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On tensor categories attached to cells in affine Weyl groups II

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Abstract.

We prove a weak version of Lusztig's Conjecture on explicit description of the asymptotic affine Hecke algebras in terms of convolution algebras.

§1. Introduction

Let R be a root system. Let W be the corresponding affine Weyl group, and let $\hat{W}$ be an extended affine Weyl group. Let $\mathcal{H}$ (respectively $\hat{\mathcal{H}}$) be the corresponding Hecke algebras. George Lusztig defined an asymptotic version of the Hecke algebra, the ring J, see [10]. By definition the ring J is a direct sum $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$ where summation is over the set of *two-sided cells* in the affine Weyl group. Further, G. Lusztig proved that the set of two-sided cells in W is bijective to the set of unipotent conjugacy classes in an algebraic group over $\mathbb{C}$ with root system R, see [10] IV. Moreover, he proposed a Conjecture describing rings $J_{\mathbf{c}}$ in terms of convolution algebras, see [10] IV, 10.5 (a), (b). This Conjecture was verified in many cases by Nanhua Xi, see [16, 17, 18]. In this note we give a more conceptual proof of all previously known results. Our proof also works in some new cases. In general, we prove a statement (see Theorem 4 below) which is weaker than Lusztig's Conjecture.

The proof relies on many results of G.Lusztig in [10]. Our new essential tool is the theory of *central sheaves* on affine flag manifold due to A. Beilinson, D. Gaitsgory, R. Kottwitz, see [6]. One of us used this theory to prove a part of Lusztig's Conjecture, see [4].

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§2. Recollections

2.1. Notations

Let G be an algebraic reductive connected group over the field of l-adic numbers $\bar{\mathbb{Q}}_l$. Let X denote the weight lattice of G and let $R \subset X$ denote the root system of G. Let W_f denote the Weyl group of G and let $\hat{W}$ be the extended Weyl group, that is the semidirect product of W_f and X. Let $l : \hat{W} \to \mathbb{Z}$ be the length function. Let $W \subset \hat{W}$ be the affine Weyl group, that is subgroup generated by W_f and $R \subset X$. Let $S = \{s \in W | l(s) = 1\}$ be the set of simple reflections. It is well known that pair (W, S) is a Coxeter system.

It is well known that any right W_f-coset in $\hat{W}$ contains unique shortest element. Let $\hat{W}^f \subset \hat{W}$ denote the subset of such representatives, so the set $\hat{W}^f$ is in natural bijection with $\hat{W}/W_f$.

Warning. The notations of this paper are different from the notations of [4], for example the group G is denoted by ${}^LG$ in [4].

2.2. Affine Hecke algebra

Let $\mathcal{A} = \mathbb{C}[v, v^{-1}]$. The affine Hecke algebra $\hat{\mathcal{H}}$ is a free $\mathcal{A}$-module with basis $H_w (w \in \hat{W})$ with an associative $\mathcal{A}$-algebra structure defined by $H_w H_{w'} = H_{ww'}$ if $l(ww') = l(w) + l(w')$ and $(H_s + v^{-1})(H_s - v) = 0$ if $s \in S$. The algebra $\hat{\mathcal{H}}$ is endowed with the *Kazhdan-Lusztig basis* $C_w, w \in \hat{W}$, see e.g. [10] IV 1.1. Let $h_{x,y,z} \in \mathcal{A}$ be the structure constants of $\hat{\mathcal{H}}$ with respect to this basis, that is

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z.$$

We say that (left, right or two-sided) ideal $I \subset \hat{\mathcal{H}}$ is KL-ideal if it admits an $\mathcal{A}$-basis consisting of some elements C_x. For $x, y \in \hat{W}$ we write $x \leq_L y$ (resp. $x \leq_R y, x \leq_{LR} y$) if left (resp. right, two-sided) KL-ideal generated by x is contained in left (resp. right, two-sided) KL-ideal generated by y, cf. [9]. The relations $\leq_L, \leq_R, \leq_{LR}$ are preorders. Let $\sim_L, \sim_R, \sim_{LR}$ be the associated equivalence relations. The corresponding equivalence classes are called left, right and two-sided cells, see *loc. cit.* Each two-sided cell is a union of left (resp. right) cells. The map $w \mapsto w^{-1}$ induces a bijection of the set of left cells to the set of right cells. This map induces identity on the set of two-sided cells.

A deep Theorem due to G. Lusztig (see [10] IV 4.8) states that the set of two-sided cells is bijective to the set of unipotent orbits in G.

2.3. Asymptotic Hecke algebra J

There are well defined functions $a : \hat{W} \to \mathbb{N}, \gamma : \hat{W} \times \hat{W} \times \hat{W} \to \mathbb{N}$ such that

$$v^{a(z)} h_{x,y,z} - \gamma_{x,y,z^{-1}} \in v\mathbb{Z}[v] \text{ for all } x, y, z \in \hat{W}$$

and such that for any $z \in \hat{W}$ there exist $x, y \in \hat{W}$ with $\gamma_{x,y,z} \neq 0$. The function a is constant on two-sided cells, see [10] I 5.4.

Let J be a free $\mathbb{Z}-$module with basis $t_x, x \in \hat{W}$. It has a unique structure of an associative $\mathbb{Z}-$algebra such that $t_x t_y = \sum_{z \in \hat{W}} \gamma_{x,y,z} t_{z^{-1}}$, see [10] II. It has a unit element $\sum_{t \in \mathcal{D}} t_d$ where the summation is over the set $\mathcal{D} \subset W$ of *distinguished involutions*, see *loc. cit.* Each left (resp. right) cell contains exactly one element of $\mathcal{D}$. For any two-sided cell $\mathbf{c}$ let $J_{\mathbf{c}} \subset J$ be the $\mathbb{Z}-$submodule generated by $t_x, x \in \mathbf{c}$. The submodule $J_{\mathbf{c}}$ is in fact a subalgebra; moreover $J_{\mathbf{c}} \cdot J_{\mathbf{c}'} = 0$ for $\mathbf{c} \neq \mathbf{c}'$, see [10] II, hence $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$.

We will use many times the following characterization of cells due to G. Lusztig: $w \sim_L w'$ if and only if $t_w t_{w'^{-1}} \neq 0$, see [11] 3.1 (k).

Algebras $J_{\mathbf{c}}$ are examples of *based algebras*, that is algebras over $\mathbb{Z}$ endowed with a basis over $\mathbb{Z}$ such that the structure constants in this basis are nonnegative integers. Another example of a based algebra can be constructed as follows: let F be a reductive algebraic group acting on the finite set X; then the Grothendieck group $K_F(X \times X)$ of the category of $F-$equivariant coherent sheaves on $X \times X$ is a based algebra with the basis given by classes of irreducible $F-$bundles and multiplication given by convolution, see [10] IV 10.2.

Assume for a moment that group G is simply connected. For any two-sided cell $\mathbf{c}$ let $u_{\mathbf{c}}$ be the unipotent element in G corresponding to $\mathbf{c}$ under Lusztig's bijection [10] IV 4.8. Let $F_{\mathbf{c}}$ be the Levi factor of the centralizer $Z_G(u_{\mathbf{c}})$ of $u_{\mathbf{c}}$ in G. In [10] IV 10.5 G. Lusztig conjectured that there exists a finite set $\mathbf{X}$ endowed with an action of $F_{\mathbf{c}}$ such that the based algebras $J_{\mathbf{c}}$ and $K_{F_{\mathbf{c}}}(\mathbf{X} \times \mathbf{X})$ are isomorphic as based algebras, that is the isomorphism respects bases. The aim of this note is to prove a weak form of this Conjecture; more precisely, we replace finite $F_{\mathbf{c}}-$set by a somewhat more general object — finite $F_{\mathbf{c}}-$set of *centrally extended points*, see below.

