characterization [4] states that a sequence $u = (u_n)_{n=0}^{\infty}$ on an alphabet A is k-automatic if and only if there exists a deterministic finite state automaton with output $\mathcal{M}$ such that, for all $n \in \mathbb{N}$, u_n is the output corresponding to the input string $[n]_k$. This is equivalent to saying that a sequence u is k-automatic if and only if the sets

$$L_a = \{[n]_k \mid n \in \mathbb{N},\ u_n = a\}, \quad a \in A,$$

are regular subsets of D_k^*.

2. Synchronized sequences

In the sequel, k will denote a fixed integer larger than 1. In this section, we study k-synchronized sequences. We start with some closure properties.

Proposition 2.1. *The sum of two k-synchronized sequences is a k-synchronized sequence.*

Proof. Let $u = (u_n)_{n=0}^{\infty}$ and $u' = (u'_n)_{n=0}^{\infty}$ be two k-synchronized sequences. In view of Proposition 1.1, the relations

$$\{(n, u_n, i, j) \mid n, i, j \in \mathbb{N}\},\ \{(n, i, u'_n, j) \mid n, i, j \in \mathbb{N}\},\ \{(n, i, j, i + j) \mid n, i, j \in \mathbb{N}\}$$

are k-synchronized, as well as their intersection

$$\{(n, u_n, u'_n, u_n + u'_n) \mid n \in \mathbb{N}\}\cdot$$

By projecting on the first and last coordinates we obtain that the relation $\{(n, u_n + u'_n) \mid n \in \mathbb{N}\}$ is k-synchronized, *i.e.* $(u_n + u'_n)_{n=0}^{\infty}$ is a k-synchronized sequence. □

Proposition 2.2. *The composition of two k-synchronized sequences is k-synchronized.*

Proof. Let $u = (u_n)_{n=0}^{\infty}$ and $u' = (u'_n)_{n=0}^{\infty}$ be two k-synchronized sequences and denote respectively by G_u and $G_{u'}$ their graphs. Then G_u and $G_{u'}$ are k-synchronized relations so that, in view of Proposition 1.1, the relation $(G_u \times D_k^*) \cup (D_k^* \times G_{u'})$ is k-synchronized, as well as its projection σ on the first and third coordinates. As one easily verifies, σ is the graph of the composed sequence $(u'_{u_n})_{n=0}^{\infty}$, which is, therefore, a k-synchronized sequence. □

Example 2.3. By Example 1.2, the sequence $(n)_{n=0}^{\infty}$ is k-synchronized. For any $a, b \in \mathbb{N}$, the linear sequence $(an + b)_{n=0}^{\infty}$ is k-synchronized. Indeed, it is obtained by summing a copies of the sequence $(n)_{n=0}^{\infty}$ and one copy of the constant sequence b, so that the conclusion follows by Proposition 2.1.

The following proposition shows, in particular, that the previous example can be generalized to the case of "rounded" linear sequences with rational coefficients. As usually, if α is a rational number, we denote by $\lfloor \alpha \rfloor$ the greatest integer not larger than α.

Proposition 2.4. *Let $u = (u_n)_{n=0}^{\infty}$ be a k-synchronized sequence and α a non-negative rational number. Then the sequence $u' = (\lfloor \alpha u_n \rfloor)_{n=0}^{\infty}$ is k-synchronized.*

Proof. Let $\alpha = a/b$, with $a, b \in \mathbb{N}$ and $b \neq 0$. Then one has

$$\alpha u_n = \frac{a u_n}{b}\cdot$$

Thus, u' can be obtained by composing the sequences $(a u_n)_{n=0}^{\infty}$ and $(\lfloor n/b \rfloor)_{n=0}^{\infty}$. The first of these two sequences is k-synchronized by Proposition 2.1, since it can be obtained by summing a copies of u. Hence, it suffices to prove that also the second one is k-synchronized. Indeed, its graph is given by

$$\left\{ \left(n, \left\lfloor \frac{n}{b} \right\rfloor\right) \middle| \, n \in \mathbb{N} \right\} = \bigcup_{i=0}^{b-1} \{(bm + i, m) \mid m \in \mathbb{N}\}\cdot$$

Since the sequences $(bm + i)_{m=0}^{\infty}$ are k-synchronized, and therefore their graphs $\{(m, bm + i) \mid m \in \mathbb{N}\}$ are k-synchronized relations, by Proposition 1.1 we derive that also the graph of $(\lfloor n/b \rfloor)_{n=0}^{\infty}$ is a k-synchronized relation, which concludes the proof. □

Now we shall prove that k-synchronized sequences cannot grow faster than the sequences considered in Example 2.3.

