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ON THE 2k-TH POWER MEAN VALUE OF THE GENERALIZED QUADRATIC GAUSS SUMS

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Abstract. The main purpose of this paper is using the elementary and analytic methods to study the properties of the $2k$-th power mean value of the generalized quadratic Gauss sums, and give two exact mean value formulae for $k = 3$ and 4.

1. Introduction

Let $q \geq 2$ be an integer, χ denotes a Dirichlet character modulo q. For any integer n, we define the generalized quadratic Gauss sums $G(n, \chi; q)$ as follows:

$$G(n, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{na^2}{q}\right),$$

where $e(y) = e^{2\pi i y}$. This sum is important, because it is a generalization of the classical quadratic Gauss sums $G(n, q)$, which is defined by

$$G(n; q) = \sum_{a=1}^{q} e\left(\frac{na^2}{q}\right).$$

About the properties of $G(n, \chi; q)$, some authors had studied it, and obtained many interesting results. For example, for any integer n with $(n, q) = 1$, from the general result of Cochrane and Zheng [2] we can deduce that

$$|G(n, \chi; q)| \leq 2^{\omega(q)} q^{\frac{1}{2}},$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q. The case where q is a prime is due to Weil [4]. Zhang [5] proved that for any odd prime p and integer n with $(n, p) = 1$, we have

$$\sum_{\chi \bmod p} |G(n, \chi; p)|^4 = \begin{cases} (p-1)[3p^2 - 6p - 1 + 4(\frac{n}{p})\sqrt{p}], & \text{if } p \equiv 1 \bmod 4; \\ (p-1)(3p^2 - 6p - 1), & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

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and

$$\sum_{\chi \bmod p} |G(n,\chi;p)|^6 = (p-1)(10p^3 - 25p^2 - 4p - 1), \text{ if } p \equiv 3 \bmod 4,$$

where $(\frac{n}{p})$ is the Legendre symbol.

Besides, W. Zhang and H. Liu [6] also proved the following conclusion:

Let $q \geq 3$ be a square-full number. Then for any integers $n,\ k$ with $(nk,\ q) = 1$ and $k \geq 1$, we have the identity

$$\sum_{\chi \bmod q} |G(n,\chi;q)|^4 = q \cdot \phi^2(q) \prod_{p|q} (k,\ p-1)^2 \prod_{\substack{p|q \\ (k,\ p-1)=1}} \frac{\phi(p-1)}{p-1},$$

where $\prod_{p|q}$ denotes the product over all prime divisors of q, $\phi(q)$ is the Euler function.

In this paper, we use the elementary and analytic methods to study the calculating problem of the $2k$-th power mean value of the generalized quadratic Gauss sums, and give two exact calculating formulae for $k = 3$ and 4. That is, we shall prove the following:

Theorem 1. *Let odd number $q > 1$ be a square-full number (i.e., for any prime p, $p \mid q$ if and only if $p^2 \mid q$). Then for any integer n with $(n,q) = 1$, we have the identity*

$$\sum_{\chi \bmod q} |G(n,\chi;q)|^6 = 16^{\omega(q)} \cdot q^2 \cdot \phi^2(q),$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

Theorem 2. *Let odd number $q > 1$ be a square-full number. Then for any integer n with $(n,q) = 1$, we have*

$$\sum_{\chi \bmod q} |G(n,\chi;q)|^8 = 64^{\omega(q)} \cdot q^3 \cdot \phi^2(q).$$

From our theorems we know that the estimates in reference [2] is the best one. In fact from Theorem 2 we know that there exists at least one Dirichlet character modulo q such that the inequality:

$$| G(n,\chi;q) | \geq 2^{\frac{3}{4}\omega(q)} q^{\frac{3}{8}} \phi^{\frac{1}{8}}(q).$$

For general integer $k \geq 5$, we believe that the following conclusion is correct:

Conjecture. *Let odd number $q > 1$ be a square-full number, $k \geq 2$ be an integer. Then for any integer n with $(n,q) = 1$, we have the identity*

$$\sum_{\chi \bmod q} |G(n,\chi;q)|^{2k} = 4^{(k-1)\omega(q)} \cdot q^{k-1} \cdot \phi^2(q).$$

The proposed method is supposed to be capable of proving this formula. However, the calculation will be so complex when $k \geq 5$ that such a general conclusion cannot be obtained. For general positive integer $q > 3$, it is an open problem whether there is a formula to calculate the $2k$-th power mean value of the generalized quadratic Gauss sums.