2.4.

We will need the following well known

Lemma. *Let* Γ_1 *and* Γ_2 *be two left cells lying in the same two-sided cell. Then the intersection* $\Gamma_1 \cap (\Gamma_2)^{-1}$ *is non empty.*

Proof. Let $w \in \Gamma_1$ and $w' \in \Gamma_2^{-1}$. By [11] 3.1 (l) $w \sim_{LR} w'$ if and only if there exists $y \in \hat{W}$ such that $t_w t_y t_{w'} \neq 0$. We see from the characterization of left cells above that $w \sim_L y^{-1}$ and $y \sim_L {w'}^{-1}$. Thus $y^{-1} \in \Gamma_1 \cap \Gamma_2^{-1}$. □

§3. Affine flags

3.1. Notations

Let ${}^L G$ be a split reductive algebraic group over $\mathbb{Z}$ which is Langlands dual to G. To ${}^L G$ one associates the following "loop objects" defined over $\mathbb{Z}$: the (inifinite type) group schemes $\mathbf{K}_{\mathbb{Z}}$ of maps from a formal disc to ${}^L G$, and the Iwahori group $\mathbf{I}_{\mathbb{Z}}$ of maps whose value at the origin lies in a fixed Borel; and ind-schemes $\mathfrak{Fl}_{\mathbb{Z}}$ (the affine flag variety), and $\mathfrak{Gr}_{\mathbb{Z}}$ (the affine Grassmanian). For a field $\mathfrak{k}$ we have $\mathbf{K}(\mathfrak{k}) = {}^L G(O)$, $\mathbf{I}(\mathfrak{k}) = I$, $\mathfrak{Gr}(\mathfrak{k}) = {}^L G(F)/{}^L G(O)$ and $\mathfrak{Fl}(\mathfrak{k}) = {}^L G(F)/I$ where $F = \mathfrak{k}((t))$, $O = \mathfrak{k}[[t]]$, and $I \subset {}^L G(O)$ is an Iwahori subgroup.

We fix a field $\mathfrak{k}$ which is either $\overline{\mathbb{F}}_p$ or complex numbers; we change scalars from $\mathbb{Z}$ to $\mathfrak{k}$ (and drop the subscript $\mathbb{Z}$). By the (derived) category of sheaves we will mean either the (derived) category of l-adic sheaves, $l \neq char(\mathfrak{k})$, or the (derived) category of constructible sheaves on the complex variety for $\mathfrak{k} = \mathbb{C}$. We will denote $\overline{\mathbb{Q}}_l$ by $\mathbb{C}$ in the first case.

The orbits of $\mathbf{I}$ on $\mathfrak{Fl}$, $\mathfrak{Gr}$ are finite dimensional and isomorphic to affine spaces; it is well known that orbits (called *Schubert cells*) are labelled by elements of $\hat{W}$ for $\mathfrak{Fl}$ and $\hat{W}/W_f$ for $\mathfrak{Gr}$. For $w \in \hat{W}$ (respectively $w \in \hat{W}/W_f$) let $\mathfrak{Fl}_w$, $\mathfrak{Gr}_w$ be the corresponding Schubert cells.

Let D^I be the $\mathbf{I}$–equivariant derived category of sheaves on $\mathfrak{Fl}$, and let $\mathcal{P}^I \subset D^I$ be the full subcategory of perverse sheaves. The convolution product defines a functor $* : D^I \times D^I \to D^I$; moreover, $*$ is equipped with a natural associativity constraint (cf. e.g. [7], §1.1.2-1.1.3, or [3], §7.6.1, p. 260).

Let $j_w : \mathfrak{Fl}_w \to \mathfrak{Fl}$ be the natural inclusion and let

$$L_w = j_{w!*}(\underline{\mathbb{Q}_l}[\dim \mathfrak{Fl}_w]),$$

where $\underline{\mathbb{Q}_l}$ is the constant sheaf. Simple objects in $\mathcal{P}^I$ are exhausted by $L_w, w \in \hat{W}$.

Remark. Following the standard yoga one can consider the "graded" versions of D^I_{mix}, $\mathcal{P}^I_{mix}$ of D^I, $\mathcal{P}^I$; here D^I_{mix}, $\mathcal{P}^I_{mix}$ are subcategories in the derived category of mixed l-adic sheaves if $\mathfrak{k}$ is of finite characteristic, and they are objects of the (derived) category of mixed Hodge

D-modules if $\mathfrak{k} = \mathbb{C}$. The convolution on D^I_{mix} is defined. It provides D^I_{mix} with the structure of a monoidal category, and thus equips the Grothendieck group $K(D^I_{mix})$ with an algebra structure; this algebra is isomorphic to $\mathcal{H}$.

We will not use this theory below; however, it underlies the relation between the categories considered in this note and affine Hecke algebras. Also, since the set of representations of an affine Hecke algebra injects into the set of representations of the corresponding p-adic group ${}^L G\left(\mathbb{F}_q((t))\right)$, appearance of the Langlands dual group in the statements below is a manifestation of the geometric Langlands duality.

Notice also that mixed sheaves are used in [4] (in the proof of Theorem 2); the results of this note are based on those of [4].

3.2. Central sheaves

Recall the following definition, see e.g. [4]

Definition. *Let $\mathcal{A}$ be a monoidal category, and $\mathcal{B}$ be a tensor (symmetric monoidal) category. A central functor from $\mathcal{B}$ to $\mathcal{A}$ is a monoidal functor $F : \mathcal{B} \to \mathcal{A}$ together with a functorial isomorphism*

$$\sigma_{X,Y} : F(X) \otimes Y \cong Y \otimes F(X)$$

fixed for all $X \in \mathcal{B}$, $Y \in \mathcal{A}$ subject to the following compatibilities:

(i) *For $X, X' \in \mathcal{B}$ the isomorphism $\sigma_{X,F(X')}$ coincides with the isomorphism induced by commutativity constraint in $\mathcal{B}$.*

(ii) *For $Y_1, Y_2 \in \mathcal{A}$ and $X \in \mathcal{B}$ the composition*

$$F(X) \otimes Y_1 \otimes Y_2 \overset{\sigma_{X,Y_1}\otimes id}{\longrightarrow} Y_1 \otimes F(X) \otimes Y_2 \overset{id\otimes\sigma_{X,Y_2}}{\longrightarrow} Y_1 \otimes Y_2 \otimes F(X)$$

coincides with $\sigma_{X,Y_1\otimes Y_2}$.

(iii) *For $Y \in \mathcal{A}$ and $X_1, X_2 \in \mathcal{B}$ the composition*

$$F(X_1 \otimes X_2) \otimes Y \cong F(X_1) \otimes F(X_2) \otimes Y \overset{id\otimes\sigma_{X_2,Y}}{\longrightarrow} F(X_1) \otimes Y \otimes F(X_2) \overset{\sigma_{X_1,Y}\otimes id}{\longrightarrow}$$
$$Y \otimes F(X_1) \otimes F(X_2) \cong Y \otimes F(X_1 \otimes X_2)$$

coincides with $\sigma_{X_1\otimes X_2,Y}$.

Let $\mathcal{P}_{\mathfrak{Gr}}$ be the category of $\mathbf{K}-$ equivariant perverse sheaves on $\mathfrak{Gr}$. The convolution endows $\mathcal{P}_{\mathfrak{Gr}}$ with monoidal structure and this structure naturally extends to a structure of a commutative rigid tensor category with a fiber functor, and this category is equivalent to $Rep(G)$, see [6, 14]; [3], §5.3, pp 199–215. We will identify $Rep(G)$ with $\mathcal{P}_{\mathfrak{Gr}}$.