Proposition 2.5. *Any k-synchronized sequence grows at most linearly.*

Proof. Let $u = (u_n)_{n=0}^{\infty}$ be a k-synchronized sequence. As the relation $S = \{([n]_k, [u_n]_k) \mid n \in \mathbb{N}\}$ is right synchronized rational, it can be expressed as a finite union of products RL with $R \subseteq D_k^* \times \{\epsilon\}$ or $R \subseteq \{\epsilon\} \times D_k^*$ and L length-preserving. As S is a functional relation, any of the sets R can contain at most one pair (ϵ, v), with $v \in D_k^*$. Denote by M the maximal length of all such words v, if any exists, otherwise set $M = 0$. Then one has $|w'| \leq |w| + M$ for all $(w, w') \in S$. One derives that

$$u_n < k^{|[u_n]_k|} \leq k^{|[n]_k|+M} \leq k^{M+1} n$$

for all $n \in \mathbb{N}$. □

For instance, by the previous proposition one derives that the sequence $(n^2)_{n=0}^{\infty}$ is not k-synchronized. As the sequence $(n)_{n=0}^{\infty}$ is k-synchronized, we conclude that the class of k-synchronized sequences is not closed for termwise product. Similarly, the class of k-synchronized sequences is not closed for convolution product, since the convolution product of $(n)_{n=0}^{\infty}$ by itself grows too fast to be a k-synchronized sequence.

Our next goal is to show that the class of k-synchronized sequences is properly contained in the class of k-regular sequences and properly contains that of k-automatic sequences.

In order to prove that k-synchronized sequences are k-regular, we need to recall some notions concerning formal power series in non-commutative variables. Allouche and Shallit [1] proved that a sequence of integers $(u_n)_{n=0}^{\infty}$ is k-regular if and only if the formal power series

$$\sum_{n=0}^{\infty} u_n [n]_k$$

(in non-commutative variables) is rational on the ring $\mathbb{Z}$. Let A and B be two alphabets, $\phi : A^* \to B^*$ a partial rational function and $S = \sum_{w \in B^*} a_w w$ a formal power series on B. Then the graph of ϕ,

$$G_\phi = \{(v, \phi(v)) \mid v \in A^*\}$$

is an unambiguous rational relation and consequently, by the "Evaluation Theorem", the formal power series

$$\phi^{-1}(S) = \sum_{v \in A^*} a_{\phi(v)} v$$

is rational, too (see, *e.g.* [5]).

Now we can prove the following:

Proposition 2.6. *Let k be a positive integer. Any k-synchronized sequence is k-regular.*

Proof. As is well known, the sequence $I = (0, 1, \ldots, n, \ldots)$ is k-regular, for all $k > 0$, since its k-kernel is the set of the sequences $(i + k^j n)_{n=0}^{\infty} = i + k^j I$, with $j \geq 0$, $0 \leq i < k^j$. Consequently, for any $k \geq 2$, the formal power series

$$S_k = \sum_{n=0}^{\infty} n[n]_k$$

is rational on the ring $\mathbb{Z}$.

Now, if $u = (u_n)_{n=0}^{\infty}$ is a k-synchronized sequence, then the partial function $\phi : D_k^* \to D_k^*$ defined by $\phi([n]_k) = [u_n]_k$ is rational and therefore the formal power series

$$\phi^{-1}(S_k) = \sum_{n=0}^{\infty} u_n[n]_k$$

is rational. This is equivalent to saying that u is a k-regular sequence. □

The inclusion of the class of k-synchronized sequences in that of k-regular sequences is strict. For instance, for any polynomial $P(x)$ of degree larger than 1 with natural coefficients, the sequence $(P(n))_{n=0}^{\infty}$ is k-regular but it is not k-synchronized, in view of Proposition 2.5. A "slowly growing" k-regular sequence which is not k-synchronized is given in the following example:

Example 2.7. The sequence $(f_n)_{n=0}^{\infty}$ defined by

$$f_0 = 0, \quad f_n = \lfloor \log_k n \rfloor, \ n \geq 1,$$

is k-regular [1] but it is not k-synchronized. Indeed, intersecting its graph with the k-synchronized relation $\{k^n \mid n \geq 0\} \times \mathbb{N}$ one obtains the relation $\{(k^n, n) \mid n \geq 0\}$ which, up to the order of coordinates, is the graph of the sequence $(k^n)_{n=0}^{\infty}$. Thus, if f_n would be a k-synchronized sequence, then the sequence $(k^n)_{n=0}^{\infty}$ would be k-synchronized, too, contradicting Proposition 2.5.

