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ON TENSOR PRODUCT GRAPHS

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Abstract

The tensor product $G \oplus H$ of graphs G and H is the graph with point set $V(G) \times V(H)$ where (u_1, v_1) adj (u_2, v_2) if, and only if, u_1 adj u_2 and v_1 adj v_2. We obtain a characterization of graphs of the form $G \oplus H$ where G or H is K_2.

1. Notation and preliminary results

As usual, let K_p denote the complete graph on p points, C_n a cycle of length n. For a connected graph G, nG is the graph with n components each being isomorphic to G.

REMARK. The tensor product $G \oplus H$ is also called *conjunction* (Harary (1969)), and *Kronecker product* (Weichsel 1963)).

Weichsel (1963) has proved Theorem 1 and Corollary 1.1.

THEOREM 1. (Weichsel). *For connected graphs G and H the product $G \oplus H$ is connected if and only if either G or H contains an odd cycle.*

COROLLARY 1.1. (Weichsel). *If G and H are connected graphs with no odd cycles then $G \oplus H$ has exactly two components.*

We now prove

COROLLARY 1.2. *For a connected graph G with no odd cycles, $G \oplus K_2 = 2G$.*

PROOF. Let $\{a_i\}$ be the point set of G and K_2 be the line $b_1 b_2$. By Corollary 1.1, $G \oplus K_2$ has exactly two components, say G_1 and G_2. If for some point a_{i_0} in G, $(a_{i_0}, b_1) \in G_1$ then $(a_{i_0}, b_2) \in G_2$. For, if there is a path $(a_{i_0}, b_1)(a_{i_1}, b_2) \cdots (a_{i_k}, b_1)(a_{i_0}, b_2)$ then k is even and G has the odd cycle $a_{i_0} a_{i_1} \cdots a_{i_k} a_{i_0}$, provided the points a_{i_r}, $r = 0, 1, \cdots, k$ are all distinct. Suppose $a_{i_r} = a_{i_s}$ for some r and s, and $r < s$. Let s be the smallest such integer. Clearly, if r is even (odd) then s is odd (even) and G has an odd cycle $a_{i_r} a_{i_{r+1}} \cdots a_{i_{s-1}} a_{i_r}$. Now, the function $f: G \to G_1$ defined by

$$f(a_i) = (a_i, b_1) \quad \text{if} \quad (a_i, b_1) \in G_1$$
$$= (a_i, b_2) \quad \text{if} \quad (a_i, b_1) \notin G_1$$

is clearly an isomorphism. Similarly, $G \cong G_2$.

THEOREM 2. *For a graph* G, $G = H \otimes K_2$ *if and only if the following conditions* I–IV *are true.*

I. *G has an even number of points and lines.*

II. *G has no odd cycles.*

III. *If G is connected, the following should hold.*

(a) *G has a cycle, say*

$C_{2n}: x_1 x_2 \cdots x_{2n} x_1$ *where $n > 1$ is odd.*

(b) *Let G_1 be the graph obtained from G by removing all lines of C_{2n}. The components of G_1 should be of the following two types only.*

Type I. *Components E such that for $1 \leqq r \leqq n$ the point x_r of C_{2n} belongs to E if and only if the point x_{n+r} belongs to a component E' ($\neq E$) isomorphic to E.*

Type II. Components F such that for $1 \leqq r \leqq n$ the point x_r of C_{2n} belongs to F if and only if $x_{n+r} \in F$. Further, F should satisfy I, II and III. That is, F should have an even number of points and lines, and should contain a cycle C_{2m} where $m > 1$ is odd, etc.

IV. *Suppose G is disconnected. Then the components of G which are not of the form mentioned in* III *should be in isomorphic pairs.*

PROOF. Let $G = H \otimes K_2$, $V(H) = \{a_i\}$ and K_2 be the line $b_1 b_2$. Clearly condition I is true. We observe that points of G are labeled alternately with the elements of the sets $V_1 = \{a_i\} \times \{b_1\}$ and $V_2 = \{a_i\} \times \{b_2\}$. This will not be possible if G has odd cycles, and thus II is true.