2. Several lemmas

To complete the proof of our theorems, we need the following several lemmas.

Lemma 1. *For any integer $q \geq 1$, we have the identity*

$$G(1;q) = \frac{1}{2}\sqrt{q}(1+i)\left(1+e^{\frac{-\pi i q}{2}}\right) = \begin{cases} \sqrt{q}, & \text{if } q \equiv 1 \pmod 4; \\ 0, & \text{if } q \equiv 2 \pmod 4; \\ i\sqrt{q}, & \text{if } q \equiv 3 \pmod 4; \\ (1+i)\sqrt{q}, & \text{if } q \equiv 0 \pmod 4. \end{cases}$$

Proof. This is a remarkable formula of Gauss. See Theorem 9.16 of [1]. □

Lemma 2. *Let p be an odd prime and $\alpha \geq 2$ be an integer. Then for any integer n with $(p,n) = 1$, we have the identity*

$$\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{nb^2}{p^\alpha}\right) = 0.$$

Proof. First we know that for any positive integers $q \geq 2$ and integer n with $(n,q) = 1$, we have the identity

$$\sum_{u=0}^{q-1} e\left(\frac{un}{q}\right) = 0.$$

From this identity and the properties of reduce residue system we have

$$\sideset{}{'}\sum_{b=1}^{p^\alpha} e\left(\frac{nb^2}{p^\alpha}\right) = \sum_{u=0}^{p-1}\sideset{}{'}\sum_{v=1}^{p^{\alpha-1}} e\left(\frac{n(up^{\alpha-1}+v)^2}{p^\alpha}\right) = \sum_{u=0}^{p-1}\sideset{}{'}\sum_{v=1}^{p^{\alpha-1}} e\left(\frac{2nuvp^{\alpha-1}+v^2}{p^\alpha}\right)$$

$$= \sideset{}{'}\sum_{v=1}^{p^{\alpha-1}} e\left(\frac{v^2}{p^\alpha}\right)\sum_{u=0}^{p-1} e\left(\frac{2nuv}{p}\right) = 0.$$

This proves Lemma 2. □

Lemma 3. *Let m, $n \geq 2$ and u be three integers with $(m,n) = 1$ and $(u,mn) = 1$. Then for any character $\chi = \chi_1\chi_2$ with χ_1 mod m and χ_2 mod n, we have the identity*

$$G(u,\chi;mn) = \chi_1(n)\chi_2(m)G(un,\chi_1;m)G(um,\chi_2;n).$$

Proof. See Lemma 6 of [6]. □

Lemma 4. *Let p be an odd prime, $\alpha \geq 2$ and n be two integers with $(n,p) = 1$. Then we have*

$$\sum_{\chi \bmod p^\alpha} |G(n,\chi;p^\alpha)|^6 = 16\phi^2\left(p^\alpha\right)p^{2\alpha}.$$

Proof. From the definition of $G(n,\chi;p^\alpha)$ we have

$$|G(n,\chi;p^\alpha)|^2 = \sum_{a=1}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' \chi(a)\overline{\chi}(b) e\left(\frac{n(a^2-b^2)}{p^\alpha}\right) = \sum_{a=1}^{p^\alpha}{}' \chi(a) \sum_{b=1}^{p^\alpha}{}' e\left(\frac{nb^2(a^2-1)}{p^\alpha}\right).$$