Now, we shall prove that k-automatic sequences are k-synchronized.

Proposition 2.8. *Let k be a positive integer. A sequence of natural numbers is k-automatic if and only if it is k-synchronized and takes on only finitely many values.*

Proof. As proved in [1] a k-regular sequence is k-automatic if and only if it takes on only finitely many values. Since by Proposition 2.6 any k-synchronized sequence is k-regular, we conclude that a k-synchronized sequence which takes on only finitely many values is k-automatic.

Conversely, let $u = (u_n)_{n=0}^{\infty}$ be a k-automatic sequence of natural numbers and consider the set

$$S = \{([n]_k, [u_n]_k) \mid n \in \mathbb{N}\}\cdot$$

We can decompose S as

$$S = \bigcup_{a \in A} L_a \times \{[a]_k\}$$

where A is the set of the values taken on by the sequence u and, for any $a \in A$, $L_a = \{[n]_k \mid n \in \mathbb{N},\ u_n = a\}$. Since u is k-automatic, the set A is finite and, for any $a \in A$, L_a is regular. Thus, S is a recognizable relation and therefore it is a right synchronized rational relation. This is equivalent to saying that u is k-synchronized. ☐

We remark that the inclusion of the class of k-automatic sequences in the class of k-synchronized sequences is strict. Indeed, there are k-synchronized sequences which are not bounded, such as, *e.g.* $(n)_{n=0}^{\infty}$.

3. Factors of synchronized sequences

In this section we study factors of k-synchronized sequences.

Next proposition shows that the equality of factors in a k-synchronized sequence is described by a k-synchronized relation.

Proposition 3.1. *Let $u = (u_n)_{n=0}^{\infty}$ be a k-synchronized sequence. Then the relation*

$$\gamma_u = \{(i, j, h) \in \mathbb{N}^3 \mid h > 0,\ u[i, i+h-1] = u[j, j+h-1]\}$$

is k-synchronized.

Proof. Let G_u be the graph of the sequence u. Consider the relations

$$\begin{aligned}
\sigma_1 &= \{(i, i', i'', j, j', j'', h) \in \mathbb{N}^7 \mid i' = i + h\},\\
\sigma_2 &= \{(i, i', i'', j, j', j'', h) \in \mathbb{N}^7 \mid (i', i'') \in G_u\},\\
\sigma_3 &= \{(i, i', i'', j, j', j'', h) \in \mathbb{N}^7 \mid j' = j + h\},\\
\sigma_4 &= \{(i, i', i'', j, j', j'', h) \in \mathbb{N}^7 \mid (j', j'') \in G_u\},\\
\sigma_5 &= \{(i, i', i'', j, j', j'', h) \in \mathbb{N}^7 \mid i'' = j''\}.
\end{aligned}$$

They can all be obtained by a Cartesian product of G_u or one of the relations $\Delta_{\mathbb{N}}$ and ρ_1 considered in Example 1.2 by a suitable number of copies of $\mathbb{N}$, possibly permuting the coordinates. Consequently, by Proposition 1.1, they are k-synchronized relations, as well as their intersection σ_6. One has

$$\sigma_6 = \bigcap_{i=1}^{5} \sigma_i = \{(i, i+h, u_{i+h}, j, j+h, u_{j+h}, h) \mid i, j, h \in \mathbb{N}, u_{i+h} = u_{j+h}\}$$

and therefore the relation

$$\sigma_7 = \{(i,j,h) \in \mathbb{N}^3 \mid u_{i+h} = u_{j+h}\}$$

is a projection of σ_6. By Proposition 1.1, σ_7 is k-synchronized, too.