In [6] a functor $Z : Rep(G) = \mathcal{P}_{\mathfrak{Gr}} \to \mathcal{P}^I(\mathfrak{Fl})$ was constructed. It enjoys the following properties:

(i) We have a natural isomorphism of functors $\pi_* \circ Z \cong id$, where $\pi : \mathfrak{Fl} \to \mathfrak{Gr}$ is the projection.

(ii) For $\mathcal{F} \in \mathcal{P}_{\mathfrak{Gr}}$, $\mathcal{G} \in \mathcal{P}^I$ we have $\mathcal{G} * Z(\mathcal{F}) \in \mathcal{P}^I$.

(iii) Z is endowed with a natural structure of a central functor from the tensor category $\mathcal{P}_{\mathfrak{Gr}}$ to the monoidal category D^I.

(iv) A unipotent automorphism (monodromy) $\mathfrak{M}$ of the tensor functor Z is given; centrality isomorphism from (iii) commutes with $\mathfrak{M}$.

3.3. Monoidal category $\mathcal{A}_{\mathbf{c}}$

For a subset $S \subset W$ let $\mathcal{P}^I_S$ denote the Serre subcategory of $\mathcal{P}^I$ with simple objects L_w, $w \in S$. Let $\hat{W}_{\leq \mathbf{c}} = \bigcup_{\mathbf{c}' \leq_{LR} \mathbf{c}} \mathbf{c}'$ and $\hat{W}_{<\mathbf{c}} = \bigcup_{\mathbf{c}' <_{LR} \mathbf{c}} \mathbf{c}'$. We abbreviate $\mathcal{P}^I_{\leq \mathbf{c}} = \mathcal{P}^I_{\hat{W}_{\leq \mathbf{c}}}$ and $\mathcal{P}_{<\mathbf{c}} = \mathcal{P}_{\hat{W}_{<\mathbf{c}}}$. Let $\mathcal{P}^I_{\mathbf{c}}$ denote the Serre quotient category $\mathcal{P}^I_{\leq \mathbf{c}}/\mathcal{P}^I_{<\mathbf{c}}$.

For any object $X \in D^I$ and integer i let $H^i(X) \in \mathcal{P}^I$ denote i−th perverse cohomology. For any $X, Y \in \mathcal{P}^I_{\mathbf{c}}$ let us define truncated convolution $X \bullet Y \in \mathcal{P}^I_{\mathbf{c}}$ by $X \bullet Y = H^{a(\mathbf{c})}(X * Y) \mod \mathcal{P}^I_{<\mathbf{c}}$. Let $\mathcal{M}_{\mathbf{c}}$ be the full subcategory of $\mathcal{P}^I_{\mathbf{c}}$ consisting of semisimple objects. It follows from the Decomposition Theorem [2] that the functor $\bullet$ preserves category $\mathcal{M}_{\mathbf{c}}$. The fact the convolution of pure perverse sheaves is pure implies (see [12], 2.6) that the Grothendieck group $K(\mathcal{M}_{\mathbf{c}})$ with the multiplication induced by $\bullet$ is isomorphic to the algebra $J_{\mathbf{c}}$. In [12] a natural associativity constraint was constructed for $\bullet$. Let $\mathbb{I}_{\mathbf{c}} = \bigoplus_{d \in \mathbf{c} \cap \mathcal{D}} L_d \in \mathcal{M}_{\mathbf{c}}$ (recall that $\mathcal{D}$ is the set of distinguished involutions). It is clear that $\mathbb{I}_{\mathbf{c}} \bullet X \simeq X \bullet \mathbb{I}_{\mathbf{c}} \simeq X$ for any $X \in \mathcal{M}_{\mathbf{c}}$. Thus a choice of an isomorphism $\mathbb{I}_{\mathbf{c}} \bullet \mathbb{I}_{\mathbf{c}} \to \mathbb{I}_{\mathbf{c}}$ defines a structure of a monoidal category on $\mathcal{M}_{\mathbf{c}}$, see [12]. We will fix such a choice for the rest of this paper.

Let $\mathcal{A}_{\mathbf{c}}$ be the full subcategory of $\mathcal{P}^I_{\mathbf{c}}$ consisting of all subquotients of $L_w * Z(\mathcal{F}) \mod \mathcal{P}^I_{<\mathbf{c}}$ where $w \in \mathbf{c}$ and $\mathcal{F} \in \mathcal{P}_{\mathfrak{Gr}}$. The following Proposition is proved in [4], Proposition 2.

Proposition. *Restriction of $\bullet$ to $\mathcal{A}_{\mathbf{c}}$ takes values in $\mathcal{A}_{\mathbf{c}}$, is exact in each variable, and it equips $\mathcal{A}_{\mathbf{c}}$ with a structure of a monoidal category with unit object $\mathbb{I}_{\mathbf{c}}$.*

It is clear from the definitions that Lusztig's category $\mathcal{M}_{\mathbf{c}}$ is a monoidal subcategory of $\mathcal{A}_{\mathbf{c}}$ consisting of semisimple objects in $\mathcal{A}_{\mathbf{c}}$.

3.4. Some results from [4]

Let $d \in \mathbf{c}$ be a Duflo involution. Let $\mathcal{A}_d \subset \mathcal{A}_{\mathbf{c}}$ be the full subcategory consisting of all subquotients of $L_d * Z(\mathcal{F})$, $\mathcal{F} \in \mathcal{P}_{\mathfrak{Gr}}$. This category is endowed with a functor $\mathrm{Res}_d : Rep(G) \to \mathcal{A}_d$ defined by $\mathrm{Res}_d(\mathcal{F}) = L_d * Z(\mathcal{F}) \mod \mathcal{P}^I_{<\mathbf{c}}$. The functor Res_d has natural automorphism $\mathfrak{M}_d$

induced by the automorphism $\mathfrak{M}$ of monodromy. The following Theorem is proved in [4] Theorems 1 and 2:

Theorem. (a) *The category $\mathcal{A}_d$ has a natural structure of a tensor category with unit object L_d, functor Res_d has a natural structure of a tensor functor and $\mathfrak{M}_d$ is an automorphism of the tensor functor Res_d.*

(b) *Moreover, there exists a subgroup $H_d \subset G$, a unipotent element $N_d \in G$ commuting with H_d, an equivalence of tensor categories $\Phi_d : Rep(H_d) \to \mathcal{A}_d$, and a natural transformation of functors $Res^G_{H_d} \simeq \Phi_d \circ Res_d$ which interwines the tensor automorphism $\mathfrak{M}_d$ with the action of N_d. The pair (H_d, N_d) is unique up to a simultaneous conjugacy. The element N_d is conjugate to $u_{\mathbf{c}}$.*

It is proved in [13] that the intersection $\mathbf{c} \cap \hat{W}^f$ consists of a unique *canonical* left cell which we will denote $\Gamma_{\mathbf{c}}$ (recall that $\hat{W}^f$ is a set of shortest representatives of right W_f−cosets in $\hat{W}$). In particular, there exists a unique distinguished involution $d = d^f \in \mathbf{c} \cap \hat{W}^f$. We call d^f a *canonical* distinguished involution.