Let G be connected. Then by Theorem 1, H is connected and contains an odd cycle say, $C_n = a_1 a_2, \cdots, a_n a_1$. This implies that G contains the cycle,

$$C_{2n} = (a_1, b_1)(a_2, b_2)(a_3, b_1) \cdots (a_n, b_1)(a_1, b_2) \cdots (a_n, b_2)(a_1, b_1).$$

Consider the graph G_1 obtained from G by removing all lines of C_{2n}. In a relabeling of the points of C_{2n} as $x_1 x_2 \cdots x_{2n} x_1$ we find that if (a_r, b_1) is labeled as x_r then (a_r, b_2) is labeled as x_{n+r} (we make use of this fact in our proof without any further reference). Let E be a component of G_1 and $x_r = (a_r, b_1)$ be a point of C_{2n} such that $x_r \in E$ and $x_{n+r} = (a_r, b_2)$ belongs to a different component E' of G_1. We now show that (i) a point (a_i, b_1) (not necessarily on C_{2n}) belongs to E if and only if (a_i, b_2) belongs to E', and (ii) $E \cong E'$.

Let $(a_i, b_1) \in E$. Then in E there is a path

$$(a_i, b_1)(a_{i_1}, b_2)(a_{i_2}, b_1) \cdots (a_{i_k}, b_2)(a_r, b_1).$$

This implies that there is a path

$$(a_i, b_2)(a_{i_1}, b_1)(a_{i_2}, b_2) \cdots (a_{i_k}, b_1)(a_r, b_2)$$

and hence $(a_i, b_2) \in E'$, since $(a_r, b_2) \in E'$. Likewise, $(a_j, b_1) \in E'$ implies $(a_j, b_2) \in E$. Clearly, the function $f: E \to E'$ defined by

$$f(a_i, b_1) = (a_i, b_2) \quad \text{if} \quad (a_i, b_1) \in E$$

$$f(a_i, b_2) = (a_i, b_1) \quad \text{if} \quad (a_i, b_2) \in E$$

is an isomorphism.

Suppose now F is a component of G_1 and $x_r = (a_r, b_1)$ is a point of C_{2n} such that both x_r and $x_{n+r} = (a_r, b_2)$ belong to F. We show that in general, a point (a_i, b_1) (not necessarily on C_{2n}) belongs to F if and only if $(a_i, b_2) \in F$. Let $(a_i, b_1) \in F$. Then there is a path

$$(a_r, b_1)(a_{i_1}, b_2) \cdots (a_{i_k}, b_2)(a_i, b_1).$$

This implies that there is a path

$$(a_r, b_2)(a_{i_1}, b_1) \cdots (a_{i_k}, b_1)(a_i, b_2)$$

and since $(a_r, b_2) \in F$, $(a_i, b_2) \in F$. Likewise, $(a_j, b_2) \in F$ implies $(a_j, b_1) \in F$. It follows now that $F = H_1 \oplus K_2$ where H_1 is the subgraph of H induced by the points $V(H_1) = \{a_i : (a_i, b_1) \in F\}$, and hence F should satisfy I, II and III. This proves that G_1 can have only two types of components as mentioned in III. If G is disconnected, a component G_i of G which is not of the form mentioned in III corresponds to a component H_i of H which does not have an odd cycle and so $H_i \oplus K_2 = 2H_i = 2G_i$ by Corollary 1.2. Thus G should have such components in pairs and such components are even in number.

Conversely, let the conditions I–IV hold good for a graph G having $2m$ points. Let $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2\}$. We label the points of G with $2m$ elements of $A \times B$ such that

(i) no two elements of $A \times \{b_1\}$ or $A \times \{b_2\}$ are adjacent, and
(ii) the function $f: G \to G$ defined by

$$f(a_i, b_1) = (a_i, b_2)$$

$$f(a_i, b_2) = (a_i, b_1)$$

is an isomorphism. The given conditions I–IV ensure that this is indeed possible. Suppose now G is connected. By hypothesis it contains a cycle C_{2n} where n is odd. We label the points of this cycle successively with the $2n$ elements

$$(a_1, b_1)(a_2, b_2)(a_3, b_1) \cdots (a_n, b_1)(a_1, b_2) \cdots (a_n, b_2).$$