Then by this formula and the orthogonality relation for character sums modulo p^α we may get

$$\begin{aligned} &\sum_{\chi \bmod p^\alpha} |G(n,\chi;p^\alpha)|^6 \\ = \ & \phi(p^\alpha) \sum_{a=1}^{p^\alpha}{}' \sum_{b=1}^{p^\alpha}{}' \sum_{\substack{c=1 \\ abc\equiv 1 \bmod p^\alpha}}^{p^\alpha}{}' \left(\sum_{u=1}^{p^\alpha}{}' e\left(\frac{nu^2(a^2-1)}{p^\alpha}\right)\right) \\ &\times \left(\sum_{v=1}^{p^\alpha}{}' e\left(\frac{nv^2(b^2-1)}{p^\alpha}\right)\right) \left(\sum_{w=1}^{p^\alpha}{}' e\left(\frac{nw^2(c^2-1)}{p^\alpha}\right)\right). \end{aligned} \tag{1}$$

Let $(a^2-1,\ p^\alpha) = p^m$. If $m \le \alpha - 2$, note that $(n(a^2-1)/p^m,\ p) = 1$, then from Lemma 2 we have

$$\sum_{u=1}^{p^\alpha}{}' e\left(\frac{nu^2(a^2-1)}{p^\alpha}\right) = p^m \sum_{u=1}^{p^{\alpha-m}}{}' e\left(\frac{nu^2(a^2-1)/p^m}{p^{\alpha-m}}\right) = 0.$$

If $m = \alpha$, then

$$\sum_{u=1}^{p^\alpha}{}' e\left(\frac{nu^2(a^2-1)}{p^\alpha}\right) = \phi(p^\alpha).$$

If $m = \alpha - 1$, then $a = rp^{\alpha-1} \pm 1$, $1 \le r \le p-1$. Note that for any prime p with $p \nmid n$, by Theorem 7.5.4 of [3] we have

$$G(n;p) = \left(\frac{n}{p}\right) G(1;p). \tag{2}$$

Then from (2) and Lemma 1 we may get

$$\begin{aligned} \sum_{u=1}^{p^\alpha}{}' e\left(\frac{nu^2(a^2-1)}{p^\alpha}\right) &= p^{\alpha-1} \sum_{u=1}^{p}{}' e\left(\frac{nu^2(a^2-1)/p^{\alpha-1}}{p}\right) \\ &= p^{\alpha-1}\left[\left(\frac{\pm 2rn}{p}\right) G(1;p) - 1\right]. \end{aligned}$$

Note that the number of the solutions of the congruent equation $1 \le a,\ b,\ c \le p^\alpha - 1$ with $p^\alpha \mid a^2-1$, $p^\alpha \mid b^2-1$, $p^\alpha \mid c^2-1$ and $abc \equiv 1 \bmod p^\alpha$ are 4, the number of the solutions of the congruent equation $1 \le a,\ b,\ c \le p^\alpha - 1$ with $p^\alpha \mid a^2-1$, $p^\alpha \mid b^2-1$, $p^{\alpha-1} \parallel c^2-1$ and $abc \equiv 1 \bmod p^\alpha$ are 0, and

$$\sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|a^2-1}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ p^{\alpha-1}\|b^2-1\\ abc\equiv 1 \bmod p^\alpha}}^{p^\alpha} \sideset{}{'}\sum_{\substack{c=1\\ p^{\alpha-1}\|c^2-1}}^{p^\alpha} \left(\sideset{}{'}\sum_{u=1}^{p^\alpha} e\left(\frac{nu^2(a^2-1)}{p^\alpha}\right)\right)$$

$$\times \left(\sideset{}{'}\sum_{v=1}^{p^\alpha} e\left(\frac{nv^2(b^2-1)}{p^\alpha}\right)\right)\left(\sideset{}{'}\sum_{w=1}^{p^\alpha} e\left(\frac{nw^2(c^2-1)}{p^\alpha}\right)\right)$$