Theorem. (see [4] Theorem 3) (a) *The set of irreducible objects of $\mathcal{A}_{d^f}$ is $\{L_w | w \in \Gamma_{\mathbf{c}} \cap (\Gamma_{\mathbf{c}})^{-1}\}$.*

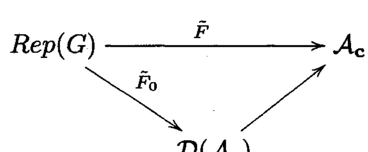
(b) *The subgroup H_{d^f} contains a maximal reductive subgroup of the centralizer $Z_G(u_{\mathbf{c}})$.*

3.5. Central action of $Rep(F_{\mathbf{c}})$

Consider the functor $\tilde{F} : Rep(G) = \mathcal{P}_{\mathfrak{G}r} \to \mathcal{A}_{\mathbf{c}}$ defined by $\tilde{F}(\mathcal{F}) = Z(\mathcal{F}) * \mathbb{I}_{\mathbf{c}} \mod \mathcal{P}^I_{<\mathbf{c}}$. It is easy to see from 3.2 that the functor $\tilde{F}$ has a natural structure of a central functor. Moreover, this functor has a canonical tensor unipotent automorphism $\mathfrak{M}$ (monodromy) commuting with the centrality isomorphism.

Theorem 1. *There exists a central functor $F : Rep(Z_G(u_{\mathbf{c}})) \to \mathcal{A}_{\mathbf{c}}$ such that $\tilde{F} = F \circ Res^G_{Z_G(u_{\mathbf{c}})}$. Moreover, automorphism $\mathfrak{M}$ is induced by the action of $u_{\mathbf{c}}$ on $Res^G_{Z_G(u_{\mathbf{c}})}$.*

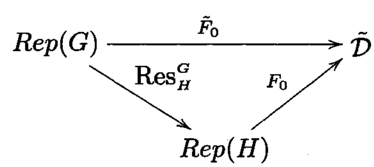
Proof. Let $\mathcal{D}(\mathcal{A}_{\mathbf{c}})$ denote the Drinfeld double of the monoidal category $\mathcal{A}_{\mathbf{c}}$, see e. g. [8]. By the universal property of double the functor $\tilde{F}$ can be factorized as



where $\tilde{F}_0$ is a braided functor. Recall that the unit object of $\mathcal{D}(\mathcal{A}_{\mathbf{c}})$ is $\mathbb{I}_{\mathbf{c}}$ endowed with the centrality isomorphism induced by the unity isomorphisms:

$$\mathbb{I}_{\mathbf{c}} \bullet X \cong X \cong X \bullet \mathbb{I}_{\mathbf{c}}.$$

We remark that $\mathbb{I}_{\mathbf{c}}$ considered as an object of $\mathcal{D}(\mathcal{A}_{\mathbf{c}})$ is irreducible: it is easy to see from Lemma 2.4 that any subobject of $\mathbb{I}_{\mathbf{c}}$ in $\mathcal{A}_{\mathbf{c}}$ does not lie in the center of $\mathcal{A}_{\mathbf{c}}$ even on the level of $K-$theory. Now consider the full subcategory $\tilde{\mathcal{D}} \subset \mathcal{D}(\mathcal{A}_{\mathbf{c}})$ consisting of all subquotients of objects $\tilde{F}_0(X)$, $X \in Rep(G)$. Then the conditions of Proposition 1 [4] are satisfied for the pair $(\tilde{\mathcal{D}}, \tilde{F}_0)$. Consequently, the functor $\tilde{F}_0$ factors through the restriction functor Res^G_H



for some subgroup $H \subset G$ and the action of $\mathfrak{M}$ is given by some unipotent element $u \in Z_G(H)$. Theorem 2 of [4] identifies u with $u_{\mathbf{c}}$. Hence the subgroup H is contained in $Z_G(u_{\mathbf{c}})$. Without loss of generality we can assume that $H = Z_G(u_{\mathbf{c}})$. We set the functor F to be equal to the composition

$$Rep(Z_G(u_{\mathbf{c}})) \xrightarrow{F_0} \tilde{\mathcal{D}} \to \mathcal{D}(\mathcal{A}_{\mathbf{c}}) \to \mathcal{A}_{\mathbf{c}}.$$

The Theorem is proved. □

Let us restrict F to the semisimple part of the category $Rep(Z_G(u_{\mathbf{c}}))$, that is to the category $Rep(F_{\mathbf{c}})$ where $F_{\mathbf{c}}$ is the maximal reductive factor of $Z_G(u_{\mathbf{c}})$.

Proposition. *For any $X \in Rep(F_{\mathbf{c}})$ the object $F(X) \in \mathcal{A}_{\mathbf{c}}$ is semisimple.*

Proof. We can assume that X is simple. Let $Y \in Rep(G)$ be an object such that X is a subquotient of $\mathrm{Res}^G_{F_{\mathbf{c}}}(Y)$. The object $\tilde{F}(Y)$ carries the monodromy filtration; by Gabber's Theorem (see [1], Theorem 5.1.2) it coincides with the weight filtration, so the associated graded object $gr\tilde{F}(Y)$ is semisimple by [2], 5.4.6. By the Theorem 1 we get the same filtration from the action of $u_{\mathbf{c}}$ on $F(\mathrm{Res}^G_{Z_G(u_{\mathbf{c}})})$. But the object X is a direct summand of $gr\ \mathrm{Res}^G_{Z_G(u_{\mathbf{c}})}(Y)$ with respect to this filtration. □

As a corollary we get

Theorem 2. *The functor F restricts to a central functor $F : Rep(F_{\mathbf{c}}) \to \mathcal{M}_{\mathbf{c}}$.*

§4. Canonical cell

4.1. Module categories

In this subsection we review basic theory of module categories. A more detailed exposition will appear in [15]. We will work over a fixed field k.

Let $\mathcal{C}$ be an abelian monoidal category with biexact tensor product and with unit object $\mathbf{1}$.

Definition. *A module category $\mathcal{M}$ over $\mathcal{C}$ is an abelian category $\mathcal{M}$ endowed with*

1) *An exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$,*

2) *Functorial* associativity isomorphisms $V \otimes (V' \otimes M) \simeq (V \otimes V') \otimes M$ *for any $V, V' \in \mathcal{C}, M \in \mathcal{M}$,*

3) *Functorial* unit isomorphisms $\mathbf{1} \otimes M \to M$ *for any $M \in \mathcal{M}$ subject to the usual pentagon and triangle axioms: the following diagrams where all arrows are associativity and unit isomorphisms commute:*

$$
\begin{array}{ccc}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes M & \\
\swarrow & & \searrow \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes M & & (V_1 \otimes V_2) \otimes (V_3 \otimes M) \\
\downarrow & & \downarrow \\
V_1 \otimes ((V_2 \otimes V_3) \otimes M) & \longrightarrow & V_1 \otimes (V_2 \otimes (V_3 \otimes M))
\end{array}
$$

$$
\begin{array}{ccc}
(V \otimes \mathbf{1}) \otimes M & \longrightarrow & V \otimes (\mathbf{1} \otimes M) \\
& \searrow \quad \swarrow & \\
& V \otimes M &
\end{array}
$$

The notions of module functors, and, in particular, equivalences of module categories are defined in the obvious way.

Remark. Module categories over general monoidal categories were considered by L. Crane and I. Frenkel, see [5]. The name comes from considering the notion of a monoidal category as categorification of the notion of a ring. Module categories seem to be of importance in Conformal Field Theory where they are implicitly considered in the context of Boundary Conformal Field Theory.