The above isomorphism takes the cycle into itself and the subgraph G_1 described

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in III into itself. In case G_1 is disconnected the isomorphism maps a component of type I into an isomorphic component of the same type, while a component of type II is mapped onto itself. The labeling of the points of G is illustrated in Figure 1, where for convenience we write i for a_i and i' for b_i. In this labeling we observe that (a_i, b_1) is adjacent to (a_j, b_2) if and only if (a_i, b_2) is adjacent to (a_j, b_1). Now, consider the graph H constructed on the points set A as follows. In H, a_i adj a_j if and only if (a_i, b_1) adj (a_j, b_2). It follows now that $G = H \oplus K_2$. The graph G in Figure 1 is the tensor product of H and K_2 in Figure 3. If G is disconnected and if a component G_i of G is of the form mentioned in III, then $G_i = H_i \oplus K_2$ for some graph H_i as above. If a component G_j of G is not of this form then by hypothesis they are in isomorphic pairs and $G_j \oplus K_2 = 2G_j$ by Corollary 1.2. This completes the proof of the Theorem.

G:

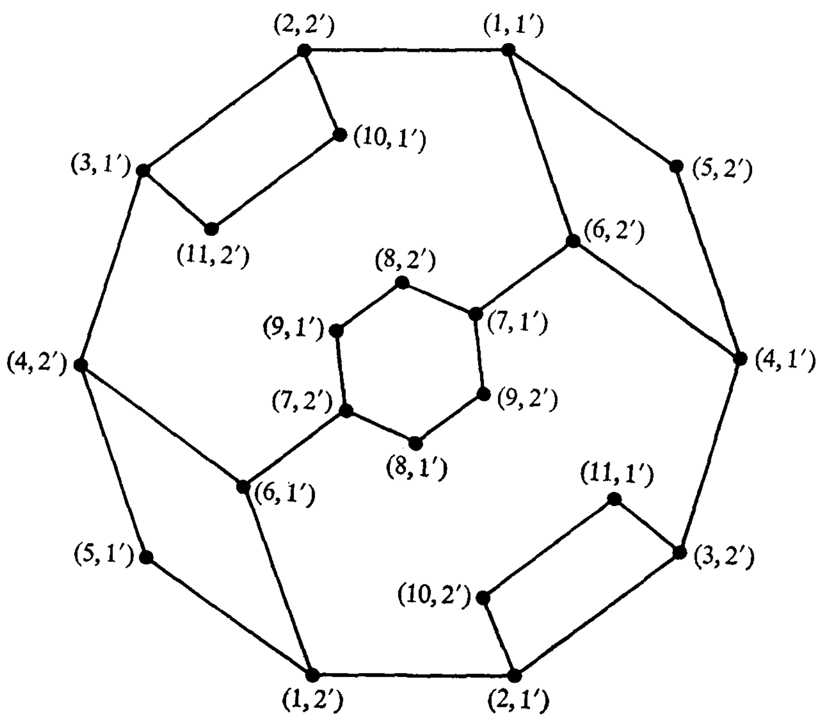

Fig. 1

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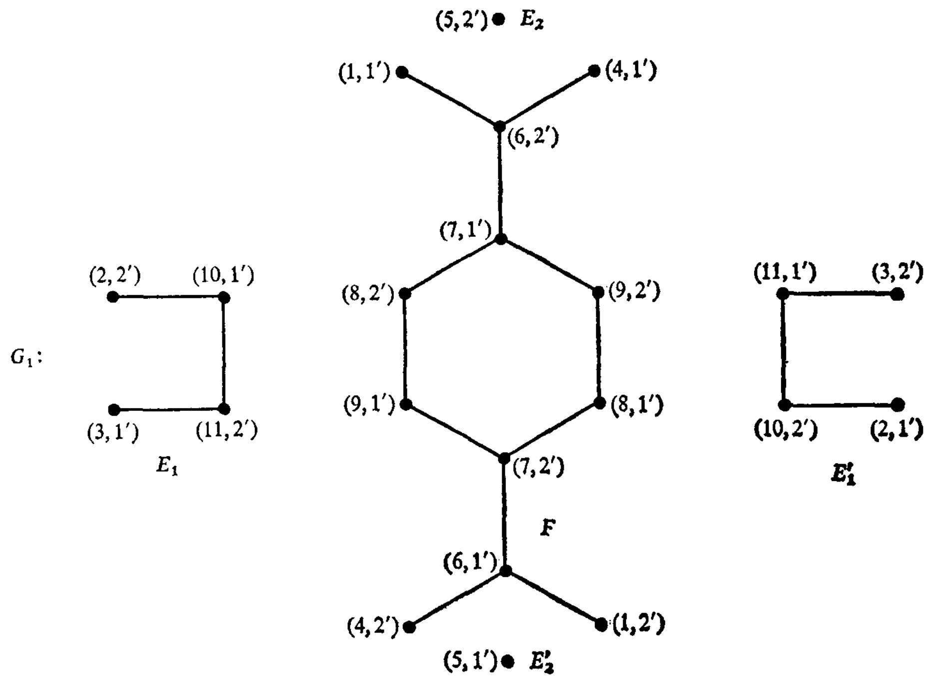