$$= \phi\left(p^\alpha\right) \sideset{}{'}\sum_{\substack{a=1\\ p^\alpha|a^2-1}}^{p^\alpha} \sideset{}{'}\sum_{\substack{b=1\\ p^{\alpha-1}\|b^2-1\\ abc\equiv 1 \bmod p^\alpha}}^{p^\alpha} \sideset{}{'}\sum_{\substack{c=1\\ p^{\alpha-1}\|c^2-1}}^{p^\alpha} \left(\sideset{}{'}\sum_{v=1}^{p^\alpha} e\left(\frac{nv^2(b^2-1)}{p^\alpha}\right)\right)\left(\sideset{}{'}\sum_{w=1}^{p^\alpha} e\left(\frac{nw^2(c^2-1)}{p^\alpha}\right)\right)$$

$$= 4\phi\left(p^\alpha\right)p^{2(\alpha-1)}\sum_{r=1}^{p-1}\left[\left(\frac{-1}{p}\right)G^2(1;p)+1\right]$$

$$-\,4\phi\left(p^\alpha\right)p^{2(\alpha-1)}\sum_{r=1}^{p-1}\left[\left(\frac{-2rn}{p}\right)+\left(\frac{2rn}{p}\right)\right]G(1;p)$$

$$= 4\phi\left(p^\alpha\right)p^{2(\alpha-1)}(p^2-1).$$

So combining the above several cases and (1) we have

$$\sum_{\chi \bmod p^\alpha} \left|G(n,\chi;p^\alpha)\right|^6$$

$$= 4\phi^4\left(p^\alpha\right) + 12\phi^2\left(p^\alpha\right)p^{2(\alpha-1)}(p^2-1)$$

$$+\,\phi\left(p^\alpha\right)p^{3(\alpha-1)} \sum_{\substack{r=1\\ (rp^{\alpha-1}\pm1)(sp^{\alpha-1}\pm1)(tp^{\alpha-1}\pm1)\equiv 1 \bmod p^\alpha}}^{p-1}\ \sum_{s=1}^{p-1}\ \sum_{t=1}^{p-1} \left[\left(\frac{\pm 2rn}{p}\right)G(1;p)-1\right]$$

$$\times\left[\left(\frac{\pm 2sn}{p}\right)G(1;p)-1\right]\left[\left(\frac{\pm 2tn}{p}\right)G(1;p)-1\right]$$

$$= 4\phi^4\left(p^\alpha\right) + 12\phi^2\left(p^\alpha\right)p^{2(\alpha-1)}(p^2-1)$$

$$+\,\phi\left(p^\alpha\right)p^{3(\alpha-1)} \sum_{\substack{r=1\\ r+s+t\equiv 0 \bmod p}}^{p-1}\ \sum_{s=1}^{p-1}\ \sum_{t=1}^{p-1} \left[\left(\frac{2rn}{p}\right)G(1;p)-1\right]$$

$$\times\left[\left(\frac{2sn}{p}\right)G(1;p)-1\right]\left[\left(\frac{2tn}{p}\right)G(1;p)-1\right]$$

$$+\,3\phi\left(p^\alpha\right)p^{3(\alpha-1)} \sum_{\substack{r=1\\ s+t\equiv r \bmod p}}^{p-1}\ \sum_{s=1}^{p-1}\ \sum_{t=1}^{p-1} \left[\left(\frac{2rn}{p}\right)G(1;p)-1\right]$$

$$\times\left[\left(\frac{-2sn}{p}\right)G(1;p)-1\right]\left[\left(\frac{-2tn}{p}\right)G(1;p)-1\right]$$

$$= 4\phi^4\left(p^\alpha\right)+12\phi^2\left(p^\alpha\right)p^{2(\alpha-1)}(p^2-1)$$

$$+4\phi\left(p^\alpha\right)p^{3(\alpha-1)}\sum_{\substack{r=1\\ r+s+t\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\left[\left(\frac{2rn}{p}\right)G(1;p)-1\right]$$

$$\times\left[\left(\frac{2sn}{p}\right)G(1;p)-1\right]\left[\left(\frac{2tn}{p}\right)G(1;p)-1\right]. \tag{3}$$