Of course the category $\mathcal{C}$ is a module category over itself with associativity and unit isomorphisms induced by ones in tensor category $\mathcal{C}$. Another example can be obtained as follows. Let $A \in \mathcal{C}$ be an associative algebra with unit, that is associative multiplication $A \otimes A \to A$ is defined and there is an inclusion $\mathbf{1} \to A$ satisfying unit axioms. Then category $Mod_{\mathcal{C}}(A)$ of *right* A−modules in the category $\mathcal{C}$ has an obvious structure of a module category.

We will say that a module category $\mathcal{M}$ is generated by objects $M_1, M_2, \ldots \in \mathcal{M}$ over $\mathcal{C}$ if any object of $\mathcal{M}$ is a subquotient of $V \otimes M_i$ for some $V \in \mathcal{C}$. We will say that $\mathcal{M}$ is finitely generated over $\mathcal{C}$ if there exists finitely many (equivalently one) objects $M_1, \ldots \in \mathcal{M}$ such that $\mathcal{M}$ is generated by them over $\mathcal{C}$.

Assume from now on that the category $\mathcal{C}$ is rigid. Then there exists a canonical isomorphism $Hom(V \otimes M, N) \cong Hom(M, V^* \otimes N)$ for any $V \in \mathcal{C}, M, N \in \mathcal{M}$.

Now assume that both categories $\mathcal{C}$ and $\mathcal{M}$ are semisimple. For any two objects $M, N \in \mathcal{M}$ the functor $\mathcal{C} \to Vec_k$, $V \mapsto Hom(V \otimes M, N)$ is representable by an ind-object $\underline{\mathrm{Hom}}(M, N)$ of $\mathcal{C}$. By Yoneda's Lemma $\underline{\mathrm{Hom}}$ is a bifunctor $\mathcal{M}^{op} \times \mathcal{M} \to$ ind-objects of $\mathcal{C}$.

Lemma. *Assume that $\omega : \mathcal{M} \to \mathcal{C}$ is an exact faithful tensor functor. Then for any $M, N \in \mathcal{M}$ $\underline{\mathrm{Hom}}(M, N) \in \mathcal{C}$.*

Proof. It is clear that the map $\underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(\omega(M), \omega(N))$ is an imbedding. □

Assume that for any $M, N \in \mathcal{M}$ the ind-object $\underline{\mathrm{Hom}}(M, N)$ is an object of $\mathcal{C}$. For any three objects $M, N, K \in \mathcal{M}$ a functorial and associative multiplication $\underline{\mathrm{Hom}}(N, K) \otimes \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M, K)$ is defined (note that the order of factors is opposite to the intuitive one). In particular, for any object $M \in \mathcal{C}$ the object $\underline{\mathrm{Hom}}(M, M)$ has a natural structure of an associative algebra in $\mathcal{C}$. Assume that $\underline{\mathrm{Hom}}(M, X) \neq 0$ for any $X \in \mathcal{M}$, that is the category $\mathcal{M}$ is generated by M over $\mathcal{C}$. It is easy to see that the functor $F_M : \mathcal{M} \to Mod_{\mathcal{C}}(A)$, $F_M(X) = \underline{\mathrm{Hom}}(M, X)$ is a tensor functor. Moreover, we claim that this functor is an equivalence of categories. The proof is straightforward: first one shows that the functor F_M induces an isomorphism on Hom's for objects of the form $V \otimes M$, $V \in \mathcal{C}$, and then one uses the fact that any object of $\mathcal{M}$ admits a resolution by objects of the form $V \otimes M$. Summarizing we get the following

Proposition. *Let $\mathcal{C}$ be a semisimple rigid monoidal category and let $\mathcal{M}$ be a semisimple module category over $\mathcal{C}$. Assume that there exists an exact faithful module functor $\omega : \mathcal{M} \to \mathcal{C}$. Then the category*

$\mathcal{M}$ is equivalent to the category $Mod_{\mathcal{C}}(A)$ for some associative algebra A. Moreover one can choose $A = \underline{\mathrm{Hom}}(M, M)$ for any object $M \in \mathcal{C}$ generating $\mathcal{M}$ over $\mathcal{C}$.

Let $\mathcal{M} = Mod_{\mathcal{C}}(A)$ be a module category. Consider the category $Fun(\mathcal{M}, \mathcal{M})$ consisting of module functors $\mathcal{M} \to \mathcal{M}$. It is clear that the category $Fun(\mathcal{M}, \mathcal{M})$ is a monoidal category with tensor product induced by the composition of functors and identity functor as unit object. One shows easily that the monoidal category $Fun(\mathcal{M}, \mathcal{M})$ is equivalent to the category of $A - A$ bimodules in $\mathcal{C}$ with the obvious monoidal structure.

4.2. Module categories over $Rep(H)$

In this subsection we specialize ourselves to the case when $\mathcal{C} = Rep(H)$ for some reductive group H over an algebraically closed field k of characteristic zero.

Examples. (i) $Rep(H)$ with the associativity and unit isomorphisms induced from those in the monoidal category $Rep(H)$ is of course a module category over $Rep(H)$.

(ii) More generally, let X be a variety endowed with an $H-$action. The category $Coh_H(X)$ of coherent $H-$equivariant sheaves on X is a module category. We get example (i) by letting X =point.

(iii) Let $1 \to \mathbb{G}_m \to \tilde{H} \to H \to 1$ be a central extension of H whose kernel is identified with the multiplicative group $\mathbb{G}_m$ (we will call such a data just "a central extension"); of course, such an extension is necessarily the pushforward if a central extension $1 \to C \to \tilde{H}' \to H \to 1$ under a homomorphism $C \to \mathbb{G}_m$ for a finite cyclic group C. Then the category $Rep^1(\tilde{H})$ of representations V of $\tilde{H}$ such that $\mathbb{G}_m$ acts on V via identity character is a module category over $Rep(H)$. We will also consider the category $Rep^{-1}(\tilde{H})$ of representations of $\tilde{H}$ on which $\mathbb{G}_m$ acts via character $x \mapsto x^{-1}$.

We will say that a module category $\mathcal{C}$ has a *quasifiber functor* if there exists a faithful exact module functor $\omega : \mathcal{C} \to Rep(H)$. The quasifiber functor if it exists is not unique: for any $V \in Rep(H)$ and quasifiber functor ω the functor $M \mapsto \omega(M) \otimes V$ is again a quasifiber functor.

Example. (iv) Let $H' \subset H$ be a subgroup of finite index. Let $1 \to \mathbb{G}_m \to \tilde{H}' \to H' \to 1$ be a central extension. The category $Rep^1(\tilde{H}')$ is a module category over $Rep(H)$ with the $Rep(H)-$action which factors through the restriction functor $Rep(H) \to Rep(H')$. Let $V_0 \in Rep^{-1}(\tilde{H}')$ be a fixed object. It is easy to see that the functor $V \mapsto Ind_{H'}^{H}(V \otimes V_0), V \in Rep^1(\tilde{H}')$ is a quasifiber functor ($\mathbb{G}_m$ acts trivially on $V \otimes V_0$ so $V \otimes V_0$ can be considered as a representation of H').

Example (iv) reduces to the example (i) with $X = H/H'$ if the central extension splits.

Example. (iv′) Finite sums of categories considered in Example (iv) admit the following invariant description.

A finite *H-set of centrally extended points* is the following collection of data:

(a) A finite set X with an H action;

(b) For any $x \in X$ a central extension $\mathbb{G}_m \to \tilde{H}(x) \to H(x)$ of the stabilizer $H(x) = Stab_H(x)$. These should be *equivariant* under the action of H, i.e. for every $g \in G$ an isomorphism of $i^g_x : \tilde{H}(x) \widetilde{\longrightarrow} \tilde{H}(gx)$ identical on $\mathbb{G}_m$ and covering the map $C_g : H(x) \to H(g(x))$ (conjugation by x) should be given. i^x_g should coincide with the conjugation by g when $g \in H(x)$ and should satisfy $i^x_{g_1 g_2} = i^{g_2(x)}_{g_1} \circ i^x_{g_2}$.