Fig. 2

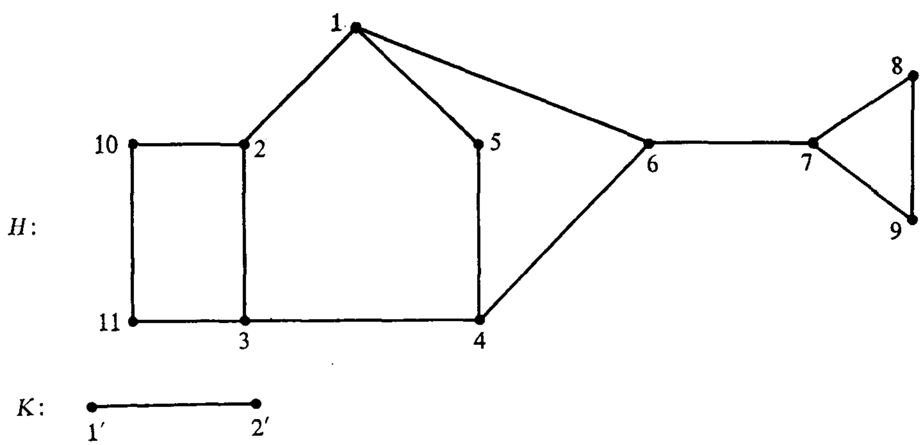

Fig. 3

For any two graphs G_1 and G_2 we have

$$G_1 \oplus G_2 = G_1 \oplus \bigcup_i e_i = \bigcup_i G_1 \oplus e_i$$

where $\{e_i\}$ is the set of lines of G_2. Each product graph $G_1 \oplus e_i$ is isomorphic to $G_1 \times K_2$. Also, the graphs $G_1 \oplus e_i$ are line disjoint and the number of common

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points (if any) between any two of these graphs is equal to that in G_1. These observations lead to

THEOREM 3. *A necessary condition for a graph G to be the tensor product of two graphs is that G is the line disjoint union of a number of graphs of the form $H \oplus K_2$ for some graph H, and the number of common points (if any) between any two of these graphs must be that in H.*

For example, the graph G in Figure 4 is the line disjoint union of $K_{1,3} \otimes K_2$ and $K_{1,3} \oplus K_2$ as illustrated. The number of common points between these two graphs is that in $K_{1,3}$, namely 4 and $G = K_{1,3} \otimes P_3$, where P_3 is a path of length 2.

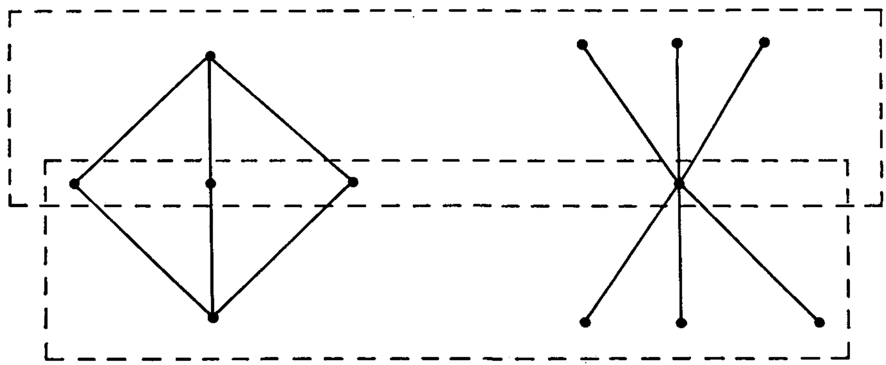

Fig. 4

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