One has

$$(\mathbb{N}^2 \times \rho_2') \setminus (\sigma_7 \times \mathbb{N}) = \{(i,j,h',h) \in \mathbb{N}^4 \mid h' < h, u_{i+h'} \neq u_{j+h'}\}$$

and the projection of this relation on the first, second and fourth coordinates is given by

$$\begin{aligned}\{(i,j,h) \in \mathbb{N}^3 \mid \exists h' \in \mathbb{N}, h' < h, u_{i+h'} \neq u_{j+h'}\} &= \\ = \{(i,j,h) \in \mathbb{N}^3 \mid u[i,i+h-1] \neq u[j,j+h-1]\} \\ = \mathbb{N}^3 \setminus \gamma_u.\end{aligned}$$

By Proposition 1.1, one derives that $\mathbb{N}^3 \setminus \gamma_u$ and, consequently, its complement γ_u, are k-synchronized relations. □

Let

$$u = u_0 u_1 \cdots u_n \cdots$$

be an infinite word. For any $n \in \mathbb{N}$, the n-th *separator* of u is the shortest word w, if any exists, such that

$$w = u[n, n+|w|-1] \quad \text{and} \quad w \neq u[i, i+|w|-1], \ 0 \leq i < n.$$

In other terms, the n-th separator is the shortest factor of u which appears for the first time at "position" n.

Garel [9] proved that if u is an ultimately periodic infinite word, then there are only finitely many n such that the n-th separator exists, while if u is not ultimately periodic, then the n-th separator exists for all $n \in \mathbb{N}$. Thus, if u is an infinite word which is not ultimately periodic, one can consider the sequence of natural numbers $(s_n)_{n=0}^{\infty}$, where, for all $n \in \mathbb{N}$, s_n is the length of the n-th separator of u. This sequence will be called the *separator sequence* of u.

As an application of Proposition 3.1, we shall prove that the separator sequence of a k-synchronized sequence is k-synchronized, too.

Proposition 3.2. *Let $u = (u_n)_{n=0}^{\infty}$ be a k-synchronized sequence which is not ultimately periodic. Then the separator sequence $(s_n)_{n=0}^{\infty}$ of u is k-synchronized, too.*

Proof. Let ρ_2' and γ_u be defined as in Example 1.2 and Proposition 3.1, respectively. One has

$$\gamma_u \cap (\rho_2' \times \mathbb{N}) = \{(i,n,h) \in \mathbb{N}^3 \mid i < n, \ h > 0 \ u[i,i+h-1] = u[n,n+h-1]\}\cdot$$

By projecting this relation on the last two coordinates, one obtains the relation

$$\delta_1 = \{(n,h) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\}) \mid \exists i < n,\ u[i, i+h-1] = u[n, n+h-1]\}.$$

By Proposition 1.1, δ_1 is k-synchronized, as well as the relation

$$\begin{aligned}\delta_2 &= \big(\mathbb{N} \times (\mathbb{N} \setminus \{0\})\big) \setminus \delta_1 \\ &= \{(n,h) \in \mathbb{N}^2 \mid h > 0,\ u[i, i+h-1] \neq u[n, n+h-1] \text{ for } 0 \leq i < n\}\cdot\end{aligned}$$

Now, consider the relation

$$\delta_3 = \{(n,h) \in \mathbb{N}^2 \mid \exists t < h, (n,t) \in \delta_2\}\cdot$$

It is k-synchronized, since it can be obtained by projecting on the first and third coordinates the relation $(\mathbb{N} \times \rho_2') \cap (\delta_2 \times \mathbb{N})$. By the definition of separator, it is clear that s_n is equal to the least $h > 0$ such that $(n,h) \in \delta_2$. Thus, one has $s_n = h$ if and only if $(n,h) \in \delta_2 \setminus \delta_3$. We conclude that the graph of the separator sequence $(s_n)_{n=0}^{\infty}$ is the k-synchronized relation $\delta_2 \setminus \delta_3$. □

We recall that Garel [9] proved that the separator sequences of sequences generated by iteration of circular uniform morphism of modulus k are k-regular. This result can be viewed as a very particular case of the previous proposition. Indeed, a sequence generated by iteration of a circular uniform morphism of modulus k is k-automatic and therefore, by Proposition 2.8, is k-synchronized. Thus, Proposition 3.2 ensures that its separator sequence is k-synchronized and, in particular, k-regular, in view of Proposition 2.6.

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