Let $\mathbf{X}$ be a finite set of centrally extended points. An *equivariant sheaf* on $\mathbf{X}$ is a sheaf $\mathcal{F}$ of finite dimensional $\mathbb{C}$-vector spaces on the underlying set X together with

(a) a projective H-equivariant structure on $\mathcal{F}$.

(b) For every $x \in X$ an action of $\tilde{H}(x)$ on the stalk $\mathcal{F}_x$, comprising an object of $Rep^1(\tilde{H}(x))$.

The data (a) and (b) should be compatible, i.e. (b) should be H-equivariant, and the projective action of $H(x)$ arising from (b) must coincide with the one arising from (a).

Equivariant sheaves on $\mathbf{X}$ obviously form a category, which we denote by $Coh_H(\mathbf{X})$.

Choosing a set of representatives x_i for H-orbits on X we see that the data of a centrally extended set with underlying equivariant set X is equivalent to a collection $\tilde{H}_i$ of central extensions $\mathbb{G}_m \to \tilde{H}_i \to H_i = H(x_i)$. The category $Coh_H(\mathbf{X})$ is then canonically equivalent to the direct sum $\oplus_i Rep^1(\tilde{H}_i)$.

Theorem 3. *Let $\mathcal{M}$ be a semisimple module category over $Rep(H)$ finitely generated over $Rep(H)$. Assume that $\mathcal{M}$ admits a quasifiber functor. Then $\mathcal{M}$ is equivalent to $Coh_H(\mathbf{X})$ for some centrally extended finite H-set $\mathbf{X}$ (i.e. to a finite direct sum of some categories of the type described in Example* (iv) *above).*

Proof. By Proposition 4.1 the module category $\mathcal{M}$ is equivalent to the module category $Mod_{Rep(H)}(A)$ for some finite dimensional H-algebra A.

Lemma. *Semisimplicity of $\mathcal{M}$ implies semisimplicity of A as an algebra in the category of vector spaces.*

Proof. Consider the regular representation A_{reg} of A as an object of $Mod_{Rep(H)}(A)$. Let $r(A)$ be the Jacobson radical of A. It is clear that $r(A)$ is $H-$invariant, hence $r(A)$ is subobject of A_{reg} in $Mod_{Rep(H)}(A)$. Suppose $A_{reg} = r(A) \oplus A_1$ for some $A_1 \in Mod_{Rep(H)}(A)$. Applying the forgetful functor $Mod_{Rep(H)}(A) \to Mod(A)$ to A_{reg} we would get a complement to $r(A)$, which is impossible unless $r(A) = 0$. □

Now let $A \ni 1 = \sum e_i$ is the decomposition of 1 in the sum of minimal central orthogonal idempotents. The group H acts on the set $\{e_i\}$. We may assume that this action is transitive. Let $H_1 \subset H$ be the stabilizer of e_1, the subgroup of finite index in H. The algebra e_1Ae_1 is isomorphic to the matrix algebra and the group H_1 acts on e_1Ae_1. We can choose a projective representation V of H_1 and an isomorphism $e_1Ae_1 \cong End(V)$. It is clear that $A \cong Ind_{H_1}^H(e_1Ae_1) = Ind_{H_1}^H End(V)$.

The projective action of H_1 on V comes from an action of a central extension $\tilde{H}_1$ of H_1. Let us consider the corresponding category from example (iv) $Rep^1(\tilde{H}_1)$. The representation V can be viewed as an object of this category and one easily calculates $\underline{\mathrm{Hom}}(V,V) = Ind_{H_1}^H End(V)$. The Theorem is proved. □

4.3. Module category corresponding to the canonical cell

For any subset $S \subset \mathbf{c}$ let $\mathcal{M}_S \subset \mathcal{M}_\mathbf{c}$ denote the full Serre subcategory with simple objects L_w, $w \in S$. Let $\Gamma \subset \mathbf{c}$ be the canonical right cell, see [13]. Let $\mathcal{M}_\Gamma \subset \mathcal{M}_\mathbf{c}$ be the corresponding subcategory. By the definition of a right cell we have $\mathcal{M}_\Gamma \bullet \mathcal{M}_\mathbf{c} \subset \mathcal{M}_\Gamma$. Define on $\mathcal{M}_\Gamma$ a structure of a module category over $Rep(F_\mathbf{c})$ by the formula $V \otimes M = F(V) \bullet M$ where F is a functor from Theorem 2. Note that due to the centrality of functor F we have $F(Rep(F_\mathbf{c})) \bullet \mathcal{M}_\Gamma = \mathcal{M}_\Gamma \bullet F(Rep(F_\mathbf{c})) \subset \mathcal{M}_\Gamma$ so this is well defined. We claim that this category admits a quasifiber functor. Indeed, let $\{w_1, w_2, \dots\} \subset \Gamma^{-1}$ be a set of representatives of all *right* cells contained in $\mathbf{c}$ (such a set exists by the Lemma 2.4 and is finite since Lusztig proved (see [10] II 2.2) that the number of cells in an affine Weyl group is finite). Consider the functor $\mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma\cap\Gamma^{-1}}$, $X \mapsto X \bullet (\oplus L_{w_i})$. Recall that in [4] the monoidal category $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}$ was identified with $Rep(F_\mathbf{c})$, see 3.4. It is a simple exercise to check that this functor is module functor with the module structure induced by the associativity isomorphism in $\mathcal{M}_\mathbf{c}$, and it is clear that it is exact and faithful. So this is quasifiber functor, and we can apply Theorem 3. We get

Proposition. *The category $\mathcal{M}_\Gamma$ as a module category over $Rep(F_\mathbf{c})$ is equivalent to the category $Coh_{F_\mathbf{c}}(\mathbf{X})$ of coherent sheaves on a finite $F_\mathbf{c}$–set $\mathbf{X}$ of possibly centrally extended points.*

Note that the inclusion $\mathcal{M}_{\Gamma\cap\Gamma^{-1}} \subset \mathcal{M}_\Gamma$ gives us a distinguished point $\mathbf{0} \in \mathbf{X}$ which is just a usual (not centrally extended) point fixed by the $F_\mathbf{c}$–action.

§5. Square of a finite set

5.1. Monoidal category $Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$

Consider the category $Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$ consisting of all module functors $Coh_{F_\mathbf{c}}(\mathbf{X}) \to Coh_{F_\mathbf{c}}(\mathbf{X})$. It is a monoidal category with the tensor product induced by the composition of functors and unit object equal to the identity functor. Since the category $Coh_{F_\mathbf{c}}(\mathbf{X})$ is semisimple, any functor $F \in Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$ has left and right adjoint functors F^* and ${}^*F$. Observe that adjoint of tensor functor has a natural structure of a module functor and hence $F^*, {}^*F \in Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$. Standard properties of adjoint functors show that F^* and ${}^*F$ are right and left duals of F in the monoidal category $Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$. So the category $Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$ is rigid.