From the properties of the Legendre symbol (see reference [1]) we know that

$$\sum_{\substack{r=1\\ r+s\pm t\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\left(\frac{r}{p}\right)\left(\frac{s}{p}\right)\left(\frac{t}{p}\right)=\sum_{r=1}^{p-1}\sum_{s=1}^{p-1}\left(\frac{\mp rs(r+s)}{p}\right)$$

$$= \sum_{r=1}^{p-1}\sum_{s=1}^{p-1}\left(\frac{\mp rs^2(rs+s)}{p}\right)=\sum_{r=1}^{p-1}\left(\frac{\mp r(r+1)}{p}\right)\sum_{s=1}^{p-1}\left(\frac{s}{p}\right)=0, \tag{4}$$

$$\sum_{\substack{r=1\\ r+s\pm t\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\left(\frac{r}{p}\right)\left(\frac{s}{p}\right)=\sum_{r=1}^{p-1}\sum_{s=1}^{p-1}\left(\frac{rs}{p}\right)-\sum_{\substack{r=1\\ r+s\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\left(\frac{rs}{p}\right)$$

$$= -\sum_{r=1}^{p-1}\left(\frac{-r^2}{p}\right)=-\left(\frac{-1}{p}\right)(p-1), \tag{5}$$

$$\sum_{\substack{r=1\\ r+s\pm t\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}\left(\frac{r}{p}\right)=\sum_{r=1}^{p-1}\sum_{s=1}^{p-1}\left(\frac{r}{p}\right)-\sum_{\substack{r=1\\ r+s\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\left(\frac{r}{p}\right)=0, \tag{6}$$

$$\sum_{\substack{r=1\\ r+s\pm t\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}\sum_{t=1}^{p-1}1=\sum_{r=1}^{p-1}\sum_{s=1}^{p-1}1-\sum_{\substack{r=1\\ r+s\equiv 0 \bmod p}}^{p-1}\sum_{s=1}^{p-1}=(p-1)(p-2). \tag{7}$$

Note that $(\frac{-1}{p})G^2(1;p)=p$, from (3), (4), (5), (6) and (7) we may get

$$\sum_{\chi \bmod p^\alpha}|G(n,\chi;p^\alpha)|^6 = 4\phi^4\left(p^\alpha\right)+12\phi^2\left(p^\alpha\right)p^{2(\alpha-1)}(p^2-1)$$

$$+12\phi^2\left(p^\alpha\right)p^{2\alpha-1}-4\phi^2\left(p^\alpha\right)p^{2(\alpha-1)}(p-2)$$

$$= 16\phi^2\left(p^\alpha\right)p^{2\alpha}.$$

This proves Lemma 4. □

3. Proof of theorems

In this section, we shall complete the proof of our theorems. We only prove Theorem 1. Similarly, we can deduce Theorem 2. In fact if q is an odd square-full number, let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of q into prime powers, then $\alpha_i \geq 2$, $i = 1, 2, \ldots, k$. Then for any integer n with $(n, q) = 1$, from Lemma 3, Lemma 4 and the properties of Dirichlet character we have

$$\begin{aligned} \sum_{\chi \bmod q} |G(n, \chi; q)|^6 &= \prod_{p^\alpha \| q} \left[\sum_{\chi \bmod p^\alpha} |G(nq/p^\alpha, \chi; p^\alpha)|^6 \right] \\ &= \prod_{p^\alpha \| q} \left[8\phi^2 \left(p^\alpha\right) p^{2\alpha} \right] \\ &= 8^{\omega(q)} \cdot q^2 \cdot \phi^2 \left(q\right), \end{aligned}$$

where $\prod_{p^\alpha \| q}$ denotes that $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

This completes the proof of Theorem 1.

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