Lemma. *The category $Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$ is semisimple.*

Proof. Let us choose an $F_\mathbf{c}$–algebra A and an equivalence $Coh_{F_\mathbf{c}}(\mathbf{X}) \to Mod_{Rep(F_\mathbf{c})}(A)$. Then the category $Fun_{F_\mathbf{c}}(\mathbf{X}, \mathbf{X})$ is equivalent to the category of $A - A$ bimodules in $Rep(F_\mathbf{c})$, or to the category of $A \otimes A^{op}$–modules in $Rep(F_\mathbf{c})$ where A^{op} is A with the opposite multiplication. The latter category is clearly semisimple since $A \otimes A^{op}$ is a semisimple algebra. □

Note that in semisimple monoidal category left and right duals coincide, so in the future we will not distinguish left and right duals.

Remark. For an H-set X it is easy to construct an equivalence $Fun_H(X, X) \cong Coh_H(X \times X)$. Let us spell out a generalization of this statement to centrally extended H-sets.

Recall that for two central extensions $1 \to \mathbb{G}_m \to \tilde{H}_i \to H \to 1$, $i = 1, 2$ their product is defined by $\tilde{H}_{12} = \tilde{H}_1 \times_H \tilde{H}_2 / \mathbb{G}_m$, where $\mathbb{G}_m$ is embedded antidiagonally; also for a central extension $\tilde{H}$ the opposite central extension $\tilde{H}'$ is the same group with the same homomorphism to H but with the identification of its kernel with $\mathbb{G}_m$ replaced by the opposite one (composition of the original one with the map $x \mapsto x^{-1}$).

Now for two centrally extended H-sets $\mathbf{X}$, $\mathbf{Y}$ one can define their product in the obvious manner: the underlying equivariant set is $X \times$

Y, where X, Y are equivariant sets underlying $\mathbf{X}$ and $\mathbf{Y}$; the central extension $\tilde{H}(x,y)$ is the product of restrictions of $\tilde{H}(x)$ and of $\tilde{H}(y)$ to $H(x,y)$. For a centrally extended H-set $\mathbf{X}$ we obtain the *opposite* centrally extended set $\mathbf{X}'$ replacing each of the central extensions $\mathbb{G}_m \to \tilde{H}(x) \to H(x)$ by the opposite one.

If $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are centrally extended H-sets with underlying H-sets X, Y, Z, then for $\mathcal{F} \in Coh_H(X \times Y')$, $\mathcal{G} \in Coh_H(Y \times Z)$ the sheaf $\mathcal{F} \boxtimes \underline{\mathbb{C}} \otimes \underline{\mathbb{C}} \otimes \mathcal{G}$ on $X \times Y \times Z$ carries a natural structure of an equivariant sheaf on $\mathbf{X} \times \mathbf{Y}_0 \times \mathbf{Z}$ (here $\mathbf{Y}_0$ is Y equipped with the trivial (split) centrally extended structure). Thus the *convolution* $\mathcal{F} * \mathcal{G} = \mathrm{pr}_{13*}(\mathcal{F} \boxtimes \underline{\mathbb{C}} \bigotimes \underline{\mathbb{C}} \boxtimes \mathcal{G})$ (where $\mathrm{pr}_{13} : X \times Y \times Z \to X \times Z$ is the projection) carries the structure of an equivariant sheaf on $\mathbf{X} \times \mathbf{Z}$. In particular, for $\mathbf{X} = \mathbf{Y} = \mathbf{Z}$ we get a monoidal structure on $Coh_H(\mathbf{X} \times \mathbf{X}')$; and for $\mathbf{X} = \mathbf{Y}$, and $\mathbf{Z}$ being the point with the split central extension we get a monoidal functor of $Coh_H(\mathbf{X} \times \mathbf{X}') \to Fun_H(\mathbf{X})$. It is easy to see that this functor is an equivalence.

5.2. Monoidal functor G

We have a monoidal functor $G : \mathcal{M}_{\mathbf{c}}^{op} \to Fun_{F_{\mathbf{c}}}(\mathbf{X}, \mathbf{X})$, $G(X) = ? \bullet X$ where $\mathcal{M}_{\mathbf{c}}^{op}$ is $\mathcal{M}_{\mathbf{c}}$ with the *opposite* tensor product. It is clear that G is exact and faithful.

The main result of this section is the following

Theorem 4. *The functor G is a tensor equivalence*

$$\mathcal{M}_{\mathbf{c}}^{op} \to Fun_{F_{\mathbf{c}}}(\mathbf{X}, \mathbf{X}).$$

Corollary. *Suppose that any subgroup of finite index in $F_{\mathbf{c}}$ has no nontrivial projective representations. Then Lusztig's Conjecture holds for the cell* $\mathbf{c}$.

5.3. A result of G. Lusztig

The following result cited from [10] II Proposition 1.4 is cruicial for the proof of Theorem 4.

Proposition. (a) *Assume that $L_x \bullet L_y$, $x, y \in \mathbf{c}$ contains as a direct summand L_d, $d \in \mathcal{D}$. Then $x = y^{-1}$ and the multiplicity of L_d in $L_x \bullet L_y$ is one.*

(b) *For any $x \in \mathbf{c}$ the truncated convolution $L_x \bullet L_{x^{-1}}$ contains L_d for a uniquely defined $d \in \mathcal{D} \cap \mathbf{c}$.*

5.4. Proof of the Theorem 4.

Since the category $Fun_{F_{\mathbf{c}}}(\mathbf{X}, \mathbf{X})$ is semisimple it is enough to prove the following statements:

(i) Any functor from $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ appears as a direct summand of $G(L)$, $L\in\mathcal{M}_{\mathbf{c}}^{op}$.

(ii) For any $w\in\mathbf{c}$ the functor $G(L_w)$ is irreducible.

(iii) For $w,w'\in\mathbf{c}$ an isomorphism $G(L_w)=G(L_{w'})$ implies $w=w'$.

5.4.1. We begin with the following

Lemma. (a) *For any $w\in\Gamma$ the functor $G(L_w)$ is irreducible. Moreover G induces an equivalence $\mathcal{M}_\Gamma\to\{$module functors $Rep(F_{\mathbf{c}})\to\mathcal{M}_\Gamma\}$.*

(b) *For any $w\in\Gamma^{-1}$ the functor $G(L_w)$ is irreducible. Moreover, G induces an equivalence $\mathcal{M}_{\Gamma^{-1}}\to\{$module functors $\mathcal{M}_\Gamma\to Rep(F_{\mathbf{c}})\}$.*

Proof. (a) It is clear that $L_v\bullet L_w=0$ for any $w\in\Gamma$, $v\in\Gamma-\Gamma\cap\Gamma^{-1}$. So the functor $G(L_w)$ can be considered as a functor $Rep(F_{\mathbf{c}})=\mathcal{M}_{\Gamma\cap\Gamma^{-1}}\to\mathcal{M}_\Gamma$. For any module category $\mathcal{M}$ over $Rep(F_{\mathbf{c}})$ the map $f\mapsto f(\mathbf{1})$ defines an equivalence of categories $Fun_{Rep(F_{\mathbf{c}})}(Rep(F_{\mathbf{c}}),\mathcal{M})\to\mathcal{M}$. In our case $G(L_w)(\mathbf{1})=L_{d^f}\bullet L_w=L_w$ and (a) is proved.

(b) Let us first check that for $w\in\Gamma^{-1}$ we have

$$G(L_w)^*\cong G(L_{w^{-1}}),\qquad (*)$$

where $G(L_w)^*$ is the functor adjoint to $G(L_w)$.

For $w\in\Gamma^{-1}$ the functor $G(L_w)$ maps $\mathcal{M}_\Gamma$ to $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}=Rep(F_{\mathbf{c}})\subset\mathcal{M}_\Gamma$. Thus $G(L_w)^*$ sends L_v to zero unless $v\in\Gamma\cap\Gamma^{-1}$; hence part (a) of the Lemma implies that $G(L_w)^*$ is isomorphic to $G(L)$ for some $L\in\mathcal{M}_\Gamma$. To see that $L\cong L_{w^{-1}}$ it is enough to check that for $v\in\Gamma$ the space $Hom((G(L_v),G(L_w)^*))$ is one dimensional if $v=w^{-1}$, and is zero otherwise. We have $Hom((G(L_v),G(L_w)^*))=Hom(G(L_w)\circ G(L_v),Id_{\mathcal{M}_\Gamma})$. Notice that $G(L_w)\circ G(L_v)$ preserves the direct summand $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}\subset\mathcal{M}_\Gamma$ and is zero on its complement. It follows that

$$Hom(G(L_w)\circ G(L_v),Id_{\mathcal{M}_\Gamma})=Hom(G(L_w)\circ G(L_v)|_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}},Id_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}}).$$

Since $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}\cong Rep(F_{\mathbf{c}})$ by 3.4, and the category of module functors from $Rep(H)$ to $Rep(H)$ (considered as the free module over itself) is equivalent to $Rep(H)$, we see that

$$Hom(G(L_w)\circ G(L_v),Id_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}})=Hom_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}}(L_w\bullet L_v,L_{d^f}),$$

thus $(*)$ follows from Proposition 5.3.

Irreducibility of $G(L_w)$ follows from $(*)$ and part (a), because the dual object of an irreducible object (in the category of functors) is irreducible. It remains to check that any module functor $\phi:\mathcal{M}_\Gamma\to Rep(F_{\mathbf{c}})$ is isomorphic to the one coming from some $L\in\mathcal{M}_{\Gamma^{-1}}$. Consider ϕ as an endofunctor $\mathcal{M}_\Gamma$ (i.e. take its composition with the imbedding

$Rep(F_{\mathbf{c}}) = \mathcal{M}_{\Gamma \cap \Gamma^{-1}} \hookrightarrow \mathcal{M}_{\Gamma}$); then by (a) the adjoint functor ϕ^* is isomorphic to $G(L)$ for $L \in \mathcal{M}_{\Gamma}$, so this statement also follows from $(*)$. □

We can now prove (i).

Corollary. *Any irreducible functor from $Fun_{Rep(F_{\mathbf{c}})}(\mathbf{X}, \mathbf{X})$ appears as a direct summand of $G(L_{w'} \bullet L_w), w \in \Gamma, w' \in \Gamma^{-1}$.*

Proof. We need to prove that any irreducible functor

$$f \in Fun_{Rep(F_{\mathbf{c}})}(\mathbf{X}, \mathbf{X})$$

is a direct summand of a composite functor $Coh_{F_{\mathbf{c}}}(\mathbf{X}) \to Rep(F_{\mathbf{c}}) \to Coh_{F_{\mathbf{c}}}(\mathbf{X})$. For this we choose any functor $g : Coh_{F_{\mathbf{c}}}(\mathbf{X}) \to Rep(F_{\mathbf{c}})$ such that the composition $g \circ f$ is nonzero. Let g^* be the adjoint functor to g. Then f is evidently a direct summand of $(g^* \circ g) \circ f$ which admits the required factorisation $g^* \circ (g \circ f)$. □

5.4.2. **Lemma.** *For any $w \in \mathbf{c} - \mathcal{D}$ we have $Hom(G(L_w), id) = 0$.*

Proof. We note that L_w as an object of $\mathcal{M}_{\mathbf{c}}$ is a direct summand of $L_u \bullet L_v$ where $u \in \Gamma^{-1}$ and $v \in \Gamma$ (this follows easily for the example from Theorem 1.8 in [10] II). Assume that $Hom(G(L_w), id) \neq 0$ and hence $Hom(G(L_u \bullet L_v), id) \neq 0$. Then we get a nonzero transformation $G(L_u) \to G(L_v)^*$; since both functors are irreducible by Lemma 5.4.1 they are actually isomorphic. On the other hand, Proposition 5.3 provides a non-zero transformation $G(L_u) \circ G(L_u^{-1}) \to Id$, which also yields an isomorphism $G(L_u)^* \cong G(L_{u^{-1}})$. Thus $G(L_{u^{-1}}) \cong G(L_v)$, and by Lemma 5.4.1 this yields $L_{u^{-1}} \cong L_v$, so $v = u^{-1}$. Furthermore, $dimHom(G(L_u \bullet L_v), id) = dimHom(G(L_v), G(L_u)^*) = 1$; and by Proposition 5.3 the object $L_u \bullet L_v$ contains L_d for a uniquely defined $d \in \mathcal{D}$. Since $Hom(G(L_d), id) \neq 0$ we have that $Hom(L_w, id) = 0$ if $w \neq d$. The Lemma is proved. □

Now we can prove (ii)

Corollary. *For any $w \in \mathbf{c}$ the functor $G(L_w)$ is irreducible.*

Proof. Consider the adjoint functor $G(L_w)^*$. By Lemma 5.4.1 any summand of $G(L_w)^*$ appears as a direct summand of $G(L_{w'})$. For such w' we have a non-zero transformation $G(L_{w'}) \circ G(L_w) = G(L_w \bullet L_{w'}) \to Id$, and by Lemma 5.4.2 and Proposition 5.3 this is possible only when $w' = w^{-1}$. If $G(L_w)$ is reducible this implies that $dimHom(G(L_w) \circ G(L_{w^{-1}}), id) > 1$. On the other hand $dimHom(G(L_w) \circ G(L_{w^{-1}}), id) = dimHom(G(L_{w^{-1}} \bullet L_w), id) = 1$ by Proposition 5.3 and Lemma 5.4.2, and we get a contradiction. □

5.4.3. We can now prove (iii). Assume that $G(L_w) = G(L_{w'})$. Then $G(L_w) \circ G(L_{w^{-1}}) = G(L_{w'}) \circ G(L_{w^{-1}})$. Since $Hom(G(L_w) \circ G(L_{w^{-1}}), id) \neq 0$ we have that $Hom(G(L_{w'}) \circ G(L_{w^{-1}}), id) \neq 0$ and by the Proposition 5.3 $w' = (w^{-1})^{-1} = w$.

Since we proved (i) (ii) and (iii) the Theorem 4 is proved. □

5.5. Examples

The Corollary 5.2 can be applied in the following cases:

(a) Let G be a simply connected group and let $\mathbf{c}$ be the lowest cell. In this case $u_{\mathbf{c}} = e \in G$ and $F_{\mathbf{c}} = G$. In this case Corollary 5.2 is a result of [16].

(b) Let $G = GL_n$. In this case all groups $F_{\mathbf{c}}$ are connected and have no nontrivial projective representations (these groups are products of various GL_m). In this case we get a result of [18].

(c) Let G be a simply connected group of rank 2. In this case one easily verifies that the condition of Corollary 5.2 is satisfied and we get a result of [17].

(d) Let G be a simple simply connected group. Let $\mathbf{c}$ be the subregular cell, that is the cell corresponding to the subregular nilpotent orbit. Again one easily verifies that the condition of Corollary 5.2 is satisfied except if G is of type C_n. In the latter case $F_{\mathbf{c}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where one of the factors comes from the center of G. One can exlude centrally extended points in this case by considering a reductive group $G_1 = G \times T/(z, -1)$ where T is the one dimensional torus, $z \in G$ is the unique nontrivial central element and $-1 \in T$ is the unique nontrivial involutive element. So we get another result of [17].

Finally note that centrally extended points naturally appear in the description of truncated convolution categories for simple non simply-connected groups, see in [18] 8.3 example with $G = PSL_2$